Letter

Synchronization and amplification enabled by diversity in nonlinear optical systems and the analogy with converse symmetry breaking for coupled oscillators

Auro M. Perego*

Aston Institute of Photonic Technologies, Aston University, Birmingham B4 7ET, United Kingdom

(Received 17 May 2022; accepted 15 September 2022; published 26 September 2022)

In this Research Letter it is shown how certain optical wave amplification phenomena, which do not obey standard phase-matching conditions and are generally classified as non-Hermitian phase matching or gain-through-loss(-filtering) processes, can be considered as particular cases of the more general phenomenon of converse symmetry breaking predicted and observed in various networks of coupled oscillators. It is shown that the abovementioned optical amplification processes are possible thanks to a phase-locking dynamics enabled by asymmetry or heterogeneity in the coupled-modes equations.

DOI: 10.1103/PhysRevA.106.L031505

Symmetry is a defining principle of science which underpins the possibility and structure of the laws of physics [1]. It is connected to the elegant construction of Hamiltonian mechanics and of modern particle physics and is associated with conserved quantities [2]. A system is said to be symmetric with respect to a certain transformation if some of its features or its evolution equations are invariant under that particular transformation. Symmetric systems may undergo symmetry breaking when one of the possible states allowed by the symmetry is selected by the dynamical evolution, resulting in a final state possessing a reduced symmetry compared with the system itself.

Symmetry breaking is indeed at the heart of a variety of important phenomena playing a relevant role in particle physics [3,4], in superconductivity [5,6], in lasing [7], in the existence of chimera states [8], and in the process of pattern formation in spatially extended systems [9].

A further paradigmatic example is the replica symmetry breaking (RSB) phenomenon, consisting in the fact that identical systems, under identical conditions, can exhibit different values of observable quantities. RSB has been originally studied in the framework of the spin-glass theory [10] and subsequently in nonlinear optical systems too [11–14].

Recently, the concept of converse symmetry breaking (CSB) has been introduced. CSB consists in the fact that, in certain situations, stable and phase-locked symmetric states are possible only if the system itself is *not* symmetric. This concept has been presented first by studying a network of coupled oscillators, showing that the symmetric synchronous dynamical state where all the oscillators are phase locked to each other can be achieved only when a certain degree of heterogeneity among the elements of the network is present, which translates into an asymmetry in the evolution equations [15]. Heterogeneity indeed results in a powerful tool to increase synchronization in oscillator networks [16] and favors homogeneous states preventing the onset of instabilities [17]. CSB has been demonstrated experimentally in a network of coupled electromechanical oscillators [18]. It has been shown theoretically to enable an increased stability of the synchronous state of national power grid systems too [19]. These findings have substantially advanced research in the complex dynamics of coupled oscillators and their synchronization properties, which have been an active subject of continuous research for many decades [20].

An apparently unrelated research line has been growing in nonlinear optics where the amplification of coupled modes in optical parametric oscillators with quadratic nonlinearity and in optical parametric amplifiers and in a nonlinear optical resonator with cubic Kerr nonlinearity, under certain conditions (absence of phase matching), can be achieved only when structural asymmetry is introduced in the system [21–34] (see also Ref. [35] for an extension of the concept to Bose-Einstein condensates). Heterogeneity, in the nonlinear optical systems studied so far, is mediated by optical filtering or by other forms of selective dissipation that act differently on the coupled modes inducing an asymmetry in the energy loss rate and phase shift between them. Counterintuitively, dissipation asymmetry for two small-amplitude coupled modes (called signal and idler waves in nonlinear optics jargon) enables energy transfer to the latter from a powerful pump field when this would not be possible under standard circumstances. In certain cases [24,25] it is the asymmetric phase change induced by the presence of asymmetric dissipation, or the anticrossing of cavity resonances caused by coupling with an auxiliary mode as in passive driven coupled nonlinear microresonators [32,33], that leads to sideband amplification. When the amplification process is caused by losses, this has been referred to as non-Hermitian phase matching [27], to highlight the breaking of Hermiticity of the system Hamiltonian due to dissipation, or as gain through loss [21] to highlight the counterintuitive effect of how gain can be achieved from its opposite effect.

Despite substantial efforts focusing on the characterization of the optical gain and on its possible technological uses in different geometries and devices, a systematic analysis of the

*a.perego1@aston.ac.uk

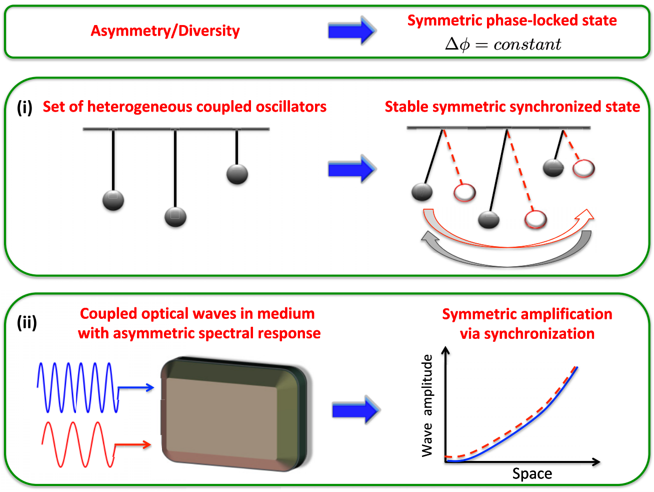

FIG. 1. Schematic of the synchronization-through-diversity concept: (i) In a set of coupled heterogeneous oscillators, CSB consists in the existence of a stable synchronized state which would be otherwise impossible for a set in which the oscillators are identical; and (ii) two optical modes synchronize and are amplified during propagation in a nonlinear medium with an asymmetric spectral response, while for a symmetric response function this does not occur.

phase dynamics in asymmetry-induced optical amplification has not been reported so far to the best of our knowledge. The main goal of this Research Letter is to show how, and to which extent, amplification of optical waves enabled by heterogeneity can be, by analogy, described in terms of the CSB framework. Based on the fact that sideband equations describing the evolution of optical modes correspond indeed to a set of heterogeneous coupled oscillators, we will show that the amplification process is associated with a synchronization (phase locking) between the different modes, while by eliminating diversity no synchronization or amplification would take place. A symmetric synchronized state mediated by heterogeneity, in a regime of parameters where the symmetric system would be unlocked, is indeed the defining feature of CSB. This analogy is summarized schematically in Fig. 1 showing how locking mediated by oscillator diversity leads to wave amplification. We will illustrate the concept considering first the case of a medium with cubic nonlinearity and then the case of a medium with quadratic nonlinearity. When a powerful monochromatic light wave (pump) propagates inside a material which possess cubic Kerr nonlinearity, if the dispersive properties of the material compensate for the nonlinear phase shift, energy can be transferred from the pump to two symmetrically located (in frequency with respect to the pump) spectral sidebands called the signal and the idler, respectively. This phenomenon is sometimes called modulation instability of the pump wave [36–38] (as the sideband amplification results in a modulation of the input wave amplitude and eventually in its destruction), and it is the principle underpinning the operation of the fiber optical parametric amplifier [39,40]. We will focus here on the linear transient stage of the sideband evolution process, where the sideband power is much smaller than the pump power, so that the latter field is basically not affected by the dynamics and its evolution equation can be neglected. The evolution of the amplitudes $a_{s,i}$ of signal and idler waves detuned by $\pm\omega$ from the pump frequency, propagating along an optical fiber spatial coordinate z under the assumption that $|a_{s,i}|^2 \ll P$ (where P is the pump wave power), reads

$$\frac{\partial a_s}{\partial z} = i\frac{\beta_2}{2}\omega^2 a_s + i\gamma P(a_s + a_i^*) + D_s(\omega)a_s,$$
$$\frac{\partial a_i}{\partial z} = i\frac{\beta_2}{2}\omega^2 a_i + i\gamma P(a_i + a_s^*) + D_i(\omega)a_i. \quad (1)$$

Here, β_2 is the group velocity dispersion coefficient (we assume higher-order dispersion to be negligible), γ is the nonlinearity coefficient of the fiber, and $D_s(\omega)$ and $D_i(\omega)$ describe nonhomogeneity, generally depending on frequency, and quantify the effects of dissipation [$\mathrm{Re}(D_{s,i})$] and associated phase shift [$\mathrm{Im}(D_{s,i})$] induced, for instance, by the presence of a spectral filter. In general the real and imaginary parts of $D_{s,i}$ have to satisfy constraints imposed by Kramers-Kronig relations in order for causality to be satisfied (frequency-dependent dissipation entails the presence of a frequency-dependent phase effect) [41].

Taking the complex conjugate of the equation for a_i, we can rewrite the system of equations (1) in the following form:

$$\dot{a}_s = i\Omega_s a_s + ica_i^*,$$
$$\dot{a}_i^* = -i\Omega_i^* a_i^* - ica_s. \quad (2)$$

Equations (2) describe two coupled oscillators (indeed the evolution equation of a generic harmonic oscillator can be recast into the form $\dot{a} = i\Omega a$, where the dot denotes the first derivative; see, e.g., Ref. [42]).

The two oscillators have complex frequencies $\Omega_s = \frac{\beta_2}{2}\omega^2 + \gamma P - iD_s$ and $\Omega_i^* = \frac{\beta_2}{2}\omega^2 + \gamma P + iD_i^*$, respectively—the imaginary part being associated with dissipation—and interact through the pump wave with coupling parameter $c = \gamma P$. Most importantly, for $D_s = D_i$, Eqs. (2) are symmetric (invariant) with respect to the transformation $(a_{s,i} \rightarrow a_{i,s}^*, i \rightarrow -i)$.

From the eigenvalues $\lambda_\pm$ of the 2×2 matrix constituted by the coefficients of Eqs. (2) it is possible to infer whether the spectral sidebands are amplified or not. If at least one of the two eigenvalues has a positive real part, then both sidebands detuned by frequency $\pm\omega$ experience exponential growth with the characteristic exponent, or gain, given by $\lambda_c = \max[\mathrm{Re}(\lambda_+), \mathrm{Re}(\lambda_-)]$. In the nondissipative limit [$\mathrm{Re}(D_{s,i}) = 0$], corresponding to the assumption that both modes are sufficiently far away from the point of maximum losses to neglect dissipation (an approximation used, for instance, in Refs. [24,25]), the eigenvalues read

$$\lambda_\pm = \tfrac{1}{2}[iD_- \pm \sqrt{-(D_+ + \beta_2\omega^2)(D_+ + 4c + \beta_2\omega^2)}], \quad (3)$$

with $D_+ = \mathrm{Im}(D_s) + \mathrm{Im}(D_i)$ and $D_- = \mathrm{Im}(D_s) - \mathrm{Im}(D_i)$. When $D_{s,i} = 0$, amplification is possible only if the sign of β_2 is negative (anomalous dispersion regime) provided that $|\omega| < \sqrt{\frac{4\gamma P}{|\beta_2|}}$. In this scenario, which corresponds to the usual operational regime of parametric amplifiers, signal and idler phases are locked and synchronization occurs; however, this is not the case we are focusing on in this Research Letter. The normal dispersion scenario ($\beta_2 > 0$) instead is the one we are

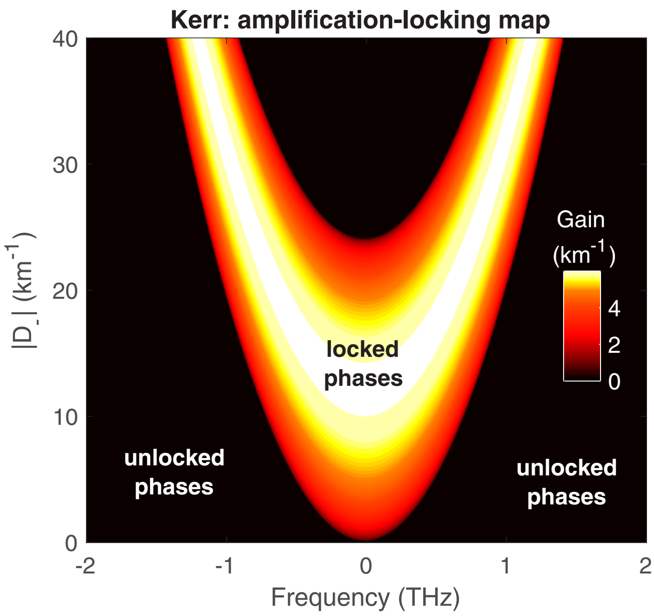

FIG. 2. Phase-locking and amplification map for the Kerr medium: The gain λ_c is plotted vs frequency and diversity parameter $|D_-|$. Phase locking and amplification (nonblack areas) occur only when $|D_-| > 0$. Parameters used are as follows: $P = 3$ W, $\gamma = 2$ W^{-1}/km, $\beta_2 = 0.5$ ps^2/km, $\text{Im}(D_s) = 0$.

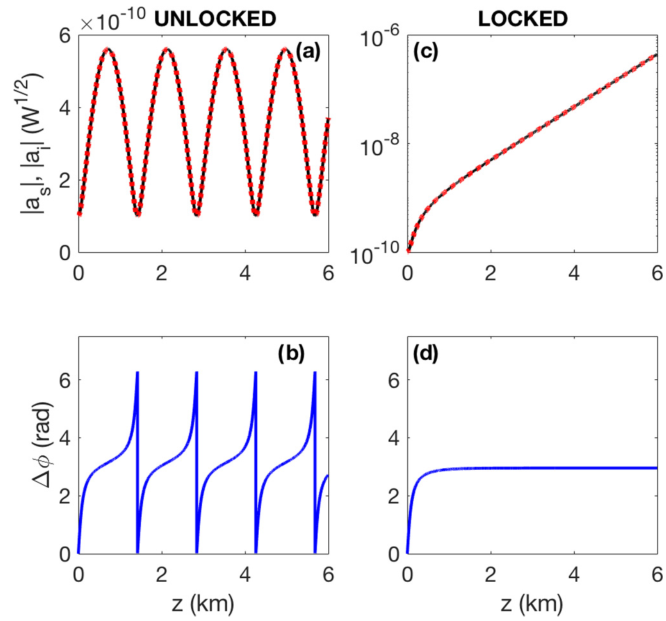

FIG. 3. Sideband evolution for the Kerr medium. In the symmetric system the evolution is plotted vs spatial coordinate z (the signal is shown with the black curve, and the idler is shown with the dotted red curve) in (a), while the phase difference $\Delta\phi$ (modulo 2π) is shown in (b): Sidebands oscillate without experiencing amplification, and the free evolution of the relative phase shows that the modes are unlocked. In (c) and (d) the same quantities are depicted for the asymmetric scenario, showing in this case exponential amplification and clear phase locking. Parameters used are as follows: $P = 3$ W, $\gamma = 2$ W^{-1}/km, $\beta_2 = 0.5$ ps^2/km, $\omega = 2\pi \times 0.2$ rad/ps, $\text{Im}(D_s) = 0$; $\text{Im}(D_i) = 0$ in (a) and (b), and $\text{Im}(D_i) = -1$ km^{-1} in (c) and (d).

interested in, to establish the analogy with CSB. In particular, for $D_{s,i} = 0$ and $\beta_2 > 0$, no amplification occurs and signal and idler phases are unlocked.

However, oscillator heterogeneity, $D_s \neq D_i$, breaks the symmetry and may lead to amplification even when $\beta_2 > 0$.

In Fig. 2 a map of the locking and amplification area is plotted versus frequency and diversity parameter $|D_-|$. In Fig. 3 the evolution of wave amplitudes and relative phase, $\Delta\phi(z) = \phi_s(z) - \phi_i(z)$, is shown in one symmetric case and in one asymmetric case resulting in an unlocked state without amplification and in a locked state with exponentially growing amplitudes, respectively.

What is important to stress here is that amplification caused by diversity occurs in correspondence to a phase synchronization between the coupled waves: Phase locking between the two light waves occurs in normal dispersion ($\beta_2 > 0$) mediated by heterogeneity, while in the absence of heterogeneity no locking would occur. This point is connected to the defining feature of the CSB, where indeed phase locking and synchronization in an ensemble of coupled oscillators occur thanks to heterogeneity (asymmetry) for a parameter set for which this would be impossible in a symmetric system scenario. As in the CSB case, not every arbitrary asymmetry will induce a locked state, but there is an optimal choice of the parameters for which locking occurs and is maximized. Implementation of this phase-diversity scheme can be realized by changing locally (in frequency) the dispersion via the introduction of a spectral filter which entails a frequency-dependent phase response through Kramers-Kronig relations. For frequencies sufficiently detuned from the filter maximum losses, this would result in a purely dispersive heterogeneity. This has been demonstrated theoretically and experimentally in passive fiber cavities by using notch filters [24,25]. A further possibility is considering concatenated fiber links, each one followed by fiber Bragg grating causing a selective phase shift only on the signal or idler.

We consider now another paradigmatic scenario of wave amplification in nonlinear optics: the optical parametric oscillator (OPO) with quadratic nonlinearity. In this case the equations governing the dynamics of signal and idler complex-conjugate wave amplitudes A_s and A_i^* interacting through a powerful pump field can be written (assuming that the signal and idler amplitudes remain small compared with the pump one and that the latter basically remains undepleted) in the following normalized coupled-oscillators form:

$$\frac{dA_s}{dz} = i\omega_s A_s + ic_s A_i^*,$$
$$\frac{dA_i^*}{dz} = -i\omega_i^* A_i^* - ic_i^* A_s. \tag{4}$$

Here, $\omega_s = -\frac{\Delta k}{2} - iD_s$ and $\omega_i^* = -\frac{\Delta k}{2} + iD_i^*$ are the frequencies, Δk is the mismatch parameter, and $c_s = \sqrt{\mu_s} A_p$ and $c_i = \sqrt{\mu_i} A_p^*$ are coupling parameters that depend on the pump amplitude A_p. $\mu_{s,i}$ are proportionality factors which we assume to be equal to μ for both modes considering the degenerate OPO case. As in the Kerr case, $D_{s,i}$ describe nonhomogeneity and here are taken to be real to illustrate an example of a purely dissipative regime (considering, in particular, the lossy case

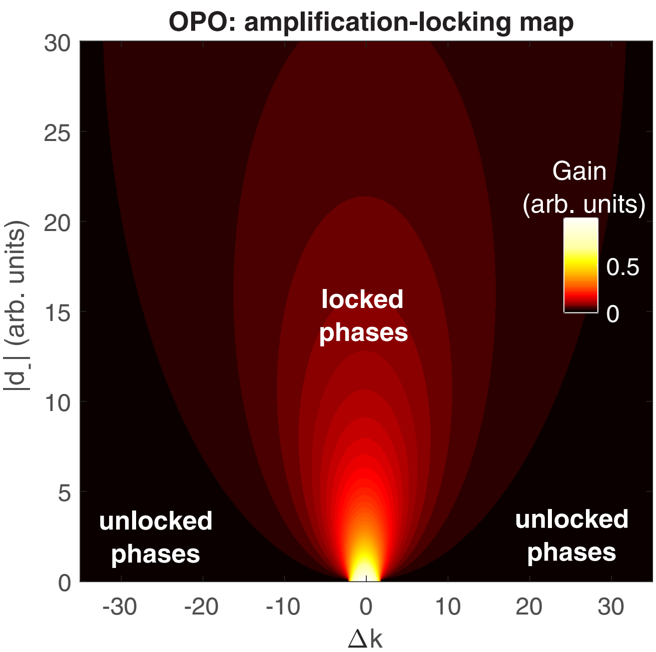

FIG. 4. Amplification-locking map for the quadratic nonlinear medium: The gain λ_c is plotted vs mismatch parameter Δk and diversity parameter $|d_-|$. Locking and amplification occur in the nonblack areas of the plot. Parameters used are as follows: $\Delta k = 4$, $\mu|A_p|^2 = 1$, $\mathrm{Re}(D_s) = 0$.

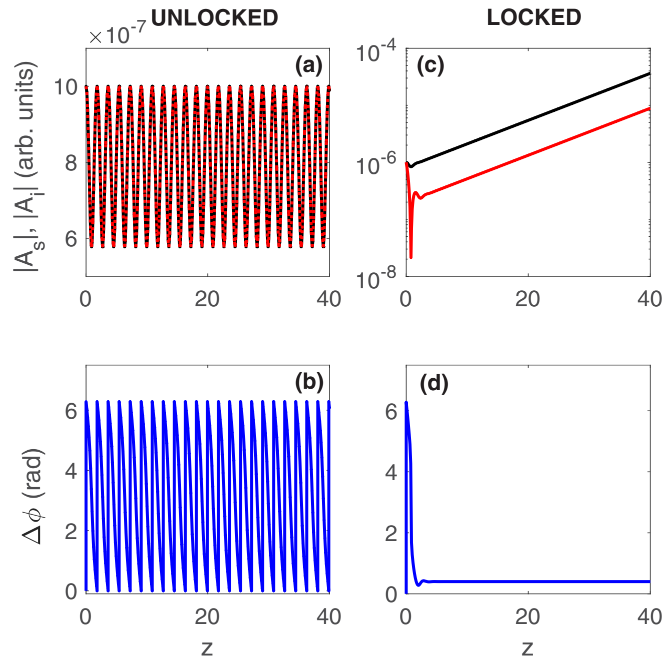

FIG. 5. The amplitude of the signal and idler modes and their relative phase evolution (modulo 2π) are plotted vs the evolution coordinate for the symmetric (unlocked) quadratic nonlinearity OPO in (a) and (b), respectively. In (c) and (d) the corresponding quantities are shown for an asymmetric system where locking and amplification are taking place. Signal and idler amplitudes are plotted in black and red, respectively. Parameters used are as follows: $\Delta k = 4$, $\mu|A_p|^2 = 1$, $\mathrm{Re}(D_s) = 0$; $\mathrm{Re}(D_i) = 0$ in (a) and (b), and $\mathrm{Re}(D_i) = -1.5$ in (c) and (d).

$D_{s,i} \leqslant 0$). The system of equations (4) is symmetric under the transformation ($A_{s,i} \to A^*_{i,s}$, $i \to -i$). The condition for amplification is determined by the existence of eigenvalues with real positive parts. In this case the eigenvalues read

$$\lambda_{1,2} = \tfrac{1}{2}[d_+ \pm \sqrt{(i\Delta k - d_-)^2 + 4\mu|A_p|^2}], \tag{5}$$

where $d_+ = \mathrm{Re}(D_s) + \mathrm{Re}(D_i)$ and $d_- = \mathrm{Re}(D_s) - \mathrm{Re}(D_i)$. The gain is defined as $\lambda_c = \max[\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)]$. In the scenario where $D_{s,i} = 0$, amplification and phase locking occur only when the mismatch satisfies $|\Delta k| < 2\sqrt{\mu|A_p|^2}$; the nontrivial balanced scenario $D_s = D_i < 0$ enables synchronization in an even smaller region of $|\Delta k|$. Amplification and phase locking for $|\Delta k| > 2\sqrt{\mu|A_p|^2}$ are impossible in the symmetric scenario. However, by introducing nonhomogeneity and breaking the system symmetry, e.g., when the dissipation is unbalanced for signal and idler waves ($D_s \neq D_i$), phase locking and amplification become possible even for the forbidden values of Δk. A locking map for the quadratic nonlinearity OPO is shown in Fig. 4, where it is clearly evident how locking in an asymmetric system $|d_-| \neq 0$ is possible for much larger values of the mismatch parameter Δk compared with the symmetric case $|d_-| = 0$.

Figure 5 illustrates the evolution of the OPO wave amplitudes and relative phases for the locked [Figs. 5(a) and 5(b)] and unlocked [Figs. 5(c) and 5(d)] scenarios, respectively.

Regarding practical implementation of the dissipative heterogeneity, in the scenarios described in a series of works on non-Hermitian phase matching [27–30], signal and idler waves are spatially separated, which enables the engineering of asymmetry through building or manipulating separated waveguides featuring different losses, e.g., through the use of fiber Bragg gratings. In other works, instead, losses on the idler have been introduced by specific doping [26].

The optical amplification phenomena enabled by diversity share with CSB the key feature that the phase difference between different oscillators or modes is locked (constant) and that symmetric synchronization occurs thanks to heterogeneity or asymmetry in the system components. This is the main finding of this Research Letter, and it is rooted in the mathematical equivalence of the optical signal and idler sideband equations with those of coupled oscillators. It should be mentioned that in the optical wave amplification scenarios we may have not only a scenario where the resulting phase-locked system state is symmetric (same amplitude for both modes) as is shown in Fig. 3(c), but also a partial symmetry in the phase-locked synchronized dynamics, as the two oscillators' amplitudes can be different from each other [see Fig. 5(c)]. On the other hand, it is important to stress that for optical amplification the description presented here refers to the transient of the amplification process (where sideband amplitude is much smaller than the pump field which couples them together and also considers the pump field to be undepleted). In the CSB cases considered so far, the synchronized dynamics is a feature of a stable system attractor [15,19].

In Ref. [15], emphasis has been put on the fact that CSB may play an important role in pattern formation. For the optical amplification case we can distinguish between two different scenarios. In the propagative regime when the Kerr or quadratic medium is not embedded in a cavity, it is more challenging to achieve a stationary nontrivial asymptotic state and hence pattern formation, also because during sideband growth, pump depletion occurs. A stable attractor can be reached instead when sideband amplification occurs in a cavity with external driving. In the latter case the flow of energy from the pump to the sidebands is compensated by the external injection at the pump frequency, and stable roll patterns (a train of light pulses) drifting at a constant speed can be formed in the time domain [24,25]. A full treatment of the nonlinear regime which accounts for the depleted-pump case requires considering an additional equation for the pump field, effectively resulting in a variable coupling strength between the two oscillator modes; furthermore, accounting for nonlinear terms in both sideband and pump equations is needed when their amplitudes grow substantially. The study of the phase dynamics and locking in the nonlinear regime is left to future investigations.

To summarize, in this Research Letter it has been shown that phase synchronization of optical modes achieved through diversity (asymmetry) between them can be interpreted as a particular case of CSB. Concerning further developments in photonics and nonlinear optics, we envisage the possibility of generalization and application of this approach to systems where a large number of optical modes are involved such as waveguide arrays and multicore optical fibers. Another relevant possibility is the study of the interplay between CSB and RSB in random lasers and in mode-locked lasers, with heterogeneity provided by disorder [11–14], leveraging the similarity of the evolution equations to the ones presented in this Research Letter. As the concepts presented here rely on the universal formalism of coupled oscillators, it is expected that they could be further generalized and adapted to a variety of systems across different scientific disciplines to make synchronization and locking processes that are more robust and/or occur in unexpected configurations. Furthermore, these concepts could have potential applications in solving consensus-related problems in complex systems where a collective and cooperative behavior between the different individual components is required.

The author acknowledges support by the Royal Academy of Engineering through the Research Fellowship scheme and is grateful to Prof. Robert S. MacKay for stimulating discussions.

[1] D. J. Gross, The role of symmetry in fundamental physics, Proc. Natl. Acad. Sci. USA **93**, 14256 (1996).

[2] E. Noether, Invariante Variationsprobleme, Nachr. Ges. Wiss. Goettingen, Math.-Phys. Kl. **1918**, 235 (1918).

[3] P. W. Higgs, Broken Symmetries and the Masses of Gauge Bosons, Phys. Rev. Lett. **13**, 508 (1964).

[4] F. Englert and R. Brout, Broken symmetry and the mass of gauge vector mesons, Phys. Rev. Lett. **13**, 321 (1964).

[5] Y. Nambu, Quasi-Particles and Gauge Invariance in the Theory of Superconductivity, Phys. Rev. **117**, 648 (1960).

[6] J. Goldstone, Field theories with "superconductor" solutions, Nuovo Cimento **19**, 154 (1961).

[7] R. Graham and H. Haken, Laserlight - first example of a second-order phase transition far away from thermal equilibrium, Z. Phys. **237**, 31 (1970).

[8] M. J. Panaggio and D. M. Abrams, Chimera states: coexistence of coherence and incoherence in networks of coupled oscillators, Nonlinearity **28**, R67 (2015).

[9] M. C. Cross and P. C. Hohenberg, Pattern formation outside of equilibrium, Rev. Mod. Phys. **65**, 851 (1993).

[10] M. Mezard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1986).

[11] F. Antenucci, C. Conti, A. Crisanti, and L. Leuzzi, General Phase Diagram of Multimodal Ordered and Disordered Lasers in Closed and Open Cavities, Phys. Rev. Lett. **114**, 043901 (2015).

[12] F. Antenucci, A. Crisanti, M. I. nez Berganza, A. Marruzzo, and L. Leuzzi, Statistical mechanics models for multimode lasers and random lasers, Philos. Mag. **96**, 704 (2016).

[13] N. Ghofraniha, I. Viola, F. Di Maria, G. Barbarella, G. Gigli, L. Leuzzi, and C. Conti, Experimental evidence of replica symmetry breaking in random lasers, Nat. Commun. **6**, 6058 (2015).

[14] A. L. Moura, P. I. R. Pincheira, A. S. Reyna, E. P. Raposo, A. S. L. Gomes, and C. B. de Araújo, Replica Symmetry Breaking in the Photonic Ferromagneticlike Spontaneous Mode-Locking Phase of a Multimode Nd:YAG Laser, Phys. Rev. Lett. **119**, 163902 (2017).

[15] T. Nishikawa and A. E. Motter, Symmetric States Requiring System Asymmetry, Phys. Rev. Lett. **117**, 114101 (2016).

[16] Y. Zhang, J. L. Ocampo-Espindola, I. Z. Kiss, and A. E. Motter, Random heterogeneity outperforms design in network synchronization, Proc. Natl. Acad. Sci. USA **118** (2021).

[17] Z. G. Nicolau, D. J. Case, E. B. van der Wee, M. M. Driscoll, and A. E. Motter, Heterogeneity-stabilized homogeneous states in driven media, Nat. Commun. **12**, 4486 (2021).

[18] F. Molnar, T. Nishikawa, and A. E. Motter, Network experiment demonstrates converse symmetry breaking, Nat. Phys. **16**, 351 (2020).

[19] F. Molnar, T. Nishikawa, and A. E. Motter, Asymmetry underlies stability in power grids, Nat. Commun. **12**, 1457 (2021).

[20] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2003).

[21] A. M. Perego, S. K. Turitsyn, and K. Staliunas, Gain through losses in nonlinear optics, Light Sci. Appl. **7**, 43 (2018).

[22] T. Tanemura, Y. Ozeki, and K. Kikuchi, Modulational Instability and Parametric Amplification Induced by Loss Dispersion in Optical Fibers, Phys. Rev. Lett. **93**, 163902 (2004).

[23] K. Xu, H. Liu, Y. Dai, J. Wu, and J. Lin, Synthesis of broadband and flat parametric gain by idler loss in optical fiber, Opt. Commun. **285**, 790 (2012).

[24] F. Bessin, A. M. Perego, K. Staliunas, S. K. Turitsyn, A. Kudlinski, M. Conforti, and A. Mussot, Gain-through-filtering enables tuneable frequency comb generation in passive optical resonators, Nat. Commun. **10**, 4489 (2019).

[25] A. M. Perego, A. Mussot, and M. Conforti, Theory of filter-induced modulation instability in driven passive optical resonators, Phys. Rev. A **103**, 013522 (2021).
[26] J. Ma, J. Wang, P. Yuan, G. Xie, K. Xiong, Y. Tu, X. Tu, E. Shi, Y. Zheng, and L. Qian, Quasi-parametric amplification of chirped pulses based on a Sm^{3+}-doped yttrium calcium oxyborate crystal, Optica **2**, 1006 (2015).
[27] R. El-Ganainy, J. I. Dadap, and R. M. Osgood, Optical parametric amplification via non-Hermitian phase matching, Opt. Lett. **40**, 5086 (2015).
[28] Q. Zhong, A. Ahmed, J. I. Dadap, R. M. Osgood Jr., and R. El-Ganainy, Parametric amplification in quasi-PT symmetric coupled waveguide structures, New J. Phys. **18**, 125006 (2016).
[29] A. Ahmed, X. Meng, Q. Zhong, R. El-Ganainy, R. M. Osgood, and J. I. Dadap, On-chip non-Hermitian optical parametric amplifiers with a large bandwidth, J. Opt. Soc. Am. B **38**, 2160 (2021).
[30] N. Quesada, E. Adjei, R. El-Ganainy, and A. M. Brańczyk, Non-Hermitian engineering for brighter broadband pseudothermal light, Phys. Rev. A **100**, 043805 (2019).
[31] A. Roy, S. Jahani, Q. Guo, A. Dutt, S. Fan, M.-A. Miri, and A. Marandi, Nondissipative non-Hermitian dynamics and exceptional points in coupled optical parametric oscillators, Optica **8**, 415 (2021).
[32] X. Xue, Y. Xuan, P.-H. Wang, Y. Liu, D. E. Leaird, M. Qi, and A. M. Weiner, Normal-dispersion microcombs enabled by controllable mode interactions, Laser Photon. Rev. **9**, L23 (2015).
[33] Y. Liu, Y. Xuan, X. Xue, P.-H. Wang, S. Chen, A. J. Metcalf, J. Wang, D. E. Leaird, M. Qi, and A. M. Weiner, Investigation of mode coupling in normal-dispersion silicon nitride microresonators for Kerr frequency comb generation, Optica **1**, 137 (2014).
[34] H. Hu, L. Zhang, C. Zhang, Y. Chen, J. Xu, and X. Zhang, Lumped dissipation induced quasi-phase matching for broad and flat optical parametric processes, IEEE Photon. J. **11**, 1 (2019).
[35] S. Wüster and R. El-Ganainy, Non-Hermitian matter-wave mixing in Bose-Einstein condensates: Dissipation-induced amplification, Phys. Rev. A **96**, 013605 (2017).
[36] V. Zakharov and L. Ostrovsky, Modulation instability: The beginning, Phys. D (Amsterdam) **238**, 540 (2009).
[37] V. Bespalov and V. Talanov, Filamentary structure of light beams in nonlinear liquids, Zh. Eksp. Teor. Fiz. **3**, 307 (1966).
[38] K. Tai, A. Hasegawa, and A. Tomita, Observation of Modulational Instability in Optical Fibers, Phys. Rev. Lett. **56**, 135 (1986).
[39] J. Hansryd, P. Andrekson, M. Westlund, J. Li, and P.-O. Hedekvist, Fiber-based optical parametric amplifiers and their applications, IEEE J. Sel. Top. Quantum Electron. **8**, 506 (2002).
[40] A. Hasegawa and W. Brinkman, Tunable coherent IR and FIR sources utilizing modulational instability, IEEE J. Quantum Electron. **16**, 694 (1980).
[41] V. Lucarini, J. Saarinen, K. E. Peiponen, and E. M. Vartiainen, *Kramers-Kronig Relations in Optical Materials Research* (Springer, New York, 2005).
[42] V. García-Morales and K. Krischer, The complex Ginzburg-Landau equation: an introduction, Contemp. Phys. **53**, 79 (2012).