Timelike correlations and quantum tensor product structure

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The state-space structure for a composite quantum system is postulated among several mathematically consistent possibilities that are compatible with a local quantum description. For instance, the unentangled Gleason's theorem allows a state space that includes density operators as a proper subset among all possible composite states. However, bipartite correlations obtained in Bell-type experiments from this broader state space are, in fact, quantum simulable [Barnum *et al.*, Phys. Rev. Lett. **104**, 140401 (2010)], and hence, such spacelike correlations are no good for making a distinction among different compositions. In this work we analyze the communication utilities of these different composite models and show that they can lead to distinct utilities in a simple communication game involving two players. Our analysis thus establishes that a beyond quantum composite structure can lead to beyond quantum correlations in the timelike scenario and hence welcomes new principles to isolate the quantum correlations from the beyond quantum ones. We also prove a no-go theorem that the classical information carrying capacity of different such compositions cannot be greater than that of the corresponding quantum composite systems.

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I. INTRODUCTION

The tensor product postulate of quantum mechanics, also called the "zeroth" axiom in the literature [1], describes the Hilbert space of a composite system as the tensor product of the components' Hilbert spaces [2–4]. A recent study, however, logically derived this postulate from the state postulate and the measurement postulate rather than taking it as an independent one [5]. Nevertheless, within this tensor product structure, the unentangled Gleason's theorem assigns state spaces for the composite systems that include density operators (the quantum states) as a proper subset [6–8]. In fact, assuming individual systems' descriptions to be quantum, several mathematical models are possible for the composite state and effect spaces that yield a consistent outcome probability. Exploring this broader class of theories helps us to compare and contrast the information-processing capabilities of quantum theory with other theories and gain insights into the origin of such capabilities.

The framework of generalized probability theory (GPT) [9–14] is well suited to studying these different composite models. Physical constraints, such as no signaling and local tomography, limit the composite state spaces to be constrained within two extremes [15–18]: the minimal tensor product composition containing only separable states and the maximal tensor product composition containing beyond quantum states that are positive on product tests (POPT) and compatible with the unentangled Gleason's theorem. The corresponding effect spaces are specified in accordance with the "no-restriction" hypothesis [19] that includes all the mathematically consistent effects in the theory.

A natural question is whether these different composite models can lead to stronger than quantum correlations that will in turn make them distinct from quantum state space and put an embargo on their physical existence. In this work, we ask and answer how the information-processing capabilities of composite systems change when one uses different mathematical structures to describe composition. For the bipartite case a negative response comes through the work of Barnum *et al.* [20], which states that no such composition can produce any beyond quantum spacelike correlations in a Bell-type experimental scenario. While a maximal composition involving more than two subsystems can yield stronger than quantum correlation in a typical Bell scenario [21], recently, in Ref. [22] it was shown that even in the bipartite case stronger than quantum correlations are possible if the typical classical input–classical output Bell scenario is generalized to a quantum input–classical output semiquantum scenario [23]. Although this quantum input scenario disallows all the compositions having beyond entangled states, it requires trustworthy verifiers in producing some predetermined unentangled quantum inputs [22]. In a completely different approach, recently, the authors in [24] showed that the bipartite minimal composition can yield a stronger than quantum correlation if a timelike scenario is considered. More specifically, it has been shown that a communication game played between two timelike separated players—a sender and a receiver in the sender's causal future—cannot be won perfectly by communicating between two elementary quantum systems (qubits) if the composite state space is considered to be the standard quantum one, whereas the game becomes perfectly winnable if the composition is assumed to be the minimal one. Although the minimal composition consisting of only separable states cannot produce any nonlocal correlation in the Bell-like scenario, the existence of beyond quantum effects in this theory results in beyond quantum correlations in the timelike scenario. This result may lead to the impression that beyond quantum effects are necessary to obtain beyond quantum timelike correlations. In this work we, however, show that such an intuition is, in fact, not true. More specifically, we show that the maximal composition that allows only product effects but permits beyond quantum states can also yield beyond quantum correlations in the timelike scenario. Thus, while Barnum *et al.*'s result [20] shows that spacelike correlations are no good to establish the beyond quantum nature of the bipartite maximal composition, our result establishes that timelike correlations do serve the purpose here. We then proceed to prove a no-go theorem that although the maximal composition allows beyond quantum timelike correlations, the classical information carrying capacity of such models cannot be more than that of the corresponding quantum composite systems. In fact, we prove a generic result regarding the information capacity of composite systems in the GPT framework.

II. PRELIMINARIES

A. Framework of the GPT

We start by briefly recalling the framework of the GPT. For a detailed overview of this framework we refer to Refs. [9–14]. In the recent past several interesting results have been reported within this framework [25–31]. A GPT is specified by a list of system types and the composition rules specifying the combination of several systems, where a system S is specified by identifying the three-tuple $(\Omega_S, \mathcal{E}_S, \mathcal{T}_S)$ of the state space, the effect space, and the set of transformations. In a prepare-and-measure scenario, which will be considered in this work, it is sufficient to describe only Ω_S and $\mathcal{E}_S$.

State space Ω_S. A state ω_S for a system S is a mathematical object that yields outcome probabilities for all the measurements that can possibly be carried out on the system. The collection of all allowed states forms the state space Ω_S, and generally, it is considered to be a compact-convex set embedded in some real vector space V. Convexity ensures that if ω_1 and ω_2 are allowed states, then their classical mixture $p\omega_1 + (1-p)\omega_2$ is also a valid state. On the other hand, compactness ensures that there is no physical distinction between states that can be prepared exactly and states that can be prepared to arbitrary accuracy [32]. The extreme points of the set Ω_S are called pure states or states of maximal knowledge.

Effect space $\mathcal{E}_S$. An effect e is a linear functional acting on V such that $e : \Omega_S \to [0,1]$. The unit effect is defined by $u(\omega) = 1,\ \forall\, \omega \in \Omega_S$. The set of all proper effects $\mathcal{E}_S \equiv \{e\,|\,0 \leqslant e(\omega) \leqslant 1,\ \forall\, \omega \in \Omega_S\}$ is the convex hull of the zero effect, the unit effect, and the extremal effects and is embedded in the vector space $V^\star$ dual to V. A measurement $\mathcal{M}$ is a collection of effects that sum to the unit effect, i.e., $\mathcal{M} \equiv \{e_i \in \mathcal{E}_S \mid \sum_i e_i = u\}$.

State and effect cones. Sometimes it is mathematically convenient to work with the notion of unnormalized states and effects. The set of unnormalized states $V_+ \subset V$ is the conical hull of Ω_s, i.e., $r\omega \in V_+$ for $r \geqslant 0$ and $\omega \in \Omega_S$. The set of unnormalized effects is its dual cone $V_+^\star \subset V^\star$, i.e., $V_+^\star \equiv \{e\,|\,e(\omega) \geqslant 0,\ \forall\, \omega \in V_+\}$. The formulation generally assumes the no-restriction hypothesis, which demands that the state and effect cones are dual to each other [11].

Composite system. Given two systems with state spaces $\Omega_A \subset V_A$ and $\Omega_B \subset V_B$, the state space Ω_{AB} for the composite systems is embedded in the vector space V_{AB}, which is the tensor product of the component vector spaces, i.e., $V_{AB} = V_A \otimes V_B$ [5]. Although the choice of Ω_{AB} is not unique, the no-signaling principle and tomographic locality postulate [9] bound the choices within two extremes: the minimal tensor product space and maximal tensor product space [15]. More formally,

$$\Omega_{AB}^{\min} \equiv \left\{\omega_{AB} = \sum_i p_i \omega_A^i \otimes \omega_B^i \middle| \omega_A^i \in \Omega_A,\right.$$
$$\left.\omega_B^i \in \Omega_B;\, p_i \geqslant 0 \sum_i p_i = 1\right\},$$
$$\Omega_{AB}^{\max} \equiv \{\omega_{AB} \in V_{AB} | 1 \geqslant e_A \otimes e_B(\omega_{AB}) \geqslant 0,$$
$$\forall e_A \in \mathcal{E}_A \& e_B \in \mathcal{E}_B\}.$$

It is not hard to see that the cone $(V_{AB}^{\min})_+$ is isomorphic to the dual cone $(V_{AB}^{\max})_+$. Therefore, in accordance with the no-restriction hypothesis for the case of minimal composition, the effect cone $(V_{AB}^{\min})_+^\star \cong (V_{AB}^{\max})_+$, and for the case of maximal composition, the effect cone $(V_{AB}^{\max})_+^\star \cong (V_{AB}^{\min})_+$. The symbol $\cong$ denotes isomorphism.

B. Quantum theory: A GPT

Quantum theory can be seen as a special instance of a GPT. State space of a d-level quantum system associated with complex Euclidean space $\mathbb{C}^d$ is the set of density operators acting on $\mathbb{C}^d$, i.e., $\Omega(\mathbb{C}^d) \equiv \mathcal{D}(\mathbb{C}^d)$. The set $\mathcal{D}(\mathbb{C}^d)$ is a convex compact set embedded in $\mathbb{R}^{d^2-1}$. The unnormalized state cone is the set of all non-negative operators $\mathcal{P}(\mathbb{C}^d) := \{\lambda\rho\,|\,\lambda \geqslant 0\,\&\,\rho \in \mathcal{D}(\mathbb{C}^d)\}$, which is also the unnormalized effect cone. In other words, quantum theory is self-dual. The minimal composition of two quantum systems associated with Hilbert spaces $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$ allows only separable states; we call it the SEP composition and denote the resulting system as the triplet $S_{\mathrm{SEP}}^{AB} \equiv [\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\mathrm{SEP}}]$. Formally, the state space for the SEP composition is given by

$$\Omega_{\mathrm{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) := \left\{\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i \middle| p_i \geqslant 0\right.$$
$$\left.\& \sum_i p_i = 1;\, \rho_A^i \in \mathcal{D}(\mathbb{C}^{d_A}),\, \rho_B^i \in \mathcal{D}(\mathbb{C}^{d_B})\right\}.$$

Since $\Omega_{\mathrm{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ contains only separable states, the corresponding effect space $\mathcal{E}_{\mathrm{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ contains effects that are not allowed in quantum theory. Entanglement witness operators yielding positive probability on separable states are valid effects in this composition, although they are not allowed in quantum theory. On the other extreme, the maximal composition, which we will call $\overline{\text{SEP}}$, and the resulting system denoted as $S^{AB}_{\overline{\text{SEP}}} \equiv [\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\overline{\text{SEP}}}]$, has the state space

$$\begin{aligned}\Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) := \{W_{AB} \in \text{Herm}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) |\\ \text{Tr}(W_{AB}) = 1, \text{Tr}[W_{AB}(\pi_A \otimes \pi_B)] \geqslant 0\\ \forall \pi_A \in \mathcal{P}(\mathbb{C}^{d_A}), \pi_B \in \mathcal{P}(\mathbb{C}^{d_B})\}.\end{aligned}$$

Here $\text{Herm}(\mathcal{X})$ denotes the set of Hermitian operators acting on the space $\mathcal{X}$, and normalization demands $\text{Tr}(W_{AB}) = 1\, \forall\, W_{AB} \in \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$. The unnormalized effect cone corresponding to $\mathcal{E}_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ is identical to the unnormalized state cone corresponding to the set $\Omega_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$. For the quantum case $S^{AB}_Q \equiv [\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_Q]$ we have $\Omega_Q(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) = \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, and the effect cone is identical to the state cone which represents the self-duality of quantum theory. The following set-inclusion relations are immediate:

$$\begin{aligned}\Omega_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \Omega_Q(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}),\\ \mathcal{E}_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \mathcal{E}_Q(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \subset \mathcal{E}_{\text{SEP}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}).\end{aligned}$$

In between SEP and $\overline{\text{SEP}}$, many other compositions can be defined by appending (deducting) suitable states (effects). Among these, quantum composition is the only one that is self-dual.

C. Operational notions of dimension

The dimension of the vector space V in which the set Ω_S is embedded is a well-defined concept, but it does not carry any operational signature. However, an operationally motivated notion of dimension can be defined through the concept of state distinguishability. For the purpose of our work, in the following, we recall a few relevant definitions [33].

Definition 1. Perfect distinguishability. Two states $\omega_1, \omega_2 \in \Omega_S$ are perfectly distinguishable whenever there exists some measurement $\mathcal{M} = \{e_1, e_2 \in \mathcal{E}_S \,|\, e_1 + e_2 = u\}$ such that $e_i(\omega_j) = \delta_{ij}$.

For instance, two quantum states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$ are perfectly distinguishable if and only if they are orthogonal, a fact which follows from the seminal no-cloning theorem [34]. On the other hand, in discrete classical probability theory the state spaces are simplexes, and any two extreme points are perfectly distinguishable [35].

Definition 2. Operational dimension. Operational dimension $\mathbb{O}(S)$ of a system S is the maximum cardinality of the set of states $\Omega_n := \{\omega_1, \dots, \omega_n\} \subset \Omega_S$ such that all the states in Ω_n are perfectly distinguishable in a single measurement.

For instance $\mathbb{O}(\mathbb{C}^d) = d$, although the dimension of the vector space in which $\mathcal{D}(\mathbb{C}^d)$ is embedded is $d^2 - 1$. The operational dimension of a system quantifies its classical information carrying capacity [9,36] (see also [37,38]); that is, by sending a system with operational dimension $\mathbb{O}(S)$ through a noiseless channel a sender can send $\log_2 \mathbb{O}(S)$ bits of classical information to a receiver.

Definition 3. Information dimension. The information dimension $\mathbb{I}(S)$ of a system S is the maximum cardinality of the set of states $\Omega_n := \{\omega_1, \dots, \omega_n\} \subset \Omega_S$ such that all the states in Ω_n are pairwise perfectly distinguishable.

Note that in defining $\mathbb{O}(S)$ a single measurement is allowed to distinguish the states in the set Ω_n. On the other hand, $\mathbb{I}(S)$ deals with the pairwise distinguishability, and for different pairs of states $\{\omega_i, \omega_j\}$ in Ω_n, different measurements $\mathcal{M}_{ij}$ can be performed to distinguish the pairs. Therefore, it clearly follows that $\mathbb{I}(\star) \geqslant \mathbb{O}(\star)$ for an arbitrary GPT system, and accordingly, one can define a quantity called dimension mismatch, $\Delta(\star) := \mathbb{I}(\star) - \mathbb{O}(\star)$. For classical and quantum systems it follows from simple arguments that both these dimensions are equal. However, as shown in [33], for the hypothetical toy model of box world ($\square$) the information dimension is strictly greater than the operational dimension. While $\mathbb{I}(\square) = 4$, one has that $\mathbb{O}(\square) = 2$.

III. RESULTS

As already mentioned, the state space of the maximal composition strictly contains the quantum state space, i.e., $\mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \subset \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$. In particular, an entanglement witness operator $W \notin \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, whereas $W \in \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$. Although the state space of the $\overline{\text{SEP}}$ theory is bigger than the quantum state space, the "nonlocal strength" of the bipartite system $S^{AB}_{\overline{\text{SEP}}}$ is no more than S^{AB}_Q. This follows from a generic result by Barnum *et al.* [20], who proved that any no-signaling bipartite input-output probability distribution $P(ab|xy)$ obtained from $S^{AB}_{\overline{\text{SEP}}}$ can also be obtained from S^{AB}_Q; here a and b denote Alice's and Bob's outputs corresponding to their respective inputs x and y. For a state $W \in \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ the correlation $P(ab|xy)$ is obtained as

$$P(ab|xy) = \text{Tr}\left[W\left(\pi^a_x \otimes \pi^b_y\right)\right],$$

$$\pi^a_x \in \mathcal{P}(\mathbb{C}^{d_A}), \quad \sum_a \pi^a_x = \mathbf{1}_{d_A} \& \pi^b_y \in \mathcal{P}(\mathbb{C}^{d_B}), \quad \sum_b \pi^b_y = \mathbf{1}_{d_B}.$$

As pointed out in [21], the result of Barnum *et al.* can be seen as follows. According to Choi-Jamiołkowski (CJ) isomorphism [39,40], any $W \in \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}) \setminus \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ can be written as $[\mathcal{I} \otimes \Lambda](\phi^+)$, where Λ is a positive map, $\mathcal{I}$ is the identity map, and ϕ^+ is the projector on the maximally entangled state. Furthermore, any such witness can also be written as $[\mathcal{I} \otimes \Lambda_{tp}](\psi)$, where Λ_{tp} is positive and trace preserving and ψ is a projector onto a pure bipartite state [41]. Therefore, we have

$$\begin{aligned}P(ab|xy) &= \text{Tr}\left[W\left(\pi^a_x \otimes \pi^b_y\right)\right]\\ &= \text{Tr}\left\{\left[\mathcal{I} \otimes \Lambda_{tp}\right](\psi)(\pi^a_x \otimes \pi^b_y)\right\}\\ &= \text{Tr}\left\{\psi\left(\pi^a_x \otimes \Lambda^\star_{tp}[\pi^b_y]\right)\right\}\\ &= \text{Tr}\left[\psi\left(\pi^a_x \otimes \tilde{\pi}^b_y\right)\right].\end{aligned}$$

Here $\Lambda^\star$ is the adjoint map of Λ, and since the adjoint of a positive trace-preserving map is positive and unital, $\{\tilde{\pi}^b_y := \Lambda^\star_{tp}[\pi^b_y]\}_b$ forms a valid quantum measurement.

We will now proceed to show that the system $S^{AB}_{\overline{\text{SEP}}}$ can yield a stronger than quantum correlation in the timelike domain.

At this point we would like to mention that the study of the stronger than timelike correlation was introduced in a recent paper by Dall'Arno *et al.*, in which the authors proposed an interesting principle called no hypersignaling [42]. However, we will follow a little different approach as studied in [24] and recall below a communication game introduced there.

Pairwise distinguishability game $\mathcal{P}_D^{[n]}$. The game involves two players (Alice and Bob) and a referee. In each run of the game, the referee provides a classical message η to Alice, randomly chosen from some set of messages $\mathcal{N}$, where $|\mathcal{N}| := n$. In the same run Bob is asked a question $\mathbb{Q}(\eta, \eta')$: whether the message given to Alice is η or η', where $\eta' \neq \eta$. The winning condition demands Bob answer all questions correctly. Alice can help Bob by sending some information about the message she received. It is not hard to see that perfect winning demands Alice to encode the message on the states of some physical system that are pairwise distinguishable. With this game we are now in a position to prove one of our main results.

Theorem 1. The game $\mathcal{P}_D^{[8]}$ cannot be won if Alice uses the system $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_Q]$ to encode her message, whereas the $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}]$ system yields a perfect winning strategy.

Proof. Perfect winning of the game $\mathcal{P}_D^{[n]}$ requires Alice to communicate to Bob a physical system which has an information dimension of at least n. For the two-qubit system $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_Q]$, the information dimension is the same as its operational dimension, which is 4, and therefore, the $\mathcal{P}_D^{[8]}$ game cannot be won perfectly by communicating two qubits.

We now provide an explicit strategy to win the game $\mathcal{P}_D^{[8]}$ using the system $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}]$. Let Alice use the set $\$[8] \equiv \{\Phi^{\pm}, \Psi^{\pm}, \overline{\Phi^{\pm}}, \overline{\Psi^{\pm}}\} \subset \Omega_{\overline{\text{SEP}}}(\mathbb{C}^2, \mathbb{C}^2)$ of eight different states to encode her messages, where $\chi := |\chi\rangle\langle\chi|, |\Phi^{\pm}\rangle := (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\Psi^{\pm}\rangle := (|01\rangle \pm |10\rangle)/\sqrt{2}$, and $\overline{\chi} := \mathcal{I} \otimes \mathrm{T}(\chi)$, with $\mathcal{I}$ denoting the identity map and T denoting the transposition map (in the computational basis). It remains to be shown that the states in $\$[8]$ are pairwise distinguishable with measurements constituted by the effects from the set $\mathcal{E}_{\overline{\text{SEP}}}(\mathbb{C}^2, \mathbb{C}^2)$.

Consider the pair of states Φ^+ and Ψ^+ and the measurement

$$\mathcal{M} \equiv \begin{cases} E_{\text{even}} := |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|, \\ E_{\text{odd}} = \mathcal{I} - E_{\text{even}} := |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\ \qquad + |1\rangle\langle 1| \otimes |0\rangle\langle 0|. \end{cases}$$

Clearly, $\mathcal{M}$ is a valid measurement on the system $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}]$ as $E_{\text{odd}}, E_{\text{even}} \in \mathcal{E}_{\overline{\text{SEP}}}(\mathbb{C}^2, \mathbb{C}^2)$, where E_{even} is the projector of even numbers of up spins and E_{odd} is the projector of odd numbers of up spins. A straightforward calculation yields

$$\mathrm{Tr}(\Phi^+ E_{\text{odd}}) = 1, \quad \mathrm{Tr}(\Phi^+ E_{\text{even}}) = 0,$$
$$\mathrm{Tr}(\Psi^+ E_{\text{odd}}) = 0, \quad \mathrm{Tr}(\Psi^+ E_{\text{even}}) = 1.$$

Therefore, the measurement $\mathcal{M}$ perfectly distinguishes the states Φ^+ and Ψ^+. To show the same for any pair of states in $\$[8]$, let us denote as $\mathcal{M}[U \otimes V]$ the measurement obtained from $\mathcal{M}$ through the unitary rotation $U \otimes V$, i.e. $\mathcal{M}[U \otimes V] := \{U \otimes V E_{\text{odd}} U^{\dagger} \otimes V\dagger, U \otimes V E_{\text{even}} U^{\dagger} \otimes V\dagger\}$. As shown in Table I, choosing U and V appropriately from the set

$$\left\{\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^x := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \right.$$
$$\left. A^y := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\},$$

any pair of states in $\$[8]$ can be distinguished perfectly by the measurement $\mathcal{M}[U \otimes V]$. This completes the proof. ∎

TABLE I. The unitaries required to construct the measurement $\mathcal{M}[U \otimes V]$ for pairwise distinguishability of the states in $\$[8]$ are given. The states in the horizontal upper (lower) diagonal can be distinguished from the states in the vertical upper (lower) diagonal using the corresponding unitaries. For instance, the measurement to distinguish the pair $\{\Psi^+, \Psi^-\}$ and the pair $\{\overline{\Psi^+}, \overline{\Psi^-}\}$ is given by the entry in the third row and fourth column, i.e., $\mathcal{M}[A^y \otimes A^y]$, whereas the pair $\{\overline{\Psi^+}, \Psi^+\}$ is distinguished by the measurement given in seventh row, third column, i.e., $\mathcal{M}[A^x \otimes A^x]$. NA means that a state cannot be distinguished from itself.

	Φ^+ / $\overline{\Phi^+}$	Φ^- / $\overline{\Phi^-}$	Ψ^+ / $\overline{\Psi^+}$	Ψ^- / $\overline{\Psi^-}$
Φ^+ / $\overline{\Phi^+}$	**NA**	$A^y \otimes A^y$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$
Φ^- / $\overline{\Phi^-}$	$A^y \otimes A^y$	**NA**	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$
Ψ^+ / $\overline{\Psi^+}$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	**NA**	$A^y \otimes A^y$
Ψ^- / $\overline{\Psi^-}$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	$A^y \otimes A^y$	**NA**
$\overline{\Phi^+}$	$A^x \otimes A^x$	$A^y \otimes A^y$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$
$\overline{\Phi^-}$	$A^y \otimes A^y$	$A^x \otimes A^x$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$
$\overline{\Psi^+}$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	$A^x \otimes A^x$	$A^y \otimes A^y$
$\overline{\Psi^-}$	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	$A^y \otimes A^y$	$A^x \otimes A^x$

Theorem 1 thus establishes that the $\overline{\text{SEP}}$ composition of two elementary qubits can result in a correlation that cannot be achieved with a two-qubit quantum composition. As an immediate corollary we have a lower bound on the information dimension of the system $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}]$.

Corollary 1. The information dimension of the system $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}]$ is at least 8, i.e., $\mathbb{I}[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\text{SEP}}}] \geqslant 8$.

At present we do not know whether the above bound is tight and leave this question open for future research. Rather, we proceed to find the operational dimension of the systems obtained through the $\overline{\text{SEP}}$ composition. To this aim, we first prove the following proposition.

Proposition 1. Every POPT state $W_{AB} \in \Omega_{\overline{\text{SEP}}}(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ can be written as $(\mathcal{I}_A \otimes \Lambda_{R\to B})(\rho_{AR})$, where Λ is a positive, unital map and $\rho_{AR} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_R}$ is a pure quantum state independent of W_{AB}.

Proof. We start by defining the following [43]:

$$W'_{AB} := W_{AB} + \mathbf{1}_A \otimes P_B^{\perp}, \quad P_B^{\perp} := \mathbf{1}_B - P_B,$$
$$P_B := \text{projector onto the support of } W_B,$$
$$W_B := \mathrm{Tr}_A\,(W_{AB}),\ W'_B := \mathrm{Tr}_A\,(W'_{AB}).$$

W'_{AB} is a POPT state (unnormalized) as for any separable effect $\pi_A \otimes \pi_B$ we have $\mathrm{Tr}\,[(W'_{AB})(\pi_A \otimes \pi_B)] = \mathrm{Tr}\,[(W_{AB})(\pi_A \otimes \pi_B)] + \mathrm{Tr}\,(\pi_A)\,\mathrm{Tr}\,(P_B^{\perp}\pi_B) \geqslant 0$. Thus, W'_B is a full-rank positive operator, and hence, we can define

$$W''_{AB} := [\mathbf{1}_A \otimes (W'_B)^{-1/2}]W'_{AB}[\mathbf{1}_A \otimes (W'_B)^{-1/2}],$$

where $(W'_B)^{-1/2}$, being a positive operator, implies that W''_{AB} is a POPT: $\mathrm{Tr}\,[(W''_{AB})(\pi_A \otimes \pi_B)] = \mathrm{Tr}\{W'_{AB}[\pi_A \otimes (W'_B)^{-1/2}\pi_B(W'_B)^{-1/2}]\} \geqslant 0$. Further, $\mathrm{Tr}_A\,(W''_{AB}) = \sum_i (W'_B)^{-1/2}\langle i|_A W'_{AB}|i\rangle_A (W'_B)^{-1/2} = (W'_B)^{-1/2}W'_B(W'_B)^{-1/2} = \mathbf{1}_B$.

Using the CJ isomorphism, we can write $W''_{AB} = \mathcal{I}_A \otimes \mathcal{U}_{S\to B}(|\chi^+\rangle_{AS}\langle\chi^+|)$, where $|\chi^+\rangle_{AB} := \sum_i |i\rangle_A|i\rangle_B$ is the unnormalized maximally entangled state and $\mathcal{U}_{S\to B}$ is a positive map. More explicitly, the action of $\mathcal{U}_{S\to B}$ is given by $\mathcal{U}_{S\to B}(M_S) = \mathrm{Tr}_S\,[(M_S^T \otimes \mathbf{1}_B)(W''_{SB})]$. $\mathcal{U}_{S\to B}$ is unital since $\mathcal{U}_{S\to B}(\mathbf{1}_S) = \mathrm{Tr}_S(W''_{SB}) = \mathbf{1}_B$. Furthermore, it is easy to check that $W_{AB} = (\mathbf{1}_A \otimes V_B^{\dagger})W''_{AB}(\mathbf{1}_A \otimes V_B)$, where $V_B := (W'_B)^{1/2}P_B$.

Let us now define a new completely positive, unital map

$$\mathcal{Y}_{BC\to B}(M_{BC}) := V_B^{\dagger}\langle 0|_C M|0\rangle_C V_B + V_B'^{\dagger}\langle 1|_C M|1\rangle_C V'_B,$$

where V'_B is chosen so that it satisfies the condition $V_B^{\dagger}V_B + V_B'^{\dagger}V'_B = \mathbf{1}_B$ and $\mathcal{H}_C := \mathbb{C}^2$. The above map is the adjoint of the completely positive, trace-preserving map with the Kraus operators $\{V_B \otimes |0\rangle_C, V'_B \otimes |1\rangle_C\}$; it has the property

$$\mathcal{I}_A \otimes \mathcal{Y}_{BC\to B}(M_{AB} \otimes |0\rangle_C\langle 0|) = (\mathbf{1}_A \otimes V_B^{\dagger})M_{AB}(\mathbf{1}_A \otimes V_B).$$

This further leads us to

$$\begin{aligned}
W_{AB} &= (\mathbf{1}_A \otimes V_B^{\dagger})W''_{AB}(\mathbf{1}_A \otimes V_B)\\
&= \mathcal{I}_A \otimes \mathcal{Y}_{BC\to B}(W''_{AB} \otimes |0\rangle_C\langle 0|)\\
&= \mathcal{I}_A \otimes \mathcal{Y}_{BC\to B}[(\mathcal{I}_A \otimes \mathcal{U}_{S\to B})(|\chi^+\rangle_{AS}\langle\chi^+|) \otimes |0\rangle_C\langle 0|]\\
&= (\mathcal{I}_A \otimes \mathcal{Y}_{BC\to B}) \circ (\mathcal{I}_A \otimes \mathcal{U}_{S\to B} \otimes \mathcal{I}_C)\\
&\quad \times [|\chi^+\rangle_{AS}\langle\chi^+| \otimes |0\rangle_C\langle 0|]\\
&= \mathcal{I}_A \otimes (\mathcal{Y}_{BC\to B} \circ \mathcal{U}'_{SC\to BC})[|\chi^+\rangle_{AS}\langle\chi^+| \otimes |0\rangle_C\langle 0|],
\end{aligned}$$

where $\mathcal{U}'_{SC\to BC} := \mathcal{U}_{S\to B} \otimes \mathcal{I}_C$. Let $d_S := \dim(\mathcal{H}_S)$, $\Lambda_{SC\to B} := \frac{1}{d_S}\mathcal{Y}_{BC\to B} \circ \mathcal{U}'_{SC\to BC}$, $|\psi\rangle_{ASC} := \frac{1}{\sqrt{d_S}}|\chi^+\rangle_{AS}|0\rangle_C$, and $\mathcal{H}_R := \mathcal{H}_S \otimes \mathcal{H}_C$. Thus, we have

$$W_{AB} = (\mathcal{I}_A \otimes \Lambda_{R\to B})(|\psi\rangle_{AR}\langle\psi|),$$

where $\Lambda_{R\to B}$ is the composition of a completely positive unital map and a positive unital map; therefore, it is positive and unital. This completes the proof. ■

We are now in a position to prove another important result of this work. The classical information carrying capacity of bipartite systems allowing maximal tensor product composition equals the classical capacity of quantum composition.

Theorem 2. The operational dimension of the system $[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\overline{\text{SEP}}}]$ is d_Ad_B.

Proof. The proof is similar in spirit to Lemma 24 of Ref. [37]. However, while the assumption of "transitivity" is used there, here we use Proposition 1.

Let the operational dimension of the system $[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\overline{\text{SEP}}}]$ be N. N must be lower bounded by $d := d_Ad_B$, as there exists d quantum states that can be perfectly distinguished by a single separable measurement. For instance, the set of pure states $\{|ij\rangle \mid i = 1, \dots d_A \,\&\, j = 1, \dots d_B\}$ can be distinguished by the separable measurement $\{|i\rangle\langle i| \otimes |j\rangle\langle j|\}_{i,j=1}^{d_A,d_B}$. As we have considered the operational dimension of the system $[\mathbb{C}^{d_A}, \mathbb{C}^{d_B}, \bigotimes_{\overline{\text{SEP}}}]$ to be N, there must exist N POPT states $\{W_1, \dots, W_N\}$ and a separable measurement $\{E_1, \dots, E_N \mid \sum_{i=1}^N E_i = \mathbf{1}_{AB}\}$ such that $\mathrm{Tr}(E_iW_j) = \delta_{ij}\ \forall\, i, j$. According to Proposition 1, $\forall\, j$, $W_j = (\mathcal{I} \otimes \Lambda_j)(\rho)$ for some positive, unital map $\Lambda_j : \mathcal{L}(\mathcal{H}_R) \to \mathcal{L}(\mathcal{H}_B)$ and pure state $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_R)$. Denoting the projector on the orthogonal support of ρ as $P := \mathbf{1}_{AR} - \rho$, we have

$$\begin{aligned}
d = \mathrm{Tr}(\mathbf{1}_{AB}) &= \sum_{i=1}^N \mathrm{Tr}(E_i) = \sum_{i=1}^N \mathrm{Tr}[E_i(\mathcal{I} \otimes \Lambda_i)(\mathbf{1}_{AR})]\\
&= \sum_{i=1}^N \mathrm{Tr}[E_i(\mathcal{I} \otimes \Lambda_i)(\rho + P)]\\
&= \sum_{i=1}^N \mathrm{Tr}[E_i(\mathcal{I} \otimes \Lambda_i)(\rho)] + \sum_{i=1}^N \mathrm{Tr}[E_i(\mathcal{I} \otimes \Lambda_i)(P)]\\
&= \sum_{i=1}^N \mathrm{Tr}[E_iW_i] + \sum_{i=1}^N \mathrm{Tr}[(\mathcal{I} \otimes \Lambda_i^*)(E_i)P],
\end{aligned}$$

where Λ_i^* is the adjoint map of Λ_i and hence positive. Furthermore, $(\mathbb{1} \otimes \Lambda_i^*)(E_i)$ are positive operators since E_i's are separable. Therefore, we have

$$d \geqslant \sum_{i=1}^N \mathrm{Tr}[E_iW_i] = \sum_{i=1}^N \delta_{ii} = N.$$

Since we know that $N \geqslant d$, we conclude that $N = d$. This completes the proof. ■

While in Theorem 1 we showed that the $\overline{\text{SEP}}$ composition of two elementary qubits can yield a stronger timelike correlation than their quantum composition (i.e., two qubit), Theorem 2 establishes that such a composition is not strong enough to show the superadditive feature of the information carrying capacity [44]. In this regard, a more generic result is presented in the next proposition.

Proposition 2. The operational dimension of any bipartite composition (with the normalized state space denoted as Ω_{AB}) of two elementary quantum systems $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$ is $d_A d_B$ if $(\mathbf{1}_{AB} - W_{AB})$ lies within the unnormalized state cone $\forall\, W_{AB} \in \Omega_{AB}$.

Proof. Let N be the operational dimension of the composite system. Then, there must exist a set containing N states $\{W_1, \ldots, W_N\}$ and measurement $\{E_1, \ldots, E_N \mid \sum_{i=1}^{N} E_i = \mathbf{1}_{AB}\}$ such that $\mathrm{Tr}(E_i W_j) = \delta_{ij}\ \forall\, i, j$. Manifestly, it follows that $N \geqslant d := d_A d_B$ since $\Omega_{\mathrm{SEP}} \subseteq \Omega_{AB} \subseteq \Omega_{\overline{\mathrm{SEP}}}$ and $\mathcal{E}_{\overline{\mathrm{SEP}}} \subseteq \mathcal{E}_{AB} \subseteq \mathcal{E}_{\mathrm{SEP}}$. As argued in the proof of Theorem 2, there always exist $d_A d_B$ product states that can be perfectly distinguished by a separable measurement. On the other hand,

$$\sum_{i=1}^{N} \mathrm{Tr}[E_i(\mathbf{1} - W_i)] = \sum_{i=1}^{N} \mathrm{Tr}(E_i) - \sum_{i=1}^{N} \mathrm{Tr}(E_i W_i)$$
$$= d - \sum_{i=1}^{N} \delta_{ii} = d - N.$$

Since $(\mathbf{1}_{AB} - W_{AB})$ is an unnormalized state by assumption, $d - N \geqslant 0$ or $d \geqslant N$, which completes the proof. ■

While Proposition 2 assumes elementary systems are quantum, it can, however, be further generalized within the GPT framework.

Proposition 3. Let $\mathcal{S}_{AB} \equiv (\Omega_{AB}, \mathcal{E}_{AB})$ be a composite system consisting two elementary systems $\mathcal{S}_A \equiv (\Omega_A, \mathcal{E}_A)$ and $\mathcal{S}_B \equiv (\Omega_B, \mathcal{E}_B)$ with operational dimensions N_A and N_B, respectively. The operational dimension of $\mathcal{S}_{AB}$ is $N_A N_B$ if $\exists\, \omega'_{AB} \in \Omega_{AB}$ such that $(N_A N_B \omega'_{AB} - \omega_{AB})$ is an unnormalized state $\forall\, \omega_{AB} \in \Omega_{AB}$.

Proof. The operational dimension N_{AB} of the composite system $\mathcal{S}_{AB}$ is always greater than the product of the operational dimension of the elementary systems, i.e., $N_{AB} \geqslant N_A N_B$. This simply follows from the fact that any valid composition includes the products states and product effects in its description. Now we have

$$\sum_{i=1}^{N_{AB}} e_i(N_A N_B \omega' - \omega_i) = N_A N_B \sum_{i=1}^{N_{AB}} e_i(\omega') - \sum_{i=1}^{N_{AB}} e_i(\omega_i)$$
$$= N_A N_B u(\omega') - \sum_{i=1}^{N_{AB}} \delta_{ii}$$
$$= N_A N_B - N_{AB} \geqslant 0$$

since, by assumption, $(N_A N_B \omega' - \omega_i)$ is an unnormalized state $\forall\, \omega_i$. Therefore, $N_{AB} \leqslant N_A N_B$, which completes the proof. ■

From Theorems 1 and 2 we can conclude that $\Delta[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\mathrm{SEP}}}] \geqslant 4$, where Δ refers to the dimension mismatch of the theory. On the other hand, we can also conclude that the gap between the information dimensions of the systems $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\mathrm{SEP}}}]$ and $[\mathbb{C}^2, \mathbb{C}^2, \bigotimes_{Q}]$ is at least 4, i.e.,

$$\mathbb{I}[\mathbb{C}^2, \mathbb{C}^2, \otimes_{\overline{\mathrm{SEP}}}] - \mathbb{I}[\mathbb{C}^2, \mathbb{C}^2, \otimes_{Q}] \geqslant 4.$$

Our next result shows that this gap can be increased further by considering more elementary systems.

Theorem 3. The information dimension of the system $[\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2, \bigotimes_{\overline{\mathrm{SEP}}}]$ is at least 24.

Proof. The proof is constructive and similar to the proof of Theorem 1. Consider the following set of 24 states:

$$\$[24] := \{\chi, \overline{\chi}, \overline{\overline{\chi}}\} \subset \Omega_{\overline{\mathrm{SEP}}}[\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2],$$
$$\overline{\chi} := \mathcal{I} \otimes \mathrm{T} \otimes \mathcal{I}(\chi),$$
$$\overline{\overline{\chi}} := \mathcal{I} \otimes \mathrm{T} \otimes \mathrm{T}(\chi),$$
$$|\chi\rangle \in \left\{ \begin{aligned} &|\Phi_{000}^{\pm}\rangle := \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \\ &|\Phi_{001}^{\pm}\rangle := \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \\ &|\Phi_{010}^{\pm}\rangle := \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle), \\ &|\Phi_{011}^{\pm}\rangle := \frac{1}{\sqrt{2}}(|011\rangle \pm |100\rangle) \end{aligned} \right\}.$$

We aim to show that the states in \$[24] are pairwise distinguishable by fully separable measurements of the form

$$\mathcal{M} \equiv \begin{cases} E_{\mathrm{odd}} := & |m\rangle\langle m| \otimes |n\rangle\langle n| \otimes |p\rangle\langle p| \\ & + |m\rangle\langle m| \otimes |n^{\perp}\rangle\langle n^{\perp}| \otimes |p^{\perp}\rangle\langle p^{\perp}| \\ & + |m^{\perp}\rangle\langle m^{\perp}| \otimes |n\rangle\langle n| \otimes |p^{\perp}\rangle\langle p^{\perp}| \\ & + |m^{\perp}\rangle\langle m^{\perp}| \otimes |n^{\perp}\rangle\langle n^{\perp}| \otimes |p\rangle\langle p|, \\ E_{\mathrm{even}} = & 1 - E_{\mathrm{odd}} \\ = & |m\rangle\langle m| \otimes |n\rangle\langle n| \otimes |p^{\perp}\rangle\langle p^{\perp}| \\ & + |m\rangle\langle m| \otimes |n^{\perp}\rangle\langle n^{\perp}| \otimes |p\rangle\langle p| \\ & + |m^{\perp}\rangle\langle m^{\perp}| \otimes |n\rangle\langle n| \otimes |p\rangle\langle p| \\ & + |m^{\perp}\rangle\langle m^{\perp}| \otimes |n^{\perp}\rangle\langle n^{\perp}| \otimes |p^{\perp}\rangle\langle p^{\perp}|, \end{cases}$$

where $|r\rangle$, and $|r^{\perp}\rangle$ are "up" and "down" eigenstates of spin measurement $(\hat{r} \cdot \sigma)$ along the $\hat{r}$ direction for $\hat{r} \in \{\hat{m}, \hat{n}, \hat{p}\}$. E_{odd} comprises an odd number of up spins, and E_{even} comprises an even number of up spins. Clearly, $\mathcal{M}$ is an allowed measurement as $E_{\mathrm{odd}}, E_{\mathrm{even}} \in \mathcal{E}_{\overline{\mathrm{SEP}}}(\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2)$. The m, n, and p required to distinguish between different pairs of states are given in the first column of Table II. For instance, the pair of states $\{\Phi_{000}^{+}, \Phi_{000}^{-}\}$ can be perfectly distinguished by choosing $(m, n, p) = (y, y, x)$. The measurement $\mathcal{M}_{\{\Phi_{000}^{+}, \Phi_{000}^{-}\}} \equiv \{E_{\mathrm{odd}}, E_{\mathrm{even}}\}$ is given by

$$\begin{aligned} E_{\mathrm{odd}} := & |y\rangle\langle y| \otimes |y\rangle\langle y| \otimes |x\rangle\langle x| \\ & + |y\rangle\langle y| \otimes |y^{\perp}\rangle\langle y^{\perp}| \otimes |x^{\perp}\rangle\langle x^{\perp}| \\ & + |y^{\perp}\rangle\langle y^{\perp}| \otimes |y\rangle\langle y| \otimes |x^{\perp}\rangle\langle x^{\perp}| \\ & + |y^{\perp}\rangle\langle y^{\perp}| \otimes |y^{\perp}\rangle\langle y^{\perp}| \otimes |x\rangle\langle x|, \\ E_{\mathrm{even}} = & 1 - E_{\mathrm{odd}} \\ := & |y\rangle\langle y| \otimes |y\rangle\langle y| \otimes |x^{\perp}\rangle\langle x^{\perp}| \\ & + |y\rangle\langle y| \otimes |y^{\perp}\rangle\langle y^{\perp}| \otimes |x\rangle\langle x| \\ & + |y^{\perp}\rangle\langle y^{\perp}| \otimes |y\rangle\langle y| \otimes |x\rangle\langle x| \\ & + |y^{\perp}\rangle\langle y^{\perp}| \otimes |y^{\perp}\rangle\langle y^{\perp}| \otimes |x^{\perp}\rangle\langle x^{\perp}|. \end{aligned}$$

TABLE II. Pairwise distinguishability of the set \$[24]. Using a particular separable measurement given in the first column, any state in the odd-number up-spin column can be distinguished from any state in the even-number up-spin column. For instance, the pair $\{\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^+_{001}}}\}$ (last row) is perfectly distinguishable via the separable measurement consisting of POVM elements given by $\mathcal{M} \equiv \{E_{\text{odd}}, E_{\text{even}}\}$, where E_{odd} and E_{even} are rank-four projectors comprising an odd number of spin-up outcomes and an even number of spin-up outcomes, respectively, for the Pauli measurement $(\sigma_Z, \sigma_Y, \sigma_Z) \equiv (z, y, z)$.

Measurement	Odd number of up spins	Even number of up spins
(y, y, x)	$\Phi^-_{000}, \Phi^-_{001}, \Phi^+_{010}, \Phi^+_{011}$	$\Phi^+_{000}, \Phi^+_{001}, \Phi^-_{010}, \Phi^-_{011}$
	$\overline{\Phi^+_{000}}, \overline{\Phi^+_{001}}, \overline{\Phi^-_{010}}, \overline{\Phi^-_{011}}$	$\overline{\Phi^-_{000}}, \overline{\Phi^-_{001}}, \overline{\Phi^+_{010}}, \overline{\Phi^+_{011}}$
	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^-_{010}}}, \overline{\overline{\Phi^-_{011}}}$	$\overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^-_{001}}}, \overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^+_{011}}}$
(y, x, y)	$\Phi^-_{000}, \Phi^+_{001}, \Phi^-_{010}, \Phi^+_{011}$	$\Phi^+_{000}, \Phi^-_{001}, \Phi^+_{010}, \Phi^-_{011}$
	$\overline{\Phi^-_{000}}, \overline{\Phi^+_{001}}, \overline{\Phi^-_{010}}, \overline{\Phi^+_{011}}$	$\overline{\Phi^+_{000}}, \overline{\Phi^-_{001}}, \overline{\Phi^+_{010}}, \overline{\Phi^-_{011}}$
	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^-_{001}}}, \overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^-_{011}}}$	$\overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^-_{010}}}, \overline{\overline{\Phi^+_{011}}}$
(x, x, x)	$\Phi^+_{000}, \Phi^+_{001}, \Phi^+_{010}, \Phi^+_{011}$	$\Phi^-_{000}, \Phi^-_{001}, \Phi^-_{010}, \Phi^-_{011}$
	$\overline{\Phi^+_{000}}, \overline{\Phi^+_{001}}, \overline{\Phi^+_{010}}, \overline{\Phi^+_{011}}$	$\overline{\Phi^-_{000}}, \overline{\Phi^-_{001}}, \overline{\Phi^-_{010}}, \overline{\Phi^-_{011}}$
	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^+_{011}}}$	$\overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^-_{001}}}, \overline{\overline{\Phi^-_{010}}}, \overline{\overline{\Phi^-_{011}}}$
(x, y, y)	$\Phi^-_{000}, \Phi^+_{001}, \Phi^+_{010}, \Phi^-_{011}$	$\Phi^+_{000}, \Phi^-_{001}, \Phi^-_{010}, \Phi^+_{011}$
	$\overline{\Phi^+_{000}}, \overline{\Phi^-_{001}}, \overline{\Phi^-_{010}}, \overline{\Phi^+_{011}}$	$\overline{\Phi^-_{000}}, \overline{\Phi^+_{001}}, \overline{\Phi^+_{010}}, \overline{\Phi^-_{011}}$
	$\overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^-_{011}}}$	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^-_{001}}}, \overline{\overline{\Phi^-_{010}}}, \overline{\overline{\Phi^+_{011}}}$
(y, z, z)	$\Phi^+_{000}, \Phi^-_{000}, \Phi^+_{011}, \Phi^-_{011}$	$\Phi^+_{001}, \Phi^-_{001}, \Phi^+_{010}, \Phi^-_{010}$
	$\overline{\Phi^+_{000}}, \overline{\Phi^-_{000}}, \overline{\Phi^+_{011}}, \overline{\Phi^-_{011}}$	$\overline{\Phi^+_{001}}, \overline{\Phi^-_{001}}, \overline{\Phi^+_{010}}, \overline{\Phi^-_{010}}$
	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^+_{011}}}, \overline{\overline{\Phi^-_{011}}}$	$\overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^-_{001}}}, \overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^-_{010}}}$
(z, z, y)	$\Phi^+_{000}, \Phi^-_{000}, \Phi^+_{001}, \Phi^-_{011}$	$\Phi^+_{010}, \Phi^-_{010}, \Phi^+_{011}, \Phi^-_{011}$
	$\overline{\Phi^+_{000}}, \overline{\Phi^-_{000}}, \overline{\Phi^+_{001}}, \overline{\Phi^-_{001}}$	$\overline{\Phi^+_{010}}, \overline{\Phi^-_{010}}, \overline{\Phi^+_{011}}, \overline{\Phi^-_{011}}$
	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^-_{001}}}$	$\overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^-_{010}}}, \overline{\overline{\Phi^+_{011}}}, \overline{\overline{\Phi^-_{011}}}$
(z, y, z)	$\Phi^+_{000}, \Phi^-_{000}, \Phi^+_{010}, \Phi^-_{010}$	$\Phi^+_{001}, \Phi^-_{001}, \Phi^+_{011}, \Phi^-_{011}$
	$\overline{\Phi^+_{000}}, \overline{\Phi^-_{000}}, \overline{\Phi^+_{010}}, \overline{\Phi^-_{010}}$	$\overline{\Phi^+_{001}}, \overline{\Phi^-_{001}}, \overline{\Phi^+_{011}}, \overline{\Phi^-_{011}}$
	$\overline{\overline{\Phi^+_{000}}}, \overline{\overline{\Phi^-_{000}}}, \overline{\overline{\Phi^+_{010}}}, \overline{\overline{\Phi^-_{010}}}$	$\overline{\overline{\Phi^+_{001}}}, \overline{\overline{\Phi^-_{001}}}, \overline{\overline{\Phi^+_{011}}}, \overline{\overline{\Phi^-_{011}}}$

A straightforward calculation yields

$$\mathrm{Tr}(\Phi^-_{000} E_{\text{odd}}) = 1, \quad \mathrm{Tr}(\Phi^-_{000} E_{\text{even}}) = 0,$$
$$\mathrm{Tr}(\Phi^+_{000} E_{\text{odd}}) = 0, \quad \mathrm{Tr}(\Phi^+_{000} E_{\text{even}}) = 1.$$

Therefore, the measurement $\mathcal{M}_{\{\Phi^+_{000}, \Phi^-_{000}\}}$ perfectly distinguishes the states Φ^-_{000} and Φ^+_{000}. As we show in Table II, any pair of states in \$[24] can be distinguished perfectly by such a measurement. This completes the proof. ■

Theorem 3 thus establishes that the $\mathcal{P}_D^{[24]}$ game can be won with three elementary qubits if the $\overline{\text{SEP}}$ composition is considered among them, whereas if we consider quantum composition, five elementary qubits are required.

Discussion. The need to understand quantum mechanics results in investigating theories other than itself. Comparisons among the information processing capabilities in the various theories leads to insights about the underlying cause for such capabilities which led to the motivation for this paper. While Barnum *et al.* [20] showed that in the spacelike scenario, the bipartite maximal tensor product structure of local quantum systems cannot generate beyond quantum correlations, the authors of [22] showed that in a generalized Bell scenario every beyond quantum state can produce beyond quantum correlations. In this work, we have used a different approach wherein timelike scenarios are considered instead of the traditional spacelike Bell scenarios. We have provided concrete results which can be experimentally verified and can be used as principles to single out the quantum composition rule. While Corollary 1 and Theorem 2 establish that the phenomenon of dimension mismatch occurs in the $\overline{\text{SEP}}$ composition, it has been shown [24,38] that dimension mismatch occurs in the SEP composition as well. A natural question, then, is to ask what other compositions can be ruled out using dimension mismatch. Another interesting direction to explore is relaxing the assumption of quantum subsystems. Propositions 2 and 3 provide some preliminary results in the GPT framework which may be useful in this regard. Our study forms an important piece of the quantum reconstruction program in which we seek to derive quantum theory from physical principles [9–11].

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[43] The techniques used in the proof are motivated from [41].
[44] The information carrying capacity of a composition of two elementary systems with operational dimensions d_A and d_B will exhibit a superadditive feature whenever its operational dimension $N > d_A d_B$, which accordingly results in $\log_2 N > \log_2 d_A + \log_2 d_B$.