



**Quantum speed limit of a noisy continuous-variable system**Wei Wu  and Jun-Hong An \**Lanzhou Center for Theoretical Physics, Key Laboratory of Theoretical Physics of Gansu Province, Lanzhou University, Lanzhou 730000, China*

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Setting the minimal-time bound for a quantum system to evolve between two distinguishable states, the quantum speed limit (QSL) characterizes the latent capability in speeding up of the system. It has found applications in determining the quantum superiority in many quantum technologies. However, previous results showed that such a speedup capability is generally destroyed by the environment induced decoherence in the Born-Markovian approximate dynamics. We here propose a scheme to recover the speedup capability in a dissipative continuous-variable system within the exact non-Markovian framework. It is found that the formation of a bound state in the energy spectrum of the total system consisting of the system and its environment can be used to restore the QSL to its noiseless performance. Giving an intrinsic mechanism in preserving the QSL, our scheme supplies a guideline to speed up certain quantum tasks in practical continuous-variable systems.

DOI: [10.1103/PhysRevA.106.062438](https://doi.org/10.1103/PhysRevA.106.062438)**I. INTRODUCTION**

The quantum speed limit (QSL) quantifies the maximal speed at which a quantum system evolves under the constraint of quantum mechanics. Mandelstam and Tamm showed that, for a unitary dynamics governed by a Hamiltonian  $\hat{H}$ , the minimal evolution time between two orthogonal states is  $\tau_{\text{MT}} = \pi \hbar / (2\Delta H)$  with  $(\Delta H)^2 \equiv \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$  being the energy fluctuation [1,2]. This result provides a physical explanation to Heisenberg's energy-time uncertainty relation [1,3–5]. Later, Margolus and Levitin provided an alternative QSL time in terms of the energy difference  $\tau_{\text{ML}} = \pi \hbar / (2\langle \hat{H} \rangle)$  [6,7]. Sun and Zheng derived a distinct QSL bound via the gauge invariant distance [8]. The above three independent bounds are summarized in a unified form for both Hermitian and non-Hermitian quantum systems [9].

Recently, much effort has been devoted to generalizing the concept of QSL from closed systems to open systems [10–15]. Deffner and Lutz derived a Margolus-Levitin-type bound on the minimal evolution time of open quantum systems [10]. A generalized geometric interpretation for the Margolus-Levitin-like QSL was provided by Ref. [16]. From the application perspective, the QSL in open quantum systems is closely related to the greatest efficiency of charging power in quantum batteries [17–19], the minimum operation time of quantum gates [20,21], the entropy production rate of nonequilibrium quantum thermodynamics [22–27], as well as the quantum Fisher information in noisy quantum metrology [16,28–32]. Thus, how to establish a unified QSL bound, which is valid for both unitary and nonunitary evolutions, is of importance. Using the information geometric formalism is a possible solution [5,8,9,16,33,34]. Starting from a geometric perspective, Refs. [35,36] reported QSL bounds, which

outperform the traditional bounds for both closed and open systems. As already shown in Refs. [8,9,37], the QSL can be used to quantify the potential capability of speeding up for quantum systems. Such a speedup potency plays a leading role in quantum control [5,38].

However, due to the decoherence induced by the inevitable system-environment interaction, the potency of quantum speedup generally vanishes in the Born-Markovian approximate decoherence dynamics [11–14,37,39–42]. How to preserve such a capacity is of importance in the protocol of quantum technology and quantum control. On the other hand, most of the existing studies of the QSL in open quantum systems have focused on the discrete-variable case [10–14,37,41–43]. Very few studies concentrate on the continuous-variable case, especially in the non-Markovian dynamics. In this paper, we investigate the QSL in a dissipative continuous-variable system beyond the traditional paradigm of Born-Markovian approximation treatment. A bound-state based mechanism to realize a controllable QSL time in the noisy environment is revealed.

The paper is organized as follows. The QSL for a Gaussian continuous-variable system being applicable in both the closed and open systems is derived in Sec. II. The non-Markovian decoherence effect on the QSL time is investigated in Sec. III. A mechanism to recover the ideal speedup capacity of the continuous-variable system under the non-Markovian noise is uncovered. In Sec. IV, we make a comparison of our scheme with the previous characterization schemes to the QSL in order to exhibit the universality of our result. Finally, a discussion and a summary are made in Sec. V.

**II. QSL IN A GAUSSIAN SYSTEM**

The QSL can be obtained from the viewpoint of the information geometry as follows. By introducing any kind of geodesic measure  $\mathcal{L} = \mathcal{L}(\varrho_\tau, \varrho_0)$  quantifying the lower

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distance bound between two quantum states  $\varrho_\tau$  and  $\varrho_0$ , an inequality is accordingly built as  $\mathcal{L} \leq \ell \equiv \int \sqrt{d\ell^2}$ . Here,  $d\ell^2$  denotes the squared infinitesimal length between  $\varrho_\tau$  and  $\varrho_\tau + d\varrho_\tau$ , which is regarded as the metric [15,44], and thus  $\ell$  is the length of the actual evolution path. Via introducing the time-averaged evolution speed  $\bar{v} = \ell/\tau$ , the QSL time is geometrically described as  $\tau_{\text{QSL}} \equiv \mathcal{L}/\bar{v}$ , which implies  $\tau_{\text{QSL}}/\tau = \mathcal{L}/\ell$  [5,15,16]. This result indicates that  $\tau_{\text{QSL}}/\tau$  characterizes the extent of the actual evolution path deviating from the geodesic path [8,9,37]. If  $\tau_{\text{QSL}}/\tau = 1$ , the length of the actual evolution path saturates the geodesic one and there is no more space for speeding up. In contrast, the quantum system has a potential speedup capacity as long as  $\tau_{\text{QSL}}/\tau < 1$ . The smaller the value of  $\tau_{\text{QSL}}/\tau$  is, the more speedup capability the system may possess. Therefore,  $\tau_{\text{QSL}}/\tau$  is physically a characterization of the latent capability in speeding up of the quantum system. It has been found that such a capability has important applications in quantum technologies [5,17–21,38]. It should be emphasized that the QSL bound considered in our paper is completely different from the so-called quantum brachistochrone problem [45–47]. The quantum brachistochrone problem commonly aims at designing an optimally controlled time-dependent Hamiltonian such that the shortest evolution time from a given initial state to a final one is achieved under a set of given constraints. It belongs to the research field of quantum optimal control. In this paper, we concentrate on an autonomous time-independent open system.

The Bures angle [5,11,48]

$$\mathcal{L}_B \equiv \arccos \text{Tr}(\sqrt{\sqrt{\varrho_0} \varrho_\tau \sqrt{\varrho_0}}) \quad (1)$$

is widely used to measure the geodesic length between  $\varrho_0$  and  $\varrho_\tau$ . The corresponding metric known as the so-called Fisher-Rao metric relates to the famous quantum Fisher information as  $d\ell^2 = \frac{1}{4} \mathcal{F}_Q dt^2$  [5,11,16,49]. Here, the quantum Fisher information  $\mathcal{F}_Q$  is defined by  $\mathcal{F}_Q = \text{Tr}(\varrho_t \hat{L}^2)$  with  $\hat{L}$  determined by  $\dot{\varrho}_t = (\hat{L}\varrho_t + \varrho_t \hat{L})/2$ . Then, the averaged speed  $\bar{v}$  and the QSL time  $\tau_{\text{QSL}}$  are derived as [5,11,16,50,51]

$$\bar{v} = \frac{1}{2\tau} \int_0^\tau dt \sqrt{\mathcal{F}_Q}, \quad (2)$$

$$\tau_{\text{QSL}} = \frac{\mathcal{L}_B}{\bar{v}} = \frac{\mathcal{L}_B}{\ell} \tau. \quad (3)$$

It is found that  $\mathcal{L}_B$  and  $\mathcal{F}_Q$  naturally reduce to  $\pi/2$  and  $4(\langle \psi | \hat{H}^2 | \psi \rangle - \langle \psi | \hat{H} | \psi \rangle^2)$ , respectively, in the special case of the pure states under the unitary evolution, i.e.,  $\varrho_0 = |\psi\rangle\langle\psi|$  and  $\varrho_\tau = |\psi_\perp\rangle\langle\psi_\perp|$  with  $\langle \psi | \psi_\perp \rangle = 0$ . Then,  $\tau_{\text{QSL}}$  recovers the well-known Mandelstam-Tamm bound.

Here, we consider a single-mode continuous-variable system consisting of a pair of annihilation and creation operators  $\hat{A} = \{\hat{a}, \hat{a}^\dagger\}$ . The characteristic function of the system is defined as  $\chi(\xi) \equiv \text{Tr}[\varrho_t \hat{D}(\xi)]$  [52,53], where  $\hat{D}(\xi) = \exp(\hat{A}^\dagger K \xi)$ , with  $K = \text{Diag}(1, -1)$  and  $\xi = (\xi, \xi^*)^T$ , is the Weyl displacement operator. If the characteristic function has a Gaussian form  $\chi(\xi) = \exp(-\frac{1}{4} \xi^\dagger \sigma \xi - i d^\dagger K \xi)$ , then such a continuous-variable system is called a Gaussian system. Its characteristic function is fully determined by the displacement vector  $d_t$  with  $d_t^j = \text{Tr}(\varrho_t \hat{A}_j)$  and the covariance matrix  $\sigma_t$  with  $\sigma_t^{ij} = \text{Tr}(\varrho_t \{\hat{A}_i - d_i, \hat{A}_j - d_j^*\})$ . For a

Gaussian continuous-variable system, the Bures angle reads  $\mathcal{L}_B = \arccos \sqrt{F}$ . Here,  $F$  is the quantum fidelity and is calculated by using the displacement vectors and the covariance matrices as [53–55]

$$F = \frac{2 \exp[-(d_0 - d_t)^\dagger (\sigma_0 + \sigma_t)^{-1} (d_0 - d_t)]}{\sqrt{\Gamma} + \sqrt{\Lambda} - \sqrt{(\sqrt{\Gamma} + \sqrt{\Lambda})^2 - \Pi}}, \quad (4)$$

where  $\Pi = \text{Det}(\sigma_0 + \sigma_t)$ ,  $\Gamma = \text{Det}(\mathbf{1} + K\sigma_0 K\sigma_t)$ , and  $\Lambda = \text{Det}(\sigma_0 + K)\text{Det}(\sigma_t + K)$ . On the other hand, the quantum Fisher information with respect to the evolution time for a Gaussian system is calculated by [56]

$$\mathcal{F}_Q = \frac{1}{2} \text{Vec}[\dot{\sigma}_t]^\dagger M^{-1} \text{Vec}[\dot{\sigma}_t] + 2d_t^\dagger \sigma_t^{-1} \dot{d}_t, \quad (5)$$

where  $\text{Vec}[\cdot]$  denotes the vectorization of a given matrix and  $M = \sigma_t^* \otimes \sigma_t - K \otimes K$ . From these results, as long as  $d_t$  and  $\sigma_t$  are known, the QSL in a Gaussian continuous-variable system is fully determined.

Let us first consider the QSL of a quantum harmonic oscillator in the ideal case of a unitary evolution governed by  $\hat{H}_s = \omega_0 \hat{a}^\dagger \hat{a}$ . The initial state is chosen as a coherent state, namely,  $\varrho_s(0) = \hat{D}(\alpha)|0\rangle\langle 0| \hat{D}(\alpha)^\dagger$ . It is readily derived that  $d_t = (\alpha e^{-i\omega_0 t}, \alpha^* e^{i\omega_0 t})^T$  and  $\sigma_t = \mathbf{1}$ , which lead to  $F = e^{-2|\alpha|^2[1 - \cos(\omega_0 \tau)]}$  and  $\mathcal{F}_Q = 4|\alpha|^2 \omega_0^2$ . Thus, we have  $\bar{v} = |\alpha| \omega_0$ , which is a time-independent constant, and thus

$$\frac{\tau_{\text{QSL}}}{\tau} = \frac{\arccos e^{-|\alpha|^2[1 - \cos(\omega_0 \tau)]}}{|\alpha| \omega_0 \tau}. \quad (6)$$

Equation (6) reveals that the QSL time behaves as  $\tau_{\text{QSL}}/\tau \propto \tau^{-1}$  with the actual evolution time  $\tau$ . It means  $\tau_{\text{QSL}}/\tau$  approaches to zero in the large- $\tau$  regime. Such a result implies that the harmonic oscillator has an infinite speedup capability in this noiseless case.

### III. QSL IN A NOISY ENVIRONMENT

Next, we consider a more practical situation in which the harmonic oscillator is coupled to a dissipative bosonic environment and experiences a decoherence. The Hamiltonian of the total system reads

$$\hat{H} = \hat{H}_s + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k (g_k \hat{a}^\dagger \hat{b}_k + \text{H.c.}), \quad (7)$$

where  $\hat{b}_k$  denotes the annihilation operator of the  $k$ th environmental mode with frequency  $\omega_k$ , and the parameter  $g_k$  is the coupling strength between the harmonic oscillator and the  $k$ th environmental mode. The coupling strength is further characterized by the spectral density  $J(\omega) \equiv \sum_k |g_k|^2 \delta(\omega - \omega_k)$ . We consider that  $J(\omega)$  explicitly takes the following Ohmic-family form:

$$J(\omega) = \eta \omega^s \omega_c^{1-s} e^{-\omega/\omega_c}, \quad (8)$$

where  $\eta$  is a dimensionless coupling constant,  $\omega_c$  is a cutoff frequency, and  $s$  is the so-called Ohmicity parameter. Depending on the value of  $s$ , the environment can be classified into the sub-Ohmic for  $0 < s < 1$ , the Ohmic for  $s = 1$ , and the super-Ohmic for  $s > 1$ .

Considering the environment is initially prepared in its vacuum state and using Feynman-Vernon's influence functional method to partially trace out the degrees of freedom of the

dissipative environment, we obtain an exact non-Markovian master equation for the harmonic oscillator as [57–59]

$$\dot{\rho}_t = -i[\Omega(t)\hat{a}^\dagger\hat{a}, \rho_t] + \gamma(t)(2\hat{a}\rho_t\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \rho_t\}), \quad (9)$$

where  $\Omega(t) = -\text{Im}[\dot{u}(t)/u(t)]$  is the renormalized frequency and  $\gamma(t) = -\text{Re}[\dot{u}(t)/u(t)]$  is the decay rate induced by the dissipative environment. The time-dependent coefficient  $u(t)$  is determined by

$$\dot{u}(t) + i\omega_0 u(t) + \int_0^t d\tau \mu(t - \tau)u(\tau) = 0, \quad (10)$$

with  $u(0) = 1$  and  $\mu(t) \equiv \int_0^\infty d\omega J(\omega)e^{-i\omega t}$ .

In order to compare with that of the noiseless ideal case, we still choose that the quantum harmonic oscillator is initially in a coherent state. Then, solving the master equation (9), we calculate the exact expressions of the displacement vector and the covariance matrix as  $d_t = [\alpha u(t), \alpha^* u(t)^*]^T$  and  $\sigma_t = \mathbf{1}$ . With the above expressions at hand, the time-averaged speed  $\bar{v}$  and the geodesic distance  $\mathcal{L}_B$  in the noise case are straightforwardly computed:

$$\bar{v} = \frac{1}{\tau} \int_0^\tau dt |\alpha \dot{u}(t)|, \quad (11)$$

$$\mathcal{L}_B = \arccos \sqrt{e^{-|\alpha[1-u(\tau)]|^2}}. \quad (12)$$

We consider the QSL of the system relaxing to its steady state by choosing  $\tau$  sufficiently large. The QSL time derived under such a condition reflects the equilibration efficiency of the dissipative harmonic oscillator. The controllability of this equilibration efficiency is vital in suppressing the detrimental effect of the decoherence in practical quantum technologies. If the system-environment coupling is weak and the characteristic time scale of the environmental correlation function is much smaller than that of the system, one can safely apply the Born-Markovian approximation to Eq. (10). Under such a circumstance, one calculates [58,59]  $u^{\text{MA}}(t) \simeq e^{-(\kappa+i[\omega_0+\Delta(\omega_0)])t}$ , where  $\kappa = \pi J(\omega_0)$  is the Markovian decay coefficient and  $\Delta(\omega_0) = \mathcal{P} \int_0^\infty d\omega \frac{J(\omega)}{\omega_0 - \omega}$  is the environmentally induced frequency shift. With the approximate expression of  $u^{\text{MA}}(t)$  at hand, we find, under the Born-Markovian approximation,  $\bar{v} = |\alpha| \sqrt{\kappa^2 + \omega_0^2} (1 - e^{-\kappa\tau}) / (\kappa\tau)$  and

$$\lim_{\tau \rightarrow \infty} \frac{\tau_{\text{QSL}}}{\tau} = \frac{\kappa \arccos e^{-\frac{1}{2}|\alpha|^2}}{|\alpha| \sqrt{\kappa^2 + \omega_0^2}}, \quad (13)$$

where we have dropped the contributions from the frequency shift term  $\Delta(\omega_0)$ . Equation (13) reduces to the one of the noiseless case in the limit  $\kappa \rightarrow 0$ . We here choose a large  $\tau$ , which means we focus on the QSL time from the initial coherent state to the equilibrium state. In this limit, we find that  $\bar{v}$  reduces to zero and  $\tau_{\text{QSL}}/\tau$  evolves to a time-independent value. This result implies that the speedup potency of the system is destroyed by the Born-Markovian decoherence. A similar result was also reported in several previous references [34,39,40,60].

Going beyond the Born-Markovian approximation, the results of  $\bar{v}$  and  $\tau_{\text{QSL}}/\tau$  are obtainable by numerically solving Eq. (10). However, via analyzing the long-time behavior of  $u(\tau)$ , we calculate their analytically asymptotic forms in the

limit  $\tau \rightarrow \infty$ , which shall help us to build up a more clear physical picture on our results. To this aim, we apply a Laplace transform to  $u(\tau)$  and find  $\tilde{u}(z) \equiv \int_0^\infty d\tau u(\tau)e^{-z\tau} = [z + i\omega_0 + \int_0^\infty d\omega \frac{J(\omega)}{z+i\omega}]^{-1}$ . The solution of  $u(\tau)$  is immediately obtained by applying an inverse Laplace transform to  $\tilde{u}(z)$ , which is exactly done by finding the poles of  $\tilde{u}(z)$  from the following transcendental equation:

$$y(\varpi) \equiv \omega_0 - \int_0^\infty d\omega \frac{J(\omega)}{\omega - \varpi} = \varpi, \quad (14)$$

with  $\varpi = iz$ . It is necessary to point out that the root of the above equation is just the eigenenergy of  $\hat{H}$  in the single-excitation subspace. To be more specific, we express the single-excitation eigenstate as  $|\Psi\rangle = (x\hat{a}^\dagger + \sum_k y_k \hat{b}_k^\dagger)|0, \{0_k\}\rangle$ . Substituting it into  $\hat{H}|\Psi\rangle = E|\Psi\rangle$ , one finds the energy eigenequation as  $E - \omega_0 - \sum_k g_k^2/(\omega_k - E) = 0$ , which retrieves Eq. (14) via simply replacing  $E$  by  $\varpi$ . This result implies that, although the subspaces with arbitrary excitation number are involved in the reduced dynamics, the dynamics of  $u(\tau)$  is essentially determined by the single-excitation energy spectrum characteristic of  $\hat{H}$ . Because  $y(\varpi)$  is a monotonically decreasing function in the regime  $\varpi < 0$ , Eq. (14) potentially has one isolated root  $E_b$  in this regime provided  $y(0) < 0$ . While  $y(\varpi)$  is not well analytic in the regime  $\varpi > 0$ , Eq. (14) has infinite roots in this regime and forms a continuous energy band. We call the eigenstate corresponding to the isolated eigenenergy  $E_b$  the bound state. Then, after applying the inverse Laplace transform and using the residue theorem, we obtain [58,59]

$$u(\tau) = Ze^{-iE_b\tau} + \int_0^\infty \frac{J(\omega)e^{-i\omega\tau} d\omega}{[\omega - \omega_0 - \Delta(\omega)]^2 + [2\pi J(\omega)]^2},$$

where the first term with  $Z \equiv [1 + \int_0^\infty \frac{J(\omega)d\omega}{(E_b - \omega)^2}]^{-1}$  is contributed from the potentially formed bound-state energy  $E_b$ , and the second term is from the band energy which approaches to zero in the long-time regime due to out-of-phase interference. Thus, if the bound state is absent, then we have  $u(\infty) = 0$ , which leads to a complete decoherence, while if the bound state with energy  $E_b$  is formed then we have  $u(\infty) \simeq Ze^{-iE_b\tau}$ , which implies a dissipationless dynamics. The condition of forming the bound state for the Ohmic-family spectral density is evaluated via  $y(0) < 0$  as  $\omega_0 - \eta\omega_c\Gamma(s) < 0$ , where  $\Gamma(s)$  is Euler's gamma function.

In the absence of the bound state, it is natural to expect a consistent result with that under the Born-Markovian approximation because  $u(\tau)$  approaches zero eventually. In contrast, with the long-time expression of  $u(\infty) \simeq Ze^{-iE_b\tau}$  in the presence of the bound state, we find  $\bar{v} \simeq |\alpha ZE_b|$  and

$$\frac{\tau_{\text{QSL}}}{\tau} = \frac{\arccos e^{-\frac{1}{2}|\alpha|^2[1+Z^2-2Z\cos(E_b\tau)]}}{|\alpha ZE_b|\tau}. \quad (15)$$

We see from Eq. (15) that, in the limit  $\tau \rightarrow \infty$ ,  $\bar{v}$  approaches to a nonzero value, while  $\tau_{\text{QSL}}/\tau$  reduces to zero in the form of  $\tau^{-1}$ . These results are verified by exact numerical simulations (see Fig. 1) and are completely different from those under the Born-Markovian case [34,39,40]. Compared to that of the noiseless ideal case, recovering the relation  $\tau_{\text{QSL}}/\tau \propto \tau^{-1}$  means the potency of quantum speedup is fully retrieved. In Fig. 2, we plot the long-time steady-state  $\bar{v}$  and  $\tau_{\text{QSL}}/\tau$

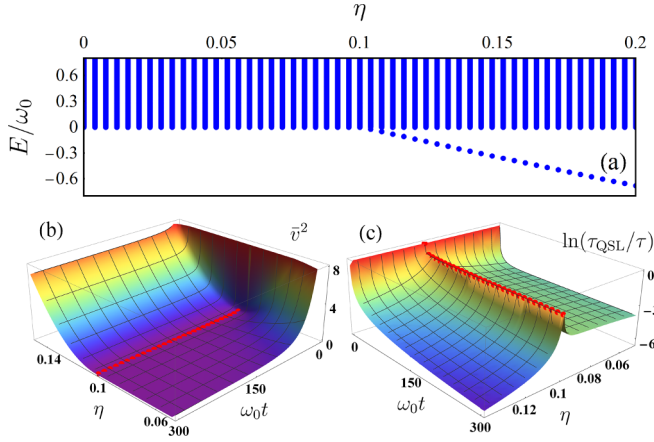


FIG. 1. (a) Energy spectrum of  $\hat{H}$  in the single-excitation subspace. Exact non-Markovian result of the average speed (b) and the QSL time (c) with different  $\eta$  and  $\omega_0 t$ . The red circles highlight the positions of the threshold coupling strength  $\eta = 0.1$  at which the bound state occurs. Other parameters are  $|\alpha| = 10$ ,  $s = 1$ , and  $\omega_c = 10\omega_0$ .

as functions of  $\eta$  and  $\omega_c/\omega_0$ . It confirms that there exists a threshold from no-speedup to speedup regimes matching well with the position of forming the bound state. Our result implies that the time-averaged quantum speed and the QSL time are controllable via engineering the energy spectrum of the whole oscillator-environment system.

#### IV. COMPARISONS WITH PREVIOUS STUDIES

In several previous articles [34,39,40], the Wigner function and the Wasserstein distance are used to calculate the QSL time in Gaussian continuous-variable systems. As displayed in Refs. [34,39,40], within the framework of the Wigner representation, the geodesic length between two Wigner distributions,  $W_\tau(\zeta)$  and  $W_0(\zeta)$  with  $\zeta = (x, p)^T = (\frac{\xi + \xi^*}{\sqrt{2}}, \frac{\xi - \xi^*}{\sqrt{2}i})^T$  being the quadrature vector, is quantified by using the Wasserstein-2 distance as

$$\begin{aligned} \mathcal{L}_W &= \|W_\tau(\zeta) - W_0(\zeta)\|_2 \\ &\equiv \left\{ \int d\zeta |W_\tau(\zeta) - W_0(\zeta)|^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (16)$$

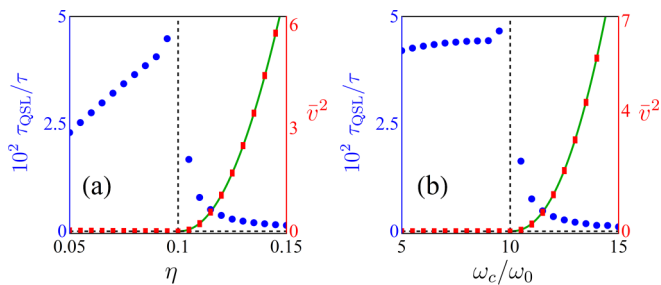


FIG. 2. (a) Steady-state average speed (red rectangles) and QSL time (blue circles) as a function of  $\eta$  (a) and  $\omega_c/\omega_0$  (b) when  $\omega_0\tau = 400$ . The green solid lines are obtained from the analytical result  $\bar{v}^2 \simeq |\alpha|^2 Z^2 E_b^2$ . Other parameters are chosen as  $|\alpha| = 10$  and  $s = 1$ .

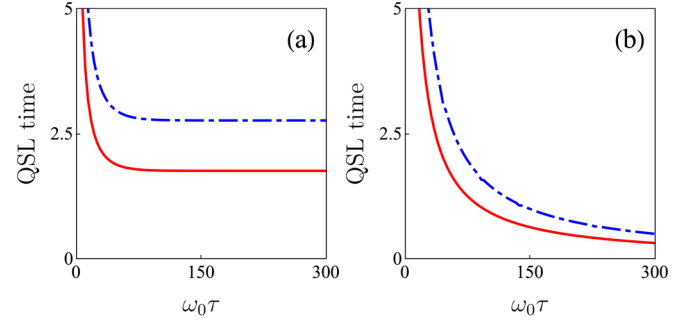


FIG. 3. (a) The QSL time  $10^2\tau_{\text{QSL}}/\tau$  (blue dot-dashed line) and  $10^2\tau_{\text{QSL}}^W/\tau$  (red solid line) are plotted as a function of  $\omega_0\tau$  in the absence of a bound state when  $\eta = 0.06$ . (b) The same as (a), but in the presence of a bound state with  $\eta = 0.12$ . Other parameters are chosen as  $|\alpha| = 10$  and  $s = 1$ .

Then the averaged evolution speed  $\bar{v}_W$  and the QSL time  $\tau_{\text{QSL}}^W$  in the Wigner space are established as [34]

$$\bar{v}_W = \frac{1}{\tau} \int_0^\tau dt \|\dot{W}_t(\zeta)\|_2, \quad (17)$$

$$\tau_{\text{QSL}}^W = \frac{\mathcal{L}_W}{\bar{v}_W}. \quad (18)$$

The above formalism was further generalized to the Wasserstein- $p$ -distance cases with  $p = 1, 2$  and  $+\infty$ , but computing these Wasserstein distances is rather complicated [34].

For our dissipative harmonic oscillator system, the exact expression of the Wigner function is given by [59,61]

$$W_t(\zeta) = \frac{e^{-\frac{1}{2}(\zeta - \tilde{d}_t)^T \tilde{\sigma}_t^{-1} (\zeta - \tilde{d}_t)}}{\pi \sqrt{|\text{Det} \tilde{\sigma}_t|}}, \quad (19)$$

where  $\tilde{d}_t = (\sqrt{2}\text{Re}[\alpha u(t)], \sqrt{2}\text{Im}[\alpha u(t)])^T$  and  $\tilde{\sigma}_t = \frac{1}{2}\sigma_t$ . With the above expression at hand, we find

$$\bar{v}_W = \frac{2}{\sqrt{\pi}\tau} \int_0^\tau dt |\alpha \dot{u}(t)|, \quad (20)$$

$$\mathcal{L}_W = \frac{2}{\sqrt{\pi}} \{1 - e^{-2|\alpha|^2 \{[\text{Re}u(\tau) - 1]^2 + [\text{Im}u(\tau) - 1]^2\}}\}^{\frac{1}{2}}. \quad (21)$$

It is immediately observed that Eq. (20) matches Eq. (11) except for a trivial prefactor  $2/\sqrt{\pi}$ . Moreover, in the limit  $\tau \rightarrow \infty$ , we find that  $\tau_{\text{QSL}}^W/\tau$  approaches to a nonzero constant in the absence of the bound state, and  $\tau_{\text{QSL}}^W/\tau \propto \tau^{-1}$  in the presence of the bound state (see Fig. 3). This conclusion is completely consistent with that of  $\tau_{\text{QSL}}$  obtained in Sec. III. It demonstrates that our bound-state-based QSL-controlling scheme is universal to different definitions of QSL time.

Next, we compare the tightness of  $\tau_{\text{QSL}}$  and  $\tau_{\text{QSL}}^W$ . Based on our numerical simulations,  $\tau_{\text{QSL}}$  is not always tighter than  $\tau_{\text{QSL}}^W$  during the whole relaxation process. However, as displayed in Fig. 3, the value of  $\tau_{\text{QSL}}$  is larger than  $\tau_{\text{QSL}}^W$  in the large- $\tau$  regime irrespective of whether the bound state is formed or not. These results mean that the QSL considered in this paper can be tighter than the previous one derived by using the Wigner function and Wasserstein-2 distance. Furthermore, it is noted that, although both our paper and the ones in Refs. [34,39,40] provide a computable way to obtain the QSL



time in Gaussian continuous-variable systems, the QSL formulation in our paper is strictly established in the differential geometry, which is more rigorous in the mathematical sense. In fact, the Fisher-Rao metric employed by us is a contractive Riemannian metric on the set of density operators. As discussed in Ref. [16], such a peculiar mathematical property may help us to find the tightest bound on the QSL time.

## V. DISCUSSION AND SUMMARY

It is necessary to emphasize that our bound-state based QSL-controlling scheme is independent of the choice of the spectral density. Although only the Ohmic-family form is considered in this paper, our result is straightforwardly generalizable to other cases without difficulties. The bound-state effect, which generally appears in the non-Markovian regime, is the crucial ingredient in our scheme of achieving a steerable QSL. How to generate the bound state is the main point in realizing our control scheme from an experimental perspective. Fortunately, thanks to the rapid development in the state-of-the-art technique of quantum optics experiments, the bound state and its dynamical effect have been observed in circuit quantum electrodynamics architecture [62] and matter-wave systems [63]. With the help of the reservoir engineering technique, the primary parameters in the spectral density  $J(\omega)$  are experimentally controllable. The spectral density of a quantum emitter acting as an open system coupling to the surface-plasmon polariton as an environment is adjustable by changing the distance between them [64,65]. For a reservoir consisting of ultracold atomic gas, the Ohmicity parameter  $s$  is tunable from the sub-Ohmic to the super-Ohmic forms by increasing the scattering length of the gas via Feshbach resonances [66]. These experimental achievements provide a

strong support to our theoretical investigations. As a final remark, our present paper is completely different from Ref. [43]. We here investigate the QSL time derived by the Fisher-Rao metric of a continuous-variable system. However, Ref. [43] considered the Fubini-Study-metric-based QSL of a dissipative two-level system. From the technical point of view, we need an exact expression of the deterministic quantum master equation to obtain the QSL. In contrast, Ref. [43] proposed a scheme to calculate the QSL time via solving the stochastic Schrödinger equation governed by an effective non-Hermitian Hamiltonian. Thus, neither the conclusions nor the methodology of Ref. [43] is directly applicable to our present paper.

In summary, by making use of an exact non-Markovian treatment, we investigate the time-averaged evolution speed and the QSL time in an open continuous-variable quantum system. It is revealed that the formation of a bound state in the energy spectrum of the whole system-environment system in the single-excitation subspace is beneficial for recovering the speedup potency of an open system, which is generally destroyed under the Born-Markovian approximation. Compared with the previous studies [34,39,40], our result provides a tighter QSL time in the dissipative continuous-variable quantum systems. Being experimentally realizable in realistic platforms, our bound-state based QSL-controlling scheme opens an avenue to control the QSL of open system via engineering the energy-spectrum characteristic of the total system consisting of the open system and its environment.

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