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Detecting quantum phase transition via magic resource in the XY spin model

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Quantum phase transition in the XY spin model with three-spin interaction is investigated using magic resource (non-stabilizerness), which is crucial in universal fault-tolerant quantum computation. The magic quantifier we employ here is defined straightforwardly via characteristic functions of quantum states, which are well defined for all dimensional quantum systems (in sharp contrast to those defined by discrete Wigner functions) and can be easily calculated. We show that the magic quantifier of both the reduced single-site spins and two-site spins of the system ground state increase to their maximum around the critical points for quantum phase transition. This indicates that the magic resource can be used to detect the critical phenomena in the XY spin model and reveals a connection between quantum phase transition in many-body systems and quantum resource in stabilizer quantum computation.

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I. INTRODUCTION

For a many-body quantum system, a variation in the coupling or an external parameter of the Hamiltonian can sometimes induce a qualitative change in the ground state of the system at absolute zero temperature. This is the so-called quantum phase transition [1,2], and the associated critical phenomena for various specific systems such as quantum spin chains have been extensively studied [3–6]. Quantum phase transition is driven by quantum fluctuation and thus is intrinsically connected to quantum resources [7–10], in sharp contrast to classical phase transition, which occurs when a system reaches a state below a critical temperature characterized by certain macroscopic order.

In the past two decades, quantum information theory has undergone rapid development [11]. Many quantum informational quantities, such as entanglement [12], quantum discord [13–17], Bell nonlocality [18], quantum coherence [19], etc., have been introduced and studied extensively, each with its own merit and usage in different contexts. In particular, some of these quantities have been employed in detecting quantum phase transition in various models [20–41]. For example, Osborne and Nielsen studied the entanglement in the transverse Ising model and showed that the next-nearest-neighbor (albeit not the nearest-neighbor) entanglement achieves its maximum value at the critical points [20,21]. Osterloh *et al.* explored the entangling resources of a spin system close to its quantum critical points and demonstrated that concurrence (a popular measure of entanglement) is maximal around the critical points [22]. By using quantum discord, Dillenschneider investigated the quantum phase transition of the transverse Ising and antiferromagnetic XXZ spin chains and showed that the amount of quantum correlations increases close to the critical points in contrast to the amount of classical correlations [32]. Li and Lin illustrated the capability of pairwise quantum discord and classical correlations in detecting quantum phase transitions at both zero and finite temperatures in the XY spin chain with three-spin interaction [39]. Ye *et al.* explored quantum phase transitions in the spin-$1/2$ XX chain with three-spin interaction in terms of local quantum Fisher information and one-way quantum deficit [41].

The magic resource is crucial for universal fault-tolerant quantum computation in the stabilizer formalism of quantum computation [42–45], just like entanglement is a crucial resource for quantum communications. Due to its importance, many different magic quantifiers have been introduced in the last few years, including the sum negativity, mana, relative entropy of magic, robustness of magic, etc. [46–55]. However, most of these quantities rely heavily on intractable optimization or the discrete Wigner formalism, which is only properly defined for odd prime power dimensional quantum systems [56–58]. Computation of these magic quantifiers are often quite hard.

Recently, a magic quantifier in terms of characteristic functions of quantum states is introduced [59]. Apart from its direct physical significance, it is well defined in all dimensions and can be easily calculated, in contrast to those defined via discrete Wigner functions. In view of the utility of magic resource in quantum computation and related foundational issues, one may expect that the magic resource may also be useful in other quantum contexts. In particular, it seems desirable to investigate how the magic quantifier will change near critical points of quantum spin systems.

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In this paper, we analyze the role magic resource plays in detecting the critical points of quantum spin systems. By analyzing the dynamics of the magic quantifier of spins in a specific XY spin model with three-spin interaction [29,39–41], we show that it displays abrupt changes around the critical points, which indicates that the magic resource may be useful in detecting the critical points. Compared with other correlation measures such as entanglement and quantum discord, the magic quantifier we employ here can be straightforwardly computed.

The paper is structured as follows. In Sec. II, we briefly review the stabilizer formalism and the magic quantifier via the characteristic functions of quantum states. In Sec. III, we calculate the magic quantifier of both single-spin and two-spin states in the XY spin model with three-spin interactions, reveal the quantitative features of magic resource in detecting the critical points. A brief comparison with other informational quantities is presented in Sec. IV. Finally, we conclude with a summary and discussion in Sec. V.

II. MAGIC RESOURCE (NON-STABILIZERNESS)

Since the magic resource is crucial for fault-tolerant universal quantum computation, its characterization and quantification have attracted considerable attention ever since the inception of stabilizer formalism of quantum error correction and computation [42–55]. Quite recently, an easily computable magic quantifier via characteristic functions of quantum states was proposed in Ref. [59], which has several nice features. In this section, we briefly review the stabilizer formalism and the associated magic quantifier in terms of characteristic functions, which will be used in detecting critical points for many-body quantum systems.

For any positive integer d, let $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$ be the ring of integers modulo d. For a d-dimensional Hilbert space K (one may simply take $K = \mathbb{C}^d$) with the computational basis $\{|j\rangle : j \in \mathbb{Z}_d\}$, the shift operator

$$X = \sum_{j=0}^{d-1} |j+1\rangle\langle j|$$

and the phase operator

$$Z = \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j|, \quad \omega = e^{2\pi i/d}$$

are the generators of the discrete Heisenberg-Weyl group $\mathcal{P}_d$, which is a subgroup of the full unitary group $\mathcal{U}(K)$ (the set of all unitary operators on K) [60].

Following the convention in Ref. [45], the discrete Heisenberg-Weyl operators are defined as

$$D_{k,l} = \tau^{kl} X^k Z^l, \quad \tau = -e^{\pi i/d},\ k, l \in \mathbb{Z}_d,$$

which are unitary and satisfy the orthogonality condition

$$\mathrm{tr}\big(D_{k,l}^\dagger D_{s,t}\big) = d\delta_{k,s}\delta_{l,t}, \quad k, l, s, t \in \mathbb{Z}_d.$$

Moreover, the set of operators

$$\left\{\frac{1}{\sqrt{d}} D_{k,l} : k, l \in \mathbb{Z}_d\right\}$$

constitutes an orthonormal basis of the d^2-dimensional operator Hilbert space $L(K)$ (which consists of all operators on the system Hilbert space K) equipped with the Hilbert-Schmidt inner product $\langle A|B\rangle = \mathrm{tr}(A^\dagger B)$.

The discrete Heisenberg-Weyl group $\mathcal{P}_d$ is not a normal (invariant) subgroup of $\mathcal{U}(K)$. Its normalizer in $\mathcal{U}(K)$ is called the Clifford group [42,43], which can be explicitly expressed as

$$\mathcal{C}_d = \{V \in \mathcal{U}(K) : V\mathcal{P}_d V^\dagger = \mathcal{P}_d\}.$$

Consequently, we have a hierarchy of groups

$$\mathcal{P}_d \subset \mathcal{C}_d \subset \mathcal{U}(K),$$

all consisting of unitary operators.

For prime dimensional systems K, any state of the form $V|0\rangle$ (with $V \in \mathcal{C}_d$) is a pure stabilizer state [45], and the set of stabilizer states is the convex hull of all pure stabilizer states. Thus quantum states which cannot be expressed as a mixture of pure stabilizer states are non-stabilizer states, which are also called magic states. Thus magic resource refers to the deviation of a state from stabilizer states.

For any quantum state (pure or mixed) ρ on K, its characteristic function is defined as [58,59]

$$c_\rho(k, l) = \mathrm{tr}(\rho D_{k,l}), \quad k, l \in \mathbb{Z}_d.$$

The state ρ is uniquely determined by its characteristic function $c_\rho(\cdot, \cdot)$.

Since $\{D_{k,l}/\sqrt{d} : k, l \in \mathbb{Z}_d\}$ constitutes an orthonormal basis of the operator space $L(K)$, any quantum state ρ on K can be expanded as

$$\rho = \sum_{k,l} \alpha_\rho(k, l)\frac{D_{k,l}}{\sqrt{d}},$$

with the expansion coefficient

$$\alpha_\rho(k, l) = \mathrm{tr}\left(\frac{D_{k,l}^\dagger}{\sqrt{d}}\rho\right) = \frac{1}{\sqrt{d}} c_\rho^*(k, l), \quad k, l \in \mathbb{Z}_d,$$

which is, up to a constant factor $1/\sqrt{d}$, actually the complex conjugate of the characteristic function of the quantum state ρ. Consequently,

$$\rho = \frac{1}{d}\sum_{k,l} c_\rho^*(k, l) D_{k,l}.$$

With the above preparations, we now recall the following magic quantifier:

$$M(\rho) = \sum_{k,l} |c_\rho(k, l)| = \sum_{k,l} |\mathrm{tr}(\rho D_{k,l})|$$

introduced in Ref. [59], which is actually the l^1 norm of the characteristic function and will play a key role in our approach to quantum phase transition.

As can be seen from the above definition, the magic quantifier $M(\rho)$ can be rather directly computed. Moreover, it possesses some nice properties, which render it to be a meaningful witness of magic resource, as elaborated in Ref. [59]. For example, it is invariant under the Clifford operations in the sense that $M(\rho) = M(V\rho V^\dagger)$ for $V \in \mathcal{C}_d$. Among all states, pure or mixed, it achieves the minimal value 1 if and only if ρ is the maximally mixed state. Among all *pure* states, it achieves the minimal value d if and only if ρ is any stabilizer state. It is convex with respect to ρ. From the last two properties, we obtain the following simple and convenient criterion for magic resource: If $M(\rho) > d$, then the quantum state ρ must be magic.

Another remarkable physical feature of the magic quantifier $M(\rho)$ is that it serves as a bridge connecting the stabilizer states and SIC fiducial states in the sense that

$$d \leqslant M(|\phi\rangle\langle\phi|) \leqslant 1 + (d+1)\sqrt{d-1}$$

for any pure state $|\phi\rangle$, with the lower bound achieved by any stabilizer state and the upper bound achieved by any SIC-POVM fiducial state (assuming its existence) [59], that is, the stabilizer states and the SIC-POVM fiducial states occupy the two extremes in terms of magic resource, which reveals an unexpected connection between stabilizerness and the outstanding Zauner conjecture for the existence of SIC-POVM [61,62]. Recall that a pure state $|f\rangle$ is called a SIC-POVM fiducial state if

$$\{|f_{k,l}\rangle = D_{k,l}|f\rangle : k, l \in \mathbb{Z}_d\}$$

constitutes a symmetric informationally complete positive operator valued measure (SIC-POVM) [61,62]. Although a plethora of analytical and numerical evidences indicate that SIC-POVM exists in any dimension (Zauner's conjecture) [61–67], the issue remains open.

In the following, we present some illustrative examples which shall be used in the paper. First, consider a qubit system, i.e., $d = 2$. For the qubit state with the standard Bloch representation

$$\rho = \tfrac{1}{2}(\mathbf{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z),$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices, since

$$D_{0,0} = \sigma_0 = \mathbf{1}, \quad D_{1,0} = \sigma_x, \quad D_{0,1} = \sigma_z, \quad D_{1,1} = -\sigma_y,$$

the characteristic function of ρ, expressed in the matrix form, reads

$$\big(c_\rho(k,l)\big) = \begin{pmatrix} c_\rho(0,0) & c_\rho(0,1) \\ c_\rho(1,0) & c_\rho(1,1) \end{pmatrix} = \begin{pmatrix} 1 & r_z \\ r_x & -r_y \end{pmatrix},$$

it follows that the magic quantifier can be readily obtained as

$$M(\rho) = 1 + |r_x| + |r_y| + |r_z|. \tag{1}$$

The n-qubit Heisenberg-Weyl group is generated by the tensor product of the constituent single-qubit Heisenberg-Weyl operators, and the magic quantifier for any n-qubit state ρ can be defined analogously as

$$M(\rho) = \sum_{\mathbf{k}} |\mathrm{tr}(\rho\sigma_{\mathbf{k}})| \tag{2}$$

with $\sigma_{\mathbf{k}}=\sigma_{k_1} \otimes \cdots \otimes \sigma_{k_n}$ for $\mathbf{k}=(k_1, \ldots, k_n),\ k_j = 0, x, y, z$.

III. DETECTING QUANTUM PHASE TRANSITION VIA MAGIC RESOURCE

The XY spin chain with three-spin interaction (XYT), which is an exactly solvable model exhibiting phase transition [3–6], is governed by the following Hamiltonian (assuming the periodic boundary condition $\sigma_\mu^j = \sigma_\mu^{N+j}$):

$$\begin{aligned} H = &-\sum_{j=1}^{N}\left(\frac{1+\gamma}{2}\sigma_x^j\sigma_x^{j+1} + \frac{1-\gamma}{2}\sigma_y^j\sigma_y^{j+1} + \lambda\sigma_z^j\right) \\ &-\sum_{j=1}^{N}\alpha\big(\sigma_x^{j-1}\sigma_z^j\sigma_x^{j+1} + \sigma_y^{j-1}\sigma_z^j\sigma_y^{j+1}\big), \end{aligned} \tag{3}$$

where N is the number of spins in the chain (assumed to be an odd number for convenience), γ describes the anisotropy of the system arising from the spin-spin interaction, λ is the external magnetic field, and α denotes the strength of the internal three-spin interaction.

As can be seen from the Hamiltonian given by Eq. (3), the model degenerates into the XY spin model when $\lambda = \alpha = 0$. The XYT model is exactly solvable. Actually, via the Jordan-Wigner transformation [1,2]

$$\begin{aligned} \sigma_x^j &= (c_j^\dagger + c_j)\prod_{i<j}(\mathbf{1} - 2c_i^\dagger c_i), \\ \sigma_y^j &= -\sqrt{-1}(c_j^\dagger - c_j)\prod_{i<j}(\mathbf{1} - 2c_i^\dagger c_i), \\ \sigma_z^j &= \mathbf{1} - 2c_j^\dagger c_j, \end{aligned}$$

where $c_j^\dagger$ and c_j are the mapped spinless fermionic creation and annihilation operators, respectively, the associated Hamiltonian can be transformed to the form

$$\begin{aligned} H = -\sum_{j=1}^{N}[&c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + \gamma(c_j^\dagger c_{j+1}^\dagger + c_j c_{j+1}) \\ &+ 2\alpha(c_{j-1}^\dagger c_{j+1} + c_{j+1}^\dagger c_{j-1}) + \lambda(1 - 2c_j^\dagger c_j)]. \end{aligned}$$

Furthermore, assuming odd N, putting $K = (N-1)/2$, and employing the discrete Fourier transform of fermionic operators determined by

$$c_j = -\frac{1}{\sqrt{N}}\sum_{k=-K}^{K}\hat{c}_k e^{i2\pi kj/N}, \quad j = 1, 2, \ldots, N,$$

one obtains the Bogoliubov transformation

$$\hat{d}_k = \hat{c}_k \cos\frac{\theta_k}{2} - i\hat{c}_{-k}^\dagger \sin\frac{\theta_k}{2}, \quad k = -K, \ldots, K,$$

where θ_k are determined via $\varepsilon_k \cos\theta_k = A_k$ with

$$A_k = \lambda - \cos\frac{2\pi k}{N} - 2\alpha\cos\frac{4\pi k}{N}, \tag{4}$$

$$\varepsilon_k = \left(A_k^2 + \gamma^2\sin^2\frac{2\pi k}{N}\right)^{1/2}. \tag{5}$$

Now via the above transformation, the original Hamiltonian H can be further diagonalized in the momentum k space as [24]

$$H = \sum_{k=-K}^{K} 2\varepsilon_k\left(\hat{d}_k^\dagger\hat{d}_k - \frac{1}{2}\right).$$

With this diagonal form, the finite-temperature equilibrium thermal state

$$\rho(T) = \frac{e^{-H/T}}{\mathrm{tr} e^{-H/T}}$$

can be easily evaluated, with T being the temperature and the Boltzmann constant k_B being taken to be 1.

The reduced two-spin states $\rho_{i,j}(T)$ of the thermal equilibrium state at two spin sites i and j can be readily obtained from $\rho(T)$ by tracing out all spins except those at sites i and j as [25,33]

$$\rho_{ij}(T) = \mathrm{tr}_{\widehat{ij}}\rho(T) = \begin{pmatrix} u_T^+ & 0 & 0 & y_T^- \\ 0 & z_T & y_T^+ & 0 \\ 0 & y_T^+ & z_T & 0 \\ y_T^- & 0 & 0 & u_T^- \end{pmatrix}, \tag{6}$$

where $\mathrm{tr}_{\widehat{ij}}$ denotes partial trace over all spins except those at sites i and j, and the matrix elements are given by

$$u_T^{\pm} = \tfrac{1}{4}\big(1 \pm 2\langle\sigma_z\rangle_T + \big\langle\sigma_z^i\sigma_z^j\big\rangle_T\big),$$
$$z_T = \tfrac{1}{4}\big(1 - \big\langle\sigma_z^i\sigma_z^j\big\rangle_T\big),$$
$$y_T^{\pm} = \tfrac{1}{4}\big(\langle\sigma_x^i\sigma_x^j\rangle_T \pm \langle\sigma_y^i\sigma_y^j\rangle_T\big).$$

The mean magnetization at temperature T is

$$\langle\sigma_z\rangle_T = \mathrm{tr}(\sigma_z\rho_i(T)) = \sum_{k=-K}^{K} \frac{A_k \tanh(\varepsilon_k/T)}{N\varepsilon_k},$$

with $\rho_i(T) = \mathrm{tr}_{\hat{i}}\rho_{i,j}(T)$ being the reduced single-spin state (at site i), and the correlation functions at temperature T are given by [4,68,69]

$$\big\langle\sigma_x^i\sigma_x^j\big\rangle_T = \mathrm{tr}\big(\sigma_x^i\sigma_x^j\rho_{i,j}(T)\big) = \begin{vmatrix} a_{-1} & a_{-2} & \cdots & a_{-r} \\ a_0 & a_{-1} & \cdots & a_{-r+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r-2} & a_{r-3} & \cdots & a_{-1} \end{vmatrix},$$

$$\big\langle\sigma_y^i\sigma_y^j\big\rangle_T = \mathrm{tr}\big(\sigma_y^i\sigma_y^j\rho_{i,j}(T)\big) = \begin{vmatrix} a_1 & a_0 & \cdots & a_{-r+2} \\ a_2 & a_1 & \cdots & a_{-r+3} \\ \vdots & \vdots & \vdots & \vdots \\ a_r & a_{r-1} & \cdots & a_1 \end{vmatrix},$$

$$\big\langle\sigma_z^i\sigma_z^j\big\rangle_T = \mathrm{tr}\big(\sigma_z^i\sigma_z^j\rho_{i,j}(T)\big) = \langle\sigma_z\rangle_T^2 - a_r a_{-r},$$

where $r = |i-j|$ and

$$a_l = -\sum_{k=-K}^{K} \frac{\big(A_k\cos\frac{2\pi kl}{N} + \gamma\sin\frac{2\pi kl}{N}\sin\frac{2\pi k}{N}\big)\tanh\big(\frac{\varepsilon_k}{T}\big)}{N\varepsilon_k},$$

for $l = 0, \pm 1, \ldots, \pm r$.

From Eq. (6), the reduced single-spin state (at site i)

$$\rho_i(T) = \mathrm{tr}_{\hat{i}}\rho_{i,j}(T) = \frac{1}{2}\begin{pmatrix} 1 + \langle\sigma_z\rangle_T & 0 \\ 0 & 1 - \langle\sigma_z\rangle_T \end{pmatrix} \tag{7}$$

of the thermal equilibrium state can be readily derived from $\rho_{i,j}(T)$ by taking further partial trace. Due to the periodic boundary condition imposed on the Hamiltonian (3), there is shift invariance and the single-spin reduced state ρ_i is independent of the site i.

As is well known, in the thermodynamic limit $N \to \infty$, the system undergoes quantum phase transition at the critical value $\alpha = \alpha_c = 1/2$ for fixed $\lambda = 0$ and any γ, and at $\lambda = \lambda_c = 1$ for fixed $\alpha = 0$ and any γ [20,39–41]. In the following discussion, we set γ to be zero. Starting from the thermal equilibrium state, by taking the limit $T \to 0$, and setting N to be a large odd number (in our numerical calculations, we set $N = 20\,001$), we can numerically approximate the ground state of the XYT model. By calculating the magic quantifier of the reduced single-spin states and two-spin states, we numerically plot their dynamics and illustrate their dramatic changes at the critical points. This indicates the physical significance of magic resource in detecting quantum phase transition.

A. Magic quantifier of single-spin states

Here we consider only a single-spin in one site of the chain described by the XYT model. By taking the limit $T \to 0$ in Eq. (7), the reduced single-spin state turns out to be

$$\rho_i = \lim_{T\to 0}\rho_i(T) = \frac{1}{2}\begin{pmatrix} 1 + \langle\sigma_z\rangle & 0 \\ 0 & 1 - \langle\sigma_z\rangle \end{pmatrix}, \tag{8}$$

where

$$\langle\sigma_z\rangle = \lim_{T\to 0}\langle\sigma_z\rangle_T = \sum_{k=-K}^{K}\frac{A_k}{N\varepsilon_k} = \sum_{k=-K}^{K}\frac{A_k}{N|A_k|}. \tag{9}$$

The last equality holds since for $\gamma = 0$, we have

$$\varepsilon_k = |A_k| = \left|\lambda - \cos\frac{2\pi k}{N} - 2\alpha\cos\frac{4\pi k}{N}\right|$$

in view of Eqs. (4) and (5).

For the reduced state given by Eq. (8), from the expression of magic quantifier for any qubit state given by Eq. (1), we immediately get

$$M(\rho_i) = 1 + |r_x| + |r_y| + |r_z| = 1 + |\langle\sigma_z\rangle|. \tag{10}$$

Substituting the expression of $\langle\sigma_z\rangle$, Eq. (9), into Eq. (10), the magic quantifier $M(\rho_i)$ can be expressed as an explicit function of the parameters N, λ, and α. We remark here that throughout the paper, all quantities involved are regarded as dimensionless.

To gain an intuitive understanding of the magic quantifier $M(\rho_i)$ in identifying quantum phase transition, we plot the magic quantifier $M(\rho_i)$ versus the three-spin interaction parameter α in Fig. 1 for $N = 20\,001$, $\lambda = 0$. As can be seen, the magic quantifier increases to its maximum around the critical point, then decreases. The critical point $\alpha_c = 1/2$ can also be recognized by the abrupt change in the derivative of the magic quantifier with respect to α, which is also plotted in Fig. 1.

Similarly, we can analyze the effectiveness of magic quantifier $M(\rho_i)$ in detecting quantum critical phenomena with regard to the external magnetic-field parameter λ. By setting $N = 20\,001$ and $\alpha = 0$, we plot the single-spin magic quantifier $M(\rho_i)$ and its derivative $dM(\rho_i)/d\lambda$ versus the external magnetic-field parameter λ in Fig. 2. We see that there is an abrupt change in the derivative of the magic quantifier with respect to λ around the critical point $\lambda_c = 1$.

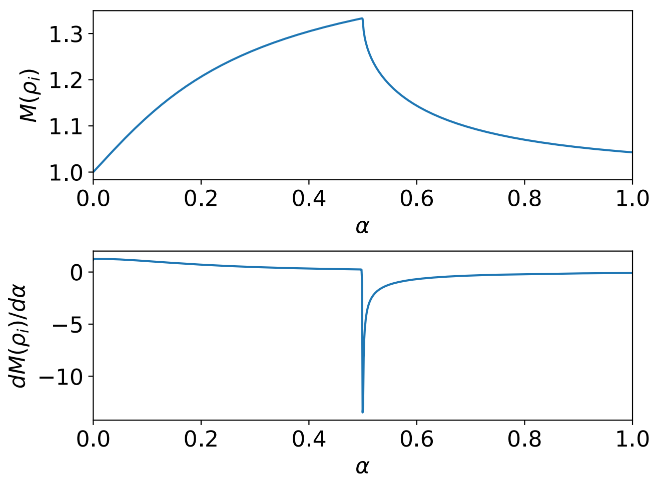

FIG. 1. Single-spin magic quantifier $M(\rho_i)$ and its first-order derivative $dM(\rho_i)/d\alpha$ versus the three-spin interaction parameter α for $N = 20\,001$, $\lambda = 0$. We see that the magic quantifier $M(\rho_i)$ achieves its maximum around the critical point $\alpha_c = 0.5$, while the derivative $dM(\rho_i)/d\alpha$ changes dramatically around the critical point $\alpha_c = 0.5$.

B. Magic quantifier of two-spin states

In the above section, we have illustrated the usage of magic resource in detecting quantum phase transition in the XYT model. Since the magic quantifier we employed can be defined for arbitrary dimensional quantum system [59], we can also analyze the role of magic resource of multispin sites in the chain. For simplicity, we only consider the reduced states consisting of two nearest neighbor spins in the chain. For other two-spin states or multispin states, the calculations can be directly adjusted, and the qualitative results are similar.

The nearest neighbor reduced two-spin state $\rho_{i,i+1}$ of the ground state at two nearest sites i and $i+1$ can be approximated from $\rho_{ij}(T)$ given in Eq. (6) by taking the limit $T \to 0$

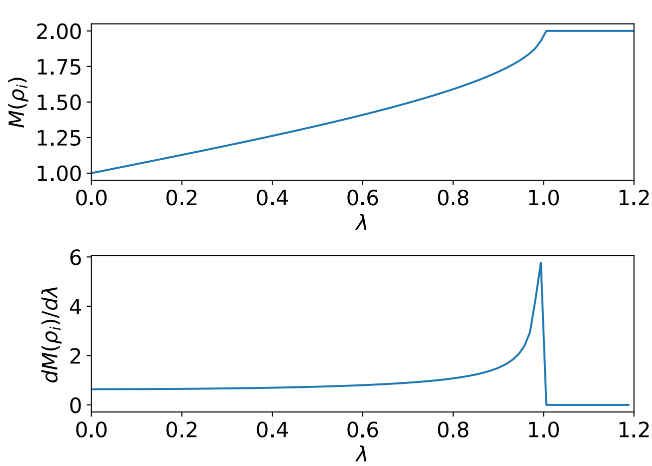

FIG. 2. Single-spin magic quantifier $M(\rho_i)$ and its derivative $dM(\rho_i)/d\lambda$ versus the external magnetic-field parameter λ for $N = 20\,001$, $\alpha = 0$. We see their distinctive behaviors around the critical point $\lambda_c = 1$.

and setting $j = i + 1$. The nearest neighbor correlation functions can be expressed as

$$\langle\sigma_x^i\sigma_x^{i+1}\rangle = \lim_{T\to 0}\langle\sigma_x^i\sigma_x^{i+1}\rangle_T = -\sum_{k=-K}^{K}\frac{A_k}{N\varepsilon_k}\cos\frac{2\pi k}{N}, \tag{11}$$

$$\langle\sigma_y^i\sigma_y^{i+1}\rangle = \lim_{T\to 0}\langle\sigma_y^i\sigma_y^{i+1}\rangle_T = -\sum_{k=-K}^{K}\frac{A_k}{N\varepsilon_k}\cos\frac{2\pi k}{N}, \tag{12}$$

$$\langle\sigma_z^i\sigma_z^{i+1}\rangle = \lim_{T\to 0}\langle\sigma_z^i\sigma_z^{i+1}\rangle_T = \langle\sigma_z\rangle^2 - \langle\sigma_x^i\sigma_x^{i+1}\rangle\langle\sigma_y^i\sigma_y^{i+1}\rangle. \tag{13}$$

For the reduced two-spin state $\rho_{i,i+1}$, which is a two-qubit state, its magic quantifier can be calculated from Eq. (2) as

$$M(\rho_{i,i+1}) = \sum_{s,t}|c_{\rho_{i,i+1}}(s,t)|,$$

where $s, t \in \{0, x, y, z\}$, and

$$(c_{\rho_{i,i+1}}(s,t)) = \left(\mathrm{tr}\left(\rho_{i,i+1}\sigma_s^i \otimes \sigma_t^{i+1}\right)\right)$$

is a 4×4 matrix composed of characteristic functions of $\rho_{i,i+1}$. After straightforward calculations, we obtain

$$\left(c_{\rho_{i,i+1}}(s,t)\right) = \begin{pmatrix} 1 & 0 & 0 & \langle\sigma_z\rangle \\ 0 & \langle\sigma_x^i\sigma_x^{i+1}\rangle & 0 & 0 \\ 0 & 0 & \langle\sigma_y^i\sigma_y^{i+1}\rangle & 0 \\ \langle\sigma_z\rangle & 0 & 0 & \langle\sigma_z^i\sigma_z^{i+1}\rangle \end{pmatrix},$$

from which the magic quantifier of $\rho_{i,i+1}$ can be readily evaluated as

$$M(\rho_{i,i+1}) = 1 + |\langle\sigma_x^i\sigma_x^{i+1}\rangle| + |\langle\sigma_y^i\sigma_y^{i+1}\rangle| + |\langle\sigma_z^i\sigma_z^{i+1}\rangle| + 2|\langle\sigma_z\rangle|.$$

Substituting the expressions given by Eqs. (9), (11)–(13) into the above equation, we get the analytical expression of magic quantifier $M(\rho_{i,i+1})$, which is also an explicit function of N, λ, and α.

To analyze the usage of the two-spin magic quantifier $M(\rho_{i,i+1})$ in detecting the critical points, we plot, in Fig. 3, $M(\rho_{i,i+1})$ and its derivative $dM(\rho_{i,i+1})/d\alpha$ versus the three-spin interaction parameter α for $N = 20\,001$ and $\lambda = 0$. While the corresponding quantity $M(\rho_{i,i+1})$ and its derivative $dM(\rho_{i,i+1})/d\lambda$ versus the magnetic-field parameter λ for $N = 20\,001$ and $\alpha = 0$ are plotted in Fig. 4. From these figures, we see that although the magic quantifier of two-spin states displays different dynamical pattern with regard to the parameters α and λ, as compared with the magic quantifier of single spin, they are also effective in detecting the quantum phase transition.

Another interesting observation from Fig. 4 is that, except for the critical point $\lambda_c = 1$, the magic quantifier also exhibits a nonsmooth change when the magnetic-field parameter λ is around 0.64, which seems unrelated to the quantum phase transition. It is certainly interesting and important to find out the implication of this unusual feature. At present we do not have a physical explanation of this numerical observation, and have to leave it as an open issue for further investigation.

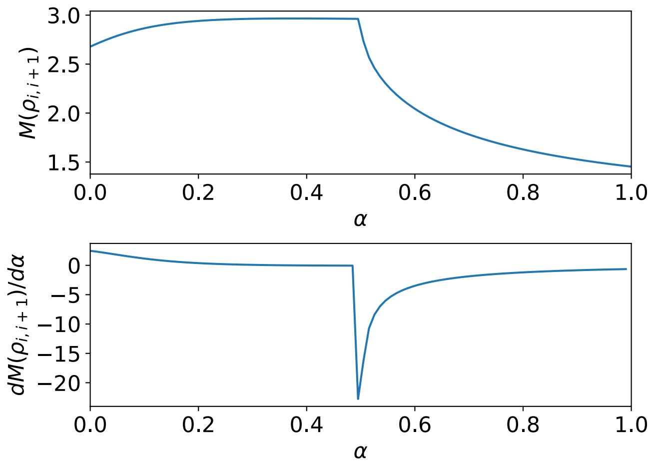

FIG. 3. Two-spin magic quantifier $M(\rho_{i,i+1})$ and its derivative $dM(\rho_{i,i+1})/d\alpha$ versus the three-spin interaction parameter α for $N = 20\,001$, $\lambda = 0$. Both exhibit apparent singular behaviors around the critical point $\alpha_c = 0.5$.

IV. COMPARISON WITH OTHER INFORMATIONAL QUANTITIES

The effectiveness of quantum correlations (e.g., entanglement and quantum discord) in revealing quantum phase transition has been verified in various models [20–38]. However, as is well known, the calculations of quantum correlations, including both the entanglement and quantum discord, are usually quite complicated due to the optimization involved, analytical results can be obtained only for quite special situations [12–17].

Compared with the above approaches, our method for employing the characteristic functions in detecting the critical points, apart from its direct physical significance connected with the magic resource in quantum computation, has the advantage that the magic quantifier $M(\rho)$ for either single or composite quantum systems can be quite easily calculated. In the XYT model, their analytical expressions can be straightforwardly derived and analyzed, and they pinpoint the critical points rather easily.

As we have mentioned before, there are some other magic quantifiers, such as the sum negativity, mana, etc. [45]. One may wonder whether these quantities can be employed to study critical points in the model we considered. Here we take the sum negativity and mana for examples. Their definitions rely heavily on the discrete Wigner function, which is well-defined only for *odd prime* power dimensional quantum systems (that is $d = p^n$ for some prime number $p > 2$) [56–58]. For the cases we studied, the single-spin is a qubit with dimension two, while the two-spin states reside in a 2^2-dimensional Hilbert space. Thus neither the sum negativity nor the mana can be applied here. Moreover, if we formally define a Wigner function for $d = 2$ following the approach there [56–58], that is, if we define

$$W_\rho(k,l) = \frac{1}{d}\mathrm{tr}(\rho A_{k,l}), \quad k,l \in \mathbb{Z}_2,$$

where $A_{k,l} = D_{k,l}A_{0,0}D_{k,l}^\dagger$, and $A_{0,0} = \frac{1}{d}\sum_{k,l} D_{k,l}$ is the discrete parity operator determined by

$$A_{0,0}|k\rangle = |-k\rangle, \quad k \in \mathbb{Z}_2,$$

then

$$A_{0,0}|0\rangle = |0\rangle, \quad A_{0,0}|1\rangle = |-1\rangle = |1\rangle,$$

it follows that $A_{0,0} = \mathbf{1}$ (the identity operator), which implies that

$$A_{k,l} = D_{k,l}A_{0,0}D_{k,l}^\dagger = \mathbf{1}.$$

Therefore, for any qubit state ρ, the Wigner function turns out to be $W_\rho(k,l) = 1/2$ for any k, l, which is useless since nothing about the quantum state can be inferred from it.

V. CONCLUSION

In this paper, we have explored the role magic resource plays in detecting the quantum phase transition in the XYT model. We have used the magic quantifier defined via the characteristic functions, which are well defined and can be rather easily calculated for all dimensional quantum systems [59]. After detailed calculations of magic quantifier for both single-spin and two-spin system, we have shown that the dynamics of magic quantifier display abrupt changes around the critical points in the XYT model, thus it may be regarded as an indicator for the critical phenomena in the studied spin systems. Actually for some other spin models, like the XXZ spin model, we have verified that the magic quantifier via the characteristic functions can still reveal the critical points.

Compared with other quantum informational quantities such as the entanglement and quantum discord, an apparent advantage of our approach is that the magic quantifier we employed can be straightforwardly calculated.

One may argue that for detecting critical points, any reasonable and continuous function of the state will do the work

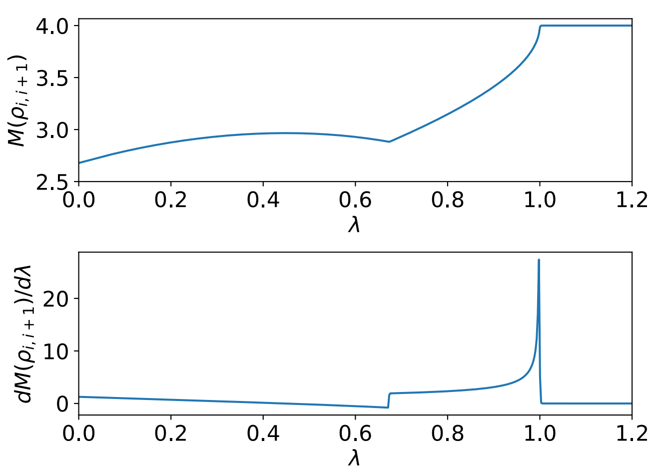

FIG. 4. Two-spin magic quantifier $M(\rho_{i,i+1})$ and its derivative $dM(\rho_{i,i+1})/d\lambda$ versus the magnetic-field parameter λ for $N = 20\,001$, $\alpha = 0$. Both exhibit apparent singular behavior around the critical point $\lambda_c = 1$. It is interesting to note that they also exhibit special behavior at $\lambda \approx 0.64$, which might indicate some unusual feature therein awaiting for further investigation.

since across the critical points, the state undergoes a radical change, which will certainly induce an abrupt change of the quantity. Why use magic resource (or entanglement, discord, coherence) to detect the critical points? The issue here is that we not only are interested in detecting critical points but also are interested in the physical meaning of the quantities. Furthermore, apart from indicating critical points, it is also desirable to study the changing behavior of magic or other quantum resources around the critical points for the purpose of quantum information processing and many-body physics.

Since magic resource is crucial for quantum computation, our present results reveal certain intrinsic connections between quantum phase transition and quantum computing, and indicate an alternative way for studying many-body systems from the perspective of quantum resource and quantum computing.

Inspired by the significant role played by magic resource in fault-tolerant quantum computation, and supported by the above observation that magic resource can be used to detect quantum phase transition in certain systems, we hope that the magic quantifier defined via characteristic functions may be a valuable figure of merit in studying many other quantum phenomena. It would be interesting to further elucidate the role of magic resource in other contexts.

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