Stationary states of open XX-spin chains

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We study an open quantum spin chain of arbitrary length with nearest neighbor XX interactions of strength g, immersed in an external constant magnetic field Δ along the z direction, whose end spins are weakly coupled to two heat baths at different temperatures. In the so-called global approach, namely, without neglecting interspin interactions, using standard weak-coupling limit techniques, we first derive the open chain master equation written in terms of fermionic mode operators. Then, we focus on the study of the dependence of the resulting open dynamics from the ratio $r \equiv g/\Delta$. By increasing r, some of the chain Bohr transition frequencies become negative; when this occurs, both the generator of the dissipative time evolution and its stationary states behave discontinuously. As a consequence, the asymptotic spin and heat flows also exhibit discontinuities, but in a different way: while source terms in the spin flow continuity equation show jumps, the heat flow instead is continuous but with discontinuous first derivatives with respect to r. These two behaviors might be experimentally accessible; in particular, they could discriminate between the global and the local approaches to open quantum spin chains. Indeed, the latter one, which neglects interspin interactions in the derivation of the master equation, does not show any kind of discontinuous behavior.

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I. INTRODUCTION

Open quantum spin chains have recently received increasing attention as instances of many-body systems driven by intrinsic interspin interactions and by suitably imposed dissipative effects at the boundaries. Specific experimental realizations have been reported in scenarios involving ultracold atoms, light-harvesting complexes, and quantum thermodynamics at large [1–21].

We dealt with such a scenario in Ref. [22] (see also Ref. [23]): there, we considered a chain in a constant magnetic field Δ along the z axis with XX interspin interactions of strength g. We did not assume any *a priori* specific irreversible modifications of the otherwise unitary chain dynamics; rather, we derived the open chain dissipative dynamics by coupling its end spins to two thermal Bosonic baths consisting of harmonic oscillators in equilibrium at possibly different temperatures. In Ref. [22], however, only energy preserving, or counterrotating, terms in the spin-bath couplings have been considered, associated with products of spin raising operators and annihilation bath operators (see Remark 1). Then, the standard weak-coupling limit techniques [24–39] have been applied in the so-called global approach that uses the spectralization of the whole chain Hamiltonian without neglecting the interspin interactions (e.g., see Refs. [40–50]). As a consequence, new effects emerged, in particular the presence of sink and source terms in the spin flow continuity equation. Interestingly, these effects are characteristic features of the global approach and disappear in other, simplified derivations of the open chain dynamics, like in the so-called local approach that neglects the interspin interaction in the application of the weak-coupling limit techniques (e.g., see Refs. [51–82]).

In the following, we approach the same physical setting by using suitable chain fermionic modes b_ℓ, $b_\ell^\dagger$ associated with the chain Bohr transition frequencies ω_ℓ, namely, differences of chain energies. The global weak-coupling limit procedure is carried out in the general case of spin-bath interactions consisting of both energy-preserving (counterrotating) and energy-nonpreserving (corotating) terms, the latter being associated with products of spin raising operators and creation bath operators. The procedure yields a dissipative irreversible time evolution generated by a master equation in Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form which exhibits discontinuities with respect to the ratio of the interspin strength to the external magnetic field. In particular, in the case of a chain-bath coupling without corotating terms, nontrivial dissipative master equations emerge only when the Bohr transition frequencies of the chain overlap with the spectrum of the baths. The underlying physical mechanism is that dissipation occurs only when Bohr transition frequencies are in resonance with bath energies; therefore, if the latter are all positive, only positive Bohr transition frequencies can give rise to dissipative effects, while the negative ones contribute to the Lamb-shift corrections to the chain Hamiltonian, only. The positivity and negativity of the Bohr transition frequencies depend on the ratio g/Δ, denoted by r in the following. In other words, when, by changing r, a Bohr transition frequency ω_ℓ becomes negative, then the corresponding contribution to the generator changes from the

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sum of a commutator with a Hamiltonian plus a dissipative term into a purely Hamiltonian one, mitigating the noisy and decohering impact of the baths. Apparently, a similar consequence as in the dissipation free subspace scenario is obtained; however, it is important to notice that, in the present context, this is an effect due not to a suitable engineering of the environment so as to benefit from a symmetry of the generator, but rather it comes from tuning either the interspin interaction strength or the external magnetic field.

In the following, we shall focus upon the case with counterrotating coupling terms only, whereby the reduced chain dynamics becomes particularly simple using the standard Jordan-Wigner fermionic representation. Fermionization indeed allows us to analytically study the chain asymptotic transport properties, as spin and heat flows, as functions of the ratio r, with the following results:

(1) When all Bohr transition frequencies are positive, there is a unique stationary state to which all initial chain states converge asymptotically in time.

(2) When a Bohr transition frequency becomes negative, the corresponding fermionic mode becomes dissipation free and time oscillations keep going undisturbed in the corresponding sectors.

(3) The resulting discontinuities of the generator affect the asymptotic behavior of spin and heat flows: both these physical quantities exhibit discontinuities; however, while the source terms in the spin flow asymptotic continuity equation present jumps, the heat flow is instead continuous, but with discontinuous first derivatives with respect to r when positive Bohr transition frequencies become negative.

The structure of the paper is as follows: in Sec. II, we first write the master equation of the open quantum spin chain by means of fermionic operators and then study the discontinuous dependence of the generator on the ratio r. In Sec. III, we first explicitly find the manifold of stationary states and then discuss the asymptotic behavior of spin and heat flows as functions of r. Conclusions are drawn in Sec. IV.

II. OPEN XX-SPIN CHAIN OF LENGTH N: FERMIONIC REPRESENTATION

We shall study a linear quantum spin 1/2 chain of length N immersed in a constant magnetic field along the z direction of intensity $\Delta > 0$, with XX nearest neighbor interactions of strength $g > 0$. With open boundary conditions, the chain Hamiltonian is thus given by

$$H = g\sum_{\ell=1}^{N-1}\left(\sigma_x^{(\ell)}\sigma_x^{(\ell+1)} + \sigma_y^{(\ell)}\sigma_y^{(\ell+1)}\right) + \Delta\sum_{\ell=1}^{N}\sigma_z^{(\ell)}, \qquad (1)$$

where $\sigma_{x,y,z}^{(\ell)}$ denote the Pauli matrices at site $1 \leqslant \ell \leqslant N$:

$$\left[\sigma_x^{(\ell)},\, \sigma_y^{(k)}\right] = \delta_{\ell k}\, 2\, i\, \sigma_z^{(\ell)},\ \left(\sigma_{x,y,z}^{(\ell)}\right)^2 = \mathbb{I}^{(\ell)}, \qquad (2)$$

with $\mathbb{I}^{(\ell)}$ the identity 2×2 matrix at site ℓ.

We turn the chain into an open one by coupling its two end spins, $j = 1$ on the left and $j = N$ on the right, to two independent free Bosonic thermal baths with Hamiltonians ($\hbar = 1$)

$$H_{\rm env}^{(j)} = \int_0^{+\infty} d\nu\, \nu\, \mathfrak{a}_j^\dagger(\nu)\, \mathfrak{a}_j(\nu), \quad j = 1, N, \qquad (3)$$

where $\mathfrak{a}_j(\nu)$, $\mathfrak{a}_j^\dagger(\nu)$ are Bosonic operators satisfying the canonical commutation relations

$$[\mathfrak{a}_j(\nu),\, \mathfrak{a}_{j'}^\dagger(\nu')] = \delta_{jj'}\, \delta(\nu - \nu')\, .$$

The coupling between the end spins and the two baths will be taken of the form

$$H_{\rm int} = \lambda \sum_{j=1,N}\left(\sigma_+^{(j)}\mathfrak{C}_j + \sigma_-^{(j)}\mathfrak{C}_j^\dagger\right), \qquad (4)$$

where $\lambda \ll 1$ is a dimensionless coupling constant,

$$\sigma_\pm^{(j)} \equiv \tfrac{1}{2}\left(\sigma_x^{(j)} \pm i\sigma_y^{(j)}\right) \qquad (5)$$

are spin ladder operators at sites 1 and N, while

$$\mathfrak{C}_j \equiv \int_0^\infty d\nu\, [h_j(\nu)\, \mathfrak{a}_j(\nu) + k_j(\nu)\, \mathfrak{a}_j^\dagger(\nu)] \qquad (6)$$

are bath operators with $h_j(\nu)$ and $k_j(\nu)$ suitable real smearing functions.

By standard fermionization methods (see Appendix A), the Hamiltonian (1) can be put in diagonal form,

$$H = \sum_{\ell=1}^{N}\omega_\ell\, b_\ell^\dagger\, b_\ell, \quad \omega_\ell := 2\,\Delta + 4g\cos\left(\frac{\ell\pi}{N+1}\right), \qquad (7)$$

apart from an unimportant constant term. In the above expression b_ℓ, $b_\ell^\dagger$ are fermionic operators, obeying the anticommutation relations

$$\{b_\ell,\, b_{\ell'}^\dagger\} = \delta_{\ell\ell'},\ \{b_\ell,\, b_{\ell'}\} = \{b_\ell^\dagger,\, b_{\ell'}^\dagger\} = 0. \qquad (8)$$

The Bohr transition frequencies ω_ℓ are in decreasing order:

$$\omega_1 \;=\; 2\,\Delta + 4g\cos\left(\frac{\pi}{N+1}\right) \geqslant \omega_2 \geqslant \cdots$$
$$\cdots \geqslant \omega_N = 2\,\Delta - 4g\cos\left(\frac{\pi}{N+1}\right). \qquad (9)$$

Notice that $\cos\frac{\ell\pi}{N+1} < 0$ when $\lfloor\frac{N+1}{2}\rfloor < \ell \leqslant N$, where $\lfloor x\rfloor$ denotes the largest integer smaller than x. Thus, depending on the ratio $r = g/\Delta$, some Bohr transition frequency can become negative. This never occurs if

$$r \leqslant \frac{1}{2\cos\frac{\pi}{N+1}}, \qquad (10)$$

since then $\omega_N \geqslant 0$. Instead, if $p + 1 > \lfloor\frac{N+1}{2}\rfloor$ and the ratio r is such that

$$\frac{1}{2\left|\cos\frac{(p+1)\pi}{N+1}\right|} \leqslant r \leqslant \frac{1}{2\left|\cos\frac{p\pi}{N+1}\right|}, \qquad (11)$$

then $\omega_\ell \geqslant 0$ for $1 \leqslant \ell \leqslant p$, while $\omega_\ell \leqslant 0$ for $p + 1 \leqslant \ell \leqslant N$. The maximum number of negative frequencies, $\lfloor\frac{N}{2}\rfloor$, is obtained when

$$r > \frac{1}{2\left|\cos\left[\left(\left\lfloor\frac{N+1}{2}\right\rfloor + 1\right)\frac{\pi}{N+1}\right]\right|}. \qquad (12)$$

As shown in Appendix A, the left and right ladder operators that couple to the baths can be written as

$$\sigma_-^{(1)} = \sum_{j=1}^{N} u_{1j}\, b_j, \quad \sigma_-^{(N)} = (-)^{\widehat{N}} \sum_{j=1}^{N} u_{Nj}\, b_j, \tag{13}$$

where $\widehat{N} = \sum_{\ell=1}^{N} b_\ell^\dagger b_\ell$ is the fermionic number operator and the coefficients

$$u_{\ell k} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\ell k\pi}{N+1}\right), \quad \ell, k = 1, 2, \dots, N, \tag{14}$$

form an orthogonal, symmetric matrix $U = [u_{k\ell}]$. Then, the coupling Hamiltonian (4) can be recast as

$$H_{\rm int} = \lambda \sum_{j=1,N} \sum_{\ell=1}^{N} u_{j\ell}(b_{j\ell}^\dagger\, \mathfrak{C}_j + b_{j\ell}\, \mathfrak{C}_j^\dagger), \tag{15}$$

where we set

$$b_{1\ell} := b_\ell, \quad b_{N\ell} := (-1)^{\hat{N}}\, b_\ell = -b_\ell\,(-1)^{\hat{N}}. \tag{16}$$

Remark 1. According to the corresponding free Hamiltonians, the chain fermionic annihilation operators b_ℓ and the Bosonic bath annihilation operators $\mathfrak{a}_j(\nu)$ evolve in time as follows:

$$b_\ell(t) := e^{itH}\, b_\ell\, e^{-itH} = e^{-i\omega_\ell t}\, b_\ell, \tag{17}$$

$$\mathfrak{a}_j(\nu, t) := e^{itH_{\rm env}}\, \mathfrak{a}_j(\nu)\, e^{-itH_{\rm env}} = e^{-i\nu t}\, \mathfrak{a}_j(\nu), \tag{18}$$

where $j = 1, N$ and $H_{\rm env} = H_{\rm env}^{(1)} + H_{\rm env}^{(N)}$.

Therefore, in the coupling Hamiltonian (15) there appear both corotating contributions $b_\ell\, \mathfrak{a}_j(\nu)$, $b_\ell^\dagger\, \mathfrak{a}_j^\dagger(\nu)$ and counterrotating ones $b_\ell\, \mathfrak{a}_j^\dagger(\nu)$, $b_\ell^\dagger\, \mathfrak{a}_j(\nu)$. By setting the smearing functions $k_j(\nu) = 0$, $j = 1, N$, only the counterrotating contributions remain. One then retrieves the physical context studied in Ref. [22] where the spin-Bose coupling is indeed of the form

$$H_{\rm int} = \lambda \sum_{j=1,N} \sum_{\ell=1}^{N} u_{j\ell} \int_0^{+\infty} d\nu\, h_j(\nu)\, [b_{j\ell}^\dagger\, \mathfrak{a}_j(\nu) + b_{j\ell}\, \mathfrak{a}_j^\dagger(\nu)]. \tag{19}$$

A. Weak-coupling limit

As we are interested in the spin chain as an open quantum system, we shall look at the reduced Markovian dynamics of the chain. Such an irreversible, dissipative time evolution is derived by tracing out the two Bosonic baths and by applying the weak-coupling limit techniques [24–39]. The latter prescriptions will be performed within the so-called global approach that does not neglect the spin interactions [22]. Concretely, we shall assume the free Boson baths coupled to the end spins 1 and N to be in equilibrium Gibbs states at temperatures $T_1 \equiv 1/\beta_1$ and $T_N \equiv 1/\beta_N$, so that the state of the environment,

$$\rho_{\rm env} = \frac{e^{-\beta_1 H_1}}{{\rm Tr}(e^{-\beta_1 H_1})} \otimes \frac{e^{-\beta_N H_N}}{{\rm Tr}(e^{-\beta_N H_N})}, \tag{20}$$

is invariant under the bath dynamics generated by $H_{\rm env}$ and exhibits thermal correlation functions of the form

$${\rm Tr}_B[\rho_{\rm env}\mathfrak{a}_j^\dagger(\nu)\mathfrak{a}_k(\nu')] = \delta_{jk}\delta(\nu - \nu')\, n_j(\nu) \tag{21}$$

$${\rm Tr}_B[\rho_{\rm env}\mathfrak{a}_j(\nu)\mathfrak{a}_k^\dagger(\nu')] = \delta_{jk}\delta(\nu - \nu')\, [1 + n_j(\nu)], \tag{22}$$

with thermal mean occupation numbers

$$n_j(\nu) = \frac{1}{e^{\beta_j \nu} - 1}. \tag{23}$$

These make sense as mean numbers only if $\nu \geqslant 0$: this is why in (3) one restricts the integration over the positive real line. Finally, the initial state of chain plus baths is chosen of the factorized form $\rho_{\rm tot}(0) = \rho(0) \otimes \rho_{\rm env}$, with $\rho(0)$ an initial state of the spin chain.

In the presence of thermal correlation functions decaying on a time scale much faster than the one typical of the spin chain, which is set by its energy spectrum, one obtains a fully physically consistent, namely, completely positive dissipative chain dynamics generated by the so-called GKSL master equation

$$\frac{\partial \rho(t)}{\partial t} = -i[H + \lambda^2 H_{\rm LS}, \rho(t)] + \lambda^2 \mathbb{D}[\rho(t)] \equiv \mathbb{L}[\rho(t)]. \tag{24}$$

Remark 2. The open chain dynamics is described by maps $\gamma_t : \rho \mapsto \rho(t) = \gamma_t[\rho]$, where γ_t is formally given by $\exp(t\mathbb{L})$. The properties of complete positivity and trace preservation require them to be of the so-called Kraus-Stinespring form

$$\gamma_t[\rho] = \sum_\alpha V_\alpha(t)\, \rho\, V_\alpha^\dagger(t), \tag{25}$$

with operators $V_\alpha(t)$ such that $\sum_\alpha V_\alpha^\dagger(t)\, V_\alpha(t) = \mathbb{I}$. Only γ_t of such form guarantee not only that chain states are sent into chain states, but also that the lifted maps $\gamma_t \otimes {\rm id}$, with id the identity map, transform into *bona fide* states all possible entangled states of the open spin chain statistically coupled to an arbitrary n-level system, not subjected to any dynamics. It turns out that for the dynamical maps γ_t to be as in (25), the generator $\mathbb{L}$ must be of the GKSL form

$$\mathbb{L}[\rho] = -i[H_{\rm eff}, \rho] + \sum_\alpha \left(L_\alpha\, \rho\, L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\right), \tag{26}$$

where $H_{\rm eff}$ is the sum of the given system Hamiltonian plus a Lamb-shift correction due to the interaction with the environment. Starting from a microscopic chain-environment interaction, a generator of the GKLS form can in general only be obtained by applying a rigorous mathematical procedure consisting of a number of approximations known under the collective name of weak-coupling limit.

The generator $\mathbb{L}$ in (24) consists of the standard commutator with the chain Hamiltonian corrected by a Lamb-shift term $H_{\rm LS} = \sum_{j=1,N} H_{\rm LS}^{(j)}$ and a purely dissipative term $\mathbb{D} = \sum_{j=1,N} \mathbb{D}^{(j)}$. Their explicit expressions are derived in Appendix B: the Lamb-shift term consists of an off-diagonal (in ℓ) contribution from all pairs of positive and negative Bohr transition frequencies ω_ℓ in (7) that satisfy $\omega_\ell + \omega_{\ell'} = 0$, for

$\ell \neq \ell'$,

$$H_{\rm LS}^{(1)} = \sum_{j=1,N} \sum_{\ell,\ell'=1}^{N} \delta_{\omega_\ell+\omega_{\ell'},0}\, u_{j\ell}\, u_{j\ell'} \times P \int_0^{+\infty} {\rm d}\nu\, \frac{h_j(\nu)\, k_j(\nu)}{\nu - \omega_\ell}\, (b_{j\ell}^\dagger\, b_{j\ell'}^\dagger + b_{j\ell'}\, b_{j\ell}), \quad (27)$$

where P denotes the principal value and of a diagonal contribution from all Bohr transition frequencies,

$$H_{\rm LS}^{(2)} = \sum_{j=1,N} \sum_{\ell=1}^{N} u_{j\ell}^2\, P \int_0^{+\infty} d\nu[1 + 2n_j(\nu)] \times \left(\frac{k_j^2(\nu)}{\nu + \omega_\ell} + \frac{h_j^2(\nu)}{\omega_\ell - \nu}\right) b_{j\ell}^\dagger\, b_{j\ell}. \quad (28)$$

The dissipative term $\mathbb{D}[\rho]$ of the generator is the sum of two terms: both of them are in GKSL form; however, the first one, $\mathbb{D}^{(1)}[\rho]$, consists of terms associated with pairs of Bohr transition frequencies ω_ℓ and $\omega_{\ell'}$ such that $\omega_\ell + \omega_{\ell'} = 0$, with $\omega_\ell \geqslant 0$. As explained in Appendix B, these pairs result from the presence of the corotating terms in the interaction Hamiltonian (4) and can exist only in correspondence to specific values of the ratio r such that

$$r = \cos\left(\frac{\pi \ell'}{N+1}\right) - \cos\left(\frac{\pi \ell}{N+1}\right). \quad (29)$$

If, for a given value of r, such pairs of Bohr transition frequencies exist, the associated GKSL dissipator reads

$$\mathbb{D}^{(1)}[\rho] = 2\pi \sum_{j=1,N} \sum_{\substack{\ell:\omega_\ell \geqslant 0 \\ \ell':\omega_{\ell'} = -\omega_\ell}}^{N} u_{j\ell}\, u_{j\ell'}\, h_j(\omega_\ell)\, k_j(\omega_\ell)[1 + 2n_j(\omega_\ell)]\left[b_{j\ell'}^\dagger\, \rho\, b_{j\ell}^\dagger - \frac{1}{2}\{b_{j\ell}^\dagger\, b_{j\ell'}^\dagger \rho\} + b_{j\ell'}\, \rho\, b_{j\ell} - \frac{1}{2}\{b_{j\ell}\, b_{j\ell'},\, \rho\}\right]. \quad (30)$$

Instead, the counterrotating terms provide diagonal (with respect to ℓ) contributions from all Bohr transition frequencies, though with different GKSL terms for positive and negative ones, selected by $\theta(\omega_\ell)$ and $\theta(-\omega_\ell)$, respectively:

$$\mathbb{D}^{(2)}[\rho] = 2\pi \sum_{j=1,N} \sum_{\ell=1}^{N} u_{j\ell}^2 \left\{\theta(\omega_\ell)\, h_j^2(\omega_\ell)\left[(1 + n_j(\omega_\ell))\left(b_{j\ell}\, \rho\, b_{j\ell}^\dagger - \frac{1}{2}\{b_{j\ell}^\dagger\, b_{j\ell},\, \rho\}\right) + n_j(\omega_\ell)\left(b_{j\ell}^\dagger\, \rho\, b_{j\ell} - \frac{1}{2}\{b_{j\ell}\, b_{j\ell}^\dagger,\, \rho\}\right)\right]\right.$$
$$\left. + \theta(-\omega_\ell)\, k_j^2(-\omega_\ell)\left[(1 + n_j(-\omega_\ell))\left(b_{j\ell}^\dagger\, \rho\, b_{j\ell} - \frac{1}{2}\{b_{j\ell}\, b_{j\ell}^\dagger,\, \rho\}\right) + n_j(-\omega_\ell)\left(b_{j\ell}\, \rho\, b_{j\ell}^\dagger - \frac{1}{2}\{b_{j\ell}^\dagger\, b_{j\ell},\, \rho\}\right)\right]\right\}. \quad (31)$$

Setting the smearing functions $k_j(\nu) = 0$, the spin-Bose coupling becomes as in (19), without corotating terms. Then, the dissipative part of the GKSL generator in (24) reduces to just $\mathbb{D}^{(2)}[\rho]$ with only positive Bohr transition frequency contributions:

$$\mathbb{D}[\rho] = \sum_{\ell:\omega_\ell \geqslant 0} \mathbb{D}_\ell[\rho]. \quad (32)$$

Therefore, the whole generator $\mathbb{L}$ splits into the sum

$$\mathbb{L}[\rho] = \sum_{\ell=1}^{N} \mathbb{L}_\ell[\rho] \quad (33)$$

of N generators, one for each fermionic mode, where

$$\mathbb{L}_\ell[\rho] = -i\big[H_\lambda^{(\ell)}, \rho\big] \quad \text{when} \quad \omega_\ell < 0 \quad (34)$$

$$\mathbb{L}_\ell[\rho] = -i\big[H_\lambda^{(\ell)}, \rho\big] + \lambda^2 \mathbb{D}_\ell[\rho] \quad \text{when} \quad \omega_\ell \geqslant 0. \quad (35)$$

Moreover, noting that, from (16), $b_{j\ell}^\dagger\, b_{j\ell} = b_\ell^\dagger b_\ell$,

$$H_\lambda^{(\ell)} = \Omega_\lambda^{(\ell)}\, b_\ell^\dagger\, b_\ell \quad \text{with} \quad \Omega_\lambda^{(\ell)} = \omega_\ell + \lambda^2\, \widetilde{\omega}_\ell \quad \text{and} \quad (36)$$

$$\widetilde{\omega}_\ell := \sum_{j=1,N} u_{j\ell}^2\, P \int_0^{+\infty} d\nu[1 + 2n_j(\nu)] \frac{h_j^2(\nu)}{\omega_\ell - \nu}, \quad (37)$$

while $\mathbb{D}_\ell = \sum_{j=1,N} \mathbb{D}_{j\ell}$, where

$$\mathbb{D}_{j\ell}[\rho] = 2\pi\, u_{j\ell}^2\, h_j^2(\omega_\ell)\left[(1 + n_j(\omega_\ell))\big(b_{j\ell}\, \rho\, b_{j\ell}^\dagger - \tfrac{1}{2}\{b_{j\ell}^\dagger\, b_{j\ell},\, \rho\}\big) + n_j(\omega_\ell)\left(b_{j\ell}^\dagger\, \rho\, b_{j\ell} - \tfrac{1}{2}\{b_{j\ell}\, b_{j\ell}^\dagger,\, \rho\}\right)\right]. \quad (38)$$

Because of the quadratic features of the $\mathbb{L}_\ell$, it is straightforward to show that, for $\ell \neq \ell'$, $\mathbb{L}_\ell \mathbb{L}_{\ell'} = \mathbb{L}_{\ell'} \mathbb{L}_\ell$; as a consequence, the reduced dynamics consists of independent, namely, commuting, single-mode dissipative time evolutions, one for each of the fermionic modes:

$$\gamma_t = e^{t\,\mathbb{L}} = \prod_{\ell=1}^{N} \gamma_t^{(\ell)}, \quad \gamma_t^{(\ell)} = \exp(t\, \mathbb{L}_\ell). \quad (39)$$

Remark 3. In Appendix C it is shown that the generator with contributions as in (35) corresponds to the one derived in Ref. [22]. However, the mode-local structure of the reduced dynamics expressed by means of the mode operators $\{b_\ell, b_\ell^\dagger\}_{\ell=1}^N$ gets lost when using the spin representation of the generator via (A3) and (A5) in Appendix A.

B. Generator discontinuities

Both dissipative contributions $\mathbb{D}^{(1)}$ in (30) and $\mathbb{D}^{(2)}$ in (31) contain step functions that select which Bohr frequencies ω_ℓ do actually contribute depending on the ratio r. More precisely, the dissipative term $\mathbb{D}^{(1)}$ contains only the step function $\theta(\omega_\ell)$, so that when ω_ℓ becomes negative, the corresponding contribution disappears from $\mathbb{D}^{(1)}$. As regards $\mathbb{D}^{(2)}$, here both negative and positive frequencies contribute. Therefore, the kind of step discontinuity is different: if, by varying r, ω_ℓ becomes negative, then its contribution, initially of the form related to the smearing function $h_j^2(\omega_\ell)$, now corresponds to the smearing function $k_j^2(-\omega_\ell)$, after suitably extending them to the entire real axis. The opposite transition occurs when $-\omega_\ell$ turns from negative to positive. Thus, the generator $\mathbb{L}$ may also become discontinuous with respect to the ratio r not because of the disappearance of some dissipative contributions, but rather because, in general, $h_j(\omega_\ell) \neq k_j(-\omega_\ell)$.

In the following, we shall concentrate on an interaction Hamiltonian (15) without corotating terms, thus on a generator $\mathbb{L}$ consisting of terms as in (35). However, unlike in Ref. [22], we shall not assume the Bohr transition frequencies to be all positive. Rather, we shall study the asymptotic properties of the dissipative dynamics when, upon varying r, Bohr transition frequencies become negative and can thus only contribute to the Lamb-shift Hamiltonian. Indeed, as soon as r, for some ℓ, becomes larger than $2|\cos\frac{\ell\pi}{N+1}|$, ω_ℓ becomes negative and the generator shows a discontinuous change since the corresponding dissipative contribution $\mathbb{D}_\ell[\rho]$ does not appear any longer in $\mathbb{D}$ [see (34) and (35)]. In the following, we shall restrict to this scenario and leave for further investigations the consequences of the discontinuities due to corotating terms in the interaction Hamiltonian.

III. STATIONARY STATES AND TRANSPORT PROPERTIES

In order to inspect how the step-discontinuities affect the physics of the open spin chain, we shall concentrate on its asymptotic transport properties. To this purpose, in the following we shall first analytically characterize the structure of the stationary states and their dependence on the ratio r. Then, we shall investigate the tendency to equilibrium and, finally, the asymptotic properties of spin and heat flows.

A. Stationary states

In Appendix D, it is shown that the stationary states of the dynamics generated by (33) which satisfy $\mathbb{L}[\rho] = 0$ form a convex set with the following structure (for the underlying theory, see Refs. [83–88]). There also their properties are discussed.

If all frequencies ω_ℓ are positive so that all contribute to the dissipative part of the generator $\mathbb{L}$, there is a unique stationary state ρ^* of the form

$$\rho^* = \prod_{\ell=1}^{N} \rho^*(\omega_\ell), \tag{40}$$

where

$$\rho^*(\omega_\ell) = [\eta^*(\omega_\ell) + \zeta^*(\omega_\ell)]\, b_\ell^\dagger\, b_\ell + \eta^*(\omega_\ell)\, b_\ell\, b_\ell^\dagger \tag{41}$$

with

$$\eta^*(\omega_\ell) = \frac{C(\omega_\ell)}{C(\omega_\ell) + \widetilde{C}(\omega_\ell)}, \tag{42}$$

$$\zeta^*(\omega_\ell) = \frac{\widetilde{C}(\omega_\ell) - C(\omega_\ell)}{C(\omega_\ell) + \widetilde{C}(\omega_\ell)} \tag{43}$$

and

$$C(\omega_\ell) = 2\pi \sum_{j=1,N} h_j^2(\omega_\ell)\, u_{j\ell}^2\, [1 + n_j(\omega_\ell)], \tag{44}$$

$$\widetilde{C}(\omega_\ell) = 2\pi \sum_{j=1,N} h_j^2(\omega_\ell)\, u_{j\ell}^2\, n_j(\omega_\ell). \tag{45}$$

Remark 4. By comparing their spectra, the faithful state ρ^* in (40) can easily be seen to correspond to the stationary state computed in Ref. [22]. Furthermore, ρ^* is faithful, for it has 2^N nonvanishing eigenvalues obtained by all possible products of $\eta^*(\omega_\ell)$ and $\eta^*(\omega_\ell) + \zeta^*(\omega_\ell)$ with different ℓ [see Appendix D for more details, in particular Eqs. (D28) and (D29)]. Such a state remains stationary even when some Bohr transition frequencies become negative and thus do not contribute with their own dissipator $\mathbb{D}_\ell$ to the generator (33). Indeed, in such a case only the commutators with the diagonal Hamiltonians contribute, and certainly the states $\rho^*(\omega_\ell)$ in (40) make all these commutators vanish.

Instead, if the Bohr transition frequencies ω_ℓ are negative for $p+1 \leqslant \ell \leqslant N$, the stationary states make a convex manifold whose extreme points are given by

$$\rho^*_{\boldsymbol{\alpha}_p} = \left(\prod_{j=1}^{p} \rho^*(\omega_j)\right)\left(\prod_{\ell=p+1}^{N} P^{(\ell)}_{\alpha_\ell}\right), \tag{46}$$

where

$$P^{(\ell)}_{\alpha_\ell} := \begin{cases} b_\ell^\dagger\, b_\ell & \alpha_\ell = 0 \\ 1 - b_\ell^\dagger\, b_\ell = b_\ell\, b_\ell^\dagger & \alpha_\ell = 1 \end{cases}. \tag{47}$$

Different stationary asymptotic states are reached by properly choosing the initial states and letting them evolve for long times. In order to better figure out the variety of possibilities opened up by the presence of negative Bohr transition frequencies, let us consider the case when r is varied such that only the smallest one, ω_N, becomes negative. When $\omega_N > 0$, all initial states ρ tend in time to ρ^*; if r makes $\omega_N < 0$, there will be initial states still asymptotically evolving into ρ^*, now being characterized by the N-mode invariant state $\rho(-|\omega_N|)$ as in (D40), and other initial states which will instead reach stationary states of the form [see (D15), (D20), and (46)]

$$\rho_{\rm stat} = \left[\prod_{\ell=1}^{N-1} \rho^*(\omega_\ell)\right][\mu\, b_N^\dagger b_N + (1-\mu)\, b_N b_N^\dagger]. \tag{48}$$

As we shall see in the next section, in the presence of negative frequencies there can also be initial states which do not converge at all, but instead keep oscillating in time.

B. Dissipation-free sectors

When the Bohr transition frequencies ω_ℓ from $\ell = p+1$ to $\ell = N$ are negative, using (39) and (36), the reduced chain dynamics reads

$$\rho \mapsto \gamma_p(t)[\rho] = \left(\prod_{\ell=1}^{p} e^{t\mathbb{L}_\ell}\right) \circ \widetilde{\mathfrak{U}}_p(t)[\rho], \tag{49}$$

where

$$\widetilde{\mathfrak{U}}_p(t)[\rho] = e^{-it\widetilde{H}_p(\lambda)}\, \rho\, e^{it\widetilde{H}_p(\lambda)}, \tag{50}$$

with $\widetilde{H}_p(\lambda)$ as in (D9). Let us introduce the projectors

$$P_p \equiv \sum_{\boldsymbol{\alpha}_p} P_{\boldsymbol{\alpha}_p} \tag{51}$$

onto the subspaces $\mathcal{H}_p$ linearly spanned by the eigenvectors of the Hamiltonian (D9) [compare (C9) in Appendix C]. They read:

$$|\mathbf{n}_p\rangle = (b^\dagger_{p+1})^{n_{p+1}} \cdots (b^\dagger_N)^{n_N}\, |\text{vac}\rangle, \tag{52}$$

where $\mathbf{n}_p = (0, 0, \dots, 0, n_{p+1}, \dots, n_N)$.

Let ρ_p be a state supported by $\mathcal{H}_p$ so that

$$P_p\, \rho_p\, P_p = \rho_p. \tag{53}$$

For $0 \leqslant \ell \leqslant p$, the invariant states $\rho^*(\omega_\ell)$ in (40) satisfy $\rho^*(\omega_\ell)|\mathbf{n}_p\rangle = \eta^*_\ell\, |\mathbf{n}_p\rangle$. Therefore, $[\rho^*(\omega_\ell), P_p] = 0$ so that operators of the form

$$\rho = \left[\prod_{\ell=1}^{p} \rho^*(\omega_\ell)\right] \rho_p \tag{54}$$

are positive and normalized, hence *bona fide* states of the open quantum chain. Furthermore, from (49) it follows that any such initial state evolves in time according to

$$\gamma_p(t)[\rho] = \left[\prod_{\ell=1}^{p} \rho^*(\omega_\ell)\right] e^{-it\widetilde{H}_p(\lambda)}\, \rho_p\, e^{it\widetilde{H}_p(\lambda)}, \tag{55}$$

with $\widetilde{H}_p(\lambda)$ as in (D9). Therefore, on one hand, not all initial states do converge to a state in the stationary manifold. On the other hand, by varying the ratio r and thus changing the sign of the frequencies, one selects fermionic modes whose states are not affected by dissipation and maintain all coherences associated with them. It is important to emphasize that such a long-time coherent behavior is not obtained by engineering the environment, as in a dissipation-free subspace scenario, but rather by suitably tuning the interspin interaction relative to the external magnetic field.

C. Generator discontinuities and spin flow

In order to investigate the physical consequences of the dissipative generator discontinuities, we shall study the asymptotic spin flow currents with respect to the convex set generated by the stationary states $\rho^*_{\boldsymbol{\alpha}_p}$ in (46).

The spin flow at site $k = 1, 2, \dots, N$ along the spin chain corresponds to the rate of change in time of the average of $\sigma_z^{(k)}$ given by the quantity

$$\begin{aligned}\frac{d}{dt}\text{Tr}\big[\sigma_z^{(k)}\rho(t)\big] &= \text{Tr}\big[\sigma_z^{(k)}\mathbb{L}[\rho(t)]\big]\\ &= \text{Tr}\big[\mathbb{L}^{\text{dual}}\big[\sigma_z^{(k)}\big]\,\rho(t)\big].\end{aligned} \tag{56}$$

From (D4), the term with the commutator gives rise to the difference of two spin currents which corresponds to the current divergence in the continuous case. Instead, the dissipative contributions can only be interpreted as sink and source terms. Our aim is to study how their asymptotic behavior and that of the currents depend the stationary states. Therefore, we assume that $\omega_p \geqslant 0$ and $\omega_{p+1} < 0$, choose $\rho(t) = \rho^*_{\boldsymbol{\alpha}_p}$ in (56), and then consider generic elements in the stationary convex set whose extremal points are the states $\rho^*_{\boldsymbol{\alpha}_p}$. In this case, $\mathbb{L}^{\text{dual}} = \mathfrak{L}_p$ [see (D4)] and the left-hand side of (56) vanishes for $\mathfrak{L}_p[\rho^*_{\boldsymbol{\alpha}_p}] = 0$. However, the mean values of currents and sink and source terms behave quite differently: while the current stationary mean values both independently vanish, the sink and source terms asymptotic mean values instead do not, but compensate each other.

1. Hamiltonian spin currents

In order to clarify these different behaviors, by using (A2) and (A5) in Appendix A, we rewrite

$$\sigma_z^{(k)} = 2\sum_{p,q=1}^{N} u_{kp}\, u_{kq}\, b_p^\dagger\, b_q - \mathbb{I}, \tag{57}$$

and, using (36), we compute

$$i\big[H(\lambda),\, \sigma_z^{(k)}\big] = 2i\sum_{p\neq q=1}^{N} u_{kp}u_{kq}\, \Omega_\lambda^{(p)}\, (b_p^\dagger b_q + b_p b_q^\dagger). \tag{58}$$

It proves convenient to express the fermionic operators $b_p^\dagger$, b_q in terms of the fermionic operators $a_j^\dagger$, a_k introduced in (A3) of Appendix A. Then, using the unitarity of the diagonalizing symmetric orthogonal matrix $U = [u_{jk}]$, one writes the spin flow at site k as minus the difference of two currents,

$$i\big[H(\lambda),\, \sigma_z^{(k)}\big] = 2i\sum_{\substack{p=1\\p\neq k}}^{N} \big(\Omega_\lambda^U\big)_{pk} (a_p\, a_k^\dagger + a_p^\dagger\, a_k) \tag{59}$$

$$= -\big(J_\lambda^{(>k)} - J_\lambda^{(<k)}\big) \qquad \text{with} \tag{60}$$

$$J_\lambda^{(<k)} := 2i\sum_{p=1}^{k-1} \big(\Omega_\lambda^U\big)_{pk}\, (a_p\, a_k^\dagger + a_p^\dagger\, a_k) \tag{61}$$

$$J_\lambda^{(>k)} := 2i\sum_{p=k+1}^{N} \big(\Omega_\lambda^U\big)_{pk}\, (a_k\, a_p^\dagger + a_k^\dagger\, a_p), \tag{62}$$

where we set $\Omega_\lambda^U = U\, \text{diag}[\Omega_\lambda^{(\ell)}]\, U$, with $\Omega_\lambda^{(\ell)}$ the Bohr transition frequencies in (36).

Notice that by switching off the coupling with the baths and thus setting $\lambda = 0$, $\Omega_\lambda^{(\ell=0)} = \omega_\ell$; hence from (A7) in Appendix A and (7) one gets that the matrix entries of Ω_0^U are

$$\left(\Omega^U_{\lambda=0}\right)_{pk} = 2\Delta\,\delta_{pk} + 2g\,(\delta_{p,k+1} + \delta_{p,k-1}). \tag{63}$$

Therefore, with $\lambda = 0$ one retrieves the standard spin currents flowing through site k

$$J_0^{(<k)} = 4ig\,(a_{k-1}\,a_k^\dagger + a_{k-1}^\dagger\,a_k), \tag{64}$$

$$J_0^{(>k)} = 4ig\,(a_k\,a_{k+1}^\dagger + a_k^\dagger\,a_{k+1}). \tag{65}$$

Remark 5. Using (A2) in Appendix A to pass to the spin formalism, the previous currents become

$$J_0^{(<k)} = 4ig\,(\sigma_-^{(k-1)}\sigma_+^{(k)} - \sigma_+^{(k-1)}\sigma_-^{(k)}), \tag{66}$$

$$J_0^{(>k)} = 4ig\,(\sigma_+^{(k)}\sigma_-^{(k+1)} - \sigma_-^{(k)}\sigma_+^{(k+1)}). \tag{67}$$

Instead, the spin currents with $\lambda \neq 0$ read

$$J_\lambda^{(<k)} = 2i\sum_{p=1}^{k-1}\left(\Omega^U_\lambda\right)_{pk}\left[\sigma_-^{(p)}\prod_{\ell=p+1}^{k-1}\left(-\sigma_z^{(\ell)}\right)\sigma_+^{(k)} - \sigma_+^{(p)}\prod_{\ell=p+1}^{k-1}\left(-\sigma_z^{(\ell)}\right)\sigma_-^{(k)}\right], \tag{68}$$

$$J_\lambda^{(>k)} = 2i\sum_{p=1}^{k-1}\left(\Omega^U_\lambda\right)_{pk}\left[\sigma_-^{(k)}\prod_{\ell=p+1}^{k-1}\left(-\sigma_z^{(\ell)}\right)\sigma_-^{(p)} - \sigma_+^{(k)}\prod_{\ell=p+1}^{k-1}\left(-\sigma_z^{(\ell)}\right)\sigma_+^{(p)}\right]. \tag{69}$$

Thus, unlike for closed spin chains, the spin currents through site k exhibit non-nearest-neighbor contributions due to the spin-spin interactions mediated by the baths: these effects indeed come from the Lamb-shift correction to the chain Hamiltonian and disappear if $\lambda = 0$.

Because of the simple product form of the stationary states $\rho^*_{\boldsymbol{\alpha}_p}$ in (46), in order to evaluate the asymptotic mean values of the currents, we reinsert the fermionic operators b_k, $b_k^\dagger$ into (61) and (62). The expressions of the currents then become

$$J_\lambda^{(<k)} = 2i\sum_{p=1}^{k-1}\sum_{j,\ell=1}^{N} u_{jp}u_{\ell k}\left(\Omega^U_\lambda\right)_{pk}(b_j\,b_\ell^\dagger + b_j^\dagger\,b_\ell), \tag{70}$$

$$J_\lambda^{(>k)} = 2i\sum_{p=k+1}^{N}\sum_{j,\ell=1}^{N} u_{jp}u_{\ell k}\left(\Omega^U_\lambda\right)_{pk}(b_j^\dagger\,b_\ell + b_j\,b_\ell^\dagger). \tag{71}$$

Since $\mathrm{Tr}[\rho^*_{\boldsymbol{\alpha}_p}(b_j\,b_\ell^\dagger + b_j^\dagger\,b_\ell)]$ does not vanish if and only if $j = \ell$ and $\sum_{j=1}^N u_{jp}u_{jk} = \delta_{pk}$, both currents have vanishing mean values with respect to the extremal stationary states $\rho^*_{\boldsymbol{\alpha}_p}$ and thus, because of (D20), with respect to any stationary state.

2. *Sink and source terms*

We now show that, unlike the currents, the sink and source terms associated with the site k do not asymptotically vanish in any of the stationary states $\rho^*_{\boldsymbol{\alpha}_p}$ associated with $\omega_\ell < 0$ from $\ell = p+1$ to $\ell = N$.

Using the fermionic commutation relations and (16), insertion of (57) into (38) yields sink and source terms at site k of the form

$$\mathfrak{S}_{j,k}\left(\{\omega_\ell\}_{\ell=1}^p\right) := \sum_{\ell=1}^p \mathbb{D}^{\mathrm{dual}}_{j\ell}\left[\sigma_z^{(k)}\right] = 2\pi\sum_{\ell=1}^p u_{j\ell}^2 u_{k\ell} h_j^2(\omega_\ell) \times \sum_{q=1}^N u_{kq}\{n_j(\omega_\ell)(b_q b_\ell^\dagger + b_\ell b_q^\dagger) \tag{72}$$

$$- [1 + n_j(\omega_\ell)]\,(b_\ell^\dagger b_q + b_q^\dagger b_\ell)\}. \tag{73}$$

Due to the quadratic structure in b_ℓ and $b_\ell^\dagger$ of all single-mode states contributing to $\rho^*_{\boldsymbol{\alpha}_p}$, when computing the mean values of the above observables with respect to such states, one is forced to set $q = \ell$. Therefore, only thermal expectations with respect to $\rho^*(\omega_\ell)$ corresponding to positive Bohr transition frequencies are involved. Remarkably, the dependence on the multiindices $\boldsymbol{\alpha}_p$ disappears; consequently, all stationary states in the convex set spanned by the extreme points $\rho^*_{\boldsymbol{\alpha}_p}$—for instance, the faithful stationary state ρ^* itself—give the same mean values to sink and source terms. Indeed, using that the symmetric real coefficients in (A6) of Appendix A are such that $u_{1\ell}^2 = u_{N\ell}^2$, from (40), (44), and (45), one computes

$$\mathfrak{Q}^*_{1,k}\left(\{\omega_\ell\}_{\ell=1}^p\right) := \mathrm{Tr}\left[\rho^*_{\boldsymbol{\alpha}_p}\,\mathfrak{S}_{1,k}\left(\{\omega_\ell\}_{\ell=1}^p\right)\right] = 4\pi\sum_{\ell=1}^p u_{k\ell}^2\,u_{1\ell}^2\,h_1^2(\omega_\ell)\,h_N^2\,(\omega_\ell) \times \frac{n_1(\omega_\ell) - n_N(\omega_\ell)}{\mathcal{N}_\ell}, \tag{74}$$

$$\mathfrak{Q}^*_{N,k}\left(\{\omega_\ell\}_{\ell=1}^p\right) := \mathrm{Tr}\left[\rho^*_{\boldsymbol{\alpha}_p}\,\mathfrak{S}_{N,k}\left(\{\omega_\ell\}_{\ell=1}^p\right)\right] = 4\pi\sum_{\ell=1}^p u_{k\ell}^2\,u_{N\ell}^2\,h_1^2(\omega_\ell)\,h_N^2(\omega_\ell) \times \frac{n_N(\omega_\ell) - n_1(\omega_\ell)}{\mathcal{N}_\ell}, \tag{75}$$

where

$$\mathcal{N}(\omega_\ell) \equiv h_1^2(\omega_\ell)[1 + 2n_1(\omega_\ell)] + h_N^2(\omega_\ell)[1 + 2n_N(\omega_\ell)]. \tag{76}$$

The above two contributions, $\mathfrak{Q}^*_{1,k}(\{\omega_\ell\}_{\ell=1}^p)$ and $\mathfrak{Q}^*_{N,k}(\{\omega_\ell\}_{\ell=1}^p)$, are the opposite of one another, as they should be in a stationary state; which of them is a sink and which a source depends on the bath temperatures and on the overall sign of the two right-hand sides of the above expressions. If all frequencies are positive and thus the sum goes from $\ell = 1$ to N, they reduce to those found in Ref. [22].

The discontinuities of the generator affect the behavior of sinks and sources as functions of the chain parameter r. Indeed, suppose one starts with positive Bohr transition frequencies $\omega_1, \ldots, \omega_p$ and varies r, turning ω_p from positive to negative while keeping $\omega_\ell > 0$ from $\ell = 1$ to $\ell = p-1$. When $\omega_p = 0$, if the bath temperatures $T_1 \neq T_N$, the difference $n_1(\omega_p) - n_N(\omega_p)$ and the coefficient $\mathcal{N}_p$ diverge as ω_p^{-1}. Therefore, by considering continuous smearing functions, $\mathfrak{S}^{(k)}_{N,p}$ depends continuously on the change from $\omega_p > 0$ to $\omega_p < 0$.

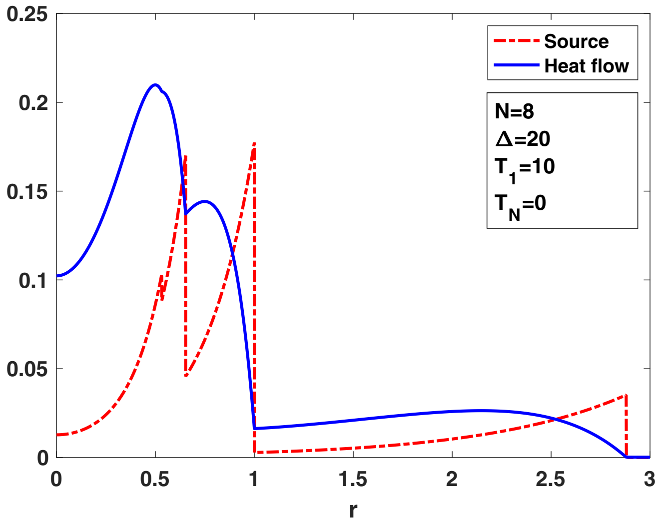

FIG. 1. $N = 8$ open spin chain with $\Delta = 20$, $T_1 = 10$, and $T_N = 0$: spin flow source term (dashed-dotted curve) through site $k = 2$ and heat flow (solid curve) as functions of r in arbitrary units; jumps (source term in the spin flow) and first derivatives discontinuities (heat flow) appear for values of r at which ω_8, ω_7, ω_6, and ω_5 progressively become negative.

According to Secs. III A and III C, the mean values of sink and source terms at a fixed number of negative Bohr transition frequencies do not depend on the stationary state with respect to which they are computed; however, they are expected to behave discontinuously when the number of negative Bohr transition frequencies changes. This behavior is shown by the dashed-dotted curve in Fig. 1, which plots the sink term at site $k = 2$ as a function of the ratio r for fixed magnetic field $\Delta = 20$, left bath temperature $T_1 = 10$, and right bath temperature $T_N = 0$: jumps appear for values of r at which the four Bohr transition frequencies $\omega_8 \leqslant \omega_7 \leqslant \omega_6 \leqslant \omega_5$ change from positive to negative. The underlying analytical features of such a behavior are illustrated by means of the following example.

Example 1. For $N = 3$ one has the following Bohr transition frequencies:

$$\omega_1 = 2\Delta(1 + r\sqrt{2}) \geqslant \omega_2 = 2\,\Delta \geqslant \omega_3 = 2\Delta(1 - \sqrt{2}r)$$

so that $\omega_{1,2} > 0$, and $\omega_3 \leqslant 0$ when $\frac{1}{\sqrt{2}} \leqslant r < +\infty$. If $r \leqslant \frac{1}{\sqrt{2}}$, $\omega_1 \geqslant \omega_2 \geqslant \omega_3 \geqslant 0$ and there exists a unique stationary state of the form

$$\rho^* = \rho^*(\omega_1)\,\rho^*(\omega_2)\,\rho^*(\omega_3), \tag{77}$$

with $\rho^*(\omega_\ell)$ as given in (41).

Instead, when $r \geqslant \frac{1}{\sqrt{2}}$ there appear two extremal stationary states,

$$\rho_0^* = \rho^*(\omega_1)\,\rho^*(\omega_2)\,b_3^\dagger b_3,\ \ \rho_1^* = \rho^*(\omega_1)\,\rho^*(\omega_2)\,b_3 b_3^\dagger. \tag{78}$$

Notice that, independently of the sign of ω_3,

$$\rho^* = \frac{\widetilde{C}(\omega_3)}{C(\omega_3) + \widetilde{C}(\omega_3)}\,\rho_0^* + \frac{C(\omega_3)}{C(\omega_3) + \widetilde{C}(\omega_3)}\,\rho_1^* \tag{79}$$

is a faithful stationary state, the main difference being that when $\omega_3 \geqslant 0$, $\rho_{0,1}^*$ are no longer stationary. When $r \geqslant 1/\sqrt{2}$ so that $\omega_3 \geqslant 0$, the sink-source asymptotic mean values computed with respect to ρ^* in (77) are

$$\mathfrak{Q}^*_{1,k}\big(\{\omega_\ell\}_{\ell=1}^3\big) = \sum_{\ell=1}^2 \lambda_k(\omega_\ell)\,\frac{n_1(\omega_\ell) - n_N(\omega_\ell)}{\mathcal{N}(\omega_\ell)} + \lambda_k(\omega_3)\,\frac{n_1(\omega_3) - n_N(\omega_3)}{\mathcal{N}(\omega_3)} \tag{80}$$

where $\lambda_k(\omega_\ell) \equiv 4\pi\, u_{k\ell}^2\, u_{1\ell}^2\, h_1^2(\omega_\ell)\, h_N^2(\omega_\ell)$. If $\omega_3 \to 0^+$, one gets

$$\lim_{\omega_3\to 0^+} \mathfrak{Q}^*_{1,k}\big(\{\omega_\ell\}_{\ell=1}^3\big) = \sum_{\ell=1}^2 \lambda_{k\ell}\,(\omega_\ell)\,\frac{n_1(\omega_\ell) - n_N(\omega_\ell)}{\mathcal{N}(\omega_\ell)} + \pi\,\nu\left(\frac{1}{\beta_1} - \frac{1}{\beta_N}\right), \tag{81}$$

where

$$\nu \equiv \frac{2\pi\, u_{k3}^2\, u_{13}^2\, h_1^2(0)\, h_N^2(0)}{\frac{h_1^2(0)}{\beta_1} + \frac{h_N^2(0)}{\beta_N}}. \tag{82}$$

When $\omega_3 < 0$, because of (78), all the asymptotic stationary sink-source terms do not depend on ω_3 and all equal $\mathfrak{Q}^*_{1,k}(\{\omega_\ell\}_{\ell=1}^2)$. Thus, if the smearing functions $h_{1,N}(0) \neq 0$, they exhibit a same jump at $\omega_3 = 0$:

$$\lim_{\omega_3\to 0^+} \mathfrak{Q}^*_{1,k}\big(\{\omega_\ell\}_{\ell=1}^3\big) - \lim_{\omega_3\to 0^-} \mathfrak{Q}^*_{1,k}\big(\{\omega_\ell\}_{\ell=1}^3\big) = 2\pi\,\nu\left(\frac{1}{\beta_1} - \frac{1}{\beta_N}\right). \tag{83}$$

D. Generator discontinuities and heat flow

The coupling of the chain end spins to the two baths makes heat coming in and out of the chain, depending on the temperature difference. According to quantum thermodynamic wisdom [41,42], the heat flow associated with the coupling of the chain to a given bath is due to the variation of the chain state due to that coupling:

$$\mathfrak{H}(t) := \mathrm{Tr}\bigg(\frac{d\rho(t)}{dt}\,H\bigg) = \mathrm{Tr}(\mathbb{L}[\rho(t)]\,H). \tag{84}$$

Because of the form (D2) of the Hamiltonian $H(\lambda)$ in the generator, only the dissipative part of the generator contributes to the heat flow:

$$\mathfrak{H}(t) := \mathrm{Tr}(\mathbb{D}[\rho(t)]\,H) = \mathrm{Tr}(\rho(t)\,\mathbb{D}^{\mathrm{dual}}[H]). \tag{85}$$

With respect to the invariant states in (46), the stationary heat flows are computed to be

$$\mathfrak{H}_1^*\big(\{\omega_\ell\}_{\ell=1}^p\big) \equiv \mathrm{Tr}\Bigg(\rho^*_{\boldsymbol{\alpha}_p}\sum_{\ell=1}^p \mathbb{D}_{1\ell}^{\mathrm{dual}}[H]\Bigg) = -\mathfrak{H}_N^*\big(\{\omega_\ell\}_{\ell=1}^0\big) = \sum_{\ell=1}^p \lambda(\omega_\ell)\,\omega_\ell\,\frac{n_1(\omega_\ell) - n_N(\omega_\ell)}{\mathcal{N}(\omega_\ell)}, \tag{86}$$

where $\lambda(\omega_\ell) \equiv 4\pi\, u_{1\ell}^2\, u_{N\ell}^2\, h_1^2(\omega_\ell)\, h_N^2(\omega_\ell)$ and $\mathcal{N}(\omega_\ell)$ is as in (76). The two heat flows need not separately vanish, but compensate each other at stationarity, and their sign, if positive, corresponds to heat flowing into the chain from the bath, or to heat flowing out of the chain and into the bath.

Like for the spin flow, if all frequencies are positive and thus the sums in (86) go from $\ell = 1$ to N, the heat flow reduces to the one found in Ref. [22]. Also, in the presence of negative Bohr transition frequencies, the asymptotic heat flow is the same for all states in the stationary manifold. However, unlike the spin flow source terms in (74) and (75), the heat flow does not depend on the chain sites; indeed, it is associated with the first and last spins being coupled to the baths. Moreover, it depends differently on the Bohr transition frequencies of the contributing modes and vanishes when $\omega_\ell \to 0$. Therefore, contrary to sink and source terms, the heat flow does not exhibit jumps when Bohr transition frequencies change sign. Indeed, in the setting of Example 1, one has

$$\lim_{\omega_3\to 0^+} \mathfrak{H}_1^*\big(\{\omega_\ell\}_{\ell=1}^3\big) = \sum_{\ell=1}^{2} \lambda(\omega_\ell)\,(\omega_\ell)\,\frac{n_1(\omega_\ell) - n_N(\omega_\ell)}{\mathcal{N}(\omega_\ell)} + \lim_{\omega_3\to 0^+} \lambda(\omega_3)\,\omega_3\left(\frac{1}{\beta_1} - \frac{1}{\beta_N}\right), \quad (87)$$

and the last limit vanishes for suitably continuous smearing functions. Such an asymptotic behavior of the heat flow as a function of the ratio r is manifest in the solid curve in Fig. 1, which shows that the heat flow exhibits discontinuities in its first derivatives with respect to r.

IV. CONCLUSIONS

We studied an arbitrarily long XX open quantum spin chain with the end spins weakly coupled to two Bosonic thermal baths by means of both energy-preserving (counterrotating) interaction terms and energy-nonpreserving (corotating) terms. Therefore, we did not assume *a priori* given dissipative boundaries, but rather derived the irreversible chain dynamics from specific microscopic couplings by means of standard weak-coupling techniques. These latter have been applied within the so-called global approach that uses the full interspin interactions. They provide a generator of the spin chain dissipative reduced dynamics that decomposes into a sum of independent generators, one for each fermionic mode and quadratic in the respective creation and annihilation operators.

Then we focused on the case where only energy-preserving (counterrotating) spin-Boson interactions are present. In such a case, since by varying the ratio $r = g/\Delta$ between the spin coupling strength g and the magnitude Δ of the external magnetic field some of the chain Bohr transition frequencies can be made negative, the associated contributions to the dissipative generator become purely unitary, thus contributing reversibly to the reduced dynamics. This phenomenon is due to the fact that the presence of the baths can give rise to dissipation only when there are resonances between the chain Bohr transition frequencies and the bath energies. Since in the case of harmonic baths in thermal equilibrium, their energies are positive, there cannot be such resonances. The associated fermionic degrees of freedom then become dissipation free; however, unlike within the decoherence-free subspace scenario, such coherent behaviors emerge not through a manipulation of the environment but rather because of a tuning of the chain physical parameters.

The discontinuities of the generator manifest themselves in the structure of the stationary states: there is a unique stationary state when all Bohr transition frequencies are positive, while, at each change of sign of one of them, the dimension of the convex set of stationary states increases by a power of 2 because each extremal state bifurcates into two new ones.

Moreover, the discontinuities in the generator reveal themselves in the peculiar behavior of asymptotic spin and heat flows. Indeed, the spin flow exhibits jumps at those values of the ratio r at which Bohr transition frequencies change signs, while the heat flow is continuous there with discontinuous first derivatives with respect to r. These two different behaviors could be experimentally investigated in any experimental setting that could implement the XX-spin chain coupled to the heat baths at different temperatures.

All discontinuities disappear if one approaches the transport properties of open quantum spin chain from the so-called local point of view that derives the master equation by neglecting the interspin couplings. Therefore, an experimental check of the emergence or not at long times of jumps (cusps) in the spin (heat) flow might provide access to the regions of applicability of the two approaches in terms of the chain physical parameters and the bath temperatures and thus possibly discriminate between them.

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APPENDIX A: DIAGONALIZATION OF THE SPIN CHAIN HAMILTONIAN

Using the spin ladder operators in (5), the Hamiltonian in (1) reads

$$H = \Delta \sum_{\ell=1}^{N} \sigma_z^{(\ell)} + 2g \sum_{\ell=1}^{N-1} (\sigma_+^{(\ell)}\sigma_-^{(\ell+1)} + \sigma_-^{(\ell)}\sigma_+^{(\ell+1)}). \quad (A1)$$

By means of the Jordan-Wigner transformation [89], one introduces the fermionic annihilation and creation operators a_j, $a_j^\dagger$ such that

$$a_j = \left[\prod_{k=1}^{j-1}\big(-\sigma_z^{(k)}\big)\right]\sigma_-^{(j)}, \quad (A2)$$

$$\sigma_-^{(j)} = \prod_{k=1}^{j-1}(-1)^{a_k^\dagger a_k}\, a_j,\ \ \sigma_z^{(j)} = 2\,a_j^\dagger\, a_j - 1. \quad (A3)$$

Then, the spin Hamiltonian (A1) becomes a fermionic one, $H = -N\,\Delta + \widetilde{H}$, where

$$\widetilde{H} = 2\Delta \sum_{j=1}^{N} a_j^\dagger a_j + 2g \sum_{j=1}^{N-1} (a_j^\dagger\, a_{j+1} + a_{j+1}^\dagger a_j). \quad (A4)$$

As shown in Ref. [22], H can then be set in diagonal form as in (7), by means of the fermionic operators

$$b_\ell := \sum_{j=1}^{N} u_{\ell j}\, a_j, \quad a_j = \sum_{\ell=1}^{N} u_{\ell j}\, b_\ell, \quad (A5)$$

with coefficients

$$u_{\ell k} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\ell k \pi}{N+1}\right), \tag{A6}$$

forming an orthogonal and symmetric matrix $U = [u_{k\ell}]$ which is such that

$$\sum_{\ell=1}^{N} u_{i\ell} u_{j\ell}\, 2\cos\left(\frac{\pi \ell}{N+1}\right) = \delta_{i,j-1} + \delta_{i,j+1}. \tag{A7}$$

It then follows from (A3) that

$$\sigma_-^{(1)} = \sum_{\ell=1}^{N} u_{1\ell}\, b_\ell, \tag{A8}$$

while, since $(-1)^{a_k^\dagger a_k}\, a_k = (1 - 2a_k^\dagger\, a_k) a_k = a_k$,

$$\sigma_-^{(N)} = (-1)^{\widehat{N}}\, a_N = (-1)^{\widehat{N}} \sum_{\ell=1}^{N} u_{N\ell}\, b_\ell, \tag{A9}$$

where $\widehat{N}$ is the number operator,

$$\widehat{N} = \sum_{\ell=1}^{N} a_\ell^\dagger a_\ell = \sum_{\ell=1}^{N} b_\ell^\dagger b_\ell = \bigotimes_{j=1}^{N} \frac{\mathbb{I}^{(j)} + \sigma_z^{(j)}}{2}. \tag{A10}$$

APPENDIX B: DERIVATION OF THE OPEN SPIN CHAIN MASTER EQUATION

Given the Hamiltonian $H_{\rm tot} = H + H_{\rm env} + H_{\rm int}$, where H is the chain Hamiltonian in (7), $H_{\rm env} = H_{\rm env}^{(1)} + H_{\rm env}^{(N)}$ the environment Hamiltonian with $H_{\rm env}^{(1,N)}$ as in (3), and $H_{\rm int}$ in (15) the chain-baths interaction, a reduced dynamics for the spin chain alone which is free from physical inconsistencies is obtained via the so-called Davies prescription [33]. Roughly speaking [37], it amounts to performing an ergodic average on top of the second-order (in λ) and Markovian approximations. It yields a dissipative dynamics consisting of a one-parameter semigroup of completely positive maps $\gamma_t = \exp(t\mathbb{L})$. The λ^2 correction

$$\widetilde{\mathbb{L}}[\rho] = -i\lambda^2 \,[H_{\rm LS}, \rho] + \lambda^2 \mathbb{D}[\rho] \tag{B1}$$

in the generator $\mathbb{L}$ (24) is obtained by means of the following ergodic average:

$$\begin{aligned}
\widetilde{\mathbb{L}}[\rho] = &-\lim_{T\to+\infty} \frac{1}{T} \int_0^T d\tau \int_0^{+\infty} dt\, e^{i\tau H} \\
&\times \mathrm{Tr}_{\rm env}\left([H_{\rm int}(t,0), [H_{\rm int},\, e^{-i\tau H}\, \rho e^{i\tau H} \otimes \rho_{\rm env}]]\right) e^{-i\tau H} \\
= &-\lim_{T\to+\infty} \frac{1}{T} \int_0^T d\tau \int_0^{+\infty} dt \\
&\times \mathrm{Tr}_{\rm env}\left([H_{\rm int}(t,\tau), [H_{\rm int}(0,\tau),\, \rho \otimes \rho_{\rm env}]]\right),
\end{aligned} \tag{B2}$$

where $\rho_{\rm env}$ is an environment equilibrium state as in (20) and

$$\begin{aligned}
&H_{\rm int}(t,\tau) \\
&= e^{i(t+\tau)H} \otimes e^{itH_{\rm env}}\, H_{\rm int}\, e^{-i(t+\tau)H} \otimes e^{-itH_{\rm env}} \\
&= \lambda \sum_{j=1,N} \sum_{\ell=1}^{N} u_{j\ell} [b_{j\ell}^\dagger(t+\tau)\, \mathfrak{C}_j(t) + b_{j\ell}(t+\tau)\, \mathfrak{C}_j^\dagger(t)].
\end{aligned} \tag{B3}$$

In the above expression, using $[H, \hat{N}] = 0$, (17) and (16), one sets

$$b_{1\ell}(t+\tau) = e^{-i(t+\tau)\omega_\ell}\, b_{1\ell}, \tag{B4}$$

$$b_{N\ell}(t+\tau) = e^{i(t+\tau)\omega_\ell}\, b_{N\ell}, \tag{B5}$$

and, by means of (18) and (6),

$$\mathfrak{C}_j(t) = \int_0^\infty d\nu\, [h_j(\nu)\, e^{-it\nu}\, \mathfrak{a}_j(\nu) + k_j(\nu)\, e^{it\nu}\, \mathfrak{a}_j^\dagger(\nu)]. \tag{B6}$$

Inserting $H_{\rm int}(t,\tau)$ and $H_{\rm int}(t,0)$ into the double commutator in (B2) yields the appearance of oscillating terms of the form

$$e^{\pm i\tau(\omega_\ell + \omega_{\ell'})}, \quad e^{\pm i\tau(\omega_\ell - \omega_{\ell'})} \tag{B7}$$

and

$$e^{\pm it(\omega_\ell + \nu)}, \quad e^{\pm it(\omega_\ell - \nu)}. \tag{B8}$$

The ergodic averages of the oscillating terms (B7) do not vanish only if $\omega_\ell + \omega_{\ell'} = 0$ and $\omega_\ell - \omega_{\ell'} = 0$. Using the explicit expressions (7) of the Bohr transition frequencies, while the second condition forces $\ell = \ell'$, the second one implies the transcendental equation

$$\cos\left(\frac{\pi \ell'}{N+1}\right) - \cos\left(\frac{\pi \ell}{N+1}\right) = \frac{\Delta}{g} = \frac{1}{r}. \tag{B9}$$

Such a relation is satisfied by $1 \leqslant \ell \neq \ell' \leqslant N$, only for specific values of the ratio r. Furthermore, the integration over t in (B2) involves the oscillating terms (B8) and yields the distributions

$$\int_0^{+\infty} dt\, e^{\pm it(\omega_\ell + \nu)} = \pi\delta(\nu + \omega_\ell) \pm iP\frac{1}{\nu + \omega}, \tag{B10}$$

$$\int_0^{+\infty} dt\, e^{\pm it(\omega_\ell - \nu)} = \pi\delta(\nu - \omega_\ell) \mp iP\frac{1}{\nu - \omega}. \tag{B11}$$

Using the correlation functions (21) and (22), one of the two integrations over the bath energies ν and ν' coming from the double commutator can be eliminated. The remaining integration, over ν, say, can be handled by means of the Dirac deltas in (B10) and (B11). Notice that, because of the positivity of the spectrum of the environment Hamiltonian, $\delta(\nu + \omega_\ell)$ selects negative Bohr transition frequencies and $\delta(\nu - \omega_\ell)$ nonnegative ones.

Instead, the principal value terms in (B10) and (B11) do not discriminate between positive and negative Bohr transition frequencies and give rise to Lamb shift corrections to the chain Hamiltonian. Collecting all these contributions, straightforward calculations finally yield the expressions (27),(28), (30), and (31).

APPENDIX C: COMPARISON WITH REF. [22]

In Ref. [22], the generator $\mathbb{L}$ in (24) is expressed in terms of Lindblad operators $A_j(\omega)$, $A_j^\dagger(\omega)$, $j=1,N$, [see expressions (C15) and (C16) below] such that the dissipative contribution reads

$$\mathbb{D}[\rho(t)] = \lambda^2 \sum_{j=1,N} \sum_{\ell:\omega_\ell \geqslant 0} \mathbb{D}^{(j)}_{\omega_\ell}[\rho], \tag{C1}$$

where

$$\mathbb{D}^{(j)}_{\omega_\ell}[\rho] = C^{(j)}_{\omega_\ell}\big[A_j(\omega_\ell)\rho A_j^\dagger(\omega_\ell) - \tfrac{1}{2}\{A_j^\dagger(\omega_\ell)A_j(\omega_\ell),\rho\}\big] \tag{C2}$$

$$+\widetilde{C}^{(j)}_{\omega_\ell}\big[A_j^\dagger(\omega_\ell)\rho A_j(\omega_\ell) - \tfrac{1}{2}\{A_j(\omega_\ell)A_j^\dagger(\omega_\ell),\rho\}\big], \tag{C3}$$

with coefficients

$$C^{(j)}_{\omega_\ell} = 2\pi\,[h_j(\omega_\ell)]^2\,[n_j(\omega_\ell)+1], \tag{C4}$$

$$\widetilde{C}^{(j)}_{\omega_\ell} = 2\pi\,[h_j(\omega_\ell)]^2\,n_j(\omega_\ell). \tag{C5}$$

Instead, the Lamb-shift correction reads

$$H_{\rm LS} = \sum_{j=1,N}\sum_{\ell=1}^{N}\big[S^{(j)}_{\omega}A_j^\dagger(\omega_\ell)A_j(\omega_\ell) + \widetilde{S}^{(j)}_{\omega_\ell}A_j(\omega_\ell)A_j^\dagger(\omega_\ell)\big], \tag{C6}$$

with coefficients

$$S^{(j)}_{\omega_\ell} = P\int_0^{+\infty} d\nu\,[h_j(\nu)]^2\frac{1+n_j(\nu)}{\omega_\ell-\nu}, \tag{C7}$$

$$\widetilde{S}^{(j)}_{\omega_\ell} = P\int_0^{+\infty} d\nu\,[h_j(\nu)]^2\frac{n_j(\nu)}{\nu-\omega_\ell}. \tag{C8}$$

Let $\mathbf{n}$ denote the N-tuple $n_1,n_2,\dots,n_N$, where $n_j=0,1$ is the occupation number of the jth mode relative to the operators b_j and $b_j^\dagger$. The eigenvectors of the Hamiltonian (1) are then given by

$$|\mathbf{n}\rangle = (b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}\cdots(b_N^\dagger)^{n_N}\,|\mathrm{vac}\rangle, \tag{C9}$$

where the vacuum $|\mathrm{vac}\rangle$ such that $b_\ell|\mathrm{vac}\rangle=0$ for all $1\leqslant\ell\leqslant N$ is the N spin tensor product vector $|\downarrow\rangle^{\otimes N}$. Notice that

$$b_\ell|\mathbf{n}\rangle = \delta_{n_\ell,1}\,(-1)^{\sum_{j=1}^{\ell-1}n_j}\sqrt{n_\ell}\,|\mathbf{n}_\ell^-\rangle, \tag{C10}$$

$$b_\ell^\dagger|\mathbf{n}\rangle = \delta_{n_\ell,0}\,(-1)^{\sum_{j=1}^{\ell-1}n_j}\sqrt{1-n_\ell}\,|\mathbf{n}_\ell^+\rangle, \tag{C11}$$

$$b_\ell^\dagger\,b_\ell|\mathbf{n}\rangle = n_\ell\,|\mathbf{n}\rangle, \tag{C12}$$

where, $\mathbf{n}_\ell^\pm$ denote the N-tuples $n_1,\dots,n_\ell\pm1,\dots,n_N$. Then, one verifies that $H\,|\mathbf{n}\rangle = E_\mathbf{n}\,|\mathbf{n}\rangle$, where

$$E_\mathbf{n} = \Delta\left(2\sum_{\ell=1}^N n_\ell - N\right) + 4g\sum_{\ell=1}^N n_\ell\,\cos\left(\frac{\ell\pi}{N+1}\right). \tag{C13}$$

Furthermore, let $\mathbf{n}_{0_\ell}$ and $\mathbf{n}_{1_\ell}$ denote the N-tuples with fixed digits $n_\ell=0,1$, respectively, at site ℓ. Then, the spin chain Bohr transition frequencies result of the form

$$\omega_\ell = E_{\mathbf{n}_{1_\ell}} - E_{\mathbf{n}_{0_\ell}} = 2\,\Delta + 4\,g\,\cos\left(\frac{\ell\pi}{N+1}\right), \tag{C14}$$

while the Lindblad operators can be expressed as

$$A_1(\omega_\ell) = u_{1\ell}\sum_{\widehat{\mathbf{n}}_\ell}(-1)^{\sum_{j=1}^{\ell-1}n_j}\,|\mathbf{n}_{0_\ell}\rangle\langle\mathbf{n}_{1_\ell}|, \tag{C15}$$

$$A_N(\omega_\ell) = u_{N\ell}\sum_{\widehat{\mathbf{n}}_\ell}(-1)^{\sum_{j=\ell+1}^{N}n_j}\,|\mathbf{n}_{0_\ell}\rangle\langle\mathbf{n}_{1_\ell}|. \tag{C16}$$

Using (C10) and (C11), one then readily shows that the matrix elements of the Lindblad operators with respect to the energy eigenbasis, $\langle\mathbf{m}|A_1(\omega_\ell)|\mathbf{n}\rangle$ and $\langle\mathbf{m}|A_N(\omega_\ell)|\mathbf{n}\rangle$, coincide with those of $u_{1\ell}\,b_{1\ell}$ and $-u_{N\ell}\,b_{N\ell}$, respectively, with $b_{j\ell}$ defined in (16). Finally, inserting $A_1(\omega_\ell)=u_{1\ell}b_{1\ell}$ and $A_N(\omega_\ell) = -u_{N\ell}b_{N\ell}$ into (36) and (38), one recovers the expressions (C2), (C3), and (C6).

APPENDIX D: STATIONARY STATES

The stationary states of the dynamics generated by (33) satisfy $\mathbb{L}[\rho]=0$ [83–88]. According to Refs. [84] and [85], in order to characterize them, one looks for two particular sets of chain operators. The first one, $\mathcal{C}^{(p)}_\mathbb{L}$, consists of all operators X that commute with all fermionic Lindblad operators b_ℓ, $b_\ell^\dagger$ appearing in the dissipators (38). Such a set is an algebra called the *commutant* of the set $\{b_\ell, b_\ell^\dagger : \ell=1,\dots,p\}$. The index p in $\mathcal{C}^{(p)}_\mathbb{L}$ signals the fact that, because of the ratio r, some Bohr transition frequencies could be negative, say, from ω_{p+1} to ω_N. In this case, the corresponding mode operators do not show up in the dissipator $\mathbb{D}$ in (32), namely, $\mathbb{D}$ reduces to the sum $\mathfrak{D}_p\equiv\sum_{\ell=1}^p\mathbb{D}_\ell$; on the contrary, the Hamiltonian does not lose any term, so that the whole generator $\mathbb{L}$ becomes

$$\mathfrak{L}_p[\rho] \equiv -i[H(\lambda),\ \rho] + \mathfrak{D}_p[\rho], \tag{D1}$$

where

$$H(\lambda)\equiv\sum_{\ell=1}^N H^{(\ell)}_\lambda, \tag{D2}$$

with $H^{(\ell)}_\lambda$ as in (36). Notice that $\mathfrak{L}_N=\mathbb{L}$. The second set of chain operators, $\mathcal{M}^{(p)}_\mathbb{L}$, used to inspect the structure of the manifold of steady state consists of all operators in $\mathcal{C}^{(p)}_\mathbb{L}$ that also commute with the number operators $b_\ell^\dagger\,b_\ell$, $\ell=p+1,\dots,N$. $\mathcal{M}^{(p)}_\mathbb{L}\subseteq\mathcal{C}^{(p)}_\mathbb{L}$. The reason for looking at these two algebras of operators can be best appreciated by considering the dual (Heisenberg) dynamics $\gamma_t^{\rm dual} = \exp\left(t\,\mathbb{L}^{\rm dual}\right)$ associated with the generator $\mathbb{L}^{\rm dual}$. In the present context where the generator is $\mathfrak{L}_p$ in (D1), its dual is defined by

$$\mathrm{Tr}(\mathfrak{L}_p[\rho]\,X) = \mathrm{Tr}\big(\rho\,\mathfrak{L}_p^{\rm dual}[X]\big), \tag{D3}$$

for all chain states ρ and chain operators X:

$$\mathfrak{L}_p^{\rm dual}[X] = i[H(\lambda),\ X] + \mathfrak{D}_p^{\rm dual}[X], \tag{D4}$$

with, from (38),

$$\mathfrak{D}_p^{\rm dual} = \sum_{\ell=1}^p\sum_{j=1,N}\mathbb{D}^{\rm dual}_{j\ell}, \tag{D5}$$

where

$$\mathbb{D}^{\rm dual}_{j\ell}[X] = 2\pi\, u^2_{j\ell}\, h^2_j(\omega_\ell)\big[(1+n_j(\omega_\ell)) \times \big(b^\dagger_{j\ell}\,X\,b_{j\ell} - \tfrac{1}{2}\{b^\dagger_{j\ell}\,b_{j\ell},\,X\}\big) + n_j(\omega_\ell)\big(b_{j\ell}\,X\,b^\dagger_{j\ell} - \tfrac{1}{2}\{b_{j\ell}\,b^\dagger_{j\ell},\,X\}\big)\big]. \quad \text{(D6)}$$

Using the contributions to the generator in (35) and their duals, it turns out that, for operators $X \in \mathcal{C}^{(p)}_{\mathbb{L}}$,

$$\sum_{\ell=1}^{p} \mathbb{L}^{\rm dual}_\ell[X] = i[H_p(\lambda),\,X] + \mathfrak{D}^{\rm dual}_p[X] = 0, \quad \text{(D7)}$$

where $H_p(\lambda) \equiv \sum_{\ell=1}^{p} H^{(\ell)}_\lambda$ and

$$\mathbb{L}^{\rm dual}_\ell[X] = i\big[H^{(\ell)}_\lambda,\,X\big] + \sum_{j=1,N} \mathbb{D}^{\rm dual}_{j\ell}[X]. \quad \text{(D8)}$$

However, $X \in \mathcal{C}^{(p)}_{\mathbb{L}}$ need not in general commute with the Hamiltonian associated with the negative Bohr transition frequencies,

$$\widetilde{H}_p(\lambda) \equiv \sum_{\ell=p+1}^{N} \Omega^{(\ell)}_\lambda b^\dagger_\ell b_\ell = H(\lambda) - H_p(\lambda), \quad \text{(D9)}$$

so that, in general, $\mathfrak{L}^{\rm dual}_p[X] \neq 0$ for $X \in \mathcal{C}^{(p)}_{\mathbb{L}}$. On the other hand, operators $X \in \mathcal{M}^{(p)}_{\mathbb{L}}$ do commute with $\widetilde{H}_p(\lambda)$, so that, for all $X \in \mathcal{M}^{(p)}_{\mathbb{L}}$,

$$\mathfrak{L}^{\rm dual}_p[X] := i\,[\widetilde{H}_p(\lambda),\,X\,] + \sum_{\ell=1}^{p} \mathbb{L}^{\rm dual}_\ell[X] = 0, \quad \text{(D10)}$$

and are thus left invariant by the dynamics. Generic chain operators X can be written as polynomials in b_ℓ and $b^\dagger_\ell$. Using the fermionic anticommutation relation (8), they can always be recast as

$$X = \sum_{\mathbf{n}} \alpha_{\mathbf{n}}\, B^{(1)}_{n_1} B^{(2)}_{n_2}, \ldots, B^{(N)}_{n_N}, \quad \text{(D11)}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ with $n_i = 0, 1, 2, 3$, and

$$B^{(\ell)}_0 = \mathbb{I},\ B^{(\ell)}_1 = b_\ell,\ B^{(\ell)}_2 = b^\dagger_\ell,\ B^{(\ell)}_3 = b^\dagger_\ell b_\ell. \quad \text{(D12)}$$

Notice that if for a given fermionic mode ℓ all three operators $B^{(\ell)}_{1,2,3}$ do appear in the generator, then, because of the relation (8), only $B^{(\ell)}_0$ can commute with all of them. Therefore, if all Bohr transition frequencies are positive, both algebras $\mathcal{C}^{(p)}_{\mathbb{L}}$ and $\mathcal{M}^{(p)}_{\mathbb{L}}$ with $p = N$ consist of multiples of the identity, only:

$$\mathcal{C}^{(N)}_{\mathbb{L}} = \{\mu\,\mathbb{I}\ :\ \mu \in \mathbb{C}\} = \mathcal{M}^{(N)}_{\mathbb{L}}. \quad \text{(D13)}$$

In such a case [84,85], if a faithful stationary state ρ^*, that is, a spin chain density matrix fulfilling $\mathbb{L}[\rho^*] = 0$ without zero eigenvalues, exists, then it is unique and any initial spin chain state converges to it asymptotically in time. Instead, if $p \neq N$ so that the fermionic modes from $\ell = p+1$ to $\ell = N$ do contribute only to the Hamiltonian terms of the generator, but not to the dissipator $\mathbb{D}$, then the operators $B^{(\ell)}_3$, $p+1 \leqslant \ell \leqslant N$, do commute with all the others in the generator. Then, the algebra $\mathcal{M}^{(p)}_{\mathbb{L}}$ is commutative since $[b^\dagger_\ell b_\ell,\ b^\dagger_{\ell'} b_{\ell'}] = 0$ for $\ell \neq \ell'$. Such an algebra is generated by the orthogonal projections

$$P_{\boldsymbol{\alpha}_p} := \prod_{\ell=p+1}^{N} P^{(\ell)}_{\alpha_\ell}, \quad \boldsymbol{\alpha}_p = (\alpha_{p+1}, \ldots, \alpha_N), \quad \text{(D14)}$$

where $\alpha_\ell = 0, 1$, and

$$P^{(\ell)}_{\alpha_\ell} := \begin{cases} b^\dagger_\ell\, b_\ell & \alpha_\ell = 0 \\ 1 - b^\dagger_\ell\, b_\ell = b_\ell\, b^\dagger_\ell & \alpha_\ell = 1 \end{cases}. \quad \text{(D15)}$$

Indeed, from $(b^\dagger_\ell b_\ell)^2 = b^\dagger_\ell b_\ell$, it follows that

$$P^{(\ell)}_{\alpha_\ell}\, P^{(\ell)}_{\alpha'_\ell} := \delta_{\alpha_\ell \alpha'_\ell}\, P^{(\ell)}_{\alpha_\ell}. \quad \text{(D16)}$$

Moreover, the index p ranges from a maximum $p = N - 1$, when only the smallest Bohr transition frequency ω_N is negative, to a minimum $p = p_{\min} \equiv \lfloor \frac{N+1}{2} \rfloor$, when all Bohr transition frequencies $\omega_{p_{\min}} \geqslant \cdots \geqslant \omega_N$ that can become negative are indeed negative. The projectors $P_{\boldsymbol{\alpha}_p}$ commute with the Lindblad operators and with the Hamiltonians in (34) and (35), thus

$$\mathfrak{L}_p\big[P_{\boldsymbol{\alpha}_p}\,\rho\,P_{\boldsymbol{\alpha}_p}\big] = P_{\boldsymbol{\alpha}_p}\,\mathfrak{L}_p[\rho]\,P_{\boldsymbol{\alpha}_p} \quad \text{(D17)}$$

for all states ρ. From (D17), it immediately follows that if ρ is stationary, such are also the states

$$\rho_{\boldsymbol{\alpha}_p} := \frac{P_{\boldsymbol{\alpha}_p}\,\rho\,P_{\boldsymbol{\alpha}_p}}{\mathrm{Tr}\big(P_{\boldsymbol{\alpha}_p}\rho\big)}, \quad \text{namely,} \quad \mathfrak{L}_p[\rho_{\boldsymbol{\alpha}_p}] = 0. \quad \text{(D18)}$$

Remark 6. Notice that, since $\mathfrak{L}^{\rm dual}_p[\mathbb{I}] = 0$, the dual dynamics $\gamma^{\rm dual}_t = \exp(t\mathbb{L}^{\rm dual})$ of γ_t in (39) preserves the identity (it is unital) and the projectors $P_{\boldsymbol{\alpha}_p}$ are left invariant by it; namely, $\mathfrak{L}^{\rm dual}_p[P_{\boldsymbol{\alpha}_p}] = 0$. However, $\mathfrak{L}_p[\mathbb{I}] \neq 0$, therefore the open chain dynamics is not unital $\gamma_t[\mathbb{I}] \neq \mathbb{I}$, $\mathfrak{L}_p[P_{\boldsymbol{\alpha}_p}] \neq 0$ and the projectors $P_{\boldsymbol{\alpha}_p}$ cannot be stationary states of γ_t.

For an arbitrarily large but finite chain, stationary states under γ_t always exist, the time average of any initial state yielding one,

$$\rho_{av} := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathrm{d}t\ \gamma_t[\rho] = \lim_{t\to+\infty} \gamma_t[\rho]. \quad \text{(D19)}$$

Suppose a stationary state ρ^* exists which is faithful, namely, without zero eigenvalues; according to Ref. [85], if the commutant $\mathcal{M}^{(p)}_{\mathbb{L}}$ is commutative as in the present case, then the stationary states form a convex subspace consisting of convex combinations of the form

$$\rho_{\rm stat} = \sum_{\boldsymbol{\alpha}_p} \mu_{\boldsymbol{\alpha}_p} \frac{P_{\boldsymbol{\alpha}_p}\,\rho^*\,P_{\boldsymbol{\alpha}_p}}{\mathrm{Tr}\big(P_{\boldsymbol{\alpha}_p}\rho^*\big)}, \quad \text{(D20)}$$

where $\mu_{\boldsymbol{\alpha}_p} \geqslant 0$ and $\sum_{\boldsymbol{\alpha}_p} \mu_{\boldsymbol{\alpha}_p} = 1$. The states $\rho^*_{\boldsymbol{\alpha}_p} := \frac{P_{\boldsymbol{\alpha}_p}\rho^* P_{\boldsymbol{\alpha}_p}}{\mathrm{Tr}(P_{\boldsymbol{\alpha}_p}\rho^*)}$ are invariant under the dissipative dynamics $\gamma_t = \exp(t\,\mathbb{L})$. Moreover, if

$$\lim_{t\to+\infty} \gamma_t[\rho] = \rho_{\rm stat}, \quad \text{then} \quad \mu_{\boldsymbol{\alpha}_p} = \mathrm{Tr}\big(P_{\boldsymbol{\alpha}_p}\,\rho\big). \quad \text{(D21)}$$

Indeed, $\mathfrak{L}^{\rm dual}_p[P_{\boldsymbol{\alpha}_p}] = 0$ yields

$$\mu_{\boldsymbol{\alpha}_p} = \lim_{t\to+\infty} \mathrm{Tr}(\gamma_t[\rho]\,P_{\boldsymbol{\alpha}_p}) = \lim_{t\to+\infty} \mathrm{Tr}\big(\rho\,\gamma^{\rm dual}_t[P_{\boldsymbol{\alpha}_p}]\big) = \mathrm{Tr}\big(\rho\,P_{\boldsymbol{\alpha}_p}\big). \quad \text{(D22)}$$

Summarizing, if all frequencies ω_ℓ are positive and contribute to the generator $\mathbb{L}$, the *commutant* $\mathcal{C}^{(N)}_{\mathbb{L}}$ is trivial and there is a unique stationary state ρ^*, all initial states tending to it when

$t \to +\infty$. If some Bohr transition frequencies are negative, then the orthogonal projectors of the commutative algebra $\mathcal{M}_{\mathbb{L}}^{(p)}$ can be used to generate all possible stationary states to which initial states may or may not converge asymptotically in time. The faithful invariant state ρ^* necessary for the construction of the convex manifold of stationary states can be easily found because of the bilinear structure of the generator $\mathbb{L}$. Indeed, it suggests to seek ρ^* of the form

$$\rho^* = \prod_{\ell=1}^{N} \rho^*(\omega_\ell), \tag{D23}$$

$$\rho^*(\omega_\ell) \equiv \eta^*(\omega_\ell)\,\mathbb{I}_\ell + \zeta^*(\omega_\ell)\, b_\ell^\dagger b_\ell \tag{D24}$$

$$= [\zeta^*(\omega_\ell) + \eta^*(\omega_\ell)]\, b_\ell^\dagger b_\ell + \eta^*(\omega_\ell)\, b_\ell b_\ell^\dagger. \tag{D25}$$

The eigenvalues of $\rho^*(\omega_\ell)$ are $\eta^*(\omega_\ell)$ and $\eta^*(\omega_\ell) + \zeta^*(\omega_\ell)$, where the coefficients $\eta^*(\omega_\ell)$ and $\zeta^*(\omega_\ell)$ are fixed by asking that $\rho^*(\omega_\ell)$ be a mode-ℓ state annihilated by $\mathbb{L}_\ell$. These constraints yield $\eta^*(\omega_\ell) \geqslant 0$, $\eta^*(\omega_\ell) + \zeta^*(\omega_\ell) \geqslant 0$, $2\eta^*(\omega_\ell) + \zeta^*(\omega_\ell) = 1$ and

$$\eta^*(\omega_\ell)\,\mathbb{L}_\ell[\mathbb{I}] + \zeta^*(\omega_\ell)\,\mathbb{L}_\ell[b_\ell^\dagger b_\ell] = 0. \tag{D26}$$

Indeed, the anticommutativity of the fermionic operators and the bilinear structure of the generator $\mathbb{L}$ is such that its action on (40) amounts to

$$\mathbb{L}[\rho^*] = \sum_{\ell=1}^{N} \left[\prod_{j=1}^{\ell-1} \rho^*(\omega_j)\right] \mathbb{L}_\ell[\rho^*(\omega_\ell)] \left[\prod_{j=\ell+1}^{N} \rho^*(\omega_j)\right]. \tag{D27}$$

The constraints (D26) then yield

$$\eta^*(\omega_\ell) = \frac{C(\omega_\ell)}{C(\omega_\ell) + \widetilde{C}(\omega_\ell)}, \tag{D28}$$

$$\zeta^*(\omega_\ell) = \frac{\widetilde{C}(\omega_\ell) - C(\omega_\ell)}{C(\omega_\ell) + \widetilde{C}(\omega_\ell)}, \tag{D29}$$

where

$$C(\omega_\ell) = 2\pi \sum_{j=1,N} h_j^2(\omega_\ell)\, u_{j\ell}^2\, [1 + n_j(\omega_\ell)], \tag{D30}$$

$$\widetilde{C}(\omega_\ell) = 2\pi \sum_{j=1,N} h_j^2(\omega_\ell)\, u_{j\ell}^2\, n_j(\omega_\ell), \tag{D31}$$

hence, from (D25),

$$\rho^*(\omega_\ell) = \frac{\widetilde{C}(\omega_\ell)}{C(\omega_\ell) + \widetilde{C}(\omega_\ell)}\, b_\ell^\dagger b_\ell + \frac{C(\omega_\ell)}{C(\omega_\ell) + \widetilde{C}(\omega_\ell)}\, b_\ell b_\ell^\dagger. \tag{D32}$$

From Remark 4, it follows that, when the ratio r is such that the Bohr transition frequencies ω_ℓ are negative for $p+1 \leqslant \ell \leqslant N$, the stationary state ρ^* can be used to generate the convex stationary manifold by means of the stationary states in (D20):

$$\rho^*_{\boldsymbol{\alpha}_p} = \left[\prod_{j=1}^{p} \rho^*(\omega_j)\right] \prod_{\ell=p+1}^{N} \frac{P_{\alpha_\ell}^{(\ell)}\, \rho^*(\omega_\ell)\, P_{\alpha_\ell}^{(\ell)}}{\mathrm{Tr}\big[P_{\alpha_\ell}^{(\ell)} \rho^*(\omega_\ell)\big]}. \tag{D33}$$

According to (D14), (D15), and (D24), one has

$$\rho^*(\omega_\ell) = \big[\eta^*(\omega_\ell) + \zeta^*(\omega_\ell)\big]\, P_0^{(\ell)} + \eta^*(\omega_\ell)\, P_1^{(\ell)}. \tag{D34}$$

It thus follows that

$$P_{\alpha_\ell}^{(\ell)}\, \rho^*(\omega_\ell)\, P_{\alpha_\ell}^{(\ell)} = \begin{cases} [\eta^*(\omega_\ell) + \zeta^*(\omega_\ell)]\, P_0^{(\ell)} & \alpha_\ell = 0 \\ \eta^*(\omega_\ell)\, P_1^{(\ell)} & \alpha_\ell = 1 \end{cases}, \tag{D35}$$

hence

$$\rho^*_{\boldsymbol{\alpha}_p} = \left[\prod_{j=1}^{p} \rho^*(\omega_j)\right] \left(\prod_{\ell=p+1}^{N} P_{\alpha_\ell}^{(\ell)}\right). \tag{D36}$$

As already noticed, $\mathbb{L}_\ell[P_{\alpha_\ell}] \neq 0$, in general; however, when $\mathbb{L}_\ell$ has no dissipative contribution then $\mathbb{L}_\ell[P_{\alpha_\ell}] = 0$ and, according to (D27), the states $\rho^*_{\boldsymbol{\alpha}_p}$ in (D36) are stationary. The quantities in (D28) and (D29) are both continuous at $\omega_\ell = 0$; indeed, the numerator and denominator in $\eta^*(\omega_\ell)$ diverge in the same way, while the numerator in $\zeta^*(\omega_\ell)$ is constant, so that

$$\lim_{\omega_\ell \to 0} \eta^*(\omega_\ell) = \tfrac{1}{2}, \quad \lim_{\omega_\ell \to 0} \zeta^*(\omega_\ell) = 0, \tag{D37}$$

and $\rho^*(\omega_\ell)$ tend to the completely mixed state

$$\lim_{\omega_\ell \to 0} \rho^*(\omega_\ell) = \frac{\mathbb{I}_\ell}{2}. \tag{D38}$$

Furthermore, the single mode contributions $\rho^*(\omega_\ell)$ and thus the faithful state ρ^* itself remain well defined when $\omega_\ell < 0$. Indeed, sending $\omega_\ell \mapsto -|\omega_\ell|$ yields

$$n(\omega_\ell) \mapsto -[1 + n(|\omega_\ell|)], \quad 1 + n(\omega_\ell) \mapsto -n(|\omega_\ell|); \tag{D39}$$

hence $C(\omega_\ell) \mapsto -\widetilde{C}(|\omega_\ell|)$, $\widetilde{C}(\omega_\ell) \mapsto -C(|\omega_\ell|)$, and from (D23), $\rho^*(\omega_\ell)$ changes into

$$\rho^*(-|\omega_\ell|) = \frac{C(|\omega_\ell|)}{C(|\omega_\ell|) + \widetilde{C}(|\omega_\ell|)}\, b_\ell^\dagger b_\ell + \frac{\widetilde{C}(|\omega_\ell|)}{C(|\omega_\ell|) + \widetilde{C}(|\omega_\ell|)}\, b_\ell b_\ell^\dagger, \tag{D40}$$

which is still a well-defined state for the ℓth fermionic mode. The main difference at negative Bohr transition frequencies is that ρ^* becomes an element of a larger convex manifold of stationary states. Indeed, to each negative Bohr transition frequency there remain associated two invariant orthogonal projections. According to (D21), different stationary asymptotic states are reached by properly choosing the initial states and letting them evolve for long times. When $\omega_N > 0$, all initial states ρ tend in time to ρ^*; if r makes $\omega_N < 0$, there will be initial states still asymptotically evolving into ρ^*, now being characterized by the N-mode invariant state $\rho(-|\omega_N|)$ as in (D40), and other initial states which will instead reach stationary states of the form [see (D15), (D20), and (46)]

$$\rho_{\rm stat} = \left[\prod_{\ell=1}^{N-1} \rho^*(\omega_\ell)\right] [\mu\, b_N^\dagger b_N + (1-\mu)\, b_N b_N^\dagger]. \tag{D41}$$

According to (D22), an initial state ρ may converge to ρ^* only if

$$\mu = \mathrm{Tr}(\rho\, b_N^\dagger b_N) = \eta^*(|\omega_N|). \tag{D42}$$

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