Characterization of nonsignaling bipartite correlations corresponding to quantum states

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Characterizing quantum correlations from physical principles is a central problem in the field of quantum information theory. Entanglement breaks bounds on correlations put forth by Bell's theorem, thus challenging the notion of *local causality* as a physical principle. A natural relaxation is to study *no-signalling* as a constraint on joint probability distributions. It was shown that, when considered with respect to so-called locally quantum observables, bipartite nonsignalling correlations never exceed their quantum counterparts; still, such correlations generally do not derive from quantum states. This leaves open the search for additional principles which identify quantum states within the larger set of (collections of) nonsignalling joint probability distributions over locally quantum observables. Here, we suggest a natural generalization of no-signalling in the form of *no-disturbance* to dilated systems. We prove that nonsignalling joint probability distributions satisfying this extension correspond with bipartite quantum states up to a choice of time orientation in subsystems.

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I. INTRODUCTION

The question whether nature admits an underlying deterministic description as in classical physics looms large over the interpretational problems which have plagued quantum theory from the outset [1]. Combining the assumption of determinism with relativity, Bell-derived nontrivial constraints on correlations arising from freely chosen local measurements on spatially separated systems, which are also known as Bell inequalities [2] (see also [3]). Entangled states in quantum mechanics violate Bell inequalities, which have since been convincingly verified in numerous Bell experiments, e.g., [4–6] (see also [7]).

Given the foundational significance of Bell inequality violations as well as the paramount importance of entanglement in quantum information theory [8–15], characterizing quantum correlations constitutes an important and ongoing research objective [16–21]. A natural starting point for such a characterization is the observation that quantum correlations obey the *no-signalling principle*: the joint probability distributions for different local measurements a, b with respective outcome sets $\{A\}$, $\{B\}$ marginalize to the same local distributions

$$\mu(A \mid a)=\sum_B \mu(A, B \mid a, b), \quad \mu(B \mid b)=\sum_A \mu(A, B \mid a, b). \tag{1}$$

Importantly, Eq. (1) depends on the set of local quantum measurements, consequently one must specify the possible choices of measurements on either subsystem to evaluate the constraints inherent to no-signalling. For instance, the Popescu-Rohrlich box correlations restrict to just two measurements on either side [22]. A physically more interesting scenario is that in which, locally, arbitrary quantum measurements are allowed. Remarkably, one can show that this already bounds the correlations to be quantum in the bipartite case [23,24]. Nevertheless, the collections of nonsignalling distributions over the set of locally quantum observables do not correspond with quantum states [25,26]. There are more nonsignalling distributions than quantum states, i.e., while the correlations are as strong as quantum ones, the underlying distributions need not derive from a quantum state.

In this paper we identify two physical principles that characterize those nonsignalling bipartite correlations which correspond with quantum states. To set the stage, we review some basic facts about correlations over nonsignalling correlations in Sec. II. In Sec. III we reformulate the problem from the perspective of contextuality and identify *no-disturbance* as the key underlying principle. This subsumes no-signalling, but makes explicit the intimate relationship with noncontextuality. Based on this, we suggest an extension of the no-disturbance principle to *dilated systems* in Sec. IV. Theorem 2 shows that this strengthened principle almost singles out quantum states. In Sec. V we identify the missing piece of data by introducing a notion of *time orientation*. Our main result, Theorem 3, proves that under a related consistency condition with respect to *unitary evolution in subsystems* correlations indeed derive from quantum states. Section VI concludes the paper.

II. NO-SIGNALLING AND LOCALLY QUANTUM OBSERVABLES

Throughout, we denote by $\mathcal{L}(\mathcal{H})$ the algebra of linear operators on some finite-dimensional Hilbert space $\mathcal{H}$, by $\mathcal{P}(\mathcal{H})$ the lattice of projections on $\mathcal{H}$, and by $\mathcal{L}(\mathcal{H})_{\rm sa}$ the real-linear space of self-adjoint (Hermitian) operators, representing the observables of a system.

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In contrast to restricted sets of observables such as those in [22,24] study no-signalling for all *locally quantum* observables. Locally, quantum here means that every local system is described in terms of an observable algebra $\mathcal{L}(\mathcal{H})_{\rm sa}$ of self-adjoint (Hermitian) operators corresponding to the respective quantum system described by the Hilbert space $\mathcal{H}$. However, rather than assuming the tensor product structure between Hilbert spaces, the composite system is described solely in terms of product observables, i.e., pairs (a_1, a_2) where $a_i \in \mathcal{L}(\mathcal{H}_i)_{\rm sa}$ for $i = 1, 2$. Using the spectral decomposition of observables $a_i = \sum_j A_j q_i^j \in \mathcal{L}(\mathcal{H}_i)_{\rm sa}$ for $A_j \in \mathbb{R}$ and $q_i^j \in \mathcal{P}(\mathcal{H}_i)$, bipartite correlations arise from measures $\mu : \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \to [0, 1]$, $\mu(\mathbb{1}) = \mu(\mathbb{1}_1, \mathbb{1}_2) = 1$ [here, $\mathbb{1}_i \in \mathcal{L}(\mathcal{H}_i)$ denotes the identity matrix]. For the later term the no-signalling constraints in Eq. (1) read

$$\mu(q_1) := \mu(q_1, \mathbb{1}_2) = \sum_j \mu\big(q_1, q_2^j\big),$$

$$\mu(q_2) := \mu(\mathbb{1}_1, q_2) = \sum_j \mu\big(q_1^j, q_2\big), \tag{2}$$

for all mutually orthogonal sets of projections $(q_i^j)_j$, i.e., $\sum_j q_i^j = \mathbb{1}_i$ and $q_i^j q_i^k = \delta_{jk}$.[1] For later reference, we point out that the structure of observables implies that (in addition to no-signalling) μ is also independent of contexts, in a sense to be made precise in Sec. III.

As a consequence, μ defines a finitely additive probability measure over the space of product projections $\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$. Gleason's theorem proves that every finitely additive probability measure over the projections $\mathcal{P}(\mathcal{H})$ derives from a quantum state [27]. Applying Gleason's theorem to the respective subsystems one therefore shows that μ extends to a linear functional $\sigma_\mu : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathbb{C}$, equivalently, that there exists a linear operator ρ_μ such that $\sigma_\mu(a) = \mathrm{tr}[\rho_\mu a]$ for every $a \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Since μ is positive, σ_μ is positive on all product observables $a_1 \otimes a_2$ with $a_1 \in \mathcal{L}(\mathcal{H}_1)_+$ and $a_2 \in \mathcal{L}(\mathcal{H}_2)_+$. We call such normalized, i.e., $\mathrm{tr}[\rho_\mu] = 1$, linear functionals *positive on pure tensor (POPT) states*.

Theorem 1. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces with $\dim(\mathcal{H}_1), \dim(\mathcal{H}_2) \geqslant 3$ finite [24,25]. There is a one-to-one correspondence between measures $\mu : \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \to [0, 1]$, which satisfy the no-signalling constraints in Eq. (2) and POPT states $\sigma_\mu : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathbb{C}$.

Crucially, a POPT state σ_μ is generally not positive. In turn, μ defines a quantum state if and only if σ_μ is positive. Despite POPT states forming a much larger set than the set of quantum states, the authors of [24] showed that correlations arising from POPT states are indistinguishable from those arising from quantum states, i.e., they can be reproduced using a quantum state.[2]

The existence of unphysical POPT states gives rise to the problem of finding a sharper classification that rules out such states. This paper offers a solution to this problem. To this end, we first argue that, rather than no-signalling, the natural principle underlying Eq. (2) is no-disturbance. We then extend the scope of the no-disturbance principle to *dilations of local systems* and show that up to a consistency condition with respect to *unitary evolution in subsystems* this establishes a one-to-one correspondence between nonsignalling joint probability distributions and bipartite quantum states.

We remark that distributions over product observables were studied before, e.g., in [25], out of the attempt to define a tensor product intrinsic to quantum logic.[3] We do not consider this problem here, but note that our result has the potential to give an alternative impulse to this research program. Another perspective on this problem was given by Wallach in [26], who considerde a generalizsation of Gleason's theorem to composite systems. Since it distracts from the physical significance of the present discussion, we study this problem in more generality elsewhere [30].

III. NONCONTEXTUALITY AND NO-DISTURBANCE

A crucial feature of locally quantum nonsignalling joint probability distributions is that they satisfy a stronger constraint than no-signalling, called no-disturbance. We introduce this notion in the following sections, before extending the no-disturbance principle to dilations in Sec. IV. Our approach naturally embeds within the analysis of the principal role of contextuality in quantum theory [31]. For a recent review of the wider subject, see [32].

A. Noncontextuality and marginalization constraints

Contextuality is a key principle in foundations, separating quantum from classical physics [16,33–37]. Moreover, it was shown to be a resource for quantum computation in various architectures [10,38,39].

At the core of contextuality lies the notion of *simultaneous measurability*, which equips the set of observables with a reflexive, symmetric, but generally nontransitive relation [37]. We call any subset of simultaneously measurable observables a *context*. The set of all contexts carries an intrinsic order relation arising from *coarse-graining* on outcomes of observables. The resulting partially ordered set is called the *partial order of contexts*. For the close connection between contextuality in this sense and the nonexistence of valuation functions in the sense of Kochen and Specker [36], we refer to [31].

In quantum theory, two observables are simultaneously measurable if and only if they commute. Consequently, contexts are given by commutative subalgebras $V \subseteq \mathcal{L}(\mathcal{H})$, which are ordered by inclusion into the corresponding partial order of contexts denoted by $\mathcal{V}(\mathcal{H})$. In this setup a quantum state becomes a collection of probability distributions

[1]For convenience, we restrict ourselves to (measures over) projective measurements. It is straightforward to extend the discussion to positive operator-valued measures (POVM), equivalently effect spaces. This allows to include the two-dimensional case in Theorem 1 (see [24,28]).

[2]More generally, multipartite correlations arising from POPT states are distinguishable from quantum correlations [23], see also [29].

[3]For a recent contribution to this problem, see [40] (and references therein).

$(\mu_V)_{V\in\mathcal{V}(\mathcal{H})}$,[4] one for every context and such that distributions are constrained *across* contexts: let $\mu_{\tilde{V}}$, μ_V be probability distributions in contexts $\tilde{V}, V \in \mathcal{V}(\mathcal{H})$ such that $\tilde{V} \subset V$, and denote by $i_{\tilde{V}V} : \mathcal{P}(\tilde{V}) \hookrightarrow \mathcal{P}(V)$ the inclusion relation between their respective projections, then $\mu_{\tilde{V}}$ is obtained from μ_V by marginalization

$$\forall q \in \mathcal{P}(\tilde{V}): \quad \mu_{\tilde{V}}(q) = \mu_V(i_{\tilde{V}V}(q)) = \mu_V|_{\tilde{V}}(q). \quad (3)$$

In other words, Eq. (3) encodes a notion of *coarse-graining* between contexts, represented by the marginalization of distributions from larger to smaller contexts. While individually inconspicuous, marginalization constraints impose a strong condition in conjunction with *noncontextuality*: the probabilities of the outcomes of an observable $a \in \tilde{V} \subset V, V'$ are independent of other observables $b \in V$, $c \in V'$, i.e., they are independent of context:

$$\forall \tilde{V}, V, V' \in \mathcal{V}(\mathcal{H}), \tilde{V} \subset V, V': \quad \mu_{V'}|_{\tilde{V}} = \mu_{\tilde{V}} = \mu_V|_{\tilde{V}}. \quad (4)$$

This noncontextuality assumption on probability distributions μ_V is at the heart of Gleason's theorem [27]; for a reformulation of the latter in this language see [30,31,41].

B. No-signalling from no-disturbance

There is a natural notion of composition for (partial orders of) contexts: their canonical product, denoted $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$, is the Cartesian product with elements (V_1, V_2) for $V_1 \in \mathcal{V}(\mathcal{H}_1)$, $V_2 \in \mathcal{V}(\mathcal{H}_2)$ and order relations such that for all $\tilde{V}_1, V_1 \in \mathcal{V}(\mathcal{H}_1)$, $\tilde{V}_2, V_2 \in \mathcal{V}(\mathcal{H}_2)$,

$$(\tilde{V}_1, \tilde{V}_2) \subseteq (V_1, V_2) :\Longleftrightarrow \tilde{V}_1 \subseteq_1 V_1 \quad \text{and} \quad \tilde{V}_2 \subseteq_2 V_2. \quad (5)$$

Restricted to product contexts, Eq. (4) says that for all $\tilde{V}_i \subset V_i, V_i' \in \mathcal{V}(\mathcal{H}_i)$ with $i = 1, 2$

$$\begin{aligned}\mu_{(V_1,V_2)}|_{(\tilde{V}_1,V_2)} &= \mu_{(\tilde{V}_1,V_2)} = \mu_{(V_1',V_2)}|_{(\tilde{V}_1,V_2)},\\ \mu_{(V_1,V_2)}|_{(V_1,\tilde{V}_2)} &= \mu_{(V_1,\tilde{V}_2)} = \mu_{(V_1,V_2')}|_{(V_1,\tilde{V}_2)}.\end{aligned} \quad (6)$$

We call the collection of constraints over all contexts $(V_1, V_2) \in \mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2)$ in Eq. (6) the *no-disturbance principle*.[5] Note that no-disturbance reduces to no-signalling in Eq. (1) when restricted to the trivial contexts $1_i := \mathbb{C}\mathbb{1}_i \subset \mathcal{L}(\mathcal{H}_i)$[6] on the respective local subsystem,

$$\begin{aligned}\mu_{(V_1,V_2)}|_{(V_1,1_2)} &= \mu_{(V_1,1_2)} = \mu_{V_1,V_2'}|_{(V_1,1_2)},\\ \mu_{(V_1,V_2)}|_{(1_1,V_2)} &= \mu_{(1_1,V_2)} = \mu_{(V_1',V_2)}|_{(1_1,V_2)}.\end{aligned} \quad (7)$$

Observe that general (possibly contextual) nonsignalling joint probability distributions over locally quantum observables correspond with collections of probability distributions $(\mu_V)_{V\in\mathcal{V}(\mathcal{H}_1)\times\mathcal{V}(\mathcal{H}_2)}$ satisfying Eq. (7). Such distributions are the natural objects of study when no further structure on observables is assumed. They are thus more general than the measures $\mu : \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \to [0, 1]$ in Eq. (2), which satisfy the more restrictive no-disturbance condition in Eq. (6).[7] In particular, no-disturbance is already implicit in [24]. With Ref. [24], we reemphasise that $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{H}_2) \neq \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a meager part of the full quantum structure of the composite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. In particular, the product system is not described by the tensor product of the individual Hilbert spaces.

Having established the crucial role of noncontextuality and no-disturbance, in the next section we extend this principle to dilations on local subsystems.

IV. NO-DISTURBANCE FOR DILATIONS

A key idea of this work is to impose the no-disturbance condition in Eq. (6) not only between locally quantum observables, but to extend its scope to dilations of (at least) one of the two subsystems. We give some motivational background for this step first.

Recall that by Theorem 1 a measure $\mu : \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \to [0, 1]$ defines a POPT state $\mu(q_1, q_2) = \sigma_\mu(q_1 \otimes q_2) = \mathrm{tr}[\rho_\mu(q_1 \otimes q_2)]$ with corresponding linear operator $\rho_\mu \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Similarly, for every $q_1 \in \mathcal{P}(\mathcal{H}_1)$ we define a positive linear operator $\rho_\mu(q_1) \in \mathcal{L}(\mathcal{H}_2)_+$ from $\mu(q_1, q_2) =: \mathrm{tr}_{\mathcal{H}_2}[\rho_\mu(q_1)q_2]$ for all $q_2 \in \mathcal{P}(\mathcal{H}_2)$. It follows from Theorem 1 that ρ_μ extends to a positive linear map $\phi_\mu : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$. On the other hand, for any single $q_1 \in \mathcal{P}(\mathcal{H}_1)$ we may assume $\rho_\mu(q_1) \in \mathcal{L}(\mathcal{H}_2)_+$ to arise via coarse-graining from some $|\psi_\mu(q_1)\rangle \in \mathcal{K} := \mathcal{H}_2 \otimes \mathcal{H}_E$ on a larger system,[8] by tracing out the extra degrees of freedom

$$\begin{aligned}\rho_\mu(q_1) &= \mathrm{tr}_{\mathcal{H}_E}[|\psi_\mu(q_1)\rangle\langle\psi_\mu(q_1)|]\\ &= \mathrm{tr}_{\mathcal{H}_E}[u^*(q_1 \otimes |\psi_\mu\rangle\langle\psi_\mu|)u].\end{aligned} \quad (8)$$

$|\psi_\mu(q_1)\rangle$ is a *purification* of $\rho_\mu(q_1)$, where the unitary $u \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_E)$ is chosen to separate the input from the ancillary system. We want to arrange the purifications for all $q_1 \in \mathcal{P}(\mathcal{H}_1)$ in an economical way. To start with, note that, since $\rho_\mu(q_1) \in \mathcal{L}(\mathcal{H}_2)_+$ for every $q_1 \in \mathcal{P}(\mathcal{H}_1)$, $[\rho_\mu(q_1^i)]_i$ defines a nonnormalized ($\sum_i \rho_\mu(q_1^i) = \rho_\mu(\mathbb{1}_1) = \mathrm{tr}_{\mathcal{H}_1}[\rho_\mu]$) positive operator-valued measure (POVM) on $\mathcal{L}(\mathcal{H}_2)$ for every set of mutually orthogonal projections $(q_1^i)_i$ in $\mathcal{L}(\mathcal{H}_1)$, i.e., for every context $V_1 \in \mathcal{V}(\mathcal{H}_1)$. We may thus write

$$\rho_\mu^{V_1}(q_1^i) = \mathrm{tr}_{\mathcal{H}_E}\big[u^*\big(q_1^i \otimes \big|\psi_\mu^{V_1}\big\rangle\big\langle\psi_\mu^{V_1}\big|\big)u\big], \quad (9)$$

for all $q_1^i \in \mathcal{P}(V_1)$ and $V_1 \in \mathcal{V}(\mathcal{H}_1)$, mimicking the purification in Eq. (8). Equation (9) is called a *dilation* of the POVM $[\rho_\mu(q_1^i)]_i$ to the projection-valued measure (PVM) $q_1 \mapsto q_1 \otimes \mathbb{1}_{\mathcal{H}_E}$. More generally, by Naimark's theorem [43,44] every POVM $\rho_\mu^{V_1}$ admits a dilation to a PVM $\varphi_\mu^{V_1} : \mathcal{P}(V_1) \to \mathcal{P}(\mathcal{K})$ such that $\rho_\mu = v^*\varphi_\mu^{V_1} v$, where $v : \mathcal{H}_2 \to \mathcal{K}$ is a linear map.[9]

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[4] Collections of probability distributions over general partial orders of contexts (not necessarily quantum) are also known as empirical models in the sheaf-theoretic framework of contextuality [33].

[5] The term appears in [42], but was used under different names before, e.g., in [16] (see also, [31,34]).

[6] The respective contexts correspond to the trivial events of observing that there is a local system.

[7] The latter additionally encodes a noncontextuality constraint in the form of an independence condition on the choice of simultaneously measurable product observables.

[8] Physically, one considers the system together with an (environmental) ancillary system.

[9] Here, we concentrate the dependence on contexts in the mapping $V_1 \mapsto \varphi_\mu^{V_1}$, by choosing $\mathcal{K}$ sufficiently large [43–45] and by absorbing any context dependence on v_{V_1} into $\varphi_\mu^{V_1}$ for every $V_1 \in \mathcal{V}(\mathcal{H}_1)$.

Importantly, the dilations $V_1 \mapsto \varphi_\mu^{V_1}$ generally depend on contexts $V_1 \in \mathcal{V}(\mathcal{H}_1)$. In contrast, recall that no-disturbance in Eq. (6) encodes noncontextuality constraints between product observables. By comparison, this suggests to extend the no-disturbance principle also to product observables with respect to the dilations $(\varphi_\mu^{V_1})_{V_1\in\mathcal{V}(\mathcal{H}_1)}$.

Definition 1. We say that the measure $\mu : \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \to [0, 1]$ satisfies *no-disturbance for dilations* if $\mu(q_1, q_2) = \mathrm{tr}_{\mathcal{H}_2}[(v^*\varphi_\mu^{V_1}(q_1)v)q_2]$ for some Hilbert space $\mathcal{K}$, linear map $v : \mathcal{H}_2 \to \mathcal{K}$, and projection-valued measures $(\varphi_\mu^{V_1})_{V_1\in\mathcal{V}(\mathcal{H}_1)}$, $\varphi_\mu^{V_1} : \mathcal{P}(V_1) \to \mathcal{P}(\mathcal{K})$ such that

$$\forall q_1 \in \mathcal{P}(V_1), V_1 \in \mathcal{V}(\mathcal{H}_1), q_2' \in \mathcal{P}(\mathcal{K}) : \mu'(q_1, q_2') := \mathrm{tr}_{\mathcal{H}_2}\big[v^*\varphi_\mu^{V_1}(q_1)q_2'v\big] \tag{10}$$

satisfies the no-disturbance principle in Eq. (6) for all product contexts in $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{K})$.

From a physical point of view, we interpret the (conditional) probability distributions $\mu_{V_2}(q_1, \cdot)$ in contexts $V_2 \in \mathcal{V}(\mathcal{H}_2)$ as states of incomplete information. We saw that this interpretation arises naturally from the explicit dilations in Eq. (9), which express the state as arising from coarse-graining of ancillary degrees of freedom. This view is further corroborated if one allows not only projective measurements on the respective subsystems $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$, but arbitrary positive operator-valued measures, e.g., as discussed in [24].

Put the other way around, if μ did not satisfy no-disturbance for dilations in Definition 1, while potentially being nondisturbing (nonsignalling) when restricted to the system $\mathcal{H}_1 \otimes \mathcal{H}_2$, it would fail to be nondisturbing (nonsignalling) *for every choice of dilations* $(\varphi_\mu^{V_1})_{V_1\in\mathcal{V}(\mathcal{H}_1)}$ to a larger system. Such measures would, at the very least, prompt a substantial revision of the concept of mixed states in terms of states of information in quantum theory. As such, the extension to dilated systems in Definition 1 is a natural one, and arguably conservative compared with other related approaches, e.g., [17,19–21].

Note also that Definition 1 does not change the fact that the composite system is described by means of the Cartesian product of contexts in Eq. (5), not the tensor product.

To state the implications of Definition 1, we remind the reader of some basic facts about Jordan algebras. Recall that the set of self-adjoint (Hermitian) matrices $\mathcal{L}(\mathcal{H})_{\mathrm{sa}}$ are closed under the *anticommutator* $\{a, b\} = ab + ba$ for all $a, b \in \mathcal{L}(\mathcal{H})_{\mathrm{sa}}$. This defines the (special) Jordan algebra $\mathcal{J}(\mathcal{H}) = (\mathcal{L}(\mathcal{H})_{\mathrm{sa}}, \{\cdot, \cdot\})$.[10] Moreover, we obtain a Jordan $*$-algebra by extending the operation $\{\cdot, \cdot\}$ to the complexified algebra $\mathcal{J}(\mathcal{H}) = \mathcal{J}(\mathcal{H})_{\mathrm{sa}} + i\mathcal{J}(\mathcal{H})_{\mathrm{sa}}$. This expresses the fact that $\mathcal{J}(\mathcal{H})_{\mathrm{sa}}$ is the self-adjoint part of $\mathcal{L}(\mathcal{H})$. Finally, a Jordan $*$-homomorphism $\Phi : \mathcal{J}(\mathcal{H}_1) \to \mathcal{J}(\mathcal{H}_2)$ is a linear map $\Phi : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ such that $\Phi(\{a, b\}) = \{\Phi(a), \Phi(b)\}$ and $\Phi^*(a) = \Phi(a^*)$ for all $a, b \in \mathcal{J}(\mathcal{H}_1)$.

We have the following key result, which extracts the Jordan structure from Definition 1.

[10]A Jordan algebra is called *special* if it is isomorphic to the subalgebra of the self-adjoint part of an associative algebra. Otherwise, it is called *exceptional*.

Theorem 2. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces with $\dim(\mathcal{H}_1), \dim(\mathcal{H}_1) \geqslant 3$ finite, and let $\mu : \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \to [0, 1]$ satisfy *no-disturbance for dilations* in Definition 1. Then the linear functional $\sigma_\mu : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathbb{C}$ in Theorem 1 is of the form

$$\sigma_\mu(a \otimes b) = \mathrm{tr}_{\mathcal{H}_2}[(v^*\Phi_\mu(a)v)b],$$

for some Hilbert space $\mathcal{K}$, linear map $v : \mathcal{H}_2 \to \mathcal{K}$, and Jordan $*$-homomorphism $\Phi_\mu : \mathcal{J}(\mathcal{H}_1) \to \mathcal{J}(\mathcal{K})$.

Proof. Since μ satisfies no-disturbance for the dilations in Definition 1, there exists a Hilbert space $\mathcal{K}$, a linear map $v : \mathcal{H}_2 \to \mathcal{K}$, and projection-valued measures $(\varphi_\mu^{V_1})_{V_1\in\mathcal{V}(\mathcal{H}_1)}$ such that $\mu(q_1, q_2) = \mathrm{tr}_{\mathcal{H}_2}[(v^*\varphi_\mu^{V_1}(q_1)v)q_2]$ for all $q_1 \in \mathcal{P}(V_1)$, $V_1 \in \mathcal{V}(\mathcal{H}_1)$, and $q_2 \in \mathcal{P}(\mathcal{H}_2)$. Moreover, the extension $\mu'(q_1, q_2') = \mathrm{tr}_{\mathcal{H}_2}[v^*\varphi_\mu^{V_1}(q_1)q_2'v]$ in Eq. (10), where $q_1 \in \mathcal{P}(V_1)$, $V_1 \in \mathcal{V}(\mathcal{H}_1)$, and $q_2' \in \mathcal{P}(\mathcal{K})$, satisfies the no-disturbance principle in Eq. (6). The constraints in Eq. (6) are equivalent to the noncontextuality constraints in Eq. (4), restricted to product contexts $\mathcal{V}(\mathcal{H}_1) \times \mathcal{V}(\mathcal{K})$. Consequently, $(\varphi_\mu^{V_1})_{V_1\in\mathcal{V}(\mathcal{H}_1)}$ does not depend on contexts, and therefore defines a map $\varphi_\mu : \mathcal{P}(\mathcal{H}_1) \to \mathcal{P}(\mathcal{K})$. It follows that φ_μ is an orthomorphism, i.e., (i) $\varphi_\mu(0) = 0$, (ii) $\varphi_\mu(1 - p) = 1 - \varphi_\mu(p)$, (iii) $\varphi_\mu(p)\varphi_\mu(q) = 0$, and (iv) $\varphi_\mu(p + q) = \varphi_\mu(p) + \varphi_\mu(q)$ all hold whenever $p, q \in \mathcal{P}(\mathcal{H}_1)$, $pq = 0$. This follows since by definition $\varphi_\mu^{V_1} : \mathcal{P}(\mathcal{H}_1) \to \mathcal{P}(\mathcal{K})$ is a projection-valued measure in every context $V_1 \in \mathcal{V}(\mathcal{H}_1)$. In particular, note that conditions (i) to (iii) hold whenever $p, q \in \mathcal{P}(\mathcal{H}_1)$, $pq = 0$, which implies $p, q \in V_1$ for some $V_1 \in \mathcal{V}(\mathcal{H}_1)$. Finally, by a result due to Bunce and Wright (Corollary 1 [46]) every orthomorphism $\varphi : \mathcal{P}(\mathcal{H}_1) \to \mathcal{P}(\mathcal{K})$ lifts to a Jordan $*$-homomorphism $\Phi : \mathcal{J}(\mathcal{H}_1) \to \mathcal{J}(\mathcal{K})$ as desired. ■

Of course, every quantum state has a purification which yields a dilation of the form in Theorem 2. Our argument works in the reverse direction: by requiring that measures μ have a noncontextual extension, i.e., that they satisfy the no-disturbance principle for at least one choice of dilations $V_1 \mapsto \varphi_\mu^{V_1}$, μ is of the form in Theorem 2. Next, we show that, in this case, μ already defines a quantum state up to a choice of time orientation in local subsystems.

V. UNITARY EVOLUTION AND TIME-ORIENTED STATES

Extending no-disturbance to dilations via Definition 1 is not quite sufficient to ensure that a POPT state $\sigma_\mu = \mathrm{tr}[\rho_\mu\, \cdot]$ becomes positive and thus a quantum state (but it almost is). The difference is best expressed in terms of the Choi-Jamiołkowski isomorphism [47]. In particular, by Choi's theorem the linear operator ρ_μ is positive if and only if a related map ϕ_μ is completely positive [45]. Moreover, by Stinespring's theorem a map is completely positive if and only if it corresponds with an algebra homomorphism (a representation) on a larger system [44], i.e., $\phi_\mu : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ is completely positive if and only if there exists a Hilbert space $\mathcal{K}$, a linear map $v : \mathcal{H}_2 \to \mathcal{K}$, and an algebra homomorphism $\Phi_\mu : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{K})$ such that $\phi_\mu(a) = v^*\Phi_\mu(a)v$. By Theorem 2, if μ satisfies no-disturbance for dilations, it is of this form with the only difference that Φ_μ is a merely a Jordan $*$-homomorphism.

The Jordan algebra $\mathcal{J}(\mathcal{H})$ does not completely determine the algebra $\mathcal{L}(\mathcal{H})$ since it lacks the antisymmetric part or *commutator* $[a,b]=ab-ba$ in the associative (and noncommutative for $\dim(\mathcal{H})\geqslant 2$) product $ab=\frac{1}{2}(ab+ba)+\frac{1}{2}(ab-ba)$ for all $a,b\in\mathcal{L}(\mathcal{H})$. In particular, the Jordan $*$-homomorphism Φ_μ is an algebra homomorphism if and only if it also preserves commutators, i.e., $\Phi([a,b])=[\Phi(a),\Phi(b)]$ for all $a,b\in\mathcal{J}(\mathcal{H})$. This property has a distinctive physical meaning in terms of the unitary evolution in local subsystems.

To see this, we describe the dynamics of a system represented by the observable algebra $\mathcal{L}(\mathcal{H})_{\rm sa}$ in terms of one-parameter groups of automorphisms $\mathbb{R}\mapsto \mathrm{Aut}[\mathcal{L}(\mathcal{H})_{\rm sa}]$. Note that $\mathcal{L}(\mathcal{H})$ possesses a canonical action on itself by conjugation $\Psi:\mathbb{R}\times\mathcal{L}(\mathcal{H})_{\rm sa}\to \mathrm{Aut}[\mathcal{L}(\mathcal{H})_{\rm sa}]$, $\Psi(t,a)b=e^{ita}be^{-ita}$ for all $a,b\in\mathcal{L}(\mathcal{H})_{\rm sa}$.[11] This action expresses the unitary evolution in the system, thus promoting the parameter t to a time parameter with a playing the role of a Hamiltonian. Note, however, that without fixing a preferred Hamiltonian the value of t has *a priori* no objective physical meaning, it is intrinsically relational. Nevertheless, the sign of t turns out to be independent of (the choice of Hamiltonian) $a\in\mathcal{L}(\mathcal{H})_{\rm sa}$ [48].

By differentiation, $\frac{d}{dt}|_{t=0}\Psi(t,a)=i[a,\cdot]$, where $[\cdot,\cdot]$ is the commutator in the ambient algebra $\mathcal{L}(\mathcal{H})$. It follows that Φ_μ preserves commutators if and only if it preserves the canonical unitary evolution Ψ of the subsystem algebras. We call Ψ the *canonical time orientation* on $\mathcal{L}(\mathcal{H})$ [48–51] and $\Psi^*(t,a)=*\circ\Psi(t,a)=\Psi(-t,a)$ (cf. Proposition 15 in [52]) the *reverse time orientation*, which corresponds with the opposite order of composition in $\mathcal{L}(\mathcal{H})$, equivalently, the opposite sign for the commutator in the respective algebras,

$$\mathcal{L}_\pm(\mathcal{H}):=(\mathcal{J}(\mathcal{H}),\Psi(\pm t,a)),$$ [12]

where we set $a\cdot_\pm b:=\frac{1}{2}a,b\pm\frac{1}{2}[a,b]$ for all $a,b\in\mathcal{J}(\mathcal{H})$. Of course, $\mathcal{L}(\mathcal{H})_+=\mathcal{L}(\mathcal{H})$.

Definition 2. We say that a measure $\mu:\mathcal{P}(\mathcal{H}_1)\times\mathcal{P}(\mathcal{H}_2)\to[0,1]$, which satisfies no-disturbance for the dilations in Definition 1, is *time-oriented with respect to $\mathcal{L}_-(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$* if

$$\begin{aligned}\forall t\in\mathbb{R}, a\in\mathcal{J}(\mathcal{H}_1)_{\rm sa}:\ \Phi_\mu\circ\Psi_1^*(t,a)&=\Phi_\mu\circ *\circ\Psi_1(t,a)\\ &=\Psi_2'(t,\Phi_\mu(a))\circ\Phi_\mu,\end{aligned}\tag{11}$$

where Φ_μ is the Jordan $*$-homomorphism in Theorem 2.

Two remarks on Definition 2 are in order. First, note that Eq. (11) requires consistency with respect to the commutators on $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{K})$. However, since $\mathcal{L}(\mathcal{H}_2)$ arises from $\mathcal{L}(\mathcal{K})$ by restriction, the time orientation Ψ_2 on $\mathcal{L}(\mathcal{H}_2)$ extends to a unique time orientation Ψ_2' on $\mathcal{L}(\mathcal{K})$. Second, we used the reverse time orientation Ψ_1^* on the first system. This choice of relative time orientation (Ψ_1^*,Ψ_2) is intrinsic to Choi's theorem [45] (see also [51]), which allows us to deduce positivity of σ_μ from complete positivity of ϕ_μ in Theorem 2.

In fact, the conjunction of Definitions 1 and 2 yields our main result.

Theorem 3. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces with $\dim(\mathcal{H}_1),\dim(\mathcal{H}_2)\geqslant 3$ finite, and let $\mu:\mathcal{P}(\mathcal{H}_1)\times\mathcal{P}(\mathcal{H}_2)\to[0,1]$ be time-oriented as by Definition 2 (and thus satisfy no-disturbance for dilations). Then μ uniquely extends to a quantum state $\sigma_\mu\in\mathcal{S}(\mathcal{H}_1\otimes\mathcal{H}_2)$.

Proof. Since μ satisfies no-disturbance for dilations in Definition 1, the dilations $(\varphi_\mu^{V_1})_{V_1\in\mathcal{V}(\mathcal{H}_1)}$ uniquely extend to a Jordan $*$-homomorphism Φ_μ by Theorem 2. In particular, Φ_μ preserves anticommutators. Moreover, since μ is time-oriented (see Definition 2), Φ_μ also preserves commutators between the algebras $\mathcal{L}_-(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$, hence, it is an algebra homomorphism. By Stinespring's theorem, this implies that the linear map $\phi_\mu:\mathcal{L}_-(\mathcal{H}_1)\to\mathcal{L}(\mathcal{H}_2)$ is completely positive. Finally, we define a linear operator $\rho_\mu\in\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ by the relation $\phi_\mu(a)=\mathrm{tr}_{\mathcal{H}_1}[\rho_\mu(a\otimes\mathbb{1}_2)]$ for all $a\in\mathcal{L}(\mathcal{H}_1)$. Note that $\phi_\mu\circ *$ is the inverse of the Choi-Jamiołkowski isomorphism [45,47], defined on the computational basis $\{|i\rangle\}_i$ in $\mathcal{H}_1$ by

$$\rho_\mu=\sum_{ij}|i\rangle\langle j|\otimes\phi_\mu(|j\rangle\langle i|)=\sum_{ij}|i\rangle\langle j|\otimes(\phi_\mu\circ *)(|i\rangle\langle j|),\tag{12}$$

where $*$ denotes the Hermitian adjoint (see also [51]). Since $\phi_\mu:\mathcal{L}_-(\mathcal{H}_1)\to\mathcal{L}(\mathcal{H}_2)$ is completely positive and since $*$ reverses time orientations by Eq. (11), it follows from Choi's theorem [45] that $\rho_\mu\in\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ is positive. Hence, $\sigma_\mu=\mathrm{tr}[\rho_\mu\ \cdot]$ defines a quantum state. By construction, σ_μ is the unique linear extension of μ, i.e., $\mu=\sigma_\mu|_{\mathcal{P}(\mathcal{H}_1)\times\mathcal{P}(\mathcal{H}_2)}$.

Conversely, it follows from the isomorphism in Eq. (12), Choi's theorem [45], and Definition 2 that every quantum state restricts to a time-oriented measure $\mu:\mathcal{P}(\mathcal{H}_1)\times\mathcal{P}(\mathcal{H}_2)\to[0,1]$. ■

We finish this section with a few remarks. We defined the canonical time orientation Ψ on $\mathcal{L}(\mathcal{H})=(\mathcal{J}(\mathcal{H}),\Psi)$ with respect to unitary symmetries. In turn, antiunitary symmetries correspond with unitary symmetries on the algebra $\mathcal{L}_-(\mathcal{H})$ (and vice versa). Recalling that every antiunitary operator is the product of a unitary and the time-reversal operator further corroborates the notion of time-oriented-ness in Definition 2.

Moreover, note that Eq. (11) is invariant under changing time orientations, equivalently, under time reversal in both subsystems (cf. [51]). On the other hand, it follows from Theorem 3 that applying time reversal to one subsystem only will generally map outside the set of bipartite quantum states. This is reminiscent of the positive partial transpose (PPT) criterion for bipartite entanglement due to Peres [56,57]. In fact, following this analogy the criterion can be given a sharp physical interpretation in terms of time orientations [50].

Finally, note that Theorem 3 represents a generalization of Gleason's theorem to composite systems. We extend this perspective to the general setting of von Neumann algebras in [30].

[11]The map Ψ "selects" unitary (as opposed to antiunitary [53,54]) symmetries on $\mathcal{J}(\mathcal{H})_{\rm sa}$. Inherent in this is an identification of self-adjoint elements in $\mathcal{J}(\mathcal{H})_{\rm sa}$ and generators of symmetries, called a *dynamical correspondence*. In other words, Ψ "selects" a canonical dynamical correspondence on $\mathcal{J}(\mathcal{H})_{\rm sa}$: $\Psi=\exp\circ\psi$ is the exponential of the (canonical) dynamical correspondence ψ on $\mathcal{L}(\mathcal{H})$. For more details, see [48,49].

[12]These are the only associative products which reduce to $\mathcal{J}(\mathcal{H})$ on the symmetric part [48,55].

VI. CONCLUSION AND OUTLOOK

We studied the physical principle of no-disturbance on joint probability distributions over product observables, which reduces to no-signalling as shown in Eq. (7). As such, we identified no-disturbance as the key principle in [24], which shows that correlations over product observables satisfying no-disturbance cannot exceed bipartite quantum correlations. However, no-disturbance is not sufficient to restrict to quantum states. Our main result, Theorem 3, resolves this issue: by extending the scope of no-disturbance to dilations in Definition 1, and by enforcing a consistency condition with respect to the unitary evolution in local subsystems in Definition 2, we show that the resulting joint probability distributions correspond with bipartite quantum states unambiguously.

Our research naturally relates to other approaches, which seek to single out quantum correlations from more general nonsignalling distributions, by imposing additional principles, e.g., [17,19–21]. Apart from the extension of the no-disturbance principle to dilated systems, which fits nicely with the usual interpretation of mixed states as states of incomplete information as argued above, our only additional assumption already has a clear physical meaning in terms of the arrow of time [50].

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