Letter

Certifying beyond quantumness of locally quantum no-signaling theories through a quantum-input Bell test

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Physical theories constrained with local quantum structure and satisfying the no-signaling principle can allow beyond-quantum global states. In a standard Bell experiment, correlations obtained from any such beyond-quantum bipartite state can always be reproduced by quantum states and measurements, suggesting the local quantum structure and no-signaling to be the axioms to isolate quantum correlations. In this Letter, however, we show that if the Bell experiment is generalized to allow local quantum inputs, then beyond-quantum correlations can be generated by every beyond-quantum state. This gives us a way to certify the beyond quantumness of locally quantum no-signaling theories and in turn suggests the requirement of additional information principles along with the local quantum structure and no-signaling principle to isolate quantum correlations. More importantly, our work establishes that the additional principle(s) must be sensitive to the quantum signature of local inputs. We also generalize our results to multipartite locally quantum no-signaling theories and further analyze some interesting implications.

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Introduction. Correlations among distant events established through the violation of Bell-type inequalities confirm the nonlocal behavior of the physical world [1–4]. Nonseparable multipartite quantum states yielding such correlations, in Schrödinger's words, are "characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought" [5]. The advent of quantum information science identifies the power of such nonlocal correlations in numerous device-independent protocols—cryptographic key distribution [6–8], randomness certification [9] and amplification [10], and dimension witness [11–13] are a few canonical examples. Cirel'son's result [14], however, establishes that the nonlocal strength of quantum correlations is limited compared to the general *no-signaling* (NS) ones [15] as depicted in the celebrated Clauser-Horne-Shimony-Holt (CHSH) inequality violation [16].

To comprehend the limited nonlocal behavior of quantum theory and to obtain a better understanding of the theory itself, researchers have proposed several approaches to compare and contrast quantum theory with other conceivable physical theories constructed within more general mathematical frameworks [17–27]. Here, we consider a class of theories wherein local measurements are described quantum mechanically, but they allow a global structure more generic than quantum theory [28–35]. Gleason-Busch's celebrated result in quantum foundations proves that any map from generalized measurements to probability distributions can be written as the trace rule with the appropriate quantum state [36,37] (see also Ref. [38]). This theorem, when appraised to the case of local observables acting on multipartite systems, hence called the unentangled Gleason's theorem, endorses the joint NS probability distributions to be obtained from some Hermitian operator called the positive over all pure tensors (POPT) state [28–31]. Although the set of POPT states is strictly larger than the set of quantum states (density operators), in a recent work, Barnum *et al.* have shown that the set of bipartite correlations attainable from the POPT states is precisely the set of quantum correlations [32]. Consequently, their result provokes a far-reaching conclusion "if nonlocal correlations beyond quantum mechanics are obtained in any experiment then quantum theory would be invalidated even locally."

In this Letter, we analyze the correlations of multipartite POPT states obtained from local measurements performed on their constituent parts by considering a generalized Bell scenario as introduced in Ref. [39]. While in the standard Bell scenario spatially separated parties receive some classical inputs and accordingly generate some classical outputs by performing local measurements on their respective parts of some composite system, recently Buscemi has generalized the scenario where the parties receive quantum inputs instead of classical variables [39]. In this generalized scenario he has shown that all entangled states exhibit nonlocality, despite some of them allowing a local-hidden-variable (LHV) model in the classical input scenario [40–42]. Considering this generalized scenario, here we show that not all correlations obtained from bipartite POPT states are quantum simulable. In fact, every beyond-quantum POPT state produces some beyond-quantum correlation in some quantum-input game. On the other hand, to illustrate the limitations of the standard Bell scenario, we show that there are POPT states which produce classical-input–classical-output correlations that are not only quantum simulable, rather simulable classically. Our result shows that the *strong* claim made by the authors in Ref. [32] will no longer be correct in this generalized Bell scenario which, as we will show, is allowed within the framework of local quantum theory. From a foundational perspective our study welcomes new information principles incorporating this generalized Bell-type scenario to isolate quantum correlations from beyond-quantum ones. We also analyze the implication of this generalized scenario for multipartite correlations and answer an open question raised in Ref. [33].

Gleason's theorem. We investigate the class of locally quantum theories studied in a series of works in the recent past [28–35]. In accordance with these works, we say that Alice is *locally quantum* if her physical system is described by a Hilbert space $\mathcal{H}_A$ with dimension d_A and her measurements M_A are given by a collection of effects corresponding to positive-operator-valued measurement (POVM) [43] operators $\{\pi_A^a\}_a$ acting on $\mathcal{H}_A$ and satisfying the constraint $\sum_a \pi_A^a = \mathbb{I}_A$, where $\forall\, a, \pi_A^a \in \mathcal{E}(\mathcal{H}_A) \subset \mathcal{L}(\mathcal{H}_A)$, with $\mathcal{E}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_A)$ respectively denoting the set of all positive operators and bounded linear operators acting on $\mathcal{H}_A$, and $\mathbb{I}_A$ is the identity operator on $\mathcal{H}_A$. The probability $p(a|M_A)$ that Alice obtains an outcome a for measurement $M_A \equiv \{\pi_A^a\}_a$ is given by a generalized probability measure $\mu : \mathcal{E}(\mathcal{H}_A) \mapsto [0, 1]$, satisfying the properties (i) $\forall\, \pi_A^a \in \mathcal{E}(\mathcal{H}_A), 0 \leqslant \mu(\pi_A^a) \leqslant 1$, (ii) $\mu(\mathbb{I}_A) = 1$, and (iii) $\mu(\sum_i \pi_A^i) = \sum_i \mu(\pi_A^i)$ for any sequence $\pi_A^1, \pi_A^2, \ldots$ with $\sum_i \pi_A^i \leqslant \mathbb{I}_A$. Each probability measure μ corresponds to a "state" in the local quantum theory. We can make the association with the familiar quantum theory in which states are described by density operators by invoking the Gleason-Busch theorem according to which any such generalized probability measure is given by a linear functional of the form $\mu(\pi_A^a) = \mathrm{Tr}(\rho_A \pi_A^a)$, for some density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$; $\mathcal{D}(\mathcal{H}_A)$ denotes the set of positive operators with unit trace on $\mathcal{H}_A$.

Interesting situations arise when the theorem is generalized to the case of local observables acting on multipartite systems. Each party is assumed to be locally quantum as described above, with the ith party performing the measurement $M_{A_i} \equiv \{\pi_{A_i}^a\}_a$. The "state" is now given by a probability measure $\mu : \times_{i=1}^n \mathcal{E}(\mathcal{H}_{A_i}) \mapsto [0, 1]$. According to the unentangled Gleason's theorem [28–31], any such functional μ satisfying the no-signaling condition is of the form $\mu(\pi_{A_1}^{a_1}, \ldots, \pi_{A_n}^{a_n}) = \mathrm{Tr}[W(\pi_{A_1}^{a_1} \otimes \cdots \otimes \pi_{A_n}^{a_n})]$, where W is a Hermitian, unit trace operator. Thus, the states of multipartite locally quantum theory are in one-to-one correspondence with the operators W. W, being positive over all pure tensors, is called a POPT state. However, positivity of W over entangled effects is not assured and such a nonpositive W can act as an entanglement witness operator [44]. The set of POPT states $\mathcal{W}(\bigotimes_i \mathcal{H}_{A_i})$ includes $\mathcal{D}(\bigotimes_i \mathcal{H}_{A_i})$ as a proper subset and a W will be called a "beyond-quantum state" (BQS) whenever $W \in \mathcal{W}(\bigotimes_i \mathcal{H}_{A_i}) \setminus \mathcal{D}(\bigotimes_i \mathcal{H}_{A_i})$. With an aim to study the correlations obtained from BQSs we briefly recall the standard Bell scenario.

Standard Bell scenario. A multipartite Bell scenario can be described as the following prover-verifier task. n distant verifiers $A_1, A_2, \ldots, A_n$ have their own source of classical indices $s_i \in \mathcal{S}_i$. With the aim to verify some global property of a composite state prepared by a powerful but untrustworthy prover, they send their respective indices as inputs to spatially separated subsystems of the composite systems. Classical outputs $a_i \in \mathcal{O}_i$ are generated from the different subsystems of the composite system and accordingly some payoff $\mathscr{P} : \times_{i=1}^n (\mathcal{S}_i \times \mathcal{O}_i) \mapsto \mathbb{R}$ is calculated. An implicit rule is that no communication is allowed among different subsystems once the game starts. Upon playing the game sufficiently many times, the input-output correlation $P \equiv \{p(a_1 \cdots a_n|s_1 \cdots s_n)\}_{s_i \in \mathcal{S}_i}^{a_i \in \mathcal{O}_i}$ is obtained. The collection of all NS correlations forms a convex polytope $\mathcal{NS}$. A correlation is called classical if and only if it is of the form $p_L(a_1 \cdots a_n|s_1 \cdots s_n) = \int_\Lambda p(\lambda) \prod_i p(a_i|s_i, \lambda) d\lambda$, where $\lambda \in \Lambda$ is some classical variable shared among the parties. A collection of such correlations also forms a convex polytope $\mathcal{L}$. On the other hand, a correlation is called quantum if it is obtained from some quantum state through local measurements, i.e., $p_Q(a_1 \cdots a_n|s_1 \cdots s_n) = \mathrm{Tr}[\rho(\bigotimes_i \pi_{s_i}^{a_i})]$ for some $\pi_{s_i}^{a_i} \in \mathcal{E}(\mathcal{H}_{A_i})$ and $\rho \in \mathcal{D}(\bigotimes_i \mathcal{H}_{A_i})$. The set of all quantum correlations $\mathcal{Q}$ forms a convex set but not a polytope. The framework of locally quantum theories allows us to define the correlation set obtained from the POPT states. Following the terminology of Ref. [33] we call such a correlation a "Gleason correlation" and denote the set as $\mathcal{GL}$. The following set of inclusion relations has been established: $\mathcal{L} \subsetneq \mathcal{Q} \subseteq \mathcal{GL} \subsetneq \mathcal{NS}$. While the first proper inclusion follows from the seminal work of Bell [1], the last one is due to Cirel'son and Popescu-Rohrlich [14,15]. On the other hand, the equality $\mathcal{Q} = \mathcal{GL}$ for bipartite correlations is established in Ref. [32]. More precisely, the authors in Ref. [32] have shown that for every POPT W_{AB} and for every local measurement $M_A = \{\pi_A^a\}_a$ and $M_B = \{\pi_B^b\}_b$, there exists a quantum state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and measurements $\tilde{M}_A = \{\tilde{\pi}_A^a\}_a$, $\tilde{M}_B = \{\tilde{\pi}_B^b\}_b$ such that $\mathrm{Tr}[W_{AB}(\pi_A^a \otimes \pi_B^b)] = \mathrm{Tr}[\rho_{AB}(\tilde{\pi}_A^a \otimes \tilde{\pi}_B^b)]$. In this classical input-output scenario we are now in a position to prove our first result that in some sense can be considered stronger than the result of Barnum *et al.*

Proposition 1. There exist beyond-quantum bipartite states yielding correlations that are classically simulable.

Proof. (Sketch) The family of operators $W_p := p\Gamma[|\phi^+\rangle\langle\phi^+|] + (1-p)\mathbb{I}/4$ is a BQS for $1/3 < p \leqslant 1$; $|\phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ and Γ denotes partial transposition. If we consider projective measurements only then a LHV description is possible whenever $p \leqslant 1/2$, whereas for generic POVMs one can have such a description for $p \leqslant 5/12$. The LHV models are motivated from the well-known constructions of Werner [40] and Barrett [41]. The explicit construction we defer to the Supplemental Material [45]. ■

The result of Barnum *et al.* [32] and our Proposition 1 depicts the limitation of the classical-input–classical-output Bell scenario to reveal the full correlation strength of BQSs. At this point a more general Bell scenario turns out to be advantageous.

Semiquantum Bell scenario. The scenario was introduced by Buscemi to establish the nonlocal behavior of all entangled quantum states [39], which has subsequently generated a plethora of research interests [46–51]. In this scenario, each of the verifiers, assumed to be *locally quantum,* has a random source of pure quantum states

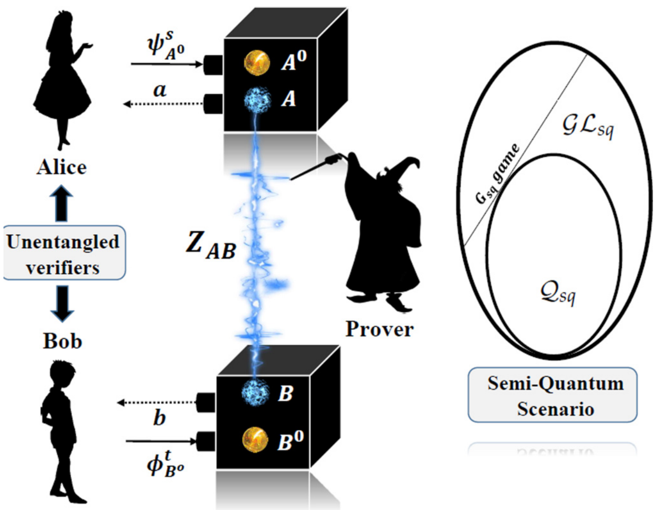

FIG. 1. A powerful but untrustworthy prover distributes a bipartite state Z_{AB} between two distant verifiers (Alice and Bob). The verifiers do not have any entanglement between them, but possess their own trusted local quantum preparation device. Such limited resourceful verifiers can verify the beyond quantumness of the state Z_{AB} provided to them (Theorem 1). The seminal Hahn-Banach separation theorem plays a crucial role in making this verification possible—the correlations produced from the bipartite quantum states form a convex-compact proper subset within the set of correlations produced from all bipartite states compatible with a local quantum description and NS principle.

$\{\{p_i(s_i), \psi^{s_i}_{A^o_i}\}|s_i \in \mathcal{S}_i\}$ (see Fig. 1). They wish to verify whether the state of a global system $W_{A_1\cdots A_n}$, prepared by a powerful but untrustworthy prover, is BQS or not. To this aim they provide their respective quantum states to the different parts of the distributed global state. The prover returns some classical index $a_i \in \mathcal{O}_i$ by performing local quantum measurements $M_{A_iA_i^o} = \{\pi^{a_i}_{A_iA^o_i}\}_{a_i}$ on the respective distributed parts of the global state and the states received from the verifiers. Accordingly, some payoff $\beta : \times_{i=1}^n(\mathcal{S}_i \times \mathcal{O}_i) \mapsto \mathbb{R}$ is given, which specifies a semiquantum game $\mathbb{G}_{\rm sq}$. From the global state $W_{A_1\cdots A_n}$, the prover can generate a correlation $P_{W_{A_1\cdots A_n}} := \{p(a_1, \dots , a_n|\psi^{s_1}, \dots , \psi^{s_n})\}$ and the expected payoff is calculated as $\mathcal{I}_{\mathbb{G}_{\rm sq}}(W_{A_1\cdots A_n}) := \sum_{s_1,a_1,\dots,s_n,a_n} \beta(s_1, a_1, \dots , s_n, a_n) \times p(a_1, \dots , a_n|\psi^{s_1}, \dots , \psi^{s_n})$. As the standard scenario, we can define the set of correlations $\mathcal{X}_{\rm sq}$ with $\mathcal{X} \in \{\mathcal{L}, \mathcal{Q}, \mathcal{GL}, \mathcal{NS}\}$ and $\mathcal{X} \subseteq \mathcal{X}_{\rm sq}$ in general. When the quantum sources consist of orthogonal quantum states, the scenario boils down to a standard Bell scenario and no distinction is possible between a bipartite entangled state and a BQS [32].

Composing POPT states. In the semiquantum Bell scenario, the prover performs local measurements $\{\pi^{a_i}_{A_iA^o_i}\}_{a_i}$ on the ith subsystem. The composite multipartite state is given by a functional $\mu_{A_1A^o_1\cdots A_nA^o_n}$ which, invoking the unentangled Gleason's theorem, corresponds to a POPT state $Z_{A_1A^o_1\cdots A_nA^o_n}$. The form of $Z_{A_1A^o_1\cdots A_nA^o_n}$ must be consistent with the states held by the verifiers and the prover. If the states held by the verifiers are pure and unentangled, then one can show that $Z_{A_1A^o_1\cdots A_nA^o_n} = \otimes_i\psi^{s_i}_{A^o_i} \otimes W_{A_1\cdots A_n}$. We leave the details to the Supplemental Material [45]. Interestingly, our next result shows that within a local quantum description the unentangled verifiers (hence weakly resourceful) can test the property "entangled versus BQS" supplied by the more resourceful prover.

Theorem 1. For every beyond-quantum state $W_{AB} \in \mathcal{W}(\mathcal{H}_A \otimes \mathcal{H}_B)$ there exists a semiquantum game $\mathbb{G}_{\rm sq}$ such that $\mathcal{I}_{\mathbb{G}_{\rm sq}}(W_{AB}) < 0$, while $\mathcal{I}_{\mathbb{G}_{\rm sq}}(\rho_{AB}) \geqslant 0, \forall\ \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Proof. At the core of our proof lies the classic Hahn-Banach separation theorem of convex analysis and the fact that for every beyond-quantum state $W_{AB} \in \mathcal{W}(\mathcal{H}_A \otimes \mathcal{H}_B)$ there exists an entangled state $\chi_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that ${\rm Tr}[W_{AB}\chi_{AB}] < 0$, whereas ${\rm Tr}[\sigma_{AB}\chi_{AB}] \geqslant 0, \forall\ \sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ [52–54]. Also note that there exist (nonunique) choices of pure states $\psi^s_A \in \mathcal{D}(\mathcal{H}_A)$ and $\psi^t_B \in \mathcal{D}(\mathcal{H}_B)$, and some real coefficients $\{\beta_{s,t}\}$ such that $\chi_{AB} = \sum_{s,t}\beta_{s,t}\psi^{s\,\rm T}_A \otimes \psi^{t\,\rm T}_B$, where T represents the transposition with respect to the computational basis. This leads us to the required game $\mathbb{G}^\chi_{\rm sq}$ where the verifiers Alice and Bob yield quantum inputs $\psi^s_{A^o}$ and $\phi^t_{B^o}$, and ask the prover to return outputs $\in\{0, 1\}$ from the distributed parts of the global state. The average payoff is calculated as $\mathcal{I} := \sum_{s,t}\beta_{s,t}p(11|\psi^s_{A^o}\psi^t_{B^o})$. The measurement $\{P^+_{uu^o}, \mathbb{I}_{uu^o} - P^+_{uu^o}\}$ is performed on the distributed parts of the global state, where $P^+_{uu^o} := |\phi^+\rangle_{uu^o}\langle\phi^+|$ with $|\phi^+\rangle_{uu^o} := \frac{1}{\sqrt{d_u}}\sum_{i=0}^{d_u-1}|ii\rangle$ and $P^+_{uu^o}$ corresponds to the outcome 1, $u \in \{A, B\}$. We therefore have

$$\begin{aligned}\mathcal{I}_{\mathbb{G}^\chi_{\rm sq}}(W_{AB}) &= \sum_{s,t}\beta_{s,t}\,{\rm Tr}\,[P^+_{AA^o}\otimes P^+_{BB^o}(\psi^s_{A^o}\otimes W_{AB}\otimes\psi^t_{B^o})]\\ &= \sum_{s,t}\beta_{s,t}\,{\rm Tr}\,[(R_A\otimes R_B)W_{AB}],\end{aligned}$$

where R_A and R_B are the effective POVMs acting on the parts of Alice's and Bob's shares of the BQS, respectively, and are given by $R_u := {\rm Tr}_{u^o}[P^+_{uu^o}(\mathbb{I}_u \otimes \psi^s_{u^o})] = \frac{1}{d_u}\psi^{s\,\rm T}_u$. Therefore, we have

$$\begin{aligned}\mathcal{I}_{\mathbb{G}^\chi_{\rm sq}}(W_{AB}) &= \frac{1}{d_Bd_A}\sum_{s,t}\beta_{s,t}\,{\rm Tr}\left[\left(\psi^{s\,\rm T}_A\otimes\psi^{t\,\rm T}_B\right)W_{AB}\right]\\ &= \frac{1}{d_Bd_A}\,{\rm Tr}\left[\left(\sum_{s,t}\beta_{s,t}\psi^{s\,\rm T}_A\otimes\psi^{t\,\rm T}_B\right)W_{AB}\right]\\ &= \frac{1}{d_Bd_A}\,{\rm Tr}\,[\chi_{AB}W_{AB}] < 0.\end{aligned}$$

On the other hand, given an arbitrary quantum state ρ_{AB}, let the measurements $M_{AA^o} \equiv \{\pi^a_{AA^o}\}_a$ and $N_{BB^o} = \{\pi^b_{BB^o}\}_b$ be performed, where $a, b \in \{0, 1\}$. The average payoff turns out to be

$$\begin{aligned}\mathcal{I}_{\mathbb{G}^\chi_{\rm sq}}(\rho_{AB}) &= \sum_{s,t}\beta_{s,t}\,{\rm Tr}\left[\pi^1_{AA^o}\otimes\pi^1_{BB^o}\left(\psi^s_{A^o}\otimes\rho_{AB}\otimes\psi^t_{B^o}\right)\right]\\ &= \sum_{s,t}\beta_{s,t}\,{\rm Tr}\left[R_{A^oB^o}\left(\psi^s_{A^o}\otimes\psi^t_{B^o}\right)\right],\end{aligned}$$

where $R_{A^oB^o} := {\rm Tr}_{AB}[(\pi^1_{AA^o}\otimes\pi^1_{BB^o})(\mathbb{I}_{A^oB^o}\otimes\rho_{AB})]$ is a positive operator. Using the linearity of trace we get

$$\begin{aligned}\mathcal{I}_{\mathbb{G}^\chi_{\rm sq}}(\rho_{AB}) &= {\rm Tr}\left[R_{A^oB^o}\left(\sum_{s,t}\beta_{s,t}\psi^s_{A^o}\otimes\psi^t_{B^o}\right)\right]\\ &= {\rm Tr}\left[R_{A^oB^o}\chi^{\rm T}_{A^oB^o}\right] \geqslant 0.\end{aligned}$$

The last inequality follows due to the fact that $\chi^{\mathrm{T}}_{A^o B^o}$ is a valid density operator, and this completes the proof. ■

Theorem 1 establishes that $\mathcal{Q}_{\text{sq}} \subsetneq \mathcal{GL}_{\text{sq}}$ in the bipartite scenario. Note that, following an argument similar to Ref. [47], it can be shown that in the semiquantum scenario, even if classical communications between different distributed parts are allowed to the prover along with the quantum entangled state ρ_{AB}, still the local statistics obtained from BQS cannot be simulated. Our result poses some interesting questions. The proper set inclusion relation $\mathcal{Q} \subsetneq \mathcal{NS}$ established in Ref. [15] has motivated several novel approaches to isolate quantum correlations from beyond-quantum ones [55–67]. Along similar lines, the proper set inclusion relation $\mathcal{Q}_{\text{sq}} \subsetneq \mathcal{GL}_{\text{sq}}$ welcomes new principle(s) to isolate the quantum correlations from beyond-quantum ones in this generalized scenario. Importantly, our Theorem 1 suggests that such principles must be sensitive to the quantum signature of local inputs [49,50].

The semiquantum scenario also has important implications while studying correlations in multipartite (involving more than two parties) scenarios. Acín *et al.* have already pointed out that the result of Barnum *et al.* does not generalize to the multipartite scenario even in the classical-input–classical-output paradigm [33]. They have provided examples of multipartite BQSs producing beyond-quantum correlations within the standard Bell scenario. They have also pointed out that a BQS of the form

$$W_{A_1\cdots A_N} = \sum_k p_k\big(\Lambda^k_{A_1} \otimes \cdots \otimes \Lambda^k_{A_N}\big)\big[\rho^k_{A_1\cdots A_n}\big], \qquad (1)$$

will not generate any classical-input–classical-output correlation that lies outside the set of correlations generated by quantum states. Here, $\{p_k\}$ is a probability distribution, $\rho^k_{A_1\cdots A_n} \in \mathcal{D}(\bigotimes_i \mathcal{H}_{A_i})$, and Λ^k_i are positive but not completely positive trace preserving maps on $\mathcal{L}(\mathcal{H}_{A_i})$ [54]. The authors in Ref. [33] have left the question open to identify the additional requirements to close the gap in their result. Our next result provides a solution to close this gap.

Theorem 2. For every BQS $W_{A_1\cdots A_N} \in \mathcal{W}(\bigotimes_{i=1}^N \mathcal{H}_{A_i})$ there exists a semiquantum game $\mathbb{G}_{\text{sq}}$ such that $\mathcal{I}_{\mathbb{G}_{\text{sq}}}(W_{A_1\cdots A_N}) < 0$, whereas $\mathcal{I}_{\mathbb{G}_{\text{sq}}}(\rho_{A_1\cdots A_N}) \geqslant 0,\ \forall \rho_{A_1\cdots A_N} \in \mathcal{D}(\bigotimes_{i=1}^N \mathcal{H}_{A_i})$.

The proof is a straightforward generalization of the proof of Theorem 1 (see Supplemental Material [45]). While Theorems 1 and 2 are just existence theorems, it is not hard to see that given an arbitrary BQS there is an efficient algorithm to construct a semiquantum game (the procedure is discussed in the Supplemental Material [45]). It is important to note that nonorthogonal quantum inputs are necessary to reveal the beyond-quantum signature of correlation for any BQS of the form of Eq. (1). This implicitly follows from the results of Barnum *et al.* [32] and Acín *et el.* [33]. It is worth mentioning that this semiquantum scenario is different from local tomography as it establishes the beyond-quantum nature of POPT states in a measurement device-independent manner where the measurement devices used to produce the classical outcomes need not be trusted [46].

Discussion. One of the earnest research endeavors in quantum theory is to understand the limited nonlocal behavior of quantum correlations. Apart from the foundational appeal, this question also has practical relevance as nonlocal correlations have been established as useful resources for several tasks. In the bipartite scenario the result of Barnum *et al.* [32] provides an answer to this question by assuming the description of local systems to be quantum. Our work, however, points out the limitation of the scenario considered in Ref. [32]. The authors there have not considered the most general bipartite scenario allowed within the unentangled Gleason-Busch framework, which assumes local quantum measurement and the no-signaling principle. Within this framework, the types of inputs allowed are not restricted to classical indices, rather they can be quantum states. Our Theorem 1 shows that all bipartite beyond-quantum states compatible with the unentangled Gleason-Busch theorem can yield beyond-quantum correlations in the quantum-input scenario, and accordingly divulges a more complex picture within the correlations zoo. Our study therefore welcomes new principle(s) to isolate the correlations obtained from quantum states, and more importantly, suggests that such a principle should take the type of inputs into consideration as the indistinguishability of nonorthogonal quantum-input states plays a crucial role in making the distinction between quantum and BQS states.

Our Theorem 2 establishes that within the quantum-input paradigm all multipartite BQSs yield beyond-quantum correlations which was known earlier only for a particular class of such states [33]. After the work of Ref. [33], Torre *et al.* have shown that when the local systems are identical qubits, any theory admitting at least one continuous reversible interaction must be identical to quantum theory [34]. However, the result in Ref. [34] has also been obtained within the classical-input–classical-output paradigm. It might be interesting to see what additional structures are required there to single out the quantum correlations in the quantum-input scenario. On the other hand, within the classical-input–classical-output paradigm, the authors in Ref. [68] and the present authors with other collaborators in Refs. [69,70] have studied beyond-quantum correlations in the timelike domain. Similar studies with quantum inputs might provide insights there.

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