Letter

Statistically significant tests of multiparticle quantum correlations based on randomized measurements

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We consider statistical methods based on finite samples of locally randomized measurements in order to certify different degrees of multiparticle entanglement in intermediate-scale quantum systems. We first introduce hierarchies of multiqubit criteria, satisfied by states which are separable with respect to partitions of different size, involving only second moments of the underlying probability distribution. Then, we analyze in detail the statistical error of the estimation in experiments and present several approaches for estimating the statistical significance based on large deviation bounds. The latter allows us to characterize the measurement resources required for the certification of multiparticle correlations, as well as to analyze given experimental data in detail.

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I. INTRODUCTION

Noisy intermediate-scale quantum (NISQ) devices involving a few dozen qubits are considered a stepping stone towards the ultimate goal of building a fault-tolerant quantum computer. While impressive achievements have been made in this direction, e.g., in terms of the precision of the individual qubit architectures [1–3], the common challenge is to scale up the considered devices and, at the same time, maintain the established accuracy [4,5]. In particular, the collective performance of the whole system of interacting qubits is of central concern in this respect.

Several approaches aimed at a verification of correlation properties of such multiparticle quantum systems have been discussed in the literature [6]. On the one hand, there are efficient protocols in terms of the required measurement resources if the experiment is expected to result in specific states, e.g., entanglement witnessing [7], self-testing [8], or direct fidelity estimation [9,10]. On the other hand, approaches which rely on few or no expectation about the underlying quantum state are usually very resource intensive and thus do not scale favorably with increasing system sizes, e.g., quantum state tomography [11,12]. Furthermore, intermediate strategies exist which do not aim for a full mathematical description of the system but rather focus on specific statistical properties. The latter can reduce the required measurement resources considerably at the expense of a nonvanishing statistical error and do not assume any prior information about the state [13–17].

Recently there has been much attention on protocols based on statistical correlations between outcomes of randomized measurements [18–37] (see Fig. 1). The latter allow one to infer several properties of the underlying system, ranging from structures of multiparticle entanglement [23,24,36], over subsystem purities [29,30], to fidelities with respect to certain target states or even another quantum device [13,33]. At the core of all those approaches is the idea to perform measurements in randomly sampled local bases leading to ensembles of measurement outcomes whose distributions provide a fingerprint of the system's correlation properties. Concerning resources required for statistically significant tests, scaling properties have been derived for the case of bipartite entanglement [28,34].

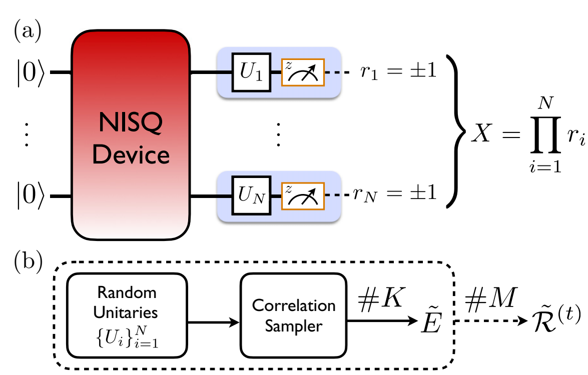

FIG. 1. Characterization of a noisy intermediate-scale quantum (NISQ) device through locally randomized measurements. (a) A measurement of N qubits in random local bases defined through the set of local unitary transformations $\{U_i\}_{i=1}^N$ resulting in a correlation sample X. (b) Repetition of the measurement protocol presented in (a) for M sets of randomly sample measurement bases and, respectively, K individual projective measurements yields an estimate of the moments (2).

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In this work we present detailed statistical methods to certify multiparticle entanglement structures in systems consisting of many qubits. First, we derive criteria in terms of second moments of randomized measurements for different forms of multiparticle entanglement allowing one to infer the entanglement depth. Second, we present several rigorous approaches for the analysis of the underlying statistical errors, based on large deviation bounds, which are of great relevance for practical experiments. As we will see, our results may directly be used in current experiments using Rydberg atom arrays or superconducting qubits [38–40].

II. MOMENTS OF RANDOM CORRELATIONS

We consider a mixed quantum state of N qubits described by the density matrix ρ. In order to characterize this state we follow a strategy based on locally randomized measurements. Each random measurement is characterized through a set of random bases $\{(|u_n^{(0)}\rangle = U_n|0_n\rangle, |u_n^{(1)}\rangle = U_n|1_n\rangle)\}_{n=1,\dots,N}$, with $\{U_n\}_{n=1,\dots,N}$ picked from the unitary group $\mathcal{U}(2)$ according to the Haar measure. Further on, we can associate to each element $(|u_n^{(0)}\rangle = U_n|0_n\rangle, |u_n^{(1)}\rangle = U_n|1_n\rangle)$, with $n \in \{1,\dots,N\}$, a direction $\boldsymbol{u}_n$ on the unit sphere S^2 with components $[\boldsymbol{u}_n]_i = \mathrm{tr}[\sigma_{\boldsymbol{u}_n}\sigma_i]$, with $i \in \{x, y, z\}$, and $\sigma_{\boldsymbol{u}_n} = U_n\sigma_z U_n^\dagger$ [see Fig. 1(a)]. One such random measurement then leads to the correlation function

$$E(\boldsymbol{u}_1,\dots,\boldsymbol{u}_N) = \langle\sigma_{\boldsymbol{u}_1}\otimes\dots\otimes\sigma_{\boldsymbol{u}_N}\rangle_\rho, \tag{1}$$

which provides a random snapshot of the correlation properties of the output state ρ. In order to get a more complete picture we consider the corresponding moments

$$\mathcal{R}^{(t)} = \frac{1}{(4\pi)^N}\int_{S^2} d\boldsymbol{u}_1\dots\int_{S^2} d\boldsymbol{u}_N[E(\boldsymbol{u}_1,\dots,\boldsymbol{u}_N)]^t, \tag{2}$$

where t is a positive integer and $d\boldsymbol{u}_i = \sin\theta_i d\theta_i d\phi_i$ denotes the uniform measure on the sphere S^2. The moments (2) are by definition invariant under local unitary transformation and thus good candidates for the characterization of multiparticle correlations.

III. MULTIPARTICLE ENTANGLEMENT CHARACTERIZATION

In a multiparticle system one defines k-separable states, with $k \in \{2,\dots,N\}$, as those states which can be written as a statistical mixture of k-fold product states $|\Psi^{(k)}\rangle = |\phi_1\rangle\otimes\dots\otimes|\phi_k\rangle$. Hence, by disproving that a state belongs to the above separability classes, one can infer different degrees of multiparticle entanglement, with the strongest form given by states which are not even 2 separable, i.e., genuinely multiparticle entangled (GME). The concept of k separability is a widely used approach to benchmark experiments [41–44] and also has been identified as a resource in quantum metrology applications [45–48].

To begin with we note the well-known criterion $\mathcal{R}^{(2)} \leqslant 1/3^N$ which holds for all N-separable (i.e., fully separable) states [21,22,49–52]. Furthermore, biseparability bounds on combinations of second moments of marginals of three-qubit systems can be formulated [36,53,54]. However, so far no useful bounds on the full N-qubit moments (2) for the detection of GME have been found. Here we close this gap and prove in Sec. I.D of the Supplemental Material (SM) [55] that all k-separable mixed states fulfill the bounds

$$\mathcal{R}^{(2)} \leqslant \frac{1}{3^{N-k+1}} \times \begin{cases} 2^{N-(2k-1)}, & N \text{ odd},\\ 2^{N-(2k-1)}+1, & N \text{ even}, \end{cases} \tag{3}$$

with $k = 2,\dots,\lfloor(N-1)/2\rfloor$. Equation (3) thus provides a hierarchy of entanglement criteria whose violation for fixed k implies that the given state is at most $(k-1)$ separable. This implies that it has an entanglement depth [63,64] of at least $\lceil N/(k-1)\rceil$, but possibly stronger bounds for the depth can be derived based on the concept of producibility; see SM [55], Sec. I.E. In any case, only states which are GME can reach the maximum value of the second moment $\mathcal{R}^{(2)}$, which is known to be attained by the N-qubit GHZ states [65,66]:

$$\mathcal{R}^{(2)}_{|\mathrm{GHZ}_N\rangle} = \frac{1}{3^N} \times \begin{cases} 2^{N-1}, & N \text{ odd},\\ 2^{N-1}+1, & N \text{ even}, \end{cases} \tag{4}$$

with $|\mathrm{GHZ}_N\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. Note that in systems consisting of larger local dimensions it is in general not true that states which maximize the corresponding generalized second moment $\mathcal{R}^{(2)}$ are GME [36,65,66].

In the following we study the performance of the criteria (3) by considering the noisy N-qubit GHZ states $\rho^{(N)}_{\mathrm{GHZ}}(p) := p\mathbb{1}/2^N + (1-p)|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N|$, which yields $\mathcal{R}^{(2)}_{\mathrm{GHZ}}(p,N) = (1-p)^2\mathcal{R}^{(2)}_{|\mathrm{GHZ}_N\rangle}$ and thus minimizes (maximizes) $\mathcal{R}^{(2)}$ for $p = 1$ ($p = 0$). We note that in practical situations where errors occur locally one can estimate the global depolarization probability p by combining local depolarization rates corresponding for instance to average gate errors (see [55], Sec. III.B). The threshold value of p up to which (3) is violated as a function of N and k thus reads

$$p^* = 1 - f(N,k)\left(\frac{3}{4}\right)^{\frac{k-1}{2}}, \tag{5}$$

where $f(N,k) = 1$ for odd N and $f(N,k) = \sqrt{(4^k + 2^{N+1})/(4 + 2^{N+1})}$ for even N (see Fig. 2). As is clear from Eq. (5), the threshold p^* is independent of N, for odd N, and coincides with the asymptotic threshold in the limit $N \to \infty$, where $f(N,k) \to 1$. The latter is strictly smaller than 1, which shows that Eq. (3) can be applied also in systems consisting of a large number of parties. Furthermore, our methods also work in the regime of low fidelities, i.e., large $p^{*'}$s, where fidelity-based witnesses fail (see Fig. 2). Furthermore, in the lower panel of Fig. 2 we analyze the performance of the criteria (3) for GHZ states with unequal amplitudes, i.e., $|\mathrm{GHZ}^{(N)}_\alpha\rangle = \sqrt{(1+\alpha)/2}|0\rangle^{\otimes N} + \sqrt{(1-\alpha)/2}|1\rangle^{\otimes N}$, with $0 \leqslant \alpha \leqslant 1$ (see also Sec. I.C of [55]). Lastly, we note that the criteria (3) become useful only for a certain minimum number of qubits depending on the value of k, e.g., GME detection is only possible for $N > 4$.

IV. ESTIMATION OF THE MOMENTS

In the following we assume that a finite sample of M random measurement bases is taken, each of which undergoes K individual projective measurements. We thus denote the outcomes of a single random measurement on N

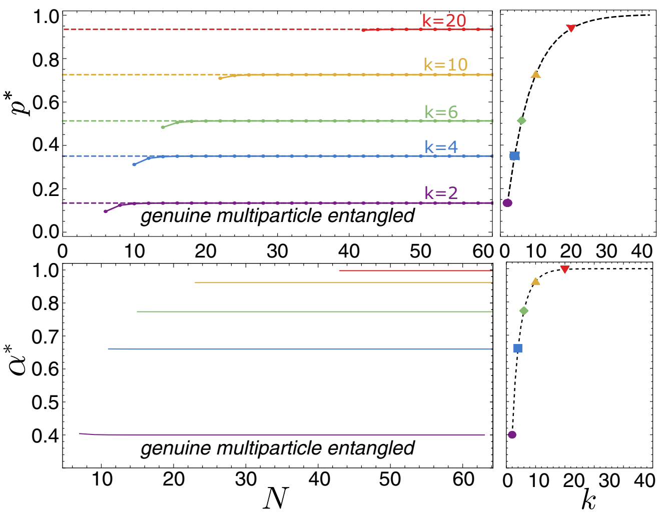

FIG. 2. Threshold values p^* (top) and α^* (bottom) up to which the noisy and the asymmetric GHZ state, $\rho_{\mathrm{GHZ}}^{(N)}(p)$ and $|\mathrm{GHZ}_{\alpha}^{(N)}\rangle$, respectively, are detected to be not 2 (violet, bottom), 4 (blue), 6 (green), 10 (yellow), and 20 separable (red, top) as a function of the number of qubits N. In the upper panel dots connected by solid lines represent values of p^* for even N; dashed lines correspond to the case of odd N. Plots in the right column show the asymptotic values of p^* and respectively α^* in the limit $N \to \infty$ as a function of the parameter k. The exemplary values corresponding to the left plot are highlighted by colored markers.

qubits by $\{r_1, \dots, r_N\}$, with $r_i = \pm 1$, and define the corresponding correlation sample as $X = \prod_{i=1}^{N} r_i$ [see Fig. 1(a)]. Given a fixed measurement basis we can thus model the binary outcomes of X through a binomially distributed random variable $\tilde{Y}$ with probability P, i.e., the probability that an even number of the measurement outcomes r_i result in -1, and K trials. The corresponding unbiased estimators $\tilde{P}_k$ of P and its kth powers, respectively, are then given by $\tilde{P}_k = \tilde{P}_{k-1}[K\tilde{P}_1 - (k-1)]/[K-(k-1)]$, with $\tilde{P}_1 = \tilde{Y}/K$ (see Sec. II.B of the SM [55]).

Further on, the unbiased estimators of the respective tth powers of Eq. (1) read

$$\tilde{E}_t = (-1)^t \sum_{k=0}^{t} (-2)^k \binom{t}{k} \tilde{P}_k, \tag{6}$$

which, in turn, allows us to define faithful estimators of the moments (2), resulting from M sampled measurement bases:

$$\tilde{\mathcal{R}}^{(t)} = \frac{1}{M} \sum_{i=1}^{M} [\tilde{E}_t]_i. \tag{7}$$

Given Eqs. (6) and (7), our goal is now to gauge the statistical error of an estimation $\tilde{\mathcal{R}}^{(t)}$ as a function of the number of subsystems N. More precisely, we aim for lower bounds on the total number of required measurement samples $M_{\mathrm{tot}} = M \times K$ needed in order to estimate $\mathcal{R}^{(t)}$ with a precision of at least δ and confidence γ, i.e., such that $\mathrm{Prob}(|\tilde{\mathcal{R}}^{(t)} - \mathcal{R}^{(t)}| \leqslant \delta) \geqslant \gamma$ for $M_{\mathrm{tot}} \geqslant M(\gamma, t)$.

In order to achieve this goal we exploit concentration inequalities which provide deviation bounds on the probability $1 - \mathrm{Prob}(|\tilde{\mathcal{R}}^{(t)} - \mathcal{R}^{(t)}| \leqslant \delta)$, i.e., the probability that the estimator deviates from the mean value by a certain margin. In Sec. II of the SM [55] we discuss three such approaches which differ in their assumptions on the random variable $\tilde{\mathcal{R}}^{(t)}$, based on the Chebyshev-Cantelli and Bernstein inequality, as well as a more general approach using Chernoff bounds [67]. For instance, for the Chebyshev-Cantelli inequality this leads to a minimal two-sided error bar of $\tilde{\mathcal{R}}^{(t)}$ that guarantees the confidence γ:

$$\delta_{\mathrm{err}}(\gamma) = \sqrt{\frac{1+\gamma}{1-\gamma} \mathrm{Var}(\tilde{\mathcal{R}}^{(t)})}, \tag{8}$$

where $\mathrm{Var}(\tilde{\mathcal{R}}^{(t)})$ denotes the variance of the estimator (7) which can be evaluated using the properties of the binomial distribution. For instance, in the case of the second moment $\mathcal{R}^{(2)}$ we find that the variance reads

$$\mathrm{Var}(\tilde{\mathcal{R}}^{(2)}) = \frac{1}{M}[A(K)\mathcal{R}^{(4)} + B(K)\mathcal{R}^{(2)} + C(K) - (\mathcal{R}^{(2)})^2], \tag{9}$$

with $A(K) = (K-2)(K-1)C(K)/2$, $B(K) = 2(K-2)C(K)$, and $C(K) = 2/[K(K-1)]$, which are determined through the properties of the binomial distribution (see Sec. II.B of the SM [55] for a derivation).

Hence the precision of an estimation of the second moment is determined through Eqs. (8) and (9) and thus depends on the state under consideration. However, by bounding the variance (9) from above we can consider a worst-case scenario and determine the required values of M and K in order to reach a precision of at least δ with confidence γ [55]. To do so, we use the conjecture that the maximum of the fourth moment $\mathcal{R}^{(4)}$, for $N > 4$, is attained by the N-qubit GHZ states. While this assumption is backed by numerical evidence we leave its proof for future investigations.

In Fig. 3(a) we present the scaling of the required number of random measurement bases M with the number of subsystems N for different values of K. First, we note that the present statistical treatment allows for an improvement over the 3^N measurement settings that are required in order to evaluate the second moment exactly using a quantum design [21–24], at the expense of a nonzero statistical error from the unitary sampling. Second, the required number of random measurement settings M depends strongly on the chosen number of projective measurements per random unitary. More precisely, the curves in Fig. 3(a) scale as $O(1.2^N)$ up to a threshold value that depends on K; beyond that the scaling changes to $O(2.25^N)$.

The minimum of $M_{\mathrm{tot}} = M \times K$ is reached for an optimal ratio between M and K which can be obtained analytically (see SM [55]) leading to $M_{\mathrm{tot}}^{(\mathrm{opt})} = M(K^{(\mathrm{opt})}) \times K^{(\mathrm{opt})}$, as presented in Fig. 3(b). We thus find that the total measurement budget follows the overall scaling law $O(1.5^N)$. Furthermore, while the required measurement resources increase slightly with higher precision, i.e., smaller δ, the asymptotic scaling remains the same. As comparison, we present in the same figure the value $M_{\mathrm{tot}}^{(\mathrm{opt})}$ obtained from the Bernstein inequality. The latter avoids the additional assumption about the upper bound on the variance (9) but scales worse with the system

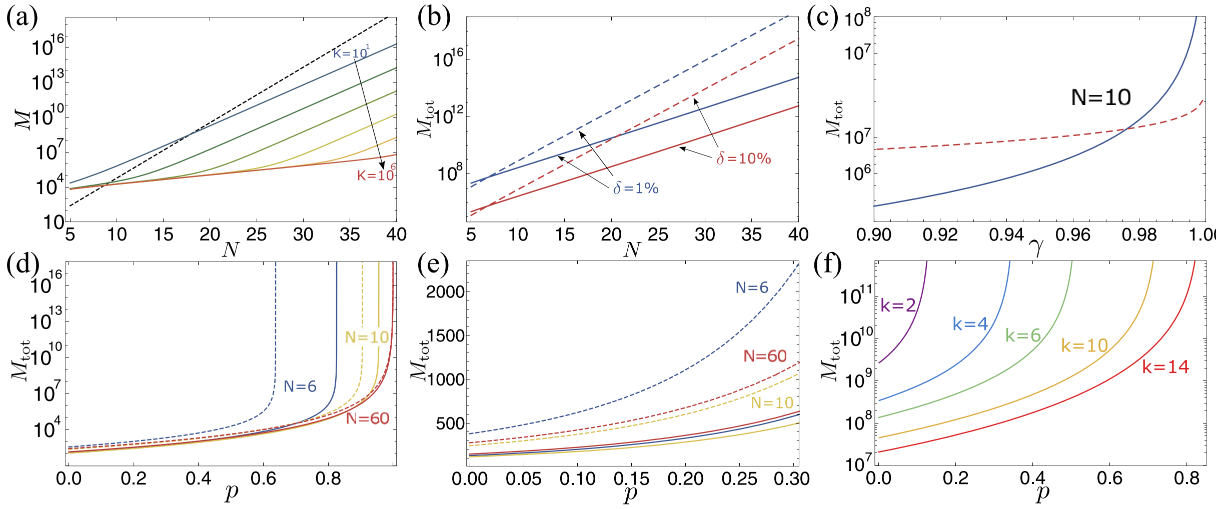

FIG. 3. (a) Number M of sampled measurement bases required to estimate $\mathcal{R}^{(2)}$ with an accuracy of at least 10% and confidence $\gamma = 90\%$ as a function of the number of subsystems N for $K = 10, 10^2, \ldots, 10^6$ (solid curves from top to bottom), based on Chebyshev-Cantelli inequality. The black dashed line indicates the required measurement settings in order to exactly determine $\mathcal{R}^{(2)}$. (b) Total measurement budget $M_{\text{tot}}^{(\text{opt})}$ required for an estimation of $\mathcal{R}^{(2)}$ with accuracy $\delta = 1\%$ (blue curve) and 10% (red curve) as a function of N obtained from Chebyshev-Cantelli (solid) and Bernstein (dashed) inequality. (c) $M_{\text{tot}}^{(\text{opt})}$ as a function of γ for $N = 10$ and $\delta = 10\%$ obtained from Chebyshev-Cantelli (solid) and Bernstein (dashed) inequality. (d),(e) Measurement budget $M_{\text{tot}}^{(\text{opt})}$ obtained from Chebyshev-Cantelli inequality required to certify with confidence $\gamma = 90\%$ that $\rho_{\text{GHZ}}^{(N)}(p)$ is entangled (solid lines) or not in the W class (dashed lines) for $N = 6$ (blue), $N = 10$ (yellow), and $N = 60$ (red) qubits as a function of the p. (e) Zoom in of (d) for $0 \leqslant p \leqslant 0.3$. (f) Same plot as in (d) but for the violation of the k-separability criteria (3), with $k = 2$ (violet, left), 4 (blue), 6 (green), 10 (yellow), and 14 (red, right), for $N = 30$.

size. On the other hand, for fixed N, the scaling of $M_{\text{tot}}^{(\text{opt})}$ with the confidence γ is improved, as illustrated in Fig. 3(c).

V. FINITE STATISTICS ENTANGLEMENT CHARACTERIZATION

In order to certify the violation of the k-separability bounds (3) one has to ensure that the statistical error δ of $\tilde{\mathcal{R}}^{(2)}$ does not exceed the amount of the observed violation. This can be ensured by choosing the total number of measurements appropriately according to the previously discussed methods. Even more, since we aim to exclude the hypothesis that the state is, e.g., k separable, we can improve our procedure by invoking upper bounds on the variances (9) for k-separable states, respectively, instead of the overall upper bound used in Figs. 3(a)–3(c). As this can only be done using the Chebyshev-Cantelli inequality we will focus on this approach in the following.

We demonstrate the above procedure using the state $\rho_{\text{GHZ}}^{(N)}(p)$ and first determine the total number of measurements $M_{\text{tot}}^{(\text{opt})}$ required to certify that it is not fully separable (i.e., $\mathcal{R}^{(2)} \leqslant 1/3^N$) and not in the class of W states [21–24] [see Figs. 3(d) and 3(e) and also Sec. I.C of [55]]. We find that already moderate numbers of $M_{\text{tot}}^{(\text{opt})} \lesssim 2000$ are enough to certify their violation for up to $N = 60$ qubits. Divergences displayed in Fig. 3(d) are due to the asymptotically decreasing difference between the true value of $\mathcal{R}^{(2)}$ and the respective bound of the targeted criterion. A similar behavior is observed for the violation of different degrees of k separability [see Fig. 3(f)]. In this case $M_{\text{tot}}^{(\text{opt})}$ is generally on a higher level due to the increasing tightness of the bounds (3) for smaller k.

VI. EXPERIMENTAL IMPLICATIONS

Lastly, in order to demonstrate the applicability of our framework, we refer to recent experiments producing GHZ states with limited fidelity [38–40,68]. For instance, in Ref. [38] a GHZ state of 11 qubits was produced with fidelity $F \approx 0.75$. By applying our formalism we can thus show that the state contains at least five- or seven-particle entanglement by performing in total of the order of 10^5 or 10^6 measurements, respectively (see Sec. III.A of the SM [55]). Note that these numbers are still moderate as compared to a full state tomography. Furthermore, we show that the 20 qubit GHZ state of fidelity $F \approx 0.44$ (see Ref. [38]) contains at least four- or five-particle entanglement by performing in total of the order of 10^7 measurements. We emphasize that such insights cannot be reached in terms of the fidelity, since fidelities up to $1/2$ can be reproduced by fully separable states.

VII. CONCLUSIONS

We have discussed statistical methods allowing for the characterization of multiparticle quantum systems based on randomized measurements. In particular, we presented criteria for the detection of different types of multiparticle correlations of N qubit systems, including genuine multiparticle entanglement, based on the lowest nonvanishing moment only. Furthermore, we carried out a detailed analysis of the involved statistical errors enabling an estimation of the statistical significance of our methods. Lastly, we applied the developed framework in order to certify different types of multiparticle entanglement based on finite statistics and discussed applications to experiments in the noisy intermediate regime.

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