Quantifying causal influence in quantum mechanics

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(Received 30 May 2021; revised 11 January 2022; accepted 23 November 2022; published 14 December 2022)

We extend Pearl's definition of causal influence to the quantum domain, where two quantum systems A, B with finite-dimensional Hilbert space are embedded in a common environment C and propagated with a joint unitary U. For a finite-dimensional Hilbert space C, we find the necessary and sufficient condition on U for a causal influence of A on B and vice versa. We introduce an easily computable measure of the causal influence and use it to study the causal influence of different quantum gates, its mutuality, and quantum superpositions of different causal orders. For two two-level atoms dipole-interacting with a thermal bath of electromagnetic waves, the space-time dependence of causal influence almost perfectly reproduces the one of reservoir-induced entanglement.

DOI: 10.1103/PhysRevA.106.062415

I. INTRODUCTION

To infer causes from effects constitutes a key task of science. Classically, causal influence (CI) is defined between random variables (RVs) that take certain values in a set of possible outcomes of randomized experiments. In a causal model, the RVs sit on vertices of a graph, and the forward-in-time-only CI in the classical world is represented by an arrow in a directed acyclic graph. According to Pearl [1] (see p. 276), a RV x has a CI on another RV y if y "listens" to x, meaning that there is a functional relationship of the form $y = f(x) + z$, where f is some function and z another RV. Apart from direct CI, correlations between x and y might arise additionally due to common causes. These can be eliminated in practice by "do-interventions," where x is randomly set by the experimenter and one examines if y reacts. The corresponding (do-)probabilities, obtained by randomized controlled experiments or via the do-calculus, are the basis of Pearl's definition of CI as well as a measure of the average causal effect [2].

Recently, there has been large interest in generalizing causal analysis to the quantum world [3–32]. One of the most exciting perspectives is to superpose different temporal orders, and hereby create "indefinite causal order." To that end, process matrices were introduced with separate input and output Hilbert spaces of each laboratory, and the possibility to "wire" CI in opposite directions (e.g., from the output of Alice to the input of Bob or vice versa) [4]. The "quantum switch" was invented [33,34] and experimentally verified [35–39]. Here a control qubit enables opposite temporal order of two quantum gates, which can improve the communication capacity of quantum channels and lead to a computational or metrological advantage [33,40,41]. Common to most of these developments is, however, that the actual CI remained unexplored, and the "indefinite causal order" refers to indefinite (i.e., superposed, or mixed) temporal orders. To study superpositions of different causal orders $x \to y$ and $y \to x$, one needs to define what is meant with a CI $x \to y$ in quantum mechanics (QM). In Refs. [12,19,30] definitions of CI in QM based on the Choi-Jamiołkowski representation of a unitary channel propagating Alice's and Bob's system were given, following earlier work in Refs. [42–44]. Here we give a clear operational definition of CI based simply on density matrices, generalizing the one by Pearl from classical statistical analysis [1,2]. We prove a theorem that gives the necessary and sufficient condition for CI in QM and introduce an easily computable measure of CI. We use it to analyze the CI of standard quantum gates, examine some of the measure's statistical properties, as well as a quantum causal switch that superposes two different CIs. At the example of a two-spin-boson model, we examine propagation of causal influence. We find that substantial CI arrives only far behind the light cone, and, surprisingly, almost perfectly in sync with reservoir-induced entanglement.

We take a conservative approach based on standard QM (using density matrices rather than process matrices) and the admission of do-interventions. Probability distributions of classical RVs are replaced by quantum states, since observables have, in general, no determined value in QM until they are measured [45–47]. On the other hand, quantum states encode all that can be known about a quantum system, and hence it is natural to base a theory of CI in QM on states: "causal influence" in QM will be understood in the sense that the final quantum state of the causally influenced system "listens to," i.e., depends on the initial quantum state of the influencer. Below we make this idea mathematically precise and introduce a measure of CI that we then explore. In principle one could base a definition of CI also on correlation functions. If all correlations are included, this is equivalent to using the quantum state, but at the same time appears to be more cumbersome and less fundamental: fundamentally the world is quantum, and quantum computers will one day probably be able to exchange quantum information without doing measurements. This motivates our attempt to base a definition of CI in quantum mechanics directly on quantum states.

II. NO-CAUSAL-INFLUENCE CONDITION

Consider two quantum systems A (Alice) and B (Bob) described by their respective density matrices ρ^A and ρ^B and a third system C, the joint environment, with a fixed initial state ρ^C that can, e.g., correspond to the physical system that propagates the CI and creates an effective interaction but also leads to decoherence, or might generate a common cause (see, e.g., [8,22]). The joint quantum system is a linear operator on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, with $\dim(\mathcal{H}_J) = d_J$, $J \in \{A, B, C\}$, where we assume d_A, d_B to be finite, whereas d_C can be infinite. We define $[d_J] := \{0, \ldots, d_J - 1\}$ and use Einstein summation convention for all repeated lower-case indices and Greek indices. Fundamentally, we consider CI to be a function of the set of quantum channels $\phi : \rho^{AB} \mapsto \rho'^{AB}$ (completely positive maps of density operators on $\mathcal{H}_A \otimes \mathcal{H}_B$) to real positive numbers. These channels can be expressed in terms of Kraus operators K^μ,

$$\rho'^{AB} = \phi(\rho^{AB}) = \mathrm{Tr}_C[K^\mu \rho^{ABC} K^{\mu\dagger}], \tag{1}$$

where A, B, C indicate the subsystems, and AB, ABC joint systems. In order for ρ'^{AB} to even be a function of ρ^{AB}, we need $\rho^{ABC} = \rho^{AB} \otimes \rho^C$. Otherwise, the same $\rho^{AB} = \mathrm{Tr}_C[\rho^{ABC}]$ can arise from different ρ^{ABC} and specifying an initial ρ^{AB} does not suffice to uniquely determine ρ'^{AB} because the channel ϕ depends on ρ^{BC} in this case. A definition of CI based on the quantum channel ϕ would then be ambiguous. On the other hand, if $\rho^{ABC} = \rho^{AB} \otimes \rho^C$, we can absorb ρ^C in the definition of ϕ, and ϕ can then be specified independently of the initial ρ^{AB}. In the simplest case there is only a single unitary Kraus operator, $K^1 = U$, but (1) allows us to also deal with the situation where the entire system is open. We can also include measurements and the collapse of states, even with postselection, as long as renormalization of the state is avoided as this would lead to nonlinearities. We must then require in addition that $\rho^{AB} = \rho^A \otimes \rho^B$. Otherwise, for an initially correlated state, a measurement by Bob collapses the state also on Alice's side and would signal CI, while it is well known that in this scenario no information can be transmitted (see, e.g., Ref. [48]). We hence assume a fully factorizing initial state, $\rho^{ABC} = \rho^A \otimes \rho^B \otimes \rho^C$, and ϕ itself is then a function of the $\{K^\mu\}$ and ρ^C, denoted as $\phi_{\{K^\mu\},\rho^C}$ if we want to make explicit the dependencies of ϕ.

In Ref. [8], a common cause of correlations between A, B in the form of an initially entangled state propagated to A, B was considered, created by an entangled state of A, C and then swapping C and B. In our approach, we deal with this situation by also including the creation of the entanglement as part of the quantum channel. In other words, distinguishing between a common cause and direct causal influence amounts, in our framework as well as in classical causal analysis, to distinguishing between a CI $C \to AB$ and a CI $A \to B$ (or mixtures thereof). Both can be done in the same framework starting from the fully factoring initial state. Indeed, also classically one would not introduce additional initial correlations between the random variables before applying the processes whose causal net of influences one wants to examine, justifying the fully factorized initial state also from a classical perspective.

If we write the initial states in an orthonormal basis, $\rho^A = \rho^A_{ij}|i\rangle\langle j|$, $\rho^B = \rho^B_{kl}|k\rangle\langle l|$, $\rho^C = \rho^C_{mn}|m\rangle\langle n|$, for $i, j \in [d_A]$, $k, l \in [d_B]$, $m, n \in [d_C]$, then (1) leads to $\rho'^{ABC}_{i'k'm',j'l'n'} = \rho^A_{ij}\rho^B_{kl}\rho^C_{mn}K^\mu_{i'k'm',ikm}K^{\mu*}_{j'l'n',jln}$, where the prime indices run over the same range as the respective nonprimed indices, and the partial trace over the systems B and C gives

$$\rho'^A_{i'j'} = F_{kl,ij}(i', j')\rho^A_{ij}\rho^B_{kl} \tag{2}$$

for Alice's final reduced state, where $F_{kl,ij}(i', j') := K^\mu_{i'k'm',ikm}K^{\mu*}_{j'k'm',jln}\rho^C_{mn}$ has the physical meaning of the propagator of the channel defined in Eq. (1) with system B traced out after the propagation.

Definition 1. Let A, B at initial time t_0 be in the state $\rho(t_0) = \rho^A \otimes \rho^B$, where ρ^B is a state set by Bob in a do-intervention. We say that B at time t_0 does not causally influence A at time t for a given initial ρ^A if and only if the reduced state $\rho'^A = \mathrm{Tr}_B[\phi(\rho(t_0))]$ of the system A after the propagation from t_0 to t is independent of ρ^B for any density matrix ρ^B. If this condition is fulfilled for any initial ρ^A, we say that system B at time t_0 does not causally influence system A at time t, shortly denoted by $B(t_0) \nrightarrow A(t)$. Otherwise we say that B at time t_0 causally influences A at time t, denoted by $B(t_0) \to A(t)$. Analogously one defines $A(t_0) \nrightarrow B(t)$ and $A(t_0) \to B(t)$, leading to a total of four possible cases. When we are not concerned with time dependence we may skip the arguments t, t_0 and simply consider initial and final states of a joint evolution. Definition 1 of no CI is similar to the one of a "semicausal map" in Ref. [42], but is based on the dependence of a final state directly on an initial state rather than a local channel. The generalization of Def. 1, as well as of Theorem II and expression (5) below to N quantum systems is given in the Appendix.

Theorem 1. Let A and B be quantum systems evolving via $\phi_{\{K^\mu\},\rho_C}$. Let $F_{kl,ij}(i', j') := K^\mu_{i'k'm',ikm}K^{\mu*}_{j'k'm',jln}\rho^C_{mn}$. Then, $B \nrightarrow A$ if and only if, for all $i, j, k, \tilde{k}, l, i', j'$ (no sum over $\tilde{k}$),

$$F_{kl,ij}(i', j') = \delta_{kl}F_{\tilde{k}\tilde{k},ij}(i', j'). \tag{3}$$

The statement for $A \nrightarrow B$ is analogous with the function $\tilde{F}_{ij,kl}(k', l') = K^\mu_{i'k'm',ikm}K^{\mu*}_{i'l'm',jln}\rho^C_{mn}$ (the propagator of channel ϕ with A traced out after propagation) replacing $F_{kl,ij}(i', j')$. This makes explicit that the CI is a function of the channel ϕ that maps ρ^{AB} to ρ'^{AB} alone. The proof of Theorem 1 is based on straightforward linear algebra and is given in the Appendix.

III. MUTUALITY

As an example, consider unitary evolution, i.e., a single Kraus operator $K^1 = U = U^A \otimes U^{BC}$, for $U^A \in \mathcal{U}(d_A)$, $U^{BC} \in \mathcal{U}(d_B d_C)$, the unitary group. Then $F_{kl,ij}(i', j') = \delta_{kl}U^A_{i'i}U^{A*}_{j'j}$, such that condition (3) is fulfilled, and, therefore, by Theorem 1, $B \nrightarrow A$ as expected. Similarly one finds $A \nrightarrow B$. However, tensor products of unitary matrices are a set of Haar measure zero in $\mathcal{U}(d_A d_B d_C)$. Unitary matrices that allow one-way CI exist as well. An example where $A \overset{\leftarrow}{\nrightarrow} B$, is given by the unitary transformation $U^{(123)}$ corresponding to the permutation $|ikm\rangle \mapsto |\mathrm{mik}\rangle$. The opposite case results from the unitary $U^{(132)}$ that permutes $|ikm\rangle \mapsto |kmi\rangle$. See Appendix for another example which does not correspond to

a permutation. Among 10^5 Haar distributed random unitary $2^3 \times 2^3$ matrices via QuTip [49] all of them permitted CI in both directions, which is the generic situation.

IV. NO TRANSITIVITY

Classical CI is not transitive (see Ref. [2], p. 237), i.e., $A(t_0) \to B(t_1)$ and $B(t_1) \to C(t_2)$ does not imply $A(t_0) \to C(t_2)$, and is hence different from logical implications. "No transitivity" straightforwardly extends to the quantum domain, as can be seen from the example of two consecutive CNOTs (from t_0 to t_1 and from t_1 to t_2) with B the control and A and C the target, respectively. The reduced density matrix of B, $\rho'^B(t_1)$, depends on ρ^A only in the off-diagonal elements (see Appendix). Similarly, starting at time t_1 with a product state $\rho^A \otimes \rho^B \otimes \rho^C$, if one applies at t_2 the CNOT gate with C the target system and B the control, one easily shows $B(t_1) \to C(t_2)$, but the reduced density matrix of system C only depends on the diagonal components of ρ'^B, which do not depend on ρ^A after the first CNOT, and therefore $A(t_0) \nrightarrow C(t_2)$.

V. MEASURE OF CAUSAL INFLUENCE

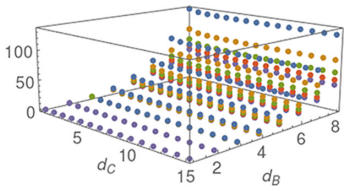
Consider two qubits A (control) and B (target) and apply the CNOT gate on a pure state. One might suspect that CNOT permits only influence from A to B. However, applying Theorem 1, we see that both CI directions are allowed. This is not merely a mathematical consequence of the definition but corresponds to a well known and real physical effect called *phase kickback* in quantum circuits [50] where the phase β in the initial state $(|0\rangle + |1\rangle) \otimes (|0\rangle + e^{i\beta}|1\rangle)/2$, ends up in the state of Alice. More generally, both reduced states after a CNOT gate on an initial product state depend on components of their partner's initial state. Nevertheless, while Alice's final state only carries Bob's dependence in the off-diagonal components, all his components after the CNOT depend on Alice's initial state [see (C2) in the Appendix]. This is in sync with the intuition that the CI from control to target is "stronger" than the other way around and motivates the introduction of a measure of CI. Although one could argue that a CNOT with reversed roles of control and target is a CNOT conjugated with local Hadamard transformations and hence expect equal influence in both directions, it is reasonable not to request invariance of a measure of CI under local prepropagation of the influencing system because varying its initial state is part of the process of examining the influence. A natural way to quantify the CI is via $|\partial \rho'^A_{i'j'}/\partial \rho^B_{hf}|$, for $h, f \in [d_B]$. We base our definition on all pure initial states ρ^A of Alice. Note that, due to linear propagation, the $\rho'^A_{i'j'}$ are holomorphic functions of the ρ^B_{hf} such that these complex derivatives are always well defined. For the same reason, the measure is independent of ρ^B.

Definition 2. Let $\rho^A = V|0\rangle\langle 0|V^\dagger$, for $V \in \mathcal{U}(d_A)$ and some fixed initial state $|0\rangle_A$ of Alice, i.e., $\rho^A_{ij} = V_{i0}V^*_{j0}$. We define the measure of causal influence from B to A as

$$I_{B\to A}(\phi) = \int d\mu(V) \sum_{hfi'j'} \left|\partial \rho'^A_{i'j'}/\partial \rho^B_{hf}\right|^2. \tag{4}$$

	$I_{B\to A}$	$I_{A\to B}$
CNOT	0.66	1.33
Csign	1.33	1.33
$\sqrt{SWAP}$	1.66	1.66
Fredkin	0.33	1.00
Toffoli	0.33	0.33

(a)



(b)

FIG. 1. (a) CI (dimensionless) for some quantum gates. The environment for the three-qubit gates is initially in state $|0\rangle_C$ and, when required, A is the control qubit and B, the target. (b) $E[I_{A\to B}]$ (dimensionless) as function of d_A, d_B and d_C [Eq. (7)], where d_A is coded in the sequence of points for each d_B, d_C with $2 \leqslant d_A \leqslant 6$ from top to bottom ($3 \leqslant d_B \leqslant 8$) or bottom to top ($d_B = 2$, coinciding on this scale) in each sequence.

Analogously, one defines $I_{A\to B}$. Notice that for either A or B corresponding to the trivial (one-dimensional) Hilbert space, there is a single physical state so the derivatives vanish and the influence is always zero. Therefore we restrict ourselves to $d_A, d_B > 1$. Based on an average rather than a maximization procedure, Def. 2 allows for straightforward evaluation. While not all do-interventions are physically realizable [21], both definitions 1 and 2 can be applied experimentally, as long as Alice and Bob can generate arbitrary local states, e.g., through measurement and local quantum channels.

For finite d_C, from (2) and the definition of F, $\partial \rho'^A_{i'j'}/\partial \rho^B_{hf} = \rho^A_{ij} F_{kl,ij}(i', j') D_{kl,hf}$ with $D_{kl,hf} \equiv \partial \rho^B_{kl}/\partial \rho^B_{hf}$. Since for all density matrices ρ, $\rho_{i_0 i_0} = 1 - \sum_{j\neq i_0} \rho_{jj}$ and the partial derivative of a complex number with respect to its complex conjugate vanishes, we have $D_{kl,hf} = (1 - \delta_{hf})\delta_{kh}\delta_{lf} - \delta_{hf}\delta_{kl}(\delta_{h0} - 1)(-1)^{\delta_{k0}}$, where, without loss of generality, we took $i_0 = 0$. Then, by algebraic computation and using invariant integration over the unitary group [51], see Appendix,

$$I_{B\to A}(\phi_{U,\rho^C}) = [F_{kl,ij}(i', j')F^*_{k_1 l_1,ij}(i', j') + F_{kl,ii}(i', j')F^*_{k_1 l_1,i_1 i_1}(i', j')]\frac{D_{kl,hf}D_{k_1 l_1,hf}}{d_A(d_A + 1)}. \tag{5}$$

In Fig. 1(a) we show the values of the CI for some common qubit gates.

VI. PROPERTIES OF THE MEASURE

The measure (4) enjoys the following important properties (see Appendix for the proofs):

(1) $I = 0$ if and only if there is no CI.

(2) For every channel $\{K^\mu\}$,

$$I_{B\to A}\left(\phi_{\{(U_1^A\otimes U^B\otimes \mathbb{I}^C)K^\mu(U_0^A\otimes \mathbb{I}^{BC})\},\rho^C}\right) = I_{B\to A}(\phi_{\{K^\mu\},\rho^C}) \tag{6}$$

for all $U_0^A, U_1^A \in \mathcal{U}(d_A)$, $U^B \in \mathcal{U}(d_B)$, $U \in \mathcal{U}(d_A d_B d_C)$, and analogously for $I_{A\to B}$. These properties are natural: after the propagation, any local action should not change the CI. And since we consider all initial states of Alice, the measure should be invariant under her local prepropagation unitaries. On the other hand, since the changes of ρ^B reflect Bob's do-interventions, $I_{B\to A}$ need not be invariant under local

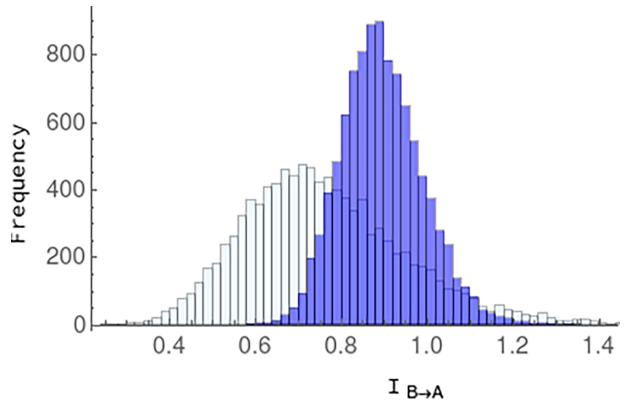


FIG. 2. Histogram of $I_{B\to A}$ (dimensionless), for 10^4 random Haar distributed unitary matrices of dimension $D = d_A d_B d_C$, for $D = 2 \times 2 \times 2$ (light blue) and $D = 3 \times 2 \times 2$ (dark blue), their mean values are 0.76 and 0.896, the standard deviations 0.19 and 0.095, and their expected values [Eq. (7)] $16/21 \simeq 0.76$ and $128/143 \simeq 0.895$, respectively.

unitaries of Bob before the propagation, and indeed, in general, $I_{B\to A}(\phi_{K^\mu(\mathbb{I}^{AC}\otimes U^B),\rho^C}) \neq I_{B\to A}(\phi_{\{K^\mu\},\rho^C})$.

(3) The natural scale of $I_{B\to A}$ is given by the Haar-measure average $E[I_{B\to A}(\phi_{U,\rho^C})] = \int d\mu(U) I_{B\to A}(\phi_{U,\rho^C})$.

$$E[I_{B\to A}(\phi_{U,\rho^C}))] = \frac{d_B - 1}{d_A^2 d_B^2 d_C^2 - 1}\big[2d_A^2 d_B^2 d_C - 2d_B^2 d_C + d_A(d_B - 2)^2\big(d_B^2 d_C^2 - 1\big)\big]. \quad (7)$$

$E[I_{A\to B}(\phi_{U,\rho^C})]$ is obtained by permuting $A \leftrightarrow B$. Figure 1(b) illustrates the behavior of $E[I_{A\to B}(\phi_{U,\rho^C})]$. Notice that it is strictly increasing with d_B, which is reasonable as one expects that the larger the Hilbert-space dimension of the influencing system, the more it can influence. A particular case of interest is such that either one of the systems A or B, or the environment has a dimension much greater than the others. $E[I_{B\to A}(\phi_{U,\rho^C})]$ tends to $2(d_B - 1)/d_C$, ∞ and $(d_B - 1)(d_B - 2)^2/d_A$ in these limits, respectively. Figure 2 shows a histogram of $I_{A\to B}$ for random Haar generated unitary matrices. The relatively narrow distribution confirms that $E[I_{B\to A}(\phi_{U,\rho^C})]$ represents a good scale for $I_{B\to A}(\phi_{U,\rho^C})$ for given Hilbert-space dimension. For three qubits, there are gates such as CNOT or CSIGN with almost twice the value of $E[I_{B\to A}(\phi_{U,\rho^C})]$.

VII. QUANTUM CAUSAL SWITCH

In analogy to the quantum switch [33,34] one can create a unitary evolution U_{sup} that superposes the CI $A \overset{\curvearrowleft}{\underset{\curvearrowright}{}} B$ and $A \overset{\curvearrowright}{\underset{\curvearrowleft}{}} B$, controlled by an ancilla qubit $|\chi^c\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi)$. In the Appendix we investigate the behavior of such a "quantum causal switch" and show how one can interpolate continuously between the two directions of the CI as function of the state of the control qubit. For almost all states of the control qubit, the CI is in both directions.

VIII. PROPAGATION OF CAUSAL INFLUENCE

To illustrate the propagation of CI within a specific exactly solvable physical model, compare it to the well-known creation of reservoir-induced entanglement, and to verify that indeed entanglement does not arrive before the CI, which would be physically unreasonable, we apply the measure to two spin-1/2s interacting with a bath of harmonic oscillators, which corresponds to $d_C = \infty$. The Hamiltonian is $H = H_{AB} + H_{bath} + H_{\text{int}}$, with $H_{\text{int}} = (S^A + S^B)\sum_k g_k q_k$. S^A, S^B are the "coupling agents" acting on $\mathcal{H}_A$ and $\mathcal{H}_B$ with eigenvalues a_0, a_1 and b_0, b_1, respectively, and q_k and g_k are the generalized coordinates and coupling constants to the kth oscillator, respectively. For degenerate-in-energy, noninteracting spins $H_{AB} = 0$, and the model becomes an exactly solvable dephasing model that can lead to reservoir-induced entanglement [52]. The Hamiltonian does not give any direct interaction between A and B, but an effective interaction is mediated by the heat bath that turns out to be the standard dipole-dipole interaction with a time-dependent reflecting retardation. The propagation of an initial product state at time t [52] leads to a CI

$$I^{\text{bath}}_{B\to A} = \tfrac{4}{3}\sin^2[(a_0 - a_1)(b_0 - b_1)\varphi(t)]e^{-2(a_0-a_1)^2 f(t)}, \quad (8)$$

where

$$\varphi(t) = \sum_k \frac{g_k^2}{2m\hbar\omega_k^2}\left(t - \frac{\sin\omega_k t}{\omega_k}\right)$$

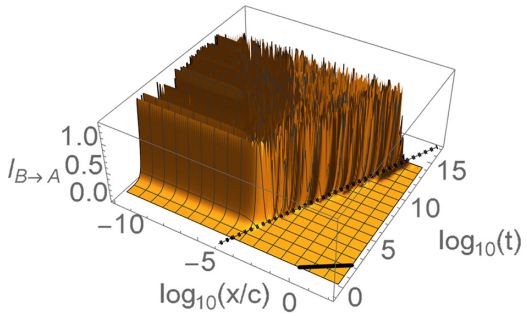
and

$$f(t) = \sum_k \frac{g_k^2(1 + 2\bar{n}_k)}{2m\hbar\omega_k^3}(1 - \cos\omega_k t),$$

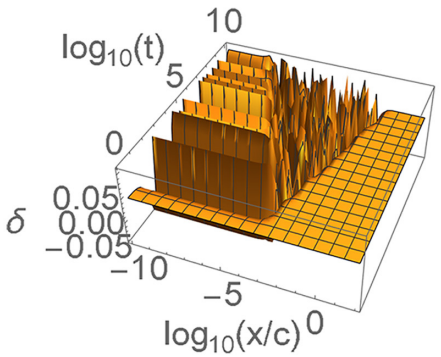
where $\bar{n}_k$ is the thermal occupation of the kth oscillator. Both $\varphi(t)$ and $f(t)$ vanish at $t = 0$ and are strictly positive for $t > 0$.

$I^{\text{bath}}_{A\to B}$ is obtained by permuting $a_i \leftrightarrow b_i$, $i = 0, 1$. The CI is invariant under $t \to -t$. For values of f and φ such that there is no causal influence, no entanglement between the qubits is generated. However, the converse is not true, i.e., not all CI generates entanglement. $I^{\text{bath}}_{B\to A}$ takes a maximum value of 4/3, periodically oscillates as function of φ and decays exponentially for large f but does not vanish exactly, whereas the generated entanglement does vanish exactly as the state approaches the fully mixed state [53–55].

For the physical example of two double quantum dots (DQDs) at distance x from each other coupled with dipole interaction to blackbody radiation, with a UV frequency cutoff $y_m = \omega_{\max}\tau$, where $\tau = \hbar/k_B T$ and T is the temperature, the functions $f(t)$ and $\varphi(t)$ can be obtained analytically (see Appendix) if one approximates $\coth \simeq 1$ for $y_m \gg 1$ [56]. Physically, it is clear that CI can only arise inside the light cone, $x = ct$, and indeed, the CI plotted for this system in Fig. 3(a) shows that there is no CI for spacelike separated points, $t < x/c$. Surprisingly, however, significant CI is generated only far behind the light cone, namely, for $t \gtrsim 10^{12}(x/c)^3$. This is reminiscent of reservoir-induced entanglement [57,58] that also arises only far behind the light-cone [56] (i.e., "Entanglement harvesting" [59–69], i.e., entanglement creation outside the light cone, is typically not possible here without "reservoir engineering"). In fact, the space-time



(a)



(b)

FIG. 3. (a) $I_{B\to A}$ (dimensionless) for two initially noninteracting DQD with $d = 10$ nm coupled with the blackbody radiation at $T = 2.73$ K with $y_m = 4250$ ($\hbar\omega_m = 1$ eV), parametrized by x/c and t. The corresponding plot for the entanglement of formation E looks almost identical (see Fig. 2 in Ref. [57]). Dotted line: $t = 10^{11}(x/c)^3$. Full line: light-cone $ct = x$. (b) Difference $\delta = 0.795\,481 I_{B\to A} - E$ (dimensionless), see text, E for the initial state $(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)/2$. Same parameters as in panel (a).

dependence of CI and entanglement of formation (EOF) E, created as long as both initial states of A and B contain components of both $|0\rangle$ and $|1\rangle$, are almost perfectly in sync, see Fig. 3(b). Minimization of $\sum_{i_t,j_x} \delta^2_{i_t,j_x}$ over a regular grid of 51×51 points in the space-time regions shown with $\delta = \lambda I_{B\to A} - E$ gives $\lambda \simeq 0.795\,481$, and the remaining differences are less than about 0.05 in absolute value over the 12 orders of magnitude of x/c considered. Under time reversal, $t \to -t$, $f(t)$ remains invariant, whereas $\varphi(-t) = -\varphi(t)$, which leads to complex-conjugate matrix elements of ρ^{AB} as usual. Formally there is hence exactly as much CI in positive time direction as in negative time direction. This can be seen already from (V) with $\rho'^{AB} \to \rho'^{AB*}$ under time reversal. Hence, reasons for the apparent purely forward CI in Nature must be sought outside quantum mechanics [70]. Indeed, causality is, even in our most fundamental established theories, implemented by hand by choosing advanced Green's functions only.

IX. CONCLUSION

In summary, we gave a definition of causal influence (CI) in the quantum world based on reduced density matrices. We derived a necessary and sufficient condition on the joint evolution operator of Alice, Bob, and an environment that a given quantum system can causally influence another one. Moreover, we introduced a measure of the CI, analyzed it in detail, showed the possibility of superposing opposite directions of CI, and applied it to particular cases of both finite- and infinite-dimensional environments. For the example of two degenerate double-quantum dots at distance x dipole-interacting with thermal blackbody radiation we found that the space-time dependence of the CI is almost perfectly in sync with the reservoir induced entanglement. Both arrive long after [$t \propto (x/c)^3$] the light cone $ct = x$. Just as entanglement measures, classicality measures and measures based on other resource theories have had a large impact in theoretical physics for the last three decades. Similarly, we hope that having a causality measure in quantum mechanics opens the path to study many new things, starting from its properties, over Bell-like inequalities, to the relationship to entanglement harvesting, the classical limit of quantum CI, and many more.

ACKNOWLEDGMENTS

We thank Lukas Fiderer and Kilian Stahl for discussions, Ronny Müller for suggesting the unitary in Eq. (B1) in the Appendix for a one-way CI, and Marten Folkertsma for discussions about invariances of the measure.

APPENDIX A: PROOF OF THEOREM 1

Let $\alpha_{ij}(i', j') := \rho^B_{kl}\rho^C_{mn}K^{\mu}_{i'k'm',ikm}K^{\mu*}_{j'k'm',jln}$. Then, from (1), $\rho'^A_{i'j'} = \rho^A_{ij}\alpha_{ij}(i', j')$. Imposing $B \nrightarrow A$ for all ρ^A, $B \nrightarrow A$ must hold in particular for $\rho^A_{ij} = \delta_{ij}\delta_{ii_0}$ and thus, $\rho'^A_{i'j'} = \alpha_{i_0i_0}(i', j')$ must be independent of (ρ^B) $\forall$ i_0. Since i_0 is arbitrary $\in [d_A]$, we need that $\alpha_{ii}(i', j')(\not\rho^B)$ $\forall$ $i \in [d_A]$, where $(\not\rho^B)$ denotes no dependence on ρ^B for all ρ^B. Moreover, consider the previous density matrix but with two nonvanishing off-diagonal components $\rho^A_{i_1j_1}$ and, due to Hermiticity, $\rho^A_{j_1i_1}$, for $i_1 \neq j_1$. Then, imposing the independence condition, it is necessary that

$$\left[\rho^A_{i_1j_1}\alpha_{i_1j_1}(i', j') + \rho^A_{j_1i_1}\alpha_{j_1i_1}(i', j')\right](\not\rho^B) \tag{A1}$$

(without implicit sum). Thus, writing $\rho^A_{i_1j_1}$ as the sum of its real part and i times its imaginary part and $\rho^A_{j_1i_1}$ as their difference, one sees that it is necessary that $(\alpha_{i_1j_1}(i', j') \pm \alpha_{j_1i_1}(i', j'))(\not\rho^B)$, leading to the conclusion that $\alpha_{ij}(i', j')(\not\rho^B)$ must hold for all i, j, i', j'. Applying the definition of $F_{kl,ij}(i', j')$,

$$\alpha_{ij}(i', j') = \rho^B_{kk}F_{kk,ij}(i', j') + \sum_{l\neq k}\rho^B_{kl}F_{kl,ij}(i', j'), \tag{A2}$$

its independence of ρ^B is fulfilled if and only if

$$\begin{aligned} F_{kk,ij}(i', j') &= F_{\tilde{k}\tilde{k},ij}(i', j') \text{ and} \\ F_{kl,ij}(i', j') &= F_{lk,ij}(i', j') = 0 \end{aligned} \tag{A3}$$

for all $k \neq l, \tilde{k}, i, j, i', j'$. The first condition comes from the trace one of the density matrices, thus, any $\rho^B_{k_0k_0}$ can be written as $\rho^B_{k_0k_0} = 1 - \sum_{k\neq k_0}\rho^B_{kk}$ and thus the first summand in (A2)

becomes

$$\sum_{k\neq k_0}\rho^B_{kk}F_{kk,ij}(i',j')+\left(1-\sum_{k\neq k_0}\rho^B_{kk}\right)F_{k_0k_0,ij}(i',j'). \tag{A4}$$

Since ρ^B_{kk} for $k\neq k_0$ can vary independently, it is necessary that $F_{kk,ij}(i',j')=F_{k_0k_0,ij}(i',j')$ for all k, then

$$\begin{aligned}\sum_k \rho^B_{kk}F_{kk,ij}(i',j') &= F_{k_0k_0,ij}(i',j')\sum_k\rho^B_{kk}\\ &= F_{k_0k_0,ij}(i',j')\mathrm{Tr}[\rho^B]=F_{k_0k_0,ij}(i',j').\end{aligned} \tag{A5}$$

Finally, the second condition comes from a similar argument derived for the independence of $\alpha_{ij}(i',j')$ with $i\neq j$.

APPENDIX B: EXAMPLE OF A SINGLE DIRECTION OF INFLUENCE

The unitary matrix (B1) in $\mathcal{U}(2\times2\times2)$ is such that $A\to B$ but $B\nrightarrow A$, i.e., it only allows one influencing direction. Moreover, $I_{A\to B}(\phi_{U,|0\rangle\langle0|_C})=1.5$ and $I_{B\to A}(\phi_{U,|0\rangle\langle0|_C})=0.0$.

$$U=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0&0\\0&0&0&0&0&0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\0&0&0&0&\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0&0\\0&0&0&0&0&0&\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{pmatrix}. \tag{B1}$$

APPENDIX C: QUBIT DENSITY MATRICES AFTER THE APPLICATION OF THE CNOT GATE

Write Alice's and Bob's initial states in the computational basis so that $\rho^A=\rho^A_{ij}|i\rangle\langle j|$, $\rho^B=\rho^B_{kl}|k\rangle\langle l|$ and $\rho^C=\rho^C_{mn}|m\rangle\langle n|$, for $i,j\in[2]$, $k,l\in[2]$, $m,n\in[2]$ and let the initial state be $\rho^A\otimes\rho^B\otimes\rho^C$ at t_0. Let ρ'^{ABC} and ρ''^{ABC} be the states after applying a CNOT gate with control B and target A and subsequently a CNOT with control B and target C, at t_1 and t_2, respectively. The reduced density matrices of B at t_1 and of C at t_2 are given by

$$\rho'^B=\begin{pmatrix}\rho^B_{00}&\rho^B_{01}2\,\mathrm{Re}\,\rho^A_{01}\\ \rho^B_{10}2\,\mathrm{Re}\,\rho^A_{10}&\rho^B_{11}\end{pmatrix},\quad \rho''^C=\begin{pmatrix}\rho^B_{00}\rho^C_{00}+\rho^B_{11}\rho^C_{11}&\rho^C_{01}\rho^B_{00}+\rho^C_{10}\rho^B_{11}\\ \rho^C_{01}\rho^B_{11}+\rho^C_{10}\rho^B_{00}&\rho^C_{00}\rho^B_{11}+\rho^C_{11}\rho^B_{00}\end{pmatrix}. \tag{C1}$$

For the application of the CNOT with A control and B target to the state $\rho^A\otimes\rho^B$, one has that the reduced density matrices are given by

$$\rho'^A=\begin{pmatrix}\rho^A_{00}&\rho^A_{01}2\,\mathrm{Re}\,\rho^B_{01}\\ \rho^A_{10}2\,\mathrm{Re}\,\rho^B_{10}&\rho^A_{11}\end{pmatrix},\quad \rho'^B=\begin{pmatrix}\rho^B_{00}\rho^A_{00}+\rho^B_{11}\rho^A_{11}&\rho^B_{01}\rho^A_{00}+\rho^B_{10}\rho^A_{11}\\ \rho^B_{01}\rho^A_{11}+\rho^B_{10}\rho^A_{00}&\rho^B_{00}\rho^A_{11}+\rho^B_{11}\rho^A_{00}\end{pmatrix}. \tag{C2}$$

APPENDIX D: INTEGRATION OVER THE UNITARY GROUP

For ϕ_{U,ρ^C}, using the definition of $D_{kl,hf}$, the CI (4) can be written as

$$I_{B\to A}(\phi_{U,\rho^C})=\int d\mu(V)\sum_{hfi'j'}|V_{i0}V^*_{j0}F_{kl,ij}(i',j')D_{kl,hf}|^2. \tag{D1}$$

Expanding the square, one has order two terms of the functions F and D and integrals of the type $\int d\mu(V)V^*_{j0}V^*_{i_10}V_{i0}V_{j_10}$.

In Ref. [51] is shown that the integrals $\int d\mu(U)U^*_{i_1j_1}\cdots U^*_{i_pj_p}U_{k_1l_1}\cdots U_{k_ql_q}$ denoted by $\langle I,J|K,L\rangle$ vanish unless $q=p$, which will be assumed to be the case. Even more, such integrals are nonvanishing if, in addition, $K=I_{\sigma_1}$ and $L=J_{\sigma_2}$, for $\sigma_1,\sigma_2\in S_p$, and since $U_{i_1j_1}\cdots U_{i_pj_p}=U_{\tau(i_1)\tau(j_1)}\cdots U_{\tau(i_p)\tau(j_p)}$ for any $\tau\in S_p$ due to the fact that it is a multiplication of complex numbers, we may assume $\sigma_1=id$, then the nonzero integrals are of the form $\langle I,J|I,J_{\sigma_2}\rangle$. Since the Haar measure is invariant under transposition, a permutation between rows and columns does not change the integral, i.e., $\langle I,J|K,L\rangle=\langle J,I|L,K\rangle$. Furthermore, the integral is affected by whether the indices take on the same or different values, but independent of what these values are, for this reason it is convenient to use the graphical representation introduced in Ref. [51]:

(1) The distinct values in the index set I are represented as dots in a column and, on its right, the distinct values of the index set J as dots in a column (since J_{σ_2} is a permutation of J, it has the same distinct values as J).

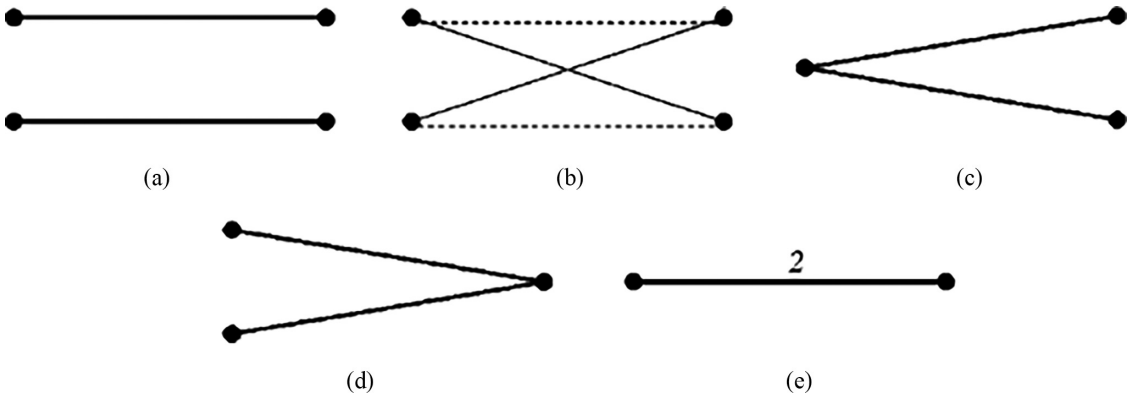


FIG. 4. [50] Nonvanishing integrals $\mathcal{I}$ for $p = 2$. Their respective values, from panel (a) to panel (e), are $\frac{1}{D^2-1} =: \mathcal{I}_{(a)}$, $\frac{-1}{D(D^2-1)} =: \mathcal{I}_{(b)}$, $\frac{1}{D(D+1)} =: \mathcal{I}_{(c)}$, $\frac{1}{D(D+1)} =: \mathcal{I}_{(d)}$, and $\frac{2}{D(D+1)} =: \mathcal{I}_{(a)}$.

(2) Factors $U^*_{i_r j_r}$ and $U_{i_r,\sigma_2(j_r)}$, for $r = 1, \ldots, p$, are represented by thin (solid) and dotted lines, respectively. The power of the matrix element, if greater than 1, will be represented above the solid line or below the dotted line, correspondingly. When a pair $U^*_{i_r j_r} U_{i_r j_r}$ occurs together, the thin and dotted lines will be replaced by a thick solid line, whose multiplicity will be understood as the power of this pair.

Using this graphical representation, the nonvanishing integrals $\mathcal{I}$ of $p = 2$ are those shown in Fig. 4 together with their values. Then, the nonvanishing integrals $\int d\mu(V) V^*_{j0} V^*_{i_1 0} V_{i0} V_{j_1 0} = \langle j i_1, 00 | i j_1, 00\rangle$, for $V \in \mathcal{U}(d_A)$, are such that $(i, j_1) = \sigma(j, i_1)$, for $\sigma \in S_2$, i.e., either

(1) $(i, j_1) = (j, i_1)$, so that the integral is $\langle i i_1, 00 | i i_i, 00\rangle$, and it takes the values

$$\langle j i_1, 00 | j i_i, 00\rangle = \begin{cases} \mathcal{I}_{(e)} = \frac{2}{d_A(d_A+1)} & \text{if } j = i_1 \\ \mathcal{I}_{(d)} = \frac{1}{d_A(d_A+1)} & \text{if } j \neq i_1, \end{cases} \tag{D2}$$

whose value can be compactly written as $(1 + \delta_{i_1 j})/[d_A(d_A + 1)]$, or

(2) $(i, j_1) = (i_1, j)$, and, since the integral is independent of what the values of the indices are, it takes the same value as in case 1.

Thus, considering both options and not overcounting the intersection,

$$\begin{aligned} \int d\mu(V) V^*_{j0} V^*_{i_1 0} V_{i0} V_{j_1 0} &= \delta_{i_1 i}\delta_{j j_1}[\delta_{j i_1}\mathcal{I}_{(e)} + (1 - \delta_{j i_1})\mathcal{I}_{(d)}] \\ &\quad + \delta_{i_1 j_1}\delta_{ij}[\delta_{j i_1}\mathcal{I}_{(e)} + (1 - \delta_{j i_1})\mathcal{I}_{(d)}] \\ &\quad - \delta_{i_1 j}\delta_{i_1 i}\delta_{i_1 j_1}\mathcal{I}_{(e)} \\ &= \frac{1}{d_A(d_A+1)}(\delta_{i_1 i}\delta_{j j_1} + \delta_{i_1 j_1}\delta_{ij}). \end{aligned} \tag{D3}$$

APPENDIX E: GENERALIZATION TO *n* QUANTUM SYSTEMS

Consider n quantum systems described by their respective density matrices ρ^1, ...,ρ^n and a system e, the joint environment, with a fixed initial state ρ^e corresponding to the physical system that propagates the causal influence and creates an effective interaction, but also leads to decoherence. The joint quantum system lies in the Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \mathcal{H}_e$, with $\dim(\mathcal{H}_r) = d_r$, $r \in \{1, \ldots, n, e\}$. Definition 1 naturally extends as $l \nrightarrow k$ if and only if, after the propagation of the initial state $\rho^1 \otimes \cdots \otimes \rho^n \otimes \rho^e$, the reduced state of system k fulfils that $\rho'^k(\not\rho^l)$ for any density matrix ρ^l describing system l.

Definition 2 is generalized as follows: Let $\rho^k = V^k|0\rangle\langle 0|V^{k\dagger}$ for, where $V^k \in \mathcal{U}(d_k)$ so that $\rho^k_{i_k j_k} = V^k_{i_k 0}(V^k_{j_k 0})^*$. Then, the causal influence from l to k is

$$I_{l\to k} = \int d\mu(V^k) \sum_{i'_k j'_k \tilde{i}_l \tilde{j}_l} \left| \frac{\partial \rho'^k_{i'_k j'_k}}{\partial \rho^l_{\tilde{i}_l \tilde{j}_l}} \right|^2. \tag{E1}$$

If the full system is closed, we consider the evolution of the systems $1, \ldots, n, e$ via a joint unitary transformation $U \in \mathcal{U}(d_1 \times \cdots \times d_n \times d_e)$. The uncorrelated initial state is mapped to $U(\rho^1 \otimes \cdots \otimes \rho^n \otimes \rho^e)U^\dagger =: \rho'^{1\cdots ne}$. Using Einstein's sum convention, we write the initial state of system r in the computational basis so that $\rho^r = \rho^A_{i_r j_r}|i_r\rangle\langle j_r|$, for $i_r, j_r \in [d_r]$ for all $r \in \{1, \ldots, n\}$ and take $\rho^e = |0\rangle\langle 0|$. Then,

$$\rho'^{1\cdots ne}_{i'_1 \ldots i'_n i'_e, j'_1 \ldots j'_n j'_e} = \rho^1_{i_1 j_1} \cdots \rho^n_{i_n j_n} U_{i'_1 \ldots i'_n i'_e, i_1 \ldots i_n 0} U^*_{j'_1 \ldots j'_n j'_e, j_1 \ldots j_n 0}. \tag{E2}$$

Then, the elements of the reduced density matrix of the system k become

$$\begin{aligned} \rho'^k_{i'_k, i'_j} = \rho^1_{i_1 j_1} \cdots \rho^n_{i_n j_n} U_{i'_1 \ldots i'_{k-1} i'_k i'_{k+1} \ldots i'_n i'_e, i_1 \ldots i_n 0} \\ \times U^*_{i'_1 \ldots i'_{k-1} j'_k i'_{k+1} \ldots i'_n i'_e, j_1 \ldots j_n 0}. \end{aligned} \tag{E3}$$

Defining

$$F_{i_1 j_1 \ldots i_n j_n}(i'_k, j'_k) = U_{i'_1 \ldots i'_{k-1} i'_k i'_{k+1} \ldots i'_n i'_e, i_1 \ldots i_n 0} U^*_{i'_1 \ldots i'_{k-1} j'_k i'_{k+1} \ldots i'_n, i'_e, j_1 \ldots j_n 0}, \tag{E4}$$

and, from (E3) and (E4),

$$\frac{\partial \rho'^k_{i'_k j'_k}}{\partial \rho^l_{\tilde{i}_l \tilde{j}_l}} = \prod_{r \neq l} \rho^r_{i_r j_r} F_{i_i j_1 \ldots i_n j_n}(i'_k, j'_k) D_{i_i j_1, \tilde{i}_l \tilde{j}_l}. \tag{E5}$$

Thus,

$$I_{l\to k} = F_{i_1 j_1 \ldots i_n j_n}(i'_k, j'_k) F^*_{i_{1_2} j_{1_2} \ldots i_{n_2} j_{n_2}}(i'_k, j'_k) D_{i_l j_l, \tilde{i}_l \tilde{j}_l} D_{i_{l_2} j_{l_2}, \tilde{i}_l \tilde{j}_l} \prod_{r, r_2 \neq k, l} \rho^r_{i_r, j_r} \rho^{r_2 *}_{i_{r_2}, j_{r_2}} \int d\mu(V^k) \rho^k_{i_k, j_k} \rho^{k*}_{i_{k_2}, j_{k_2}}, \tag{E6}$$

where the values of the integral is given by (D3).

On the other hand, applying an analogous reasoning as in the proof of Theorem 1, one finds the extension of Theorem 1: $l \nrightarrow k$ if and only if

$$\prod_{r\neq k,l} \rho^r_{i_r j_r} F_{i_1 j_1 \ldots i_l j_l \ldots i_n j_n}(i'_k, j'_k) = \delta_{i_l j_l} \left(\prod_{r\neq k,l} \rho^r_{i_r j_r} F_{i_1 j_1 \ldots \tilde{i}_l \tilde{i}_l \ldots i_n j_n}(i'_k, j'_k) \right), \tag{E7}$$

for all $i_l, j_l, \tilde{i}_l, i_k, j_k, i'_k, j'_k$.

APPENDIX F: PROOF OF PROPERTY 1 OF THE MEASURE

By Def. 1, if there is no causal influence, all the derivatives in (3) vanish and therefore I is the integral over zero, which is zero. Conversely, suppose that $\partial \rho'^A_{i'j'}/\partial \rho^B_{hf} \neq 0$ for a particular combination of indices h, f and a specific value of ρ^B_{hf}. Since $\rho'^A_{i'j'}$ is a linear function of all ρ^B_{hf}, $\partial \rho'^A_{i'j'}/\partial \rho^B_{hf}$ is independent of ρ^B_{hf}. Hence, $\partial \rho'^A_{i'j'}/\partial \rho^B_{hf} > 0$ for a set of finite measure, and hence $I_{B\to A} \neq 0$.

APPENDIX G: PROOF OF PROPERTY 2 OF THE MEASURE

The invariance follows from the following three invariances:

(1) $I_{B\to A}(\phi_{\{(U^A\otimes \mathbb{I}^{BC})K^\mu\},\rho^C}) = I_{B\to A}(\phi_{\{k^\mu\},\rho^C})$.

From (4), using $\partial \rho'^A_{i'j}/\partial \rho^B_{hf} = \rho^A_{ij} F_{kl,ij}(i', j') D_{kl,hf}$,

$$\begin{aligned} I_{B\to A}(\phi_{\{(U^A\otimes \mathbb{I}^{BC})K^\mu\},\rho^C}) &= \int d\mu(V) \sum_{hfi'j'} \left| \rho^A_{ij} U^A_{i'i'_1} K^\mu_{i_1 k' m', ikm} U^{A*}_{j'j'_1} K^{\mu *}_{j'_1 k' m', jln} \rho^C D_{kl,hf} \right|^2 \\ &= \int d\mu(V) \sum_{hfi'j'} \rho^A_{ij} U^A_{i'i'_1} K^\mu_{i_1 k' m', ikm} U^{A*}_{j'j'_1} K^{\mu *}_{j'_1 k' m', jln} \rho^C_{mn} D_{kl,hf} \rho^{A*}_{\tilde{i}\tilde{j}} U^{A*}_{i'\tilde{i}'_1} K^{\mu *}_{\tilde{i}_1 \tilde{k}' \tilde{m}', \tilde{i}\tilde{k}\tilde{m}} U^A_{j'\tilde{j}'_1} K^\mu_{\tilde{j}'_1 \tilde{k}' \tilde{m}', \tilde{j}\tilde{l}\tilde{n}} \rho^{C*}_{\tilde{m}\tilde{n}} D_{\tilde{k}\tilde{l},hf} \\ &= \int d\mu(V) \sum_{hfi'_1 j'_1} \rho^A_{ij} K^\mu_{i'_1 k' m', ikm} K^{\mu *}_{j'_1 k' m', jln} \rho^C_{mn} D_{kl,hf} \rho^{A*}_{\tilde{i}\tilde{j}} K^{\mu *}_{i'_1 \tilde{k}' \tilde{m}', \tilde{i}\tilde{k}\tilde{m}} K^\mu_{j'_1 \tilde{k}' \tilde{m}', \tilde{j}\tilde{l}\tilde{n}} \rho^{C*}_{\tilde{m}\tilde{n}} D_{\tilde{k}\tilde{l},hf} \\ &= I_{B\to A}(\phi_{\{k^\mu\},\rho^C}), \end{aligned} \tag{G1}$$

where we used that $U^A_{i'i'_1} U^{A*}_{i'\tilde{i}'_1} = U^{A\dagger}_{\tilde{i}'_1 i'} U^A_{i'i'_1} = \delta_{i'_1 \tilde{i}'_1}$ and $U^{A*}_{j'j'_1} U^A_{j'\tilde{j}'_1} = U^{A\dagger}_{j'_1 j'} U^A_{j'\tilde{j}'_1} = \delta_{j'_1 \tilde{j}'_1}$.

(2) $I_{B\to A}(\phi_{\{(\mathbb{I}^A\otimes U^B\otimes \mathbb{I}^C)K^\mu\},\rho^C}) = I_{B\to A}(\phi_{\{K^\mu\},\rho^C})$.

According to Def. 2, $I_{B\to A} = \int d\mu(V) \sum_{hfi'j'} |\partial \rho'^A_{i'j'}/\partial \rho^B_{hf}|^2$. Alice's reduced final state is obtained tracing out systems B and C after the evolution, nevertheless Alice's final state remains unaltered due to the invariance of the partial trace under final U^B.

(3) $I_{B\to A}(\phi_{U(U^A_0\otimes \mathbb{I}^{BC}),\rho^C}) = I_{B\to A}(\phi_{U,\rho^C})$.

It follows from the (Haar) integration over all possible initial Alice's states.

APPENDIX H: PROOF OF PROPERTY 3 OF THE MEASURE

From equation (5),

$$I_{B\to A}(\phi_{U,\rho^C}) = \frac{1}{d_A(d_A+1)} D_{kl,hf} D_{k_1 l_1,hf} (\delta_{i_1 i}\delta_{j j_1} + \delta_{ij}\delta_{i_1 j_1}) F_{kl,ij}(i', j') F^*_{k_1 l_1, i_1 j_1}(i', j'), \tag{H1}$$

and, applying the definition of the first moment,

$$\begin{aligned} E[I_{B\to A}(\phi_{U,\rho^C})] &= \int d\mu(U) I_{B\to A}(U) \\ &= \frac{1}{d_A(d_A+1)} \sum_{i,j,i_1,j_1} D_{kl,hf} D_{k_1 l_1,hf} (\delta_{i_1 i}\delta_{j j_1} + \delta_{ij}\delta_{i_1 j_1}) \int d\mu(U) F_{kl,ij}(i', j') F^*_{k_1 l_1, i_1 j_1}(i', j'). \end{aligned} \tag{H2}$$

Using the definition of the F functions, the integral over U can be written as

$$\sum_{k',\tilde{k}',m',\tilde{m}'} \int d\mu(U) U^*_{j'k'm',jl0} U^*_{i'\tilde{k}'\tilde{m}',i_1k_10} U_{i'k'm',ik0} U_{j'\tilde{k}'\tilde{m}',j_1l_10} = \sum_{k',\tilde{k}',m',\tilde{m}'} \langle p_1p_2, q_1q_2 | r_1r_2, s_1s_2\rangle, \tag{H3}$$

where the integrals of the form $\int d\mu(V) V^*_{i_1j_1}\cdots V^*_{i_pj_p} V_{k_1l_1}\cdots V_{k_pl_p}$ have been denoted by $\langle (i_1\ldots i_p),(j_1\ldots j_p)|(k_1\ldots k_p),(l_1\ldots l_p)\rangle =: \langle I, J|K, L\rangle$,. and where, for short, we introduced the notation $p_1 = j'k'm'$, $p_2 = i'\tilde{k}'\tilde{m}'$, $q_1 = jl0$, $q_2 = i_1k_10$, $r_1 = i'k'm'$, $r_2 = j'\tilde{k}'\tilde{m}'$, $s_1 = ik0$, and $s_2 = j_1l_10$. The nonvanishing integrals are those such that $(r_1, r_2) = \sigma_1(p_1, p_2)$ and $(s_1, s_2) = \sigma_2(q_1, q_2)$, for $\sigma_1, \sigma_2 \in S_2 = \{id, \sigma\}$. Denote $P = (p_1, p_2)$, $Q = (q_1, q_2)$, $R = (r_1, r_2)$, and $S = (s_1, s_2)$, so that the order-two integrals are of the form $\langle P, Q|R, S\rangle$ and we split them into the four (three nonequal) following (nondisjoint) sets of integrals: $\langle P, Q|P, S\rangle$, $\langle P, Q|R, Q\rangle$, and $\langle P, Q|P, Q\rangle = \langle PQ|P_\sigma Q_\sigma\rangle$. From set theory, given three sets I_1, I_2, and I_3, the number of elements of its union is

$$|I_1 \cup I_2 \cup I_3| = |I_1| + |I_2| + |I_3| - |I_1 \cap I_2| - |I_1 \cap I_3| - |I_2 \cap I_3| + |I_1 \cap I_2 \cap I_3|. \tag{H4}$$

Thus, we can write the order-two integrals in terms of the four (three nonequal) sets of integrals stated above:

$$\begin{aligned}\langle P, Q|R, S\rangle = {} & \delta_{PR}\langle P, Q|P, S\rangle + \delta_{QS}\langle P, Q|R, Q\rangle + \delta_{PR_\sigma}\delta_{QS_\sigma}\langle P, Q|P_\sigma, Q_\sigma\rangle \\ & - \delta_{PR}\delta_{QS}\langle P, Q|P, Q\rangle - \delta_{PR}\delta_{PR_\sigma}\delta_{QS_\sigma}\langle P, Q|P, Q_\sigma\rangle - \delta_{QS}\delta_{PR_\sigma}\delta_{QS_\sigma}\langle P, Q|P_\sigma, Q\rangle \\ & + \delta_{PR}\delta_{PR_\sigma}\delta_{QS}\delta_{QS_\sigma}\langle P, Q|P, Q\rangle.\end{aligned} \tag{H5}$$

The integrals in (H5), in terms of integrals depending on p_1, p_2, q_1, and q_2, are as follows:

(1) $\langle P, Q|P, S\rangle$, which can be written as

$$\begin{aligned}\langle p_1p_2, q_1q_2|p_1p_2, s_1s_2\rangle = {} & \delta_{q_1s_1}\delta_{q_2s_2}\langle p_1p_2, q_1q_2|p_1p_2, q_1q_2\rangle + \delta_{q_1s_2}\delta_{q_2s_1}\langle p_1p_2, q_1q_2|p_1p_2, q_2q_1\rangle \\ & - \delta_{q_1q_2}\delta_{q_1s_1}\delta_{q_1s_2}\langle p_1p_2, q_1q_1|p_1p_2, q_1q_1\rangle.\end{aligned} \tag{H6}$$

(2) $\langle P, Q|R, Q\rangle$, which takes the same value as $\langle Q, P|Q, R\rangle$, and therefore it is computed as in 1.

(3) $\langle P, Q|P_\sigma, Q_\sigma\rangle = \langle P, Q|P, Q\rangle = \langle p_1p_2, q_1q_2|p_1p_2, q_1q_2\rangle$.

(4) $\langle P, Q|P, Q_\sigma\rangle = \langle p_1p_2, q_1q_2|p_1p_2, q_2q_1\rangle$.

(5) $\langle P, Q|P_\sigma, Q\rangle$ which takes the same value as $\langle Q, P|Q, P_\sigma\rangle$, and therefore it is computed as in 4.

Therefore, we are left with the values of $\langle p_1p_2, q_1q_2|p_1p_2, q_1q_2\rangle$ and $\langle p_1p_2, q_1q_2|p_1p_2, q_2q_1\rangle$. From Fig. 4 we have that

$$\langle p_1p_2, q_1q_2|p_1p_2, q_1q_2\rangle = \begin{cases} \mathcal{I}_{(e)} & \text{if } p_1 = p_2 \text{ and } q_1 = q_2 \\ \mathcal{I}_{(c)} & \text{if } p_1 = p_2 \text{ and } q_1 \neq q_2 \\ \mathcal{I}_{(d)} & \text{if } p_1 \neq p_2 \text{ and } q_1 = q_2 \\ \mathcal{I}_{(a)} & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2, \end{cases} \tag{H7}$$

and

$$\langle p_1p_2, q_1q_2|p_1p_2, q_2q_1\rangle = \begin{cases} \mathcal{I}_{(e)} & \text{if } p_1 = p_2 \text{ and } q_1 = q_2 \\ \mathcal{I}_{(c)} & \text{if } p_1 = p_2 \text{ and } q_1 \neq q_2 \\ \mathcal{I}_{(d)} & \text{if } p_1 \neq p_2 \text{ and } q_1 = q_2 \\ \mathcal{I}_{(b)} & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2. \end{cases} \tag{H8}$$

Combining (H5)–(H8), a generic integral of order two can be written as

$$\begin{aligned}\langle p_1p_2, q_1q_2|r_1r_2, s_1s_2\rangle = {} & \delta_{p_1r_1}\delta_{p_2r_2}\Bigg(\delta_{q_1s_1}\delta_{q_2s_2}\left\{\delta_{q_1q_2}\frac{1+\delta_{p_1p_2}}{D(D+1)} + \left(1-\delta_{q_1q_2}\right)\left[\frac{\delta_{p_1p_2}}{D(D+1)} + \frac{1-\delta_{p_1p_2}}{D^2-1}\right]\right\} \\ & + \delta_{q_1s_2}\delta_{q_2s_1}\left\{\delta_{q_1q_2}\frac{1+\delta_{p_1p_2}}{D(D+1)} + \left(1-\delta_{q_1q_2}\right)\left[\frac{\delta_{p_1p_2}}{D(D+1)} - \frac{1-\delta_{p_1p_2}}{D(D^2-1)}\right]\right\} \\ & - \delta_{q_1q_2}\delta_{q_1s_1}\delta_{q_1s_2}\frac{1+\delta_{p_1p_2}}{D(D+1)}\Bigg) \\ & + \delta_{q_1s_1}\delta_{q_2s_2}\Bigg(\delta_{p_1r_1}\delta_{p_2r_2}\left\{\delta_{p_1p_2}\frac{1+\delta_{q_1q_2}}{D(D+1)} + \left(1-\delta_{p_1p_2}\right)\left[\frac{\delta_{q_1q_2}}{D(D+1)} + \frac{1-\delta_{q_1q_2}}{D^2-1}\right]\right\} \\ & + \delta_{p_1r_2}\delta_{p_2r_1}\left\{\delta_{p_1p_2}\frac{1+\delta_{q_1q_2}}{D(D+1)} + \left(1-\delta_{p_1p_2}\right)\left[\frac{\delta_{q_1q_2}}{D(D+1)} - \frac{1-\delta_{q_1q_2}}{D(D^2-1)}\right]\right\}\end{aligned}$$

$$
\begin{aligned}
&- \delta_{p_1p_2}\delta_{p_1r_1}\delta_{p_1r_2}\frac{1+\delta_{q_1q_2}}{D(D+1)}\Bigg)\\
&+ \left(\delta_{p_1r_2}\delta_{p_2r_1}\delta_{q_1s_2}\delta_{q_2s_1} - \delta_{p_1r_1}\delta_{p_2r_2}\delta_{q_1s_1}\delta_{q_2s_2}\right)\left(\delta_{q_1q_2}\frac{1+\delta_{p_1p_2}}{D(D+1)} + \left(1-\delta_{q_1q_2}\right)\left[\frac{\delta_{p_1p_2}}{D(D+1)} + \frac{1-\delta_{p_1p_2}}{D^2-1}\right]\right)\\
&- \delta_{p_1r_1}\delta_{p_1p_2}\delta_{p_1r_2}\delta_{q_1s_2}\delta_{q_2s_1}\left[\delta_{q_1q_2}\frac{2}{D(D+1)} + \frac{1-\delta_{q_1q_2}}{D(D+1)}\right]\\
&- \delta_{p_1r_2}\delta_{p_2r_1}\delta_{q_1s_1}\delta_{q_1q_2}\delta_{q_2s_2}\left[\delta_{p_1p_2}\frac{2}{D(D+1)} + \frac{1-\delta_{p_1p_2}}{D(D+1)}\right]\\
&+ \delta_{p_1r_1}\delta_{p_1p_2}\delta_{p_1r_2}\delta_{q_1s_1}\delta_{q_1s_2}\delta_{q_1q_2}\frac{2}{D(D+1)},
\end{aligned}
\tag{H9}
$$

and plugging it on equation (H3) and subsequently in (H2), we obtain the expected value of the measure of causal influence.

Let $\cdots$ denote the integral $\langle p_1p_2, q_1q_2|r_1r_2, s_1s_2\rangle$ assuming that the indices are written in terms of the original ones, i.e., $p_1 = j'k'm'$, etc., and denote the same integral but with a certain index taking the same value as another index writing the equality of indices inside $\ldots$ instead of the three dots, i.e., the integral $\langle p_1p_2, q_1q_2|r_1r_2, s_1s_2\rangle$ for $i_1 = i$ will be written as $i_1 = i$. Then, from equation (H2), and denoting by λ the set of indices $i'j'k'\tilde{k}'m'\tilde{m}'$,

$$
\begin{aligned}
d_A(d_A+1)E[I_{B\to A}(\phi_{U,\rho^C})] &= \sum_{\substack{\lambda i j i_1 j_1\\ k l k_1 l_1 h f}} D_{kl,hf}D_{k_1l_1,hf}\left(\delta_{i_1i}\delta_{jj_1} + \delta_{i,j}\delta_{i_1,j_1}\right)\ldots\\
&= \sum_{\substack{\lambda i j\\ k l k_1 l_1 h f}} D_{kl,hf}D_{k_1l_1,hf}\, i_1 = i,\, j_1 = j + \sum_{\substack{\lambda i i_1\\ k l k_1 l_1 h f}} D_{kl,hf}D_{k_1l_1,hf}\, j = i,\, j_1 = i_1.
\end{aligned}
\tag{H10}
$$

Since, for our purpose, the two terms in (H10) can be treated analogously, we will show the computations for the first term. We separate the cases where $h \neq f$ and $h = f$. In the former case $D_{kl,hf}$ takes the value $\delta_{kh}\delta_{lf}$ and in the latter, $(-1)^{\delta_{k0}}(\delta_{h0}-1)\delta_{kl}\delta_{hf}$. Thus, the first term summed in the above expression can be written as

$$
\sum_{\substack{\lambda i j\\ k l k_1 l_1 h}} \sum_{f:f\neq h} \delta_{kh}\delta_{lf}\delta_{k_1h}\delta_{l_1f}\, i_1 = i,\, j_1 = j + \sum_{\substack{\lambda i j\\ k l k_1 l_1 h f}} \sum_{f:f=h} (-1)^{\delta_{k0}+\delta_{k_10}}(1-\delta_{h0})\delta_{hf}\delta_{kl}\delta_{k_1l_1}\, i_1 = i,\, j_1 = j. \tag{H11}
$$

The first term of the sum (H11) can be written as

$$
\sum_{\lambda i j k l} i_1 = i,\, j_1 = j,\, k = k_1 = h,\, l = l_1 = f - \sum_{\lambda i j k} i_1 = i,\, j_1 = j,\, k = k_1 = l = l_1 = f = h, \tag{H12}
$$

and the second term is simplified as

$$
(d_B - 1)\sum_{\substack{\lambda i j\\ k k_1}} (-1)^{\delta_{k0}+\delta_{k_10}}\, i_1 = i,\, j_1 = j,\, l = k,\, l_1 = k_1. \tag{H13}
$$

Using *Mathematica*, computing the integrals via (H9), one obtains the sums in (H12) and (H13), e.g., the first term in (H12) is

$$
\frac{d_A^2 d_B\left[d_B^2 d_C\left(d_A^2 + d_C - 1\right) - 1\right]}{d_A^2 d_B^2 d_c^2 - 1},
$$

and (6) is recovered. $E[I_{A\to B}(\phi_{U,\rho^C})]$ is obtained by permuting $A \leftrightarrow B$, i.e., $E[I_{A\to B}(\phi_{U,\rho^C})](d_A, d_B, d_C) = \tau\{E[I_{B\to A}(\phi_{U,\rho^C})](d_A, d_B, d_C)\}$, where τ is the permutation $\tau(A) = B$, $\tau(B) = A$.

APPENDIX I: CAUSAL QUANTUM SWITCH

Consider unitary evolutions $U^{A\overleftrightarrow{\to}B}$ and $U^{A\overleftarrow{\nrightarrow}B}$ that permit a single influence direction indicated in their superscript. In analogy to the quantum switch one can create a unitary evolution U_{sup} that superposes the CI $A\overleftrightarrow{\to}B$ and $A\overleftarrow{\nrightarrow}B$, controlled by an ancilla qubit $|\chi^c\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi)$,

$$
U_{\text{sup}} = |0\rangle\langle 0| \otimes U^{A\overleftrightarrow{\to}B} + |1\rangle\langle 1| \otimes U^{A\overleftarrow{\nrightarrow}B}. \tag{I1}
$$

Figure 5 shows the CI from A to B and vice versa, with $d_A = d_B = 2$, for $U^1_{\text{sup}} = |0\rangle\langle 0| \otimes U^{(123)} + |1\rangle\langle 1| \otimes U^{(132)}$ depending on the control qubit, where one sees that if $|\chi^c\rangle = |0\rangle$, $I(\phi_{U_{\text{sup}},|\chi^c\rangle}) = I(\phi_{U^{(123)},|0\rangle\langle 0|_C})$ and if $|\chi^c\rangle = |1\rangle$, $I(\phi_{U_{\text{sup}},|\chi^c\rangle}) = I(\phi_{U^{(132)},|0\rangle\langle 0|_C})$, for $I \in \{I_{A\to B}, I_{B\to A}\}$. Classically, CI is usually considered one-way only (represented by an arrow in a directed acyclic graph), as time-ordering of events and forward-in-time-only causal influence is assumed. In (I1), both CI are forward in time. When time stamps of events can be exchanged, mixtures of causal influences are also possible

classically (lung cancer can be caused by smoking, but also incite people to enjoy some final cigarettes ...), but the superpositions introduced here go beyond this. More generally, the four possible CI options could be superposed, and, for open quantum systems, Kraus-operators with different causal influence.

APPENDIX J: PROPAGATION OF CAUSAL INFLUENCE

For the physical example of two double quantum dots (DQDs) at distance l from each other coupled with dipole interaction to blackbody radiation, with a UV frequency cutoff $y_m = \omega_{\max}\tau$, where $\tau = \hbar/k_B T$ and T is the temperature, the functions $f(t)$ and $\varphi(t)$ can be obtained analytically if one approximates $\coth \simeq 1$ for $y_m \gg 1$ [54]. The argument of the sin in (7) becomes

$$\frac{At}{t_0^3}\{-2\sin(y_m t_0) + \mathrm{Si}[y_m(t-t_0)] + 2\mathrm{Si}(y_m t_0) - \mathrm{Si}[y_m(t+t_0)]\} + \frac{2A}{y_m t_0^3}\sin(y_m t)\sin(y_m t_0), \tag{J1}$$

where $t_0 = l/c$ denotes the time of travel of a light signal between the two DQD and $A = \alpha_0 d^2/\pi c^2\tau^2$, d is the dipole moment of the quantum system divided by the electron charge, and $\alpha_0 \simeq \frac{1}{137}$ and c are the fine-structure constant and the speed of light in vacuum, respectively. In (J1) and onwards, both t and t_0 are in units of τ. The exponent in (6) becomes $-\alpha_0 d^2\omega_{\max}^2/3c^2$.

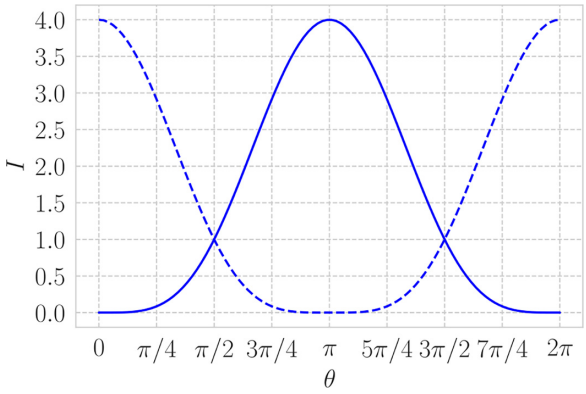


FIG. 5. $I_{A\to B}$ (dashed line) and $I_{B\to A}$ (continuous line), both dimensionless, of U^1_{sup} depending on the control qubit $|\chi^c\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$, parametrized by θ.

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