Letter

Bridging the gap between topological non-Hermitian physics and open quantum systems

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We relate observables in open quantum systems with the topology of non-Hermitian models using the Keldysh path-integral method. This allows to extract an effective Hamiltonian from the Green's function which contains all the relevant topological information and produces ω-dependent topological invariants, linked to the response functions at a given frequency. Then, we show how to detect a transition between different topological phases by measuring the response to local perturbations. Our formalism is exemplified in a one-dimensional Hatano-Nelson model, highlighting the difference between the bosonic and the fermionic case.

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Introduction. Topological phases of matter were first discovered in electronic systems [1,2] and since then, their properties in equilibrium have been thoroughly studied [3–5]. During the past decade, the exploration of topological phases in nonequilibrium and dissipative systems has attracted great interest [6,7], triggered by the observation of topological effects in photonic lattices [8–10] and quantum simulators [11–13] and more lately applications in sensing [14–17] and amplification [18–22].

Characterizing topology in out-of-equilibrium systems is complicated by the presence of intrinsic gain and loss mechanisms which must be included in the dynamics. Several approaches have been considered to partially tackle this problem. In Ref. [7], the notion of topology by dissipation was introduced, showing that the engineering of jump operators can lead to the dissipative preparation of topological states. More recently, a criterion to define topological invariants from density matrices was proposed [23,24], and an extension of topological band theory to include non-Hermitian matrices has been introduced [25–30]. Whereas part of the phenomenology can be explained in terms of non-Hermitian effective Hamiltonians, this neglects quantum jumps and the quantum noise of the dissipative dynamics and, thus, cannot consistently describe the steady state nor the experimental observables of the system [31,32]. Therefore, the field would benefit from a more complete characterization of dissipative topological phases, which accounts for gain and loss mechanisms in many-body open quantum systems, describes bosons and fermions on the same footing and links the topological properties to measurable correlation functions.

We undertake this task by using the Keldysh path-integral formalism [33], which we use to characterize nontrivial topological phases of quantum open lattices and to establish a link between topological indices defined for non-Hermitian matrices and physical observables. (i) We establish a link between Keldysh-Green's functions of quantum open lattices with gain and loss terms and non-Hermitian matrices. (ii) We present a topological characterization of nonequilibrium Green's functions that leads us to define ω-dependent topological indices. Our formalism relies on a mapping from non-Hermitian matrices to topological insulator Hamiltonians. Nontrivial topological phases correspond to directional amplification of excitations at a particular frequency. (iii) Our Letter leads to a definition of topological phase transition in quantum open lattices. In that definition, the properties of observables in the frequency domain are characterized by a frequency-dependent topological index. Thus, the same quantum open lattice can exhibit different topological properties, depending on the frequency of the measured excitations (e.g., the frequency of the measured photons in the case of a photonic lattice). (iv) We show how topological properties can be tested by measuring the response of the system to perturbations. (v) We illustrate our results by studying the bosonic and fermionic realizations of the one-dimensional (1D) Hatano-Nelson (H-N) model, and highlight the role of particle statistics in our topological characterization..

Keldysh path integral. Consider a quantum system in a lattice with particles described by bosonic or fermionic operators, $\hat{\psi}_j$ and $\hat{\psi}_j^\dagger$. The dynamics can be described by the master equation for the density-matrix operator $\hat{\rho}$ [34],

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \sum_{j,l} \gamma_{jl}^{(d)} \left(\hat{\psi}_j \hat{\rho} \hat{\psi}_l^\dagger - \frac{1}{2}\{\hat{\psi}_l^\dagger \hat{\psi}_j, \hat{\rho}\} \right) + \sum_{j,l} \gamma_{jl}^{(p)} \left(\hat{\psi}_j^\dagger \hat{\rho} \hat{\psi}_l - \frac{1}{2}\{\hat{\psi}_l \hat{\psi}_j^\dagger, \hat{\rho}\} \right), \quad (1)$$

where $\hat{H}$ is the Hamiltonian and $\gamma^{(d)}$, $\gamma^{(p)}$ are matrices describing decay and gain processes, respectively.

An alternative to the operator formalism is the Keldysh path-integral method [35]. There, a time-slicing procedure and the insertion of coherent states leads to a set of fields $\{\psi_{j,\pm}, \bar{\psi}_{j,\pm}\}$, being $\pm$ the Keldysh contour where the fields act. These fields are complex variables in the bosonic case

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and independent Grassman variables in the fermionic one (i.e., $\bar{\psi}_{j,\pm} = \psi^*_{j,\pm}$ in the bosonic case). From Eq. (1) one finds the Keldysh action [33],

$$S = \int_{-\infty}^{t_f} dt \left[\sum_j (\bar{\psi}_{j,+} i\, \partial_t \psi_{j,+} - \bar{\psi}_{j,-} i\, \partial_t \psi_{j,-}) - i\mathcal{L} \right] \quad (2)$$

is defined in terms of the Lagrangian,

$$\begin{aligned} \mathcal{L} = &-i(H_+ - H_-) \\ &+ \sum_{j,l} \gamma^{(d)}_{jl} \left(\psi_{j,+}\bar{\psi}_{l,-} - \frac{1}{2}(\bar{\psi}_{j,+}\psi_{l,+} + \bar{\psi}_{j,-}\psi_{l,-}) \right) \\ &+ \sum_{j,l} \gamma^{(p)}_{jl} \left(\bar{\psi}_{j,+}\psi_{l,-} - \frac{1}{2}(\psi_{j,+}\bar{\psi}_{l,+} + \psi_{j,-}\bar{\psi}_{l,-}) \right). \end{aligned} \quad (3)$$

Here, $H_\pm$ results from $\hat{H}$ acting on the $\pm$ branch of the Keldysh contour. Remarkably, the action in Eq. (2) has the same form, irrespective of whether it is for bosons or fermions [35,36]. Also, note that gain $\gamma^{(p)}$ and loss $\gamma^{(d)}$ couple different Keldysh contours in Eq. (3), a signature of the nonequilibrium nature of the system.

From now on we focus on the steady state of quadratic lattice models, but transient dynamics and interacting systems can also be studied using this formalism [33,37–39]. Since the system is time-translation invariant, it is useful to Fourier transform the action to frequency domain and write $H_\pm = \sum_{jl} H_{jl} \bar{\psi}_{j,\pm}\psi_{l,\pm}$. We show below that this ω dependence translates to the observables, where ω physically corresponds to the energy at which the steady state of the system is being probed.

For practical calculations it is useful to perform a Keldysh rotation. In the bosonic case it corresponds to $\psi_\pm = (\psi_c \pm \psi_q)/\sqrt{2}$, and the bosonic action becomes

$$S_b = \int_\omega \Psi^\dagger \left(\begin{array}{c|c} 0 & \omega - \mathcal{H}_A \\ \hline \omega - \mathcal{H}_R & i\Gamma \end{array} \right) \Psi, \quad (4)$$

where we have defined $\int_\omega = \int \frac{d\omega}{2\pi}$ and written the fields in vector form $\Psi = (\vec{\psi}_c, \vec{\psi}_q)$, being $\vec{\psi}_\alpha = (\psi_{1,\alpha}, \psi_{2,\alpha}, \dots)$ and $\alpha = c, q$. The different blocks in Eq. (4) are given by $\mathcal{H}_{A/R} = H \pm i\frac{\gamma^{(d)} - \gamma^{(p)}}{2}$ and $\Gamma = \gamma^{(d)} + \gamma^{(p)}$. A key observation is that the non-Hermitian matrix $\mathcal{H}_R$ correspond to the effective Hamiltonian proposed to study the short-time dynamics of dissipative systems [30].

In the fermionic case, the Keldysh rotation is slightly different [40], but importantly, it changes the sign of the gain contribution in the action. This results in the following expression for fermions:

$$S_f = \int_\omega \bar{\Psi}^T \left(\begin{array}{c|c} \omega - \mathcal{H}_R & i\Gamma \\ \hline 0 & \omega - \mathcal{H}_A \end{array} \right) \Psi, \quad (5)$$

with blocks now given by $\mathcal{H}_{A/R} = H \pm i\frac{\gamma^{(d)} + \gamma^{(p)}}{2}$ and $\Gamma = \gamma^{(d)} - \gamma^{(p)}$. We show below that the sign change in the pump term will have important consequences in the resulting topological phase diagram.

Finally, to turn the formalism into an effective calculation tool we define the generating functional from which we can obtain correlation functions by functional differentiation,

$$\begin{aligned} Z[J_c, J_q, \bar{J}_c, \bar{J}_q] = \prod_{l=1}^N \int \mathcal{D}\psi_{l,c}\mathcal{D}\bar{\psi}_{l,c}\mathcal{D}\psi_{l,q}\mathcal{D}\bar{\psi}_{l,q} e^{iS} \\ \times e^{i\int_\omega (\bar{j}_{l,c}\psi_{l,q} + \bar{j}_{l,q}\psi_{l,c} + j_{l,c}\bar{\psi}_{l,q} + j_{l,q}\bar{\psi}_{l,c})}, \end{aligned} \quad (6)$$

where we have introduced the sources $J_\alpha = (j_{1,\alpha}, j_{2,\alpha}, \dots)$. The final form of the generating functional is obtained by Gaussian integration,

$$Z[J, \bar{J}] = e^{-i\int_\omega \bar{J}^T(\omega)G(\omega)J(\omega)}. \quad (7)$$

It is a quadratic form of the sources $J = (J_c, J_q)$ with G obtained from the inverse of the action [41] [Eq. (4) for the bosonic and Eq. (5) for the fermionic cases]. In general, $G(\omega)$ is a 2×2 block matrix with entries,

$$G_{A/R} = \frac{1}{\omega - \mathcal{H}_{A/R}}, \quad G_K = G_R^{-1}(-i\Gamma)G_A^{-1}, \quad (8)$$

being $G_{A/R}$ is the advanced and retarded and G_K is the Keldysh-Green's function. From Eq. (7) it is possible to obtain all correlation functions by functional differentiation.

Concretely, here we are interested in two-point correlation functions of the form $\mathcal{M}_{jl}(\omega) = \int d\tau \langle \psi_j^\dagger(t)\psi_l(t+\tau)\rangle e^{-i\omega\tau}$, which can be expressed in terms of Green's functions as $\mathcal{M}(\omega) = \frac{i\eta}{2}[G_K(\omega) + G_A(\omega) - G_R(\omega)]$, where $\eta = \pm 1$ for bosons and fermions [33]. Remarkably, this expression can be simplified in the case of gain and loss systems [see the Supplemental Material (SM)] [42]:

$$\mathcal{M}(\omega) = G_R(\omega)\gamma^{(p)}G_A(\omega). \quad (9)$$

Note that the last term in Eq. (9) is independent of the particle statistics and relates the two-point correlations with the non-Hermitian matrices $\mathcal{H}_{A/R}$ and with the incoherent pump of particles in the system $\gamma^{(p)}$.

Topological properties. We address now the topological characterization in terms of $G_R(\omega)$, which has also been considered as a topological tool in different situations [27,43–45] and is related to the electromagnetic response in topological field theories [28,29,46].

For that, we first define the doubled Hamiltonian $\tilde{\mathcal{H}}(\omega)$,

$$\tilde{\mathcal{H}}(\omega) = \begin{pmatrix} 0 & \omega - \mathcal{H}_R \\ \omega - \mathcal{H}_A & 0 \end{pmatrix}, \quad (10)$$

which is Hermitian by construction (note that $\mathcal{H}_R^\dagger = \mathcal{H}_A$) and has a built-in chiral symmetry due to its block structure. The doubled Hamiltonian has been used as a formal technique in the classification of topological phases of non-Hermitian systems with the poing-gap criterion [19,25,26,47].

In this Letter, $\tilde{\mathcal{H}}(\omega)$ will allow us to link Hermitian topological invariants and the nonequilibrium Green's functions. This is because $\tilde{\mathcal{H}}(\omega)$ can serve us to compute the inverse of $\omega - \mathcal{H}_R$ [21] with the advantage that its eigenvalues are insensitive to the skin effect [27,48–51]. To see this, note that due to the artificial chiral symmetry, the eigenstates of $\tilde{\mathcal{H}}$ can be written as $\tilde{\mathcal{H}}\binom{u_n}{\pm v_n} = \pm\tilde{\epsilon}_n\binom{u_n}{\pm v_n}$ with $\tilde{\epsilon}_n > 0$. For example, in 1D, if $\tilde{\mathcal{H}}(\omega)$ is in a topologically nontrivial phase, zero-energy states will have $\tilde{\epsilon}_n \approx 0$ and the corresponding

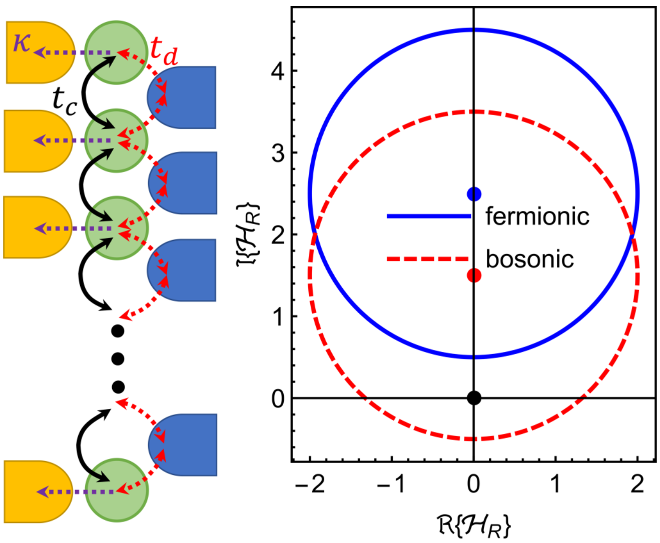

FIG. 1. (Left) Schematic for the H-N model. Sites in the array coherently couple with hopping t_c and auxiliary sites are dissipatively coupled via κ and t_d. (Right) Complex plane plot of the eigenvalues of $\mathcal{H}_R$ for the bosonic (red) and the fermionic (blue) H-N model with periodic boundary conditions (PBCs). The fermionic case never encloses the origin and remains trivial. Red and blue dots show the collapse of the eigenvalues due to the skin effect for open boundary conditions (OBCs).

vectors u_n and v_n are left and right localized edge states. By using this observation we can easily derive (see the SM [42]),

$$G_R(\omega)_{jl} = \sum_n \frac{1}{\tilde{\epsilon}_n}(v_n)_j(u_n)_l^*, \tag{11}$$

which relates the non-Hermitian Green's function and the eigenstates of the doubled Hamiltonian. Importantly, Eq. (11) indicates that topological zero-energy modes of the doubled Hamiltonian $\tilde{\mathcal{H}}(\omega)$ dominate the correlation functions [cf. Eq. (9)], through the inverse energy factor $1/\tilde{\epsilon}_n$. This relation between $\tilde{\mathcal{H}}(\omega)$ and $G_R(\omega)$ shows that one can perform the topological analysis of $\tilde{\mathcal{H}}(\omega)$ in terms of the tenfold way [52] applied to chiral symmetric Hamiltonians and directly link its topological phases with physical observables determined by $G_R(\omega)$. This classification results in a smaller number of different phases than those predicted in Ref. [26] since it is restricted to those topological properties that are directly related to observables of the quantum open lattice.

Hatano-Nelson model. The H-N model is a well-known example of topology induced by dissipation [25,53,54], although it was originally proposed to described the flux lines in superconductors.

In the bosonic version, the Hamiltonian $\hat{H} = \sum_{i,j} t_{i,j}\hat{a}_i^\dagger\hat{a}_j$ describes hopping in a lattice with a background gauge field ϕ, where $t_{i,j} = \omega_0\delta_{i,j} + t_c(e^{i\phi}\delta_{i,j-1} + e^{-i\phi}\delta_{i,j+1})$ and ω_0 describes the detuning from the cavity frequency. In addition, the particle dynamics is influenced by local loss $\gamma_{i,j}^{(d)} = \kappa\delta_{i,j}$ and nonlocal gain $\gamma_{i,j}^{(p)} = 4t_d\delta_{i,j} + 2t_d(\delta_{i,j-1} + \delta_{i,j+1})$ [see the schematic in Fig. 1(left)], which lead to anisotropic dissipative hopping and nonreciprocity. The implementation of the bosonic model can be carried out, for example, by using reservoir engineering and Floquet techniques for inducing synthetic gauge fields [19,55,56].

From Eq. (4) it is straightforward to write the different blocks of the bosonic action S_b, for the case of PBCs and OBCs, respectively). In the case of PBCs each block corresponds to

$$\mathcal{H}_{A/R} = \omega_0 + 2t_c\cos(k-\phi) \pm i\frac{\kappa - 8t_d\cos^2\left(\frac{k}{2}\right)}{2}, \tag{12}$$

$$\Gamma = \kappa + 8t_d\cos^2\left(\frac{k}{2}\right). \tag{13}$$

According to the standard classification of non-Hermitian matrices, $\mathcal{H}_R$ belongs to the AI class because it lacks all symmetries [26]. In consequence, its winding number is nonzero when the complex eigenvalues form a point gap which encloses the origin [see Fig. 1 (right)]. Importantly, the classification of $\mathcal{H}_R$ is ω independent and only indicates the presence of a topological amplification phase, neglecting the range of ω where states are amplified.

If we instead classify the doubled Hamiltonian $\tilde{\mathcal{H}}$ using the tenfold way, we find that it belongs to the AIII class due to the artificial chiral symmetry. Its topological phase is characterized by a winding number which can be written as

$$W_1(\omega) = \int_{-\pi}^{\pi}\frac{dk}{2\pi i}\partial_k\log(\omega - \mathcal{H}_R). \tag{14}$$

Note that its ω dependence naturally arises from the analysis of $G_R(\omega)$ and is physically motivated by its role in the behavior of the two-point functions in Eq. (9), which indicates that the observation of the topology depends on the energies at which the system is experimentally being probed. In addition, this is in agreement with the ω-dependent topological invariants predicted in dissipative systems [20,21,29]. Figure 2 (top) shows the value of $W_1(\omega)$ for different loss rates κ, which affects the range of frequencies which can be amplified. Figure 2 (middle) shows the eigenvalues of $\tilde{\mathcal{H}}$ for PBCs (red) with the appearance of a pair of topological boundary modes for OBCs (black dots). The lack of skin effect in the eigenvalues of $\tilde{\mathcal{H}}$ and the match between the appearance of boundary modes and changes in $W_1(\omega)$ are obvious advantages with respect to $\mathcal{H}_R$. However, the skin effect still is important for the stability of the system and its presence in the eigenvalues, crucial to understand the physics.

In analogy with Hermitian topology we can see in Fig. 2 (bottom) that for bosons and PBCs, the eigenvalues of $\tilde{\mathcal{H}}(\omega)$ vs k link a gap closure with the phase boundary.

Physically, the topological phase in the bosonic H-N model corresponds to unidirectional amplification. The ω dependence in $W_1(\omega)$ is crucial as it indicates that topological amplification happens for a finite range of frequencies only. This is interesting to relate topology in dissipative systems with its experimental detection. The simplest way consists of detecting the number of particles at each site $\langle a_j^\dagger a_j\rangle$, which in the amplification phase shows an exponential dependence with the array length [21]. This, however, does not characterize the ω dependence of $W_1(\omega)$, even if the number of particles is measured at different ω's because there is not a sharp transition as a function of ω (the gap closes continuously).

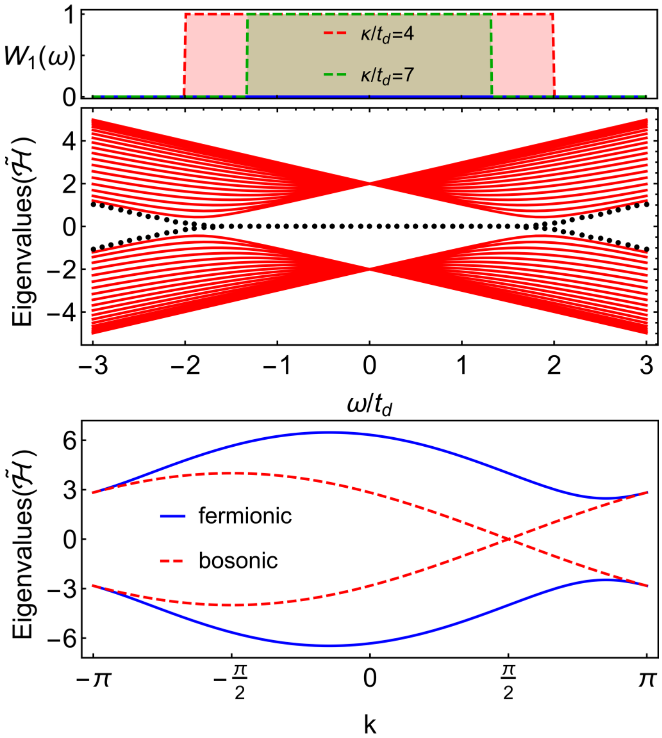

FIG. 2. (Top) $W_1(\omega)$ for different values of κ/t_d. In the fermionic case $W_1(\omega)$ is always zero (blue). (Middle) Eigenvalues of $\tilde{\mathcal{H}}$ vs ω for κ/t_d=4. The spectrum for PBCs is shown in red, whereas black dots indicate the two boundary modes with OBCs. (Bottom) Eigenvalues of $\tilde{\mathcal{H}}$ vs k for κ/t_d=4 and $\omega/t_d = 2$. All plots consider $t_c/t_d = 1$ and $\phi = \pi/2$

An alternative approach, for example, is to measure the response function to a perturbation of the frequency at site l by adding the term $H_\text{I} = \Omega_l \hat{\psi}_l^\dagger \hat{\psi}_l$ to the Hamiltonian. Then, we define a susceptibility as the variation in the excitation number at frequency ω and site j, $\chi_{jl}(\omega) = d\langle n_j(\omega)\rangle/d\Omega_l$, where $\langle n_j(\omega)\rangle = \mathcal{M}_{jj}(\omega)$, and find (see the SM [42]),

$$\chi_{jl}(\omega) = G^R_{jl}(\omega)\mathcal{M}_{lj}(\omega) + \mathcal{M}_{jl}(\omega)G^A_{lj}(\omega). \qquad (15)$$

Interestingly, Fig. 3 shows that plotting in the logarithmic scale the susceptibility between different sites allows to directly detect the critical point. This is a consequence of the topological phase transition to unidirectional amplification where signals are exponentially amplified with the number of sites and their scaling when measured at different sites is drastically affected. Importantly, this indirect detection gives very accurate results, even for small arrays (Fig. 3 gives identical results for sizes of $N = 10,\ 15,$ and 20 sites). Furthermore, we have checked its resilience to disorder in the the dissipative hopping and found that the critical point is correctly captured until it is of the same order as the hopping.

Fermions vs bosons. The symmetry class of $\mathcal{H}_R$ does not depend on whether we consider a fermionic or a bosonic lattice, however, we can show that particle statistics drastically affects topological phases. The key observation is the sign change in the pump term in the fermionic case Eq. (5), which physically accounts for Pauli exclusion as opposed to bosonic amplification. The consequences of this for the H-N model can be derived from Eq. (12), which in the fermionic case leads to $\text{Im}(H_R) \propto -\kappa - 8t_d\cos^2(k/2)$. Since $\kappa,\ t_d > 0$, $\text{Im}(H_R)$ does not change sign as a function of k, which is a necessary condition for $W_1(\omega) \neq 0$ [see Fig. 1 (right)]. Its consequences are also shown in Fig. 2, where the winding number is always zero, and the band structure remains always gapped. We, thus, conclude that the fermionic H-N model has a topologically trivial phase diagram.

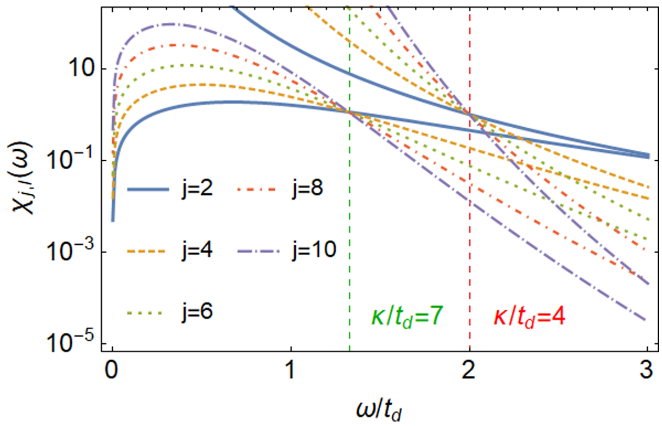

FIG. 3. Logarithmic scale plot of $\chi_{jl}(\omega)$ between sites $l = 1$ and $j = 2,\ 4,\ 6,\ 8,$ and 10. We have considered the bosonic case for an array with $N = 10$ sites, $t_c/t_d = 1$ and $\phi = \pi/2$. The crossing at a certain value of ω allows to extract the position of the critical point, indicated for the cases of $\kappa/t_d = 4$ and 7.

Formally, the limitations found in the fermionic H-N model could be surpassed if the diagonal and nondiagonal elements of the matrix $\gamma^{(p)}$ could be independently tuned, which would free the model from the $\cos^2(k/2)$ dependence in $\mathcal{H}_R$. However, from the derivation of the H-N master equation it can be shown that $\gamma^{(p)}_{jj} = 2\gamma^{(p)}_{j,j+1}$ (see the SM [42]), which accounts for the fact that dissipative couplings between sites induced by a common bath inevitably come together with local dissipation terms. This result has a clear physical meaning since the directional amplification that would result from a nontrivial topological phase is not expected to occur in a fermionic lattice.

Conclusions and outlook. We have used the Keldysh path-integral formalism to link topological properties of non-Hermitian matrices with the observables of open quantum systems. We have shown that this leads to ω-dependent topological invariants that capture certain properties of the nonequilibrium Green's functions. Our formalism also allows us to obtain a unified description of bosonic and fermionic open models, and we have unveiled fundamental differences between the topological phases of the two cases. We have applied our theoretical framework to the 1D Hatano-Nelson model, and we have explicitly shown how physical observables and response functions of that model can be used to detect nontrivial topological phases. Our theory leads to an unambiguous definition of topological phases and topological phase transitions in quantum open systems, based on their experimental signatures.

Our Letter paves the way for further applications of the Keldysh theoretical machinery [33] in the description of topological gain and loss systems. In particular, adding interactions to the theoretical framework presented here would allow us to investigate topological interacting quantum open systems. One can also apply our ideas to transient physics rather than to the steady state [57–59]. From a practical point of view, our Letter can lead to the design of quantum metrology or sensing protocols [14,16,17] by exploiting the extreme sensitivity of the system to input fields and perturbations in nontrivial topological phases. Our results are relevant for current experimental setups in photonic lattices where the H-N model could be implemented using Floquet techniques and reservoir engineering. Similar techniques may lead to the investigation of fermionic models by using, for example, arrays of coupled quantum dots [60,61].

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