

Thermal equilibrium in Gaussian dynamical semigroups

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We characterize all Gaussian dynamical semigroups in continuous-variables quantum systems of n -bosonic modes which have a thermal Gibbs state as a stationary solution. This is performed through an explicit relation between the diffusion and dissipation matrices, which characterize the semigroup dynamics, and the covariance matrix of the thermal equilibrium state. We also show that Alicki's quantum detailed-balance condition, based on a Gelfand-Naimark-Segal inner product, allows the determination of the temperature dependence of the diffusion and dissipation matrices and the identification of different Gaussian dynamical semigroups which share the same thermal equilibrium state.

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I. INTRODUCTION

In modern quantum information theory for continuous variable systems, i.e., systems described by n -bosonic modes, Gaussian channels are the standard models in most of the quantum communication protocols [1–6]. These channels are defined as those bosonic channels which transform Gaussian states into Gaussian states [7,8]. Further, these states have an exceptional role in quantum communication as, for example, they are optimal for the transmission of classical information through Gaussian bosonic quantum channels with additive capacity [9].

The most general form of a one-parameter Gaussian channel of n -bosonic modes is a Gaussian dynamical semigroup (GDS) [10,11], constituting thus the tool to describe the dynamics of all memoryless continuous-in-time Gaussian quantum channels. This is the reason why they are widely used to describe noisy quantum channels in continuous variable systems [10,12,13]. Further, GDSs are able to describe all processes which can formally be written as decomposition and production of noninteracting particles (or quasiparticles) which can be treated at least approximately as bosons [14]. In this context, GDSs are known as quasifree completely positive semigroups [15–17], which happens, for example, in damped collective modes in deep inelastic collisions [18].

The dynamics of any quantum dynamical semigroup (QDS), not necessarily Gaussian, also known as quantum Markov semigroups [19], is described by a master equation in the Lindblad form [20,21]. In an analogous way, a GDS verifies a Lindblad master equation where the unitary part of the evolution is set by a quadratic Hamiltonian which describes the n -bosonic modes. By a quadratic Hamiltonian, we mean one composed by products of any two canonical conjugate operators, positions and momenta, or the set of

self-adjoint operators that constitutes a representation of the Heisenberg canonical commutation relations of the particular bosonic system considered [22]. Meanwhile, the nonunitary part is given by Lindblad operators corresponding to complex linear functions of positions and momenta, which simplifies the nonunitary dynamics to be described only in terms of two real $2n \times 2n$ matrices: the diffusion and dissipation matrices [11,23].

Alternatively, the Weyl-Wigner representation [24] for the master equation of a GDSs can be employed and corresponds to a linear Fokker-Planck equation for the evolution of the Wigner function of the evolved states [11,23,25]. This is the quantum counterpart of the classical channels corresponding to Ornstein-Uhlenbeck processes, thus also known as Bose-Ornstein-Uhlenbeck semigroups [19], whose evolved probability distributions satisfy exactly the same Fokker-Planck equation as the GDSs.

Notably when the GDS dynamics has a steady state, this will be a dynamically invariant Gaussian state which attracts, over long times, the evolution of any initial condition [26–28]. Thus, for a given quadratic Hamiltonian, the characterization of stationary situations in GDSs corresponds to find all the diffusion and dissipation matrices that allow a stationary state [11]. Of particular importance are the stationary states which also correspond to thermal equilibrium states, characterized by a Gibbs state of a quadratic Hamiltonian. This is the main subject of our study.

Thermal equilibrium in QDSs in finite-dimensional quantum systems has its own well-established theory based on the so-called quantum detailed balance condition (QDBC), as first stated by Alicki [14,29] (see also [19]). Of note are the results in [19] where the authors prove, using the QDBC, that the evolution governed by a QDS is a gradient flow, in a particular Riemannian metric on the set of states, for the relative entropy of a state with respect to a Gibbs state. On the other hand, the study of thermal equilibrium in GDSs is scarce, which is particularly true for the multimode scenario; a prominent exception is the extension of the results for QDSs in

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finite-dimensional quantum systems to the case of a one-mode GDS performed in [19].

Here we fill this gap and give a complete characterization of all n -mode GDSs with a thermal equilibrium state. To this aim, we employ the Fokker-Planck equation for the evolution of the Wigner function and show that the thermal probability current is always null in every phase space point. This enables us to conclude that GDSs with thermal equilibrium are characterized by a set of three commuting matrices. Two of the them are the Hamiltonian matrices associated to the covariance matrix of the thermal state and the diffusion matrix of the GDS. The third one is the skew-Hamiltonian matrix associated to the dissipation matrix of the GDS. By another side, this condition neglects that different GDSs, characterized by the diffusion and dissipation matrices, may share the same thermal equilibrium state, as a consequence of the fact that the relation among the matrices does not set their temperature dependence. To circumvent this, we show that these characterizations are possible by extending Alicki's QDBC to bosonic-mode systems.

The extended QDBC leads to a master equation for a GDS with the form of a quantum master optical equation (QOME) [30] and the temperature dependence of the diffusion and dissipation matrices is established for this type of GDS. From this temperature characterization, we establish that all GDSs that lead to thermal equilibrium satisfy a QDBC if we allow an arbitrary temperature dependence for coupling constants between the system and the environment. Finally, we discriminate the Hamiltonians corresponding to the unitary part of the dynamics of a GDS which allow the occurrence of the thermalization. Although, we show that the thermalization process itself is not effected by these Hamiltonians.

The paper is organized as follows. In Sec. II we introduce the GDSs, their action on Gaussian states (Sec. II A), and the Weyl-Wigner formalism to describe its dynamics (Sec. II B). In the introduction of Sec. III, we establish the general time dependence of the first- and second-order moments in GDSs with stationary solutions. Then, in Sec. III A we set up the problem of having thermal equilibrium as stationary solutions and describe general properties that must be satisfied by GDSs with thermal equilibrium. Section III B contains one of our main results: a theorem that characterizes all the GDSs with a thermal equilibrium state. The extension of the QDBC to n -bosonic mode systems is placed in Sec. IV, where five theorems are presented. These theorems completely characterize all GDSs satisfying the detailed balance. In Sec. V, we show that the master equation of GDSs satisfying the QDBC always corresponds to a QOME; this section finishes with a discussion about entanglement properties of its thermal equilibrium state solution. We further explore the characterization of thermal equilibrium states in GDSs that satisfy a QDBC in Sec. VI, where the temperature dependence of the diffusion and dissipation matrices is developed in Sec. VI A, the high- and low-temperature limits are described in Sec. VI B, and in Sec. VI C we explain the pure diffusive regime, where the stationary solution is lost. In Sec. VII we describe the necessary structure of a quadratic Hamiltonian, governing the unitary part of the GDS, that has a thermal equilibrium state. In this section we also clarify the role of this Hamiltonian in the process of thermalization. Finally, we summarize our findings

in Sec. VIII. Some auxiliary calculations and technical proofs are presented in the Appendixes A, B, C, D, E, and H.

II. GAUSSIAN DYNAMICAL SEMIGROUPS

In the Schrödinger picture, a QDS is ruled by the Lindblad master equation (LME) [20,21]

$$\frac{d\hat{\rho}_t}{dt} = \mathcal{L}[\hat{\rho}_t] = \mathcal{L}_U[\hat{\rho}_t] + \mathcal{L}_{\text{NU}}[\hat{\rho}_t], \quad (1)$$

where

$$\mathcal{L}_U[\cdot] = -\frac{i}{\hbar}[\hat{H}_{\text{eff}}, \cdot], \quad (2a)$$

$$\mathcal{L}_{\text{NU}}[\cdot] = \frac{1}{2\hbar} \sum_{k=1}^K (2\hat{L}_k \cdot \hat{L}_k^\dagger - \hat{L}_k^\dagger \hat{L}_k \cdot - \cdot \hat{L}_k^\dagger \hat{L}_k) \quad (2b)$$

are, respectively, the infinitesimal generators of the unitary and nonunitary parts of the evolution. The operator \hat{H}_{eff} is the effective Hamiltonian of the system and \hat{L}_k ($k = 1, \dots, K$) are the Lindblad operators.

With the help of the Hilbert-Schmidt inner product $\langle \hat{A}, \hat{B} \rangle = \text{Tr}(\hat{A}^\dagger \hat{B})$, the adjoint $\bar{\mathcal{L}}$ of the superoperator \mathcal{L} is defined by

$$\langle \mathcal{L}[\hat{A}], \hat{B} \rangle = \langle \hat{A}, \bar{\mathcal{L}}[\hat{B}] \rangle, \quad (3)$$

which enables us to write the Heisenberg picture of Eq. (1) for an observable \hat{O}_t [22]:

$$\frac{d\hat{O}_t}{dt} = \bar{\mathcal{L}}[\hat{O}_t] = \bar{\mathcal{L}}_U[\hat{O}_t] + \bar{\mathcal{L}}_{\text{NU}}[\hat{O}_t], \quad (4)$$

where

$$\bar{\mathcal{L}}_U[\cdot] = \frac{i}{\hbar}[\hat{H}_{\text{eff}}, \cdot], \quad (5a)$$

$$\bar{\mathcal{L}}_{\text{NU}}[\cdot] = \frac{1}{2\hbar} \sum_{k=1}^K (2\hat{L}_k^\dagger \cdot \hat{L}_k - \hat{L}_k^\dagger \hat{L}_k \cdot - \hat{L}_k^\dagger \hat{L}_k). \quad (5b)$$

Since \mathcal{L} is time independent, the solution of (1) is formally given by $\hat{\rho}_t = e^{t\mathcal{L}}\hat{\rho}_0$, which is the evolution of an initial condition $\hat{\rho}_0$, and the set $\{\Lambda_t = e^{t\mathcal{L}}\}_{t \geq 0}$ is properly the QDS in the Schrödinger picture [31]. The solution of (4) is formally written as $\hat{O}_t = e^{t\bar{\mathcal{L}}}\hat{O}_0$, which gives the evolution of an initial condition \hat{O}_0 , where the set $\{\bar{\Lambda}_t = e^{t\bar{\mathcal{L}}}\}_{t \geq 0}$ is the Heisenberg picture version of the QDS.

The kinematics of a system with n -bosonic modes is described by a $2n$ -dimensional column vector of canonical operators,

$$\hat{\mathbf{x}} = (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)^\top, \quad (6)$$

satisfying the canonical commutation relation $[\hat{x}_j, \hat{x}_k] = i\hbar \mathbf{J}_{jk} \hat{1}$, where

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathbf{J}^{-1} = -\mathbf{J} = \mathbf{J}^\top \quad (7)$$

is a $2n \times 2n$ real antisymmetric symplectic matrix and $\mathbb{1}$ is the $n \times n$ identity matrix.

In a n -bosonic-mode system a Gaussian dynamical semigroup is a QDS with the Hamiltonian and Lindblad operators given by

$$\hat{H}_{\text{eff}} = \frac{1}{2} \hat{\mathbf{x}}^\top \mathbb{B}' \hat{\mathbf{x}} + \hat{\mathbf{x}}^\top \mathbf{J} \boldsymbol{\xi}', \quad (8a)$$

$$\hat{L}_k = \mathbf{I}_k^\top \mathbf{J} \hat{\mathbf{x}} \quad (k = 1, \dots, K), \quad (8b)$$

where $\mathbb{B}' = (\mathbb{B}')^\top$ is the Hessian matrix of the Hamiltonian, $\boldsymbol{\xi}'$ is an $(2n)$ -dimensional real column vector, and the \mathbf{I}_k 's are $(2n)$ -dimensional complex column vectors. In this case, according to [11], the superoperator in (2b) becomes¹

$$\mathcal{L}_{\text{NU}}^{\text{G}}[\hat{\rho}_t] = \text{tr} \left(\mathbb{C} \mathbf{J} \frac{\partial}{\partial \hat{\mathbf{x}}} [\hat{\mathbf{x}}^\top \hat{\rho}_t] + \frac{1}{2} \mathbb{D} \frac{\partial}{\partial \hat{\mathbf{x}}} \frac{\partial}{\partial \hat{\mathbf{x}}^\top} \hat{\rho}_t \right), \quad (9)$$

where $\frac{\partial}{\partial \hat{\mathbf{x}}} = \frac{i}{\hbar} [(\mathbf{J} \hat{\mathbf{x}}), \cdot]$ is a column vector operator whose components are $\frac{\partial}{\partial x_k} = \frac{i}{\hbar} [(\mathbf{J} \hat{\mathbf{x}})_k, \cdot]$ with $k = 1, \dots, 2n$. The matrices

$$\mathbb{D} = \hbar \text{Re}(\mathbb{F}) \quad \text{and} \quad \mathbb{C} = \text{Im}(\mathbb{F}) \quad (10)$$

are, respectively, the diffusion and dissipation matrices, both defined through the decoherence matrix

$$\mathbb{F} = \sum_{k=1}^K \mathbf{I}_k \mathbf{I}_k^\dagger, \quad (11)$$

which is composed by the vectors in the Lindblad operators (8b). The adjoint generator $\bar{\mathcal{L}}_{\text{NU}}$ in (4), using Eq. (5b) for the present case, is

$$\bar{\mathcal{L}}_{\text{NU}}^{\text{G}}[\hat{O}_t] = \text{tr} \left(\mathbb{C} \mathbf{J} \hat{\mathbf{x}} \frac{\partial}{\partial \hat{\mathbf{x}}^\top} \hat{O}_t + \frac{1}{2} \mathbb{D} \frac{\partial}{\partial \hat{\mathbf{x}}} \frac{\partial}{\partial \hat{\mathbf{x}}^\top} \hat{O}_t \right). \quad (12)$$

According to the definitions in (10), $\mathbb{D} = \mathbb{D}^\top \geq 0$ and $\mathbb{C} = -\mathbb{C}^\top$. Note also that, according to (11), $\mathbb{F} \geq 0$, and thus

$$\hbar \mathbb{F} = \mathbb{D} + i \mathbb{C} \geq 0, \quad (13)$$

which can be interpreted as a generalized fluctuation-dissipation relation [32]. For the next sections, a useful result concerning a relation between the matrices in (13) is the following lemma, which is proved in Appendix A.

Lemma 1. If $\det \mathbb{C} \neq 0$ in (13), i.e., \mathbb{C} is invertible, then both the diffusion and the decoherence matrices are invertible and strictly positive definite, that is, $\mathbb{D} > 0$ and $\mathbb{F} > 0$.

Two quantities of main importance for establishing the results of this work are the mean-value vector

$$\langle \hat{\mathbf{x}} \rangle_t = \text{Tr}(\hat{\rho}_t \hat{\mathbf{x}}) \quad (14)$$

and the (dimensionless) covariance matrix

$$\mathbb{V}_t = \frac{1}{2\hbar} \text{Tr}[\hat{\rho}_t (\hat{\mathbf{x}} - \langle \hat{\mathbf{x}} \rangle_t) (\hat{\mathbf{x}} - \langle \hat{\mathbf{x}} \rangle_t)^\top]. \quad (15)$$

Despite the evolution of the system state through a GDS can be analytically determined [28], the description for the system behavior is improved when analyzing the evolution of these two moments. Taking the temporal derivative of above equations and using the LME in (1) for the GDS, i.e., with

the operators in (8), the cyclicity of the trace together with the canonical commutation relation yield [23]

$$\frac{d\langle \hat{\mathbf{x}} \rangle_t}{dt} = \mathbb{A} \langle \hat{\mathbf{x}} \rangle_t - \boldsymbol{\xi} \quad (16)$$

and

$$\frac{d\mathbb{V}_t}{dt} = (\mathbb{A} \mathbb{V}_t + \mathbb{V}_t \mathbb{A}^\top) + \frac{\mathbb{D}}{\hbar}, \quad (17)$$

where we defined the drift matrix

$$\mathbb{A} = \mathbf{J} \mathbb{B}' - \mathbb{C} \mathbf{J}, \quad (18)$$

for \mathbb{B}' from (8a) and \mathbb{C} from (10).

By direct integration, the solutions of Eqs. (16) and (17) are, respectively,

$$\langle \hat{\mathbf{x}} \rangle_t = e^{\mathbb{A}t} \langle \hat{\mathbf{x}} \rangle_0 - \int_0^t dt' e^{\mathbb{A}(t-t')} \boldsymbol{\xi}, \quad (19a)$$

$$\mathbb{V}_t = e^{\mathbb{A}t} \mathbb{V}_0 e^{\mathbb{A}^\top t} + \frac{1}{\hbar} \int_0^t dt' e^{\mathbb{A}(t-t')} \mathbb{D} e^{\mathbb{A}^\top(t-t')}. \quad (19b)$$

If the matrix \mathbb{A} is invertible, the integral in (19a) can be explicitly performed and this solution becomes

$$\langle \hat{\mathbf{x}} \rangle_t = e^{\mathbb{A}t} (\langle \hat{\mathbf{x}} \rangle_0 - \mathbb{A}^{-1} \boldsymbol{\xi}) + \mathbb{A}^{-1} \boldsymbol{\xi}. \quad (20)$$

A. Gaussian states

The formalism presented so far describes the action of a GDS on a generic quantum state. However, the term ‘‘Gaussian’’ in the acronym ‘‘GDS’’ refers to the fact that this kind of dynamics is a quantum channel that preserves the Gaussian character of an initial Gaussian state throughout the whole evolution.

The density operator $\hat{\sigma}_t$ of a Gaussian state can be expressed as [6,33]

$$\hat{\sigma}_t = \frac{e^{-\frac{1}{2\hbar} (\hat{\mathbf{x}} - \langle \hat{\mathbf{x}} \rangle_t)^\top \mathbb{U}_t (\hat{\mathbf{x}} - \langle \hat{\mathbf{x}} \rangle_t)}}{\sqrt{\det(\mathbb{V}_t + \frac{i}{2} \mathbf{J})}}, \quad (21)$$

which is completely determined only by the moments in Eqs. (14) and (15), where the mean value is $\langle \hat{\mathbf{x}} \rangle_t = \text{Tr}(\hat{\sigma}_t \hat{\mathbf{x}})$ and the matrix \mathbb{U}_t is given by

$$\mathbb{U}_t = 2i \mathbf{J} \coth^{-1}(2i \mathbb{V}_t \mathbf{J}). \quad (22)$$

Note that $\mathbb{V}_t + \frac{i}{2} \mathbf{J} \geq 0$ is the *bona fide* condition of a covariance matrix of a quantum state [34], thus the determinant in the denominator (21) is never negative. When subjected to a GDS, the evolved state is like (21) with $\langle \hat{\mathbf{x}} \rangle_t$ and \mathbb{V}_t given in (19).

The relation between the matrices in (22) can be strengthened, which will be necessary for our future results. It is immediate from (22) that

$$\mathbf{J} \mathbb{U}_t \mathbb{V}_t = \mathbb{V}_t \mathbb{U}_t \mathbf{J} \Leftrightarrow [\mathbf{J} \mathbb{U}_t, \mathbb{V}_t \mathbf{J}] = 0; \quad (23)$$

however, we will prove this relation using well-known results in order to establish methods and notations for several future occasions. First, we use the Williamson theorem [34,35] which establishes that for every $2n \times 2n$ real symmetric and positive-definite matrix \mathbb{V}_t , i.e., $\mathbb{V}_t^\top = \mathbb{V}_t > 0$, there exists a symplectic matrix $\mathbf{S}_t \in \text{Sp}(2n, \mathbb{R})$ such that

$$\mathbf{S}_t \mathbb{V}_t \mathbf{S}_t^\top = \mathbb{I}_t \oplus \mathbb{I}_t, \quad (24)$$

¹From now on, we will use tr to denote the trace of a matrix and Tr to denote the trace of an operator.

where $\mathbb{k}_t = \text{diag}(\kappa_1(t), \dots, \kappa_n(t))$ is the symplectic spectra of \mathbb{V}_t and $\kappa_j(t) \geq \frac{1}{2}$ ($j = 1, \dots, n$) are the symplectic eigenvalues. Next, we use the following lemma, also a consequence of the Williamson theorem.

Lemma 2. A Hamiltonian matrix² $\mathbb{O}\mathbb{J}$, where \mathbb{O} is symmetric and positive definite and \mathbb{J} is in (7), is diagonalized by the similarity transformation

$$(\mathbb{Q}\mathbb{S})\mathbb{O}\mathbb{J}(\mathbb{Q}\mathbb{S})^{-1} = (\iota\mathbb{o}) \oplus (-\iota\mathbb{o}), \quad (25)$$

where $\mathbb{o} = \text{diag}(o_1, \dots, o_n)$, $o_j > 0$ ($j = 1, \dots, n$) are the symplectic eigenvalues of \mathbb{O} through \mathbb{S} , i.e., $\mathbb{S}\mathbb{O}\mathbb{S}^\top = \mathbb{o} \oplus \mathbb{o}$, and \mathbb{Q} is the complex matrix³

$$\mathbb{Q} = \mathbb{Q}^\top = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\iota\mathbb{1} \\ -\iota\mathbb{1} & \mathbb{1} \end{pmatrix} \quad (26)$$

such that $\mathbb{Q}^\top\mathbb{J}\mathbb{Q} = \mathbb{J}$ and $\mathbb{Q}^{-1} = \mathbb{Q}^\dagger$. Equivalently, we have

$$(\mathbb{Q}\mathbb{S}^{-\top})\mathbb{J}\mathbb{O}(\mathbb{Q}\mathbb{S}^{-\top})^{-1} = (\iota\mathbb{o}) \oplus (-\iota\mathbb{o}). \quad (27)$$

Returning to the proof of Eq. (23), the above lemma can be used to diagonalize the matrix $\mathbb{V}_t\mathbb{J}$, i.e.,

$$\mathbb{Q}\mathbb{S}_t\mathbb{V}_t\mathbb{J}(\mathbb{Q}\mathbb{S}_t)^{-1} = (\iota\mathbb{k}_t) \oplus (-\iota\mathbb{k}_t) =: (\mathbb{V}_t\mathbb{J})_d. \quad (28)$$

As a useful notation, the diagonal matrix $(\mathbb{V}_t\mathbb{J})_d$ defined above will be called the canonical form of $\mathbb{V}_t\mathbb{J}$. From (22), we write $\mathbb{J}\mathbb{U}_t = -2\iota g(2\iota\mathbb{V}_t\mathbb{J})$, where

$$g(x) = 2 \coth^{-1}(2x) > 0 \quad (29)$$

is a continuous function for $x > \frac{1}{2}$. Consequently, employing Eq. (28) and noting that $g(x) = -g(-x)$, we attain

$$\mathbb{Q}\mathbb{S}_t\mathbb{J}\mathbb{U}_t(\mathbb{Q}\mathbb{S}_t)^{-1} = [\iota g(\mathbb{k}_t)] \oplus [-\iota g(\mathbb{k}_t)] =: (\mathbb{J}\mathbb{U}_t)_d, \quad (30)$$

which is the canonical form of the Hamiltonian matrix $\mathbb{J}\mathbb{U}_t$. Therefore, the matrix $(\mathbb{Q}\mathbb{S}_t)^{-1}$ simultaneously diagonalizes the matrices $\mathbb{V}_t\mathbb{J}$ and $\mathbb{J}\mathbb{U}_t$ and they must commute, as we wanted to prove.

Noteworthy, the matrix \mathbb{U}_t is positive definite so the Williamson theorem can be applied. The symplectic diagonalization can be easily inferred using (27) with $\mathbb{O} = \mathbb{U}_t$ and $\mathbb{S}^{-\top} = \mathbb{S}_t$, so we get the canonical form in (30). Therefore, according to the hypothesis of the lemma the matrix that diagonalizes symplectically \mathbb{U}_t is $\mathbb{S}_t^{-\top}$, i.e.,

$$\mathbb{S}_t^{-\top}\mathbb{U}_t\mathbb{S}_t^{-1} = g(\mathbb{k}_t) \oplus g(\mathbb{k}_t), \quad (31)$$

where $g(\mathbb{k}_t) = \text{diag}(g(\kappa_1(t)), \dots, g(\kappa_n(t)))$ and $g(\kappa_j) > 0$ ($j = 1, \dots, n$) are the symplectic eigenvalues of \mathbb{U}_t .

From (31), we see that the matrix \mathbb{U}_t is finite whenever $\kappa_j(t) > \frac{1}{2}$ for all $j = 1, \dots, n$ and for each fixed value of t . In this case the density operator $\hat{\sigma}_t$ in (21) corresponds to a full-rank mixed state. Still, the representation of Gaussian states as in (21) is also valid in the limit $\kappa_j(t) \rightarrow \frac{1}{2} \forall j$, where both the matrix \mathbb{U}_t and $\det(\mathbb{V}_t + \frac{1}{2}\mathbb{J})$ diverge. In this limit, we have $\hat{\sigma}_t \rightarrow |\Psi_t\rangle\langle\Psi_t|$, where $|\Psi_t\rangle$ is an n -mode pure Gaussian state. When some but not all symplectic eigenvalues are such

that $\kappa_j(t) = 1/2$, the same divergences happen, and $\hat{\sigma}_t$ in (21) represents a rank-deficient mixed quantum state in the limit $\kappa_j(t) \rightarrow \frac{1}{2}$. In conclusion, $\hat{\sigma}_t$ in (21) is a valid representation of the density operator of any Gaussian state.

B. Wigner function and Fokker-Planck equation

In continuous-variable systems, the following sets $\{\hat{T}_\xi = e^{\frac{i}{\hbar}\hat{\mathbf{x}}^\top\mathbb{J}\xi} \mid \xi \in \mathbb{R}^{2n}\}$ and $\{\hat{R}_\mathbf{x} = (4\pi\hbar)^{-n} \int d\xi e^{\frac{i}{\hbar}\xi^\top\mathbb{J}\mathbf{x}} \hat{T}_\xi \mid \mathbf{x} \in \mathbb{R}^{2n}\}$ are basis of a vector space constituted by operators acting on the separable infinite-dimensional Hilbert space $\mathcal{H} = \otimes_{j=1}^n \mathcal{H}_j$ of the n -mode bosonic system [36]. The elements of these sets are called translations and reflections, respectively, and refer to their action on the vector operator $\hat{\mathbf{x}}$ in (6), corresponding to the Heisenberg picture, namely, $\hat{T}_\xi^\dagger \hat{\mathbf{x}} \hat{T}_\xi = \hat{\mathbf{x}} + \xi \hat{\mathbb{1}}$ and $\hat{R}_\mathbf{x}^\dagger \hat{\mathbf{x}} \hat{R}_\mathbf{x} = -\hat{\mathbf{x}} + 2\mathbf{x} \hat{\mathbb{1}}$ [24]. Translation operators are unitary $\hat{T}_\xi^\dagger = \hat{T}_\xi^{-1} = \hat{T}_{-\xi}$ and reflection operators are unitary and Hermitian, i.e., involutory operators $\hat{R}_\mathbf{x}^2 = \hat{\mathbb{1}}$. The operator \hat{T}_ξ is also known as Weyl operator and $\hat{R}_\mathbf{x}$ as Wigner operator [22].

When dealing with continuous-variable systems, the existence of unbounded operators and operators with continuous spectra may cause some mathematical difficulties. In particular, it is often difficult to find the algebra of operators that defines the domain of applicability of a given formalism. In this work, we circumvent this difficulty by applying our formalism to the algebra of all operators with a Weyl and Wigner representation. This will be particularly important for the demonstration developed in Appendixes B and C.

The Weyl and Wigner representations of an operator \hat{A} are, respectively, the Hilbert-Schmidt inner products $A(\xi) = \langle \hat{\mathbb{1}}, \hat{T}_\xi \rangle$ and $A(\mathbf{x}) = \langle \hat{\mathbb{1}}, \hat{R}_\mathbf{x} \rangle$, which are the coefficients of the expansion (also called symbols of \hat{A}) in one of the mentioned bases through the Bochner integrals [37]:

$$\hat{A} = \int \frac{d\xi}{(2\pi\hbar)^n} A(\xi) \hat{T}_\xi = \int \frac{d\mathbf{x}}{(\pi\hbar)^n} A(\mathbf{x}) \hat{R}_\mathbf{x}. \quad (32)$$

These facts are consequences of the orthogonality relations $\langle \hat{T}_\xi, \hat{T}_{\xi'} \rangle = 2^n \langle \hat{R}_\xi, \hat{R}_{\xi'} \rangle = (2\pi\hbar)^n \delta(\xi' - \xi)$. In particular, the Wigner representation of the operator vector $\hat{\mathbf{x}}$ in (6),

$$\mathbf{x} = (\hat{\mathbf{x}}, \hat{R}_\mathbf{x}) = (q_1, \dots, q_n, p_1, \dots, p_n)^\top, \quad (33)$$

is a real vector in phase space. In the following, we will use the Wigner representation of the master equation for a GDS, which is nothing more than an alternative description of the system evolution. Through this representation we will establish our first result in Theorem 1 in Sec. III B. Also, this will be important to prove the results in Appendixes B and C.

The Wigner representation of the LME in (1) for the operators in (8) (see [25,28,32]) is the Fokker-Planck equation [38]

$$\frac{dW(\mathbf{x}, t)}{dt} = -\frac{\partial}{\partial \mathbf{x}^\top} [\mathbf{v}_U(\mathbf{x}, t) + \mathbf{v}_{NU}(\mathbf{x}, t)] \quad (34)$$

²A real $2n \times 2n$ matrix \mathbb{M} is said Hamiltonian matrix iff $\mathbb{J}\mathbb{M}$ (or equivalently $\mathbb{M}\mathbb{J}$) is symmetric, where \mathbb{J} is in (7).

³The matrix \mathbb{Q} is a member of the compact symplectic group $\text{Sp}(n) := \text{Sp}(2n, \mathbb{C}) \cap \text{SU}(2n)$.

for the Wigner function⁴ $W(\mathbf{x}) = \frac{1}{(\pi\hbar)^n} \langle \hat{\rho}, \hat{R}_{\mathbf{x}} \rangle$. In the above equation, we identify the Fokker-Planck current vectors:

$$\mathbf{v}_U(\mathbf{x}, t) = (\mathbf{J}\mathbb{B}'\mathbf{x} - \boldsymbol{\xi}') W(\mathbf{x}, t), \quad (35a)$$

$$\mathbf{v}_{NU}(\mathbf{x}, t) = -\frac{1}{2} \mathbb{D} \frac{\partial}{\partial \mathbf{x}} W(\mathbf{x}, t) - \mathbb{C}\mathbf{J}\mathbf{x}, \quad (35b)$$

corresponding, respectively, to the unitary (reversible) and nonunitary (irreversible) contributions to the evolution of $W(\mathbf{x})$. Note that the first term in Eq. (34) is the Poisson bracket

$$-\frac{\partial}{\partial \mathbf{x}^\top} [\mathbf{v}_U(\mathbf{x}, t)] = [H_{\text{eff}}, W(\mathbf{x}, t)]_{\text{cl}} \quad (36)$$

between the Wigner function and the Hamiltonian $H_{\text{eff}} = \frac{1}{2} \mathbf{x}^\top \mathbb{B}' \mathbf{x} + \mathbf{x}^\top \mathbf{J} \boldsymbol{\xi}'$, which is the Wigner symbol of (8a).

When the initial state is a Gaussian state, the evolved Wigner function of $\hat{\sigma}_t$ in (21) is

$$W_G(\mathbf{x}, t) = \frac{\langle \hat{\sigma}_t, \hat{R}_{\mathbf{x}} \rangle}{(\pi\hbar)^n} = \frac{e^{-\frac{1}{2\hbar} (\mathbf{x} - \langle \hat{\mathbf{x}} \rangle_t)^\top \mathbb{V}_t^{-1} (\mathbf{x} - \langle \hat{\mathbf{x}} \rangle_t)}}{(2\pi\hbar)^n \sqrt{\det \mathbb{V}_t}}, \quad (37)$$

for $\langle \hat{\mathbf{x}} \rangle_t$ and \mathbb{V}_t in (19). This function is a multivariate Gaussian distribution, so the Fokker-Planck currents in (35) are true probability currents given by

$$\mathbf{v}_U^G(\mathbf{x}, t) = (\mathbf{J}\mathbb{B}'\mathbf{x} - \boldsymbol{\xi}') W_G(\mathbf{x}, t), \quad (38a)$$

$$\mathbf{v}_{NU}^G(\mathbf{x}, t) = \left[\frac{1}{2\hbar} \mathbb{D} \mathbb{V}_t^{-1} (\mathbf{x} - \langle \hat{\mathbf{x}} \rangle_t) - \mathbb{C}\mathbf{J}\mathbf{x} \right] W_G(\mathbf{x}, t). \quad (38b)$$

III. THERMAL EQUILIBRIUM IN GDS: GENERAL CONSIDERATIONS

For GDSs, if there exists a stationary state $\hat{\sigma}^S$, it will be unique for any initial state $\hat{\rho}_0$, i.e., $\lim_{t \rightarrow \infty} \hat{\rho}_t = \hat{\sigma}^S$ [26–28]. In particular, starting with an initial Gaussian state, the evolved state remains Gaussian throughout the whole evolution, therefore, $\hat{\sigma}^S$ is necessarily Gaussian.

The first moments and covariance matrix of the stationary state can be determined through the asymptotic behavior of Eqs. (19) and for that we resort to the Lyapunov theory of stability [39]. Note that the only way to erase all the information about any initial condition in Eqs. (19) is to admit a matrix \mathbb{A} with all of its eigenvalues with negative real part, which is the same as saying that \mathbb{A} is a *Hurwitz* matrix. Consequently, the covariance matrix of $\hat{\sigma}^S$ is a solution of the Lyapunov equation

$$\frac{d\mathbb{V}^S}{dt} = (\mathbb{A}\mathbb{V}^S + \mathbb{V}^S\mathbb{A}^\top) + \frac{\mathbb{D}}{\hbar} = 0. \quad (39)$$

Recalling that \mathbb{V}^S is strictly positive definite, $\mathbb{V}^S > 0$ and together with the Hurwitz condition over \mathbb{A} , the Lyapunov theorem [39] sets that $\mathbb{D} > 0$. In this case, the linear dynamical system in (16) is said globally asymptotically stable (AS) [39]

and the solutions in (19) attain the asymptotic values

$$\langle \hat{\mathbf{x}} \rangle^S = \mathbb{A}^{-1} \boldsymbol{\xi}', \quad (40a)$$

$$\mathbb{V}^S = \frac{1}{\hbar} \int_0^\infty dt e^{\mathbb{A}t} \mathbb{D} e^{\mathbb{A}^\top t}, \quad (40b)$$

where it is clear that any trace of the initial state disappears.

A. Gibbs states as stationary states

A stationary Gaussian state in a GDS corresponds to a thermal equilibrium state when $\hat{\sigma}^S = \hat{\sigma}^{\text{th}}$ is the Gibbs state

$$\hat{\sigma}^{\text{th}} = \frac{\hat{\sigma}^{\text{th}}}{\mathcal{Z}^{\text{th}}}, \quad \hat{\sigma}^{\text{th}} = e^{-\beta \hat{H}}, \quad \mathcal{Z}^{\text{th}} = \text{Tr}(\hat{\sigma}^{\text{th}}), \quad (41)$$

where β is the “inverse temperature” and $\hat{H} = \hat{H}_{\text{eff}}$ is the quadratic Hamiltonian in (8a) or, more generically, another quadratic Hamiltonian such that $[\hat{H}, \hat{H}_{\text{eff}}] = 0$. In the following we will establish necessary conditions over the quadratic Hamiltonians \hat{H}_{eff} and \hat{H} , which allows a GDS to have an equilibrium thermal state.

Without loss of generality, we can set the origin of the phase-space coordinates \mathbf{x} in (33) such $\langle \hat{\mathbf{x}} \rangle^{\text{th}} = \text{Tr}(\hat{\sigma}^{\text{th}} \hat{\mathbf{x}}) = -\mathbb{B}^{-1} \mathbf{J} \boldsymbol{\xi} = 0$, where the vector $\boldsymbol{\xi}$ is associated with a possible linear term of \hat{H} . Therefore, the quadratic Hamiltonian \hat{H} can be chosen as

$$\hat{H} = \frac{1}{2} \hat{\mathbf{x}}^\top \mathbb{B} \hat{\mathbf{x}}, \quad (42)$$

i.e., with $\boldsymbol{\xi} = 0$. The Hessian matrix of the Hamiltonian has to be positive definite, $\mathbb{B} > 0$, in order to fulfill the normalization condition $\text{Tr}(\hat{\sigma}^{\text{th}}) = 1$ [36]. From \hat{H}_{eff} in (8a) and \hat{H} in (42), we have

$$\begin{aligned} [\hat{H}, \hat{H}_{\text{eff}}] &= -\frac{i\hbar}{2} \hat{\mathbf{x}}^\top \mathbf{J} [\mathbf{J}\mathbb{B}, \mathbf{J}\mathbb{B}'] \hat{\mathbf{x}} - i\hbar (\boldsymbol{\xi}')^\top \mathbb{B} \hat{\mathbf{x}} = 0 \\ &\Leftrightarrow [\mathbf{J}\mathbb{B}, \mathbf{J}\mathbb{B}'] = 0 \quad \text{and} \quad \boldsymbol{\xi}' = 0. \end{aligned} \quad (43)$$

Therefore, the Hamiltonian of the free evolution of a GDS with an equilibrium thermal state must also be of the form

$$\hat{H}_{\text{eff}} = \frac{1}{2} \hat{\mathbf{x}}^\top \mathbb{B}' \hat{\mathbf{x}}. \quad (44)$$

Comparing the general form of the density operator of a Gaussian state (21) with the thermal state $\hat{\sigma}^{\text{th}}$ in (41) with \hat{H} in (42), we arrive to

$$\mathbb{U}^{\text{th}} = \hbar\beta\mathbb{B}. \quad (45)$$

Applying condition (23), we get

$$\mathbf{J}\mathbb{B}\mathbb{V}^{\text{th}} = \mathbb{V}^{\text{th}}\mathbf{B}\mathbf{J} \Leftrightarrow [\mathbf{J}\mathbb{B}, \mathbb{V}^{\text{th}}\mathbf{J}] = 0, \quad (46)$$

where \mathbb{V}^{th} is the covariance matrix of the Gibbs Gaussian state in (41). Using (22), it is clear that $\mathbb{V}^{\text{th}}\mathbf{J}$ is a function of $\mathbf{J}\mathbb{B}$, viz.,

$$\mathbb{V}^{\text{th}}\mathbf{J} = -\frac{i}{2} \coth\left(\frac{i\hbar\beta}{2} \mathbf{J}\mathbb{B}\right). \quad (47)$$

Taking into account the relation $[\mathbf{J}\mathbb{B}, \mathbf{J}\mathbb{B}'] = 0$ from (43) and that $\mathbb{V}^{\text{th}}\mathbf{J}$ is a function of $\mathbf{J}\mathbb{B}$, it is also true that

$$[\mathbf{J}\mathbb{B}', \mathbb{V}^{\text{th}}\mathbf{J}] = 0 \Leftrightarrow \mathbf{J}\mathbb{B}'\mathbb{V}^{\text{th}} = \mathbb{V}^{\text{th}}\mathbb{B}'\mathbf{J}. \quad (48)$$

⁴The Wigner function is proportional to the Wigner symbol of the density operator; this proportionality guarantees the normalization of that function as a quasiprobability density (see [24,28,36,37], for instance).

According to Williamson theorem it is possible to find a symplectic matrix \mathbf{S}^{th} such that

$$(\mathbf{S}^{\text{th}})^{\vee^{\text{th}}}(\mathbf{S}^{\text{th}})^{\top} = \mathbb{k}^{\text{th}} \oplus \mathbb{k}^{\text{th}}, \quad (49)$$

and using Eq. (25) of Lemma 2, the matrix $(\mathbb{Q}\mathbf{S}^{\text{th}})$ diagonalizes $\vee^{\text{th}}\mathbf{J}$:

$$\mathbb{Q}\mathbf{S}^{\text{th}}\vee^{\text{th}}\mathbf{J}(\mathbb{Q}\mathbf{S}^{\text{th}})^{-1} = ({}_{t}\mathbb{k}^{\text{th}}) \oplus (-{}_{t}\mathbb{k}^{\text{th}}). \quad (50)$$

However, due to (47), the same matrix $(\mathbb{Q}\mathbf{S}^{\text{th}})$ also diagonalizes $\mathbf{J}\mathbb{B}$. Using Eq. (31) one realizes that $(\mathbf{S}^{\text{th}})^{-\top}$ is the symplectic matrix that diagonalizes \mathbb{B} , i.e.,

$$(\mathbf{S}^{\text{th}})^{-\top}\mathbb{B}(\mathbf{S}^{\text{th}})^{-1} = \omega \oplus \omega, \quad \omega = \text{diag}(\omega_1, \dots, \omega_n), \quad (51)$$

where $\omega_j > 0$ ($j = 1, \dots, n$) are the symplectic eigenvalues of the Hessian matrix $\mathbb{B} > 0$ in (42). It is also possible to define, again according to Lemma 2, the canonical form of $\mathbf{J}\mathbb{B}$:

$$(\mathbf{J}\mathbb{B})_{\text{d}} = (\mathbb{Q}\mathbf{S}^{\text{th}})\mathbf{J}\mathbb{B}(\mathbb{Q}\mathbf{S}^{\text{th}})^{-1} = ({}_{t}\omega) \oplus (-{}_{t}\omega). \quad (52)$$

Finally, from (47), the relation between the symplectic spectra of \vee^{th} and \mathbb{B} is

$$\mathbb{k}^{\text{th}} = \frac{1}{2} \coth\left(\frac{\hbar\beta\omega}{2}\right) \quad (53)$$

or, equivalently, $\omega = \frac{1}{\hbar\beta} g(\mathbb{k}^{\text{th}})$ [see Eq. (29)].

For a phase space described by \mathbf{x} in (33), the classical counterpart of the Hamiltonian \hat{H} in (42) coincides with its Wigner symbol, i.e.,

$$H = \frac{1}{2}\mathbf{x}^{\top}\mathbb{B}\mathbf{x} = \langle \hat{H}, \hat{R}_{\mathbf{x}} \rangle \quad (54)$$

and the solution of the Hamilton equation $\dot{\mathbf{x}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{x}}$ is given by $\mathbf{x}(t) = \tilde{\mathbf{S}}_t \mathbf{x}(0)$ with

$$\tilde{\mathbf{S}}_t = e^{\mathbf{J}\mathbb{B}t} = (\mathbb{Q}\mathbf{S}^{\text{th}})^{-1} e^{(\mathbf{J}\mathbb{B})_{\text{d}}t} (\mathbb{Q}\mathbf{S}^{\text{th}}) \quad (55)$$

and $(\mathbf{J}\mathbb{B})_{\text{d}}$ in (52). The matrix $\tilde{\mathbf{S}}_t$ defined above will be important in Sec. III B, but here it is worth to note that it generates a Hamiltonian flow around the elliptical fixed point $\mathbf{x} = 0$ and that, from (51), ω_j are the eigenfrequencies of the Hamiltonian (54). This is a direct consequence of the positive definiteness of \mathbb{B} and, by this reason, we call positive elliptic all the Hamiltonians in (42) with $\mathbb{B} > 0$. Consequently, a positive-elliptic Hamiltonian in (42) is a necessary condition for a GDS to have an n -mode equilibrium thermal state since it is necessary for the convergence of the partition function \mathcal{Z}^{th} in (41).

Up to this point we were describing some properties of thermal states associated to quadratic Hamiltonians. In the next section, we will show the conditions over the diffusion and dissipation matrices \mathbb{D} and \mathbb{C} , respectively, which define a GDS with a thermal equilibrium state.

B. Diffusion and dissipation matrices for thermal equilibrium

The Lyapunov equation (39) for the covariance matrix of a GDS thermal-equilibrium state \vee^{th} , where \mathbb{A} is given in (18), attains a simpler form through condition (48):

$$\mathbb{C}\vee^{\text{th}} + \vee^{\text{th}}\mathbb{J}\mathbb{C} = \frac{\mathbb{D}}{\hbar}, \quad (56)$$

whose unique formal solution [see Eq. (40b)] is

$$\vee^{\text{th}} = \frac{1}{\hbar} \int_0^{\infty} dt e^{-\mathbb{C}\mathbf{J}t} \mathbb{D} e^{-\mathbf{J}\mathbb{C}t}, \quad (57)$$

where according to Lyapunov theorem [39], the matrix $-\mathbb{C}\mathbf{J}$ must be *Hurwitz* since $\vee^{\text{th}} > 0$ and $\mathbb{D} > 0$. Therefore, the dissipation matrix \mathbb{C} must be invertible and according to Lemma 1, $\mathbb{D} > 0$ and $\mathbb{F} > 0$, i.e., the diffusion and decoherence matrices in (10) and (11), respectively, of a QDS with a thermal equilibrium state must be positive definite. In the following we show that an explicit solution of the integral (57) can be obtained from the stationary condition over the Fokker-Planck equation (34) corresponding to a QDS with a thermal equilibrium state.

When a GDS has a stationary state, this is a Gaussian state $\hat{\sigma}^s$ and from the Fokker-Planck equation (34), we obtain the condition

$$-\frac{\partial}{\partial \mathbf{x}^{\top}} [\mathbf{v}_{\text{U}}^{\text{G}}(\mathbf{x}) + \mathbf{v}_{\text{NU}}^{\text{G}}(\mathbf{x})] = 0, \quad (58)$$

with $\mathbf{v}_{\text{U}}^{\text{G}}(\mathbf{x})$ and $\mathbf{v}_{\text{NU}}^{\text{G}}(\mathbf{x})$ in Eqs. (38). If this stationary state is a thermal equilibrium state, $\hat{\sigma}^s = \hat{\sigma}^{\text{th}}$, condition (58) simplifies to

$$-\frac{\partial}{\partial \mathbf{x}^{\top}} [\mathbf{v}_{\text{NU}}^{\text{th}}(\mathbf{x})] = 0 \quad (59)$$

since, due to Eq. (36), one has

$$[H_{\text{eff}}(\mathbf{x}), W^{\text{th}}(\mathbf{x})]_{\text{cl}} = \text{tr}[\mathbf{J}\mathbb{B}'\mathbf{x}\mathbf{x}^{\top}(\vee^{\text{th}})^{-1}] = 0, \quad (60)$$

where we employed Eq. (48) and the fact that $\text{tr}(\mathbb{A}) = \text{tr}(\mathbb{A}^{\top})$ for any matrix \mathbb{A} . Now we can establish the following theorem that characterizes a QDS with a thermal equilibrium state.

Theorem 1. A QDS has a thermal equilibrium state iff

$$\mathbf{v}_{\text{NU}}^{\text{th}}(\mathbf{x}) = 0. \quad (61)$$

The covariance matrix of such state is given by

$$\mathbf{J}\vee^{\text{th}} = \frac{1}{2\hbar}\mathbf{J}\mathbb{D}(\mathbf{J}\mathbb{C})^{-1}, \quad (62)$$

where

$$[\mathbf{J}\mathbb{D}, \mathbf{J}\mathbb{C}] = 0. \quad (63)$$

In order to prove our theorem, we use the divergence theorem⁵ and (59): both enable us to relate the divergence of the vector field $\mathbf{v}_{\text{NU}}^{\text{th}}(\mathbf{x})$ with the flux through the boundary $\partial\Omega$ of the region $\Omega \in \mathbb{R}^{2n}$,

$$\int_{\partial\Omega} \mathbf{n}^{\top} \mathbf{v}_{\text{NU}}^{\text{th}}(\mathbf{x}) ds = \int_{\Omega} \frac{\partial}{\partial \mathbf{x}^{\top}} [\mathbf{v}_{\text{NU}}^{\text{th}}(\mathbf{x})] d\mathbf{x}^{2n} = 0, \quad (64)$$

where \mathbf{n} is the $2n$ -dimensional real vector normal to the surface $\partial\Omega$. Since Ω has arbitrary volume, the necessary and sufficient condition in (61) is proved. From (38b), $\mathbf{v}_{\text{NU}}^{\text{th}}(\mathbf{x}) = [\frac{1}{2\hbar}\mathbb{D}(\vee^{\text{th}})^{-1} - \mathbb{C}\mathbf{J}]\mathbf{x} W^{\text{th}}(\mathbf{x}) = 0$ for any \mathbf{x} , ending up with Eq. (62). The relation in Eq. (63) follows from the requirement $\vee^{\text{th}} = (\vee^{\text{th}})^{\top}$. Note that if $[\mathbf{J}\mathbb{D}, \mathbf{J}\mathbb{C}] = 0$, then $\mathbb{D} e^{-\mathbf{J}\mathbb{C}t} = e^{-\mathbb{C}\mathbf{J}t} \mathbb{D}$ and the integration in (57) can be explicitly performed to obtain exactly the expression in (62).

⁵See, for example, Appendix A of [40].

It is worth to note that the covariance matrix \mathbb{V}^{th} of the thermal equilibrium state $\hat{\sigma}^{\text{th}}$ in (41) is completely determined by the Hessian \mathbb{B} of the Hamiltonian \hat{H} in (42). This is shown in Eqs. (49) and (53), where $(\mathbb{S}^{\text{th}})^{-\text{T}}$ symplectically diagonalizes \mathbb{B} , whose symplectic spectrum is contained in the diagonal matrix \mathfrak{w} . So, the expression in Eq. (62) simply establishes the connection between the fixed matrix \mathbb{V}^{th} and the dynamics of the GDS, that is, the one determined by the matrices \mathbb{C} and \mathbb{D} , which at the end determines $\hat{\sigma}^{\text{th}}$ as an equilibrium state. However, the Lyapunov equation (56) has common solutions [41], i.e., there are different matrices \mathbb{D} and \mathbb{C} which are able to give the same covariance matrix \mathbb{V}^{th} in (62). Each pair (\mathbb{D}, \mathbb{C}) corresponds to a different GDS which has as steady state the same thermal state $\hat{\sigma}^{\text{th}}$. In the next section we show that the QDBC determines common solutions.

Also, according to Theorem 1, a GDS has a thermal equilibrium state iff the set $\{\mathbb{J}\mathbb{V}^{\text{th}}, \mathbb{J}\mathbb{D}, \mathbb{J}\mathbb{C}\}$ is a commuting set of matrices. Therefore, there is a matrix that simultaneously diagonalizes the three matrices $\mathbb{J}\mathbb{V}^{\text{th}}$, $\mathbb{J}\mathbb{D}$, and $\mathbb{J}\mathbb{C}$ (see Appendix H). Using Lemma 2, the Hamiltonian matrix $\mathbb{J}\mathbb{V}^{\text{th}}$ is diagonalized by $(\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger}$ so, for a GDS with a thermal equilibrium state, we can always write

$$\mathbb{J}\mathbb{V}^{\text{th}} = (\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger} (\mathbb{J}\mathbb{V}^{\text{th}})_{\text{d}} ((\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger})^{-1}, \quad (65)$$

with $(\mathbb{J}\mathbb{V}^{\text{th}})_{\text{d}} = \iota \mathbb{k}^{\text{th}} \oplus (-\iota \mathbb{k}^{\text{th}})$ and \mathbb{k}^{th} in (53). Equivalently, applying the same lemma to the diagonalization of $\mathbb{J}\mathbb{D}$ we arrive at

$$\mathbb{J}\mathbb{D} = (\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger} (\mathbb{J}\mathbb{D})_{\text{d}} ((\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger})^{-1}, \quad (66)$$

where $(\mathbb{J}\mathbb{D})_{\text{d}} = (\iota \hbar \mathfrak{d}) \oplus (-\iota \hbar \mathfrak{d})$ with $\hbar \mathfrak{d}$ the $\mathbb{R}^{n \times n}$ diagonal matrix with the symplectic spectrum of \mathbb{D} in its diagonal.⁶ Because $\mathbb{J}\mathbb{C}$ is a skew-Hamiltonian matrix,⁷ its eigenvalues are real and with at least multiplicity equal to 2 [42]. So we can write

$$\mathbb{J}\mathbb{C} = (\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger} (\mathbb{J}\mathbb{C})_{\text{d}} ((\mathbb{Q}\mathbb{S}^{\text{th}})^{\dagger})^{-1}, \quad (67)$$

where $(\mathbb{J}\mathbb{C})_{\text{d}} = \mathfrak{j}\mathbb{c} \oplus \mathfrak{j}\mathbb{c}$ with $\mathfrak{j}\mathbb{c}$ a $\mathbb{R}^{n \times n}$ diagonal matrix. Therefore, using (62) we arrive to $\frac{1}{2\hbar} (\mathbb{J}\mathbb{D})_{\text{d}} (\mathbb{J}\mathbb{C})_{\text{d}}^{-1} = (\mathbb{J}\mathbb{V}^{\text{th}})_{\text{d}} = (\iota \mathbb{k}^{\text{th}}) \oplus (-\iota \mathbb{k}^{\text{th}})$ or, equivalently, to

$$\mathfrak{d}(\mathfrak{j}\mathbb{c})^{-1} = 2\mathbb{k}^{\text{th}} = \coth\left(\frac{\hbar\beta\mathfrak{w}}{2}\right) = \frac{1 + e^{-\hbar\beta\mathfrak{w}}}{1 - e^{-\hbar\beta\mathfrak{w}}}. \quad (68)$$

However, this relation says nothing about the dependence on β , the inverse temperature of the matrices \mathfrak{d} and $\mathfrak{j}\mathbb{c}$ composed by the eigenvalues of the matrices $\mathbb{J}\mathbb{D}$ and $\mathbb{J}\mathbb{C}$, respectively. In the next section we will show that a QDBC allows the determination of this dependence.

IV. GDSs SATISFYING A DETAILED BALANCE CONDITION

The notion of detailed balance is the principle governing the way thermal equilibrium is attained by classical Markov

processes [38]. It has several different quantum versions (see [19] and references therein) and in the context of QDS for finite-dimensional systems, the one due to Alicki [29] stands out because it allows the extension of time-reversal invariance of classical equilibrium to the quantum realm [19].

Inspired by the classical case in Markov processes, where the time-reversal invariance of transition probabilities is related to a particular definition of an inner product, Alicki's definition for quantum detailed balance is based on the $\hat{\rho}$ -Gelfand-Naimark-Segal ($\hat{\rho}$ -GNS) inner product in finite-dimension Hilbert spaces:

$$\langle \hat{A}, \hat{B} \rangle_{\text{GNS}} = \text{Tr}(\hat{\rho} \hat{A}^{\dagger} \hat{B}), \quad (69)$$

where $\hat{\rho}$ is a positive operator,⁸ $\hat{\rho} = \hat{\rho}^{\dagger} > 0$, and the operators \hat{A} and \hat{B} belong to a finite-dimensional C^* algebra. Thus, Alicki's QDBC relies on the notion of self-adjointness with respect to the $\hat{\rho}^{\text{th}}$ -GNS inner product, where $\hat{\rho}^{\text{th}} = e^{-\beta \hat{H}}$ is an unnormalized Gibbs state with Hamiltonian \hat{H} . A superoperator Λ is said to be self-adjoint with respect to the $\hat{\rho}$ -GNS inner product if

$$\langle \Lambda[\hat{A}], \hat{B} \rangle_{\text{GNS}} = \langle \hat{A}, \Lambda[\hat{B}] \rangle_{\text{GNS}}, \quad (70)$$

for any operators \hat{A} and \hat{B} in the C^* algebra.

The extension of Alicki's approach for continuous variable systems relies on a definition for the set of operators where the GNS inner product is well defined. In this regard, we consider operators acting on the separable Hilbert space of n -bosonic modes, $\mathcal{H} = \otimes_{j=1}^n \mathcal{H}_j$, of infinite dimension. In our case, the $\hat{\sigma}^{\text{th}}$ -GNS inner products are computed with $\hat{\sigma}^{\text{th}}$ in (41) and \hat{H} being the quadratic Hamiltonian in (42). Also, regardless of whether the operators \hat{A} and \hat{B} are bounded or unbounded, having continuous spectra or not, the domain of applicability of the $\hat{\sigma}^{\text{th}}$ -GNS inner product in (69) with $\hat{\rho} = \hat{\sigma}^{\text{th}}$ is over all operators such that the trace on this formula is finite.⁹

For the definition of the QDBC in the context of GDSs, we recall the notation of Sec. II, where $\bar{\Lambda}_t = e^{t\bar{\mathcal{L}}}$ with $\bar{\mathcal{L}}$ in (4) represents the superoperator that generates a GDS in the Heisenberg picture and $\Lambda_t = e^{t\mathcal{L}}$ with \mathcal{L} in (1), the one in the Schrödinger picture.

Definition 1. Consider the GDS $\{\bar{\Lambda}_t = e^{t\bar{\mathcal{L}}}\}_{t \geq 0}$ with the infinitesimal generator $\bar{\mathcal{L}} = \bar{\mathcal{L}}_{\text{U}} + \bar{\mathcal{L}}_{\text{NU}}^{\text{G}}$, where $\bar{\mathcal{L}}_{\text{U}}$ is defined in (5a) for the quadratic Hamiltonian in (44) and $\bar{\mathcal{L}}_{\text{NU}}^{\text{G}}$ is defined in (12). This GDS satisfies the QDBC with respect to $\hat{\sigma}^{\text{th}}$ in (41) if $\bar{\Lambda}_t$ is self-adjoint with respect to the $\hat{\sigma}^{\text{th}}$ -GNS inner product for all t . In such case, we say that $\bar{\Lambda}_t$ satisfies a $\hat{\sigma}^{\text{th}}$ -DBC.

⁸The operator $\hat{\rho}$ in Alicki's work [29] is a full-rank density operator in finite-dimensional systems. However, it is more convenient to extend the definition of a $\hat{\rho}$ -GNS inner product for unnormalized density operators $\hat{\rho}$ and, in particular, to unnormalized Gibbs states like $\hat{\sigma}^{\text{th}}$ in (41) (see [19]).

⁹The existence of the trace in (69) can be checked, for example, using $\langle \hat{A}, \hat{B} \rangle_{\text{GNS}} = \int_{\mathbb{R}^{2n}} d\xi \bar{\sigma}^{\text{th}}(\xi) (A^{\dagger} B)(\xi) = \int_{\mathbb{R}^{2n}} d\mathbf{x} \bar{\sigma}^{\text{th}}(\mathbf{x}) (A^{\dagger} B)(\mathbf{x})$, where in the integrands we have the Weyl and Wigner symbols of $\hat{\sigma}^{\text{th}}$ and $\hat{A}^{\dagger} \hat{B}$, respectively. Notwithstanding, any other representation that could be more convenient can be used.

⁶It is worth to note that \mathbb{S}^{th} simultaneously diagonalizes symplectically \mathbb{V}^{th} and \mathbb{D} .

⁷A real $2n \times 2n$ matrix \mathbb{M} is said a skew-Hamiltonian matrix iff $\mathbb{J}\mathbb{M}$ (or equivalently $\mathbb{M}\mathbb{J}$) is a skew-symmetric matrix.

The connection between the QDBC and a steady state of the GDS is in the following theorem, which is proved in Appendix B.

Theorem 2. If a GDS $\bar{\Lambda}_t$ (Heisenberg picture) satisfies a $\hat{\sigma}^{\text{th}}$ -DBC, then $\hat{\sigma}^{\text{th}}$ is invariant under the GDS Λ_t (Schrödinger picture), i.e., $\Lambda_t[\hat{\sigma}^{\text{th}}] = \hat{\sigma}^{\text{th}}$ or $\mathcal{L}[\hat{\sigma}^{\text{th}}] = 0$, equivalently.

In the theorem above, the statement $\mathcal{L}[\hat{\sigma}^{\text{th}}] = 0$ follows from $d\Lambda_t[\hat{\sigma}^{\text{th}}]/dt = \mathcal{L}[\Lambda_t[\hat{\sigma}^{\text{th}}]]$. So, if a GDS satisfies a $\hat{\sigma}^{\text{th}}$ -DBC, the quantum state $\hat{\sigma}^{\text{th}} = \hat{\sigma}^{\text{th}}/\text{Tr}(\hat{\sigma}^{\text{th}})$ is a stationary state of the evolution. Therefore, in order to attain thermal equilibrium, it is enough that the superoperator $\bar{\Lambda}_t$ of a GDS satisfies a $\hat{\sigma}^{\text{th}}$ -DBC for (the unnormalized Gibbs state) $\hat{\sigma}^{\text{th}}$ defined in (41) with \hat{H} in (42).

The equilibrium properties of a GDS can be extracted from its relation with the so-called modular automorphism group [19]:

$$\Xi_t[\hat{A}] = e^{\frac{i}{\hbar}\hat{H}t} \hat{A} e^{-\frac{i}{\hbar}\hat{H}t}, \quad (71)$$

with \hat{H} in (42) and $t \in \mathbb{C}$. Of particular relevance will be the elements of the group given by the superoperator $\Xi_{-t, \hbar\beta}[\cdot] = (\hat{\sigma}^{\text{th}})^{-1} \cdot \hat{\sigma}^{\text{th}}$, which existence is guaranteed for any finite value of β . The relation between the $\hat{\sigma}^{\text{th}}$ -DBC for a GDS and the above-defined modular group is established in the following theorem:

Theorem 3. If a GDS satisfies a $\hat{\sigma}^{\text{th}}$ -DBC, then $\bar{\Lambda}_t = e^{t\bar{\mathcal{L}}}$ and $\bar{\mathcal{L}}$ both commute with Ξ_t for all values of $t \in \mathbb{C}$.

This theorem was proved in [19] for QDS in finite-dimensional unital C^* algebras and our demonstration for GDSs follows almost the same lines (see Appendix C).

Now, due to the commutation relation between the Hamiltonians (42) and (44), $[\hat{H}, \hat{H}_{\text{eff}}] = 0$, we have that $\bar{\mathcal{L}}_U$ commutes with Ξ_t . Therefore, the commutation of $\bar{\mathcal{L}} = \bar{\mathcal{L}}_U + \bar{\mathcal{L}}_{\text{NU}}^G$ with the automorphism Ξ_t in (71) is equivalent to the following statement: $\bar{\mathcal{L}}_{\text{NU}}^G$ commutes with Ξ_t . The generator $\bar{\mathcal{L}}_{\text{NU}}^G$ in (12) for a GDS is an explicit function of both diffusion and dissipation matrices \mathbb{D} and \mathbb{C} , respectively. The properties of these matrices that stem from the fact that $\bar{\mathcal{L}}_{\text{NU}}^G$ commutes with Ξ_t is settled by the following theorem:

Theorem 4. A GDS satisfies a $\hat{\sigma}^{\text{th}}$ -DBC if and only if the diffusion and dissipation matrices, defined in Eqs. (10), are such that

$$\mathbb{D} = \tilde{\mathbf{S}}_t \mathbb{D} \tilde{\mathbf{S}}_t^\dagger \quad \text{and} \quad \mathbb{C} = \tilde{\mathbf{S}}_t \mathbb{C} \tilde{\mathbf{S}}_t^\dagger, \quad (72)$$

for $\tilde{\mathbf{S}}_t$ ($t \in \mathbb{R}$) in Eq. (55).

This means that both matrices are invariant under a congruence relation through the symplectic matrix $\tilde{\mathbf{S}}_t$.

We begin the proof first noting that, for real values of t , the operator $e^{\frac{i}{\hbar}\hat{H}t}$ in (71), with \hat{H} in (42), belongs to the metaplectic group $\text{Mp}(2n, \mathbb{R})$ of unitary operators and, consequently, is associated with the symplectic matrix $\tilde{\mathbf{S}}_t^{-1} = \tilde{\mathbf{S}}_{-t}$ defined by (55) [24,37]. So, in the Heisenberg picture, the action of these operators on the vector (6) is described by

$$\Xi_{-t}[\hat{\mathbf{x}}] = \tilde{\mathbf{S}}_{-t}^{-1} \hat{\mathbf{x}}. \quad (73)$$

Note that the above equation is equivalent to $\Xi_t[\hat{\mathbf{x}}] = \tilde{\mathbf{S}}_t \hat{\mathbf{x}}$. Using these actions, in Appendix D, we prove that

$$\Xi_{-t} \left[\frac{\partial}{\partial \hat{\mathbf{x}}^\dagger} [\cdot] \right] = \frac{\partial}{\partial \hat{\mathbf{x}}^\dagger} [\Xi_{-t}[\cdot]] \tilde{\mathbf{S}}_t, \quad (74)$$

where $\Xi_{-t} = \Xi_t^{-1}$. This relation can be equivalently rewritten as $\Xi_{-t}[\partial/\partial \hat{\mathbf{x}}[\cdot]] = \tilde{\mathbf{S}}_t^\dagger \partial/\partial \hat{\mathbf{x}}[\Xi_{-t}[\cdot]]$. Now, using Eq. (74), we get

$$\Xi_{-t} \left[\mathbf{J} \hat{\mathbf{x}} \frac{\partial}{\partial \hat{\mathbf{x}}} [\Xi_t[\cdot]] \right] = \tilde{\mathbf{S}}_t^\dagger \mathbf{J} \hat{\mathbf{x}} \frac{\partial}{\partial \hat{\mathbf{x}}} [\cdot] \tilde{\mathbf{S}}_t, \quad (75a)$$

$$\Xi_{-t} \left[\frac{\partial}{\partial \hat{\mathbf{x}}} \frac{\partial}{\partial \hat{\mathbf{x}}^\dagger} [\Xi_t[\cdot]] \right] = \tilde{\mathbf{S}}_t^\dagger \frac{\partial}{\partial \hat{\mathbf{x}}} \frac{\partial}{\partial \hat{\mathbf{x}}^\dagger} [\cdot] \tilde{\mathbf{S}}_t, \quad (75b)$$

where in (75a) we used the symplectic condition $\tilde{\mathbf{S}}_t^{-1} = -\mathbf{J} \tilde{\mathbf{S}}_t^\dagger \mathbf{J}$. Finally, inserting Eqs. (75) in (12) we attain

$$\begin{aligned} & \Xi_{-t}[\bar{\mathcal{L}}_{\text{NU}}^G[\Xi_t[\cdot]]] \\ &= \text{tr} \left(\tilde{\mathbf{S}}_t \mathbb{C} \tilde{\mathbf{S}}_t^\dagger \mathbf{J} \hat{\mathbf{x}} \frac{\partial}{\partial \hat{\mathbf{x}}^\dagger} [\hat{\mathcal{O}}_t] + \frac{1}{2} \tilde{\mathbf{S}}_t \mathbb{D} \tilde{\mathbf{S}}_t^\dagger \frac{\partial}{\partial \hat{\mathbf{x}}} \frac{\partial}{\partial \hat{\mathbf{x}}^\dagger} \hat{\mathcal{O}}_t \right), \end{aligned} \quad (76)$$

which is equal to $\bar{\mathcal{L}}_{\text{NU}}^G[\cdot]$ in (12) iff \mathbb{D} and \mathbb{C} satisfy Eqs. (72). In summary, all these prove that Eqs. (72) are equivalent to the statement that $\bar{\mathcal{L}}_{\text{NU}}^G$ commutes with Ξ_t for any real value of t . However, due to $\Xi_{t^*} = (\Xi_t)^{*/t}$, $\bar{\mathcal{L}}_{\text{NU}}^G$ must also commute with Ξ_{t^*} for any complex value t^* . We finish the demonstration of Theorem 4 noting that the commutation between $\bar{\mathcal{L}}_{\text{NU}}^G$ and Ξ_t with $t \in \mathbb{C}$ is tantamount to say that a GDS verifies a $\hat{\sigma}^{\text{th}}$ -DBC, according to Theorem 3.

As a consequence of the results in Theorem 4, the Lindblad operators in (8b) are restricted to a particular structure since the congruence relations in (72) are extended to the decoherence matrix \mathbb{F} in (11), due to the definitions in (10), that is, $\mathbb{F} = \tilde{\mathbf{S}}_t \mathbb{F} \tilde{\mathbf{S}}_t^\dagger$ for $\tilde{\mathbf{S}}_t$ ($t \in \mathbb{R}$) in Eq. (55). Explicitly, the continuous-variable version of Theorem 3 from Alicki's work [29], which deals with QDS in discrete Hilbert spaces, is a mere reformulation of our Theorem 4.

Theorem 5. A GDS satisfies a $\hat{\sigma}^{\text{th}}$ -DBC iff the Lindblad operators describing the GDS are eigenoperators of the automorphism group Ξ_{-t} , i.e.,

$$\Xi_{-t}[\hat{L}_j] = e^{i\omega_j t} \hat{L}_j, \quad (77a)$$

$$\Xi_{-t}[\hat{L}_{n+j}] = e^{-i\omega_j t} \hat{L}_{n+j} = e^{-i\omega_j t} e^{-\frac{1}{2}\hbar\beta\omega_j} \hat{L}_j^\dagger, \quad (77b)$$

with $j = 1, \dots, n$ and $\omega_j > 0$ are the eigenfrequencies (symplectic eigenvalues) of the Hessian matrix \mathbb{B} which defines, through the Hamiltonian (42), the thermal equilibrium state $\hat{\sigma}^{\text{th}}$ of the GDS.

The proof for this theorem stands on Theorem 4 and on Eq. (73), and some technical details are placed in Appendix E. In this Appendix we prove that Eqs. (72) are equivalent to write the decoherence matrix \mathbb{F} in the following characteristic form:

$$\begin{aligned} \mathbb{F} &= \sum_{j=1}^n (|s_j|^2 \bar{\mathbf{I}}_j \bar{\mathbf{I}}_j^\dagger + |r_j|^2 \bar{\mathbf{I}}_j^* (\bar{\mathbf{I}}_j^*)^\dagger) \\ &= (\mathbb{Q}\mathbf{S}^{\text{th}})^{-1} (\mathfrak{s} \oplus \mathfrak{r}) (\mathfrak{s}^* \oplus \mathfrak{r}^*) (\mathbb{Q}\mathbf{S}^{\text{th}})^{-\dagger}, \end{aligned} \quad (78)$$

where $\{\bar{\mathbf{I}}_j\}_{j=1, \dots, n}$ are the eigenvectors of $\tilde{\mathbf{S}}_t$, i.e.,

$$\tilde{\mathbf{S}}_t \bar{\mathbf{I}}_j = e^{i\omega_j t} \bar{\mathbf{I}}_j \quad (j = 1, \dots, n), \quad (79)$$

the matrix $\mathfrak{r} := \text{diag}(r_1, \dots, r_n)$ is the diagonal matrix satisfying

$$|\mathfrak{r}|^2 = \text{diag}(|r_1|^2, \dots, |r_n|^2) = e^{-\frac{1}{2}\hbar\beta\omega} |s|^2 \quad (80)$$

[see Eq. (E9) in Appendix E]; the diagonal matrix ω is defined in (51) and contains the symplectic spectrum of the Hessian matrix \mathbb{B} of the Hamiltonian in (42); the matrix $|s|^2 = \text{diag}(|s_1|^2, \dots, |s_n|^2)$ is a real diagonal matrix with a particular temperature dependence. Although Theorem 5 implies this dependence for any GDS satisfying its conditions, we will keep our track on the proof, postponing the analysis of $|s|^2$ to Sec. VI [see Eq. (94)].

Comparing the canonical form (78) with (11), the Lindblad operators $\hat{L}_k = \mathbf{I}_k^\top \mathbf{J} \hat{\mathbf{x}}$ in (8b), with $k = 1, \dots, K = 2n$, correspond to the vectors

$$\mathbf{I}_j = s_j \bar{\mathbf{I}}_j, \quad \mathbf{I}_{n+j} = r_j \bar{\mathbf{I}}_j^* \quad (j = 1, \dots, n). \quad (81)$$

From (79), $\tilde{\mathbf{S}}_t \mathbf{I}_{n+j} = \tilde{\mathbf{S}}_t \bar{\mathbf{I}}_j^* = e^{-i\omega_j t} r_j \bar{\mathbf{I}}_j^*$, consequently $\Xi_{-t}[\hat{L}_j] = \mathbf{I}_j^\top \tilde{\mathbf{S}}_t^\top \mathbf{J} \hat{\mathbf{x}} = s_j \bar{\mathbf{I}}_j^\top \tilde{\mathbf{S}}_t^\top \mathbf{J} \hat{\mathbf{x}} = e^{i\omega_j t} s_j \bar{\mathbf{I}}_j^\top \mathbf{J} \hat{\mathbf{x}} = e^{i\omega_j t} \hat{L}_j$ and $\Xi_{-t}[\hat{L}_{n+j}] = r_j \bar{\mathbf{I}}_j^\top \tilde{\mathbf{S}}_t^\top \mathbf{J} \hat{\mathbf{x}} = e^{i\omega_j t} r_j \bar{\mathbf{I}}_j^\top \mathbf{J} \hat{\mathbf{x}} = e^{-i\omega_j t} e^{-\frac{1}{2} \hbar \beta \omega_j} \hat{L}_j^\dagger$, where we used (73) and (80) and the symplectic condition $\mathbf{J} \tilde{\mathbf{S}}_t^{-1} = \tilde{\mathbf{S}}_t^\top \mathbf{J}$. With all these, we finish the proof of Theorem 5.

Here, two important observations are in order. First, the decoherence matrix (78) could be written as $\mathbb{F} = \mathbb{V} \mathbb{V}^\dagger$, where

$$\mathbb{V} = (\mathbb{Q} \mathbf{S}^{\text{th}})^{-1} (\mathfrak{s} \oplus e^{-\frac{1}{2} \hbar \beta \omega}) \mathbb{W} \quad (82)$$

and $\mathbb{W} \in \mathbb{C}^{2n \times 2n}$ is an arbitrary unitary matrix. Therefore, alternatively we can use the column vectors \mathbf{I}'_j of the matrix \mathbb{V} to define new Lindblad operators $\hat{L}'_k = (\mathbf{I}'_k)^\top \mathbf{J} \hat{\mathbf{x}}$ with $k = 1, \dots, 2n$. Notwithstanding, it is straightforward to check that transformation (82) corresponds to

$$\hat{L}_k \longrightarrow \hat{L}'_k = \sum_{j=1}^{2n} \mathbb{W}_{kj} \hat{L}_j \quad (83)$$

and the arbitrariness introduced by \mathbb{W} in (82) is equivalent to a well-known symmetry of any QDS (see, e.g., [31, Sec. 3.2.2]): the LME (1) is invariant under the unitary transformation in (83) of the Lindblad operators. Therefore, this symmetry also holds for a GDS. Thus, the semigroup dynamic associated to the new set of Lindblad operators \hat{L}'_k is exactly the same as the one generated by the old set, i.e., \hat{L}_k in (8b) with $\bar{\mathbf{I}}_k$ in (81). Second, Theorem 5 shows that it is enough to consider only n Lindblad operators to describe a GDS which satisfies a $\hat{\sigma}^{\text{th}}$ -DBC, although we are dealing with infinite-dimensional quantum systems. In this sense, GDS are like QDS in finite-dimensional systems.

Properties of the environment, characterized by the matrices \mathbb{D} and \mathbb{C} , can be extracted from the canonical form of the decoherence matrix \mathbb{F} in (78). To this end, we conveniently rewrite \mathbb{F} in (78) as

$$\mathbb{F} = \frac{\mathbb{D}}{\hbar} + \iota \mathbb{C} = (\mathbf{S}^{\text{th}})^{-1} (\mathbb{A}_r - \iota \mathbf{J} \mathbb{A}_i) (\mathbf{S}^{\text{th}})^{-\top}, \quad (84)$$

where we define the diagonal positive defined matrices

$$\begin{aligned} \mathbb{A}_r &= \frac{|s|^2 + |\mathfrak{r}|^2}{2} \oplus \frac{|s|^2 + |\mathfrak{r}|^2}{2} \\ &= \frac{|s|^2}{2} (\mathbb{1} + e^{-\hbar \beta \omega}) \oplus \frac{|s|^2}{2} (\mathbb{1} + e^{-\hbar \beta \omega}) \end{aligned} \quad (85a)$$

$$\begin{aligned} \text{and } \mathbb{A}_i &= \frac{|s|^2 - |\mathfrak{r}|^2}{2} \oplus \frac{|s|^2 - |\mathfrak{r}|^2}{2} \\ &= \frac{|s|^2}{2} (\mathbb{1} - e^{-\hbar \beta \omega}) \oplus \frac{|s|^2}{2} (\mathbb{1} - e^{-\hbar \beta \omega}), \end{aligned} \quad (85b)$$

using the notation $|s|^2 = \text{diag}(|s_1|^2, \dots, |s_n|^2)$, $|\mathfrak{r}|^2 = \text{diag}(|r_1|^2, \dots, |r_n|^2)$, and the matrix relation in (80). Therefore, the diffusion and dissipation matrices are

$$\mathbb{D} = (\mathbf{S}^{\text{th}})^{-1} \hbar \mathbb{A}_r (\mathbf{S}^{\text{th}})^{-\top}, \quad (86a)$$

$$\mathbb{C} = (\mathbf{S}^{\text{th}})^{-1} \mathbf{J}^\top \mathbb{A}_i (\mathbf{S}^{\text{th}})^{-\top}. \quad (86b)$$

Note that each pair of matrices \mathbb{D} and \mathbb{C} uniquely determine a GDS. Since these only depend on the real matrix $|s|^2$, it is enough to choose a real matrix $|s| = \text{diag}(|s_1|, \dots, |s_n|)$ instead of a complex matrix \mathfrak{s} in (78) with \mathfrak{r} in (80). In this way the relations in (81) change to

$$\mathbf{I}_j = |s_j| \bar{\mathbf{I}}_j, \quad \mathbf{I}_{n+j} = |s_j| e^{-\frac{1}{2} \hbar \beta \omega_j} \bar{\mathbf{I}}_j^* \quad (j = 1, \dots, n), \quad (87)$$

which gives the expressions for the coefficients of the Lindblad operators in (8b).

The structure of \mathbb{D} and \mathbb{C} in (86) is determined by the matrices \mathbf{S}^{th} , ω , and $|s|$. It is worth to note that, according to Eq. (52), the matrices \mathbf{S}^{th} and ω can be extracted from the ordinary diagonalization of the Hamiltonian matrix $\mathbf{J} \mathbb{B}$, where \mathbb{B} is the Hessian matrix of the Hamiltonian \hat{H} in (42), that defines the thermal equilibrium state $\hat{\sigma}^{\text{th}}$ in (41). However, we will see in Sec. VI that $|s|$ depends on the coupling constants of the system and the environment. Thus, each matrix $|s|$, corresponding to different coupling constants, defines one different pair of diffusion and dissipation matrices through Eqs. (86). The dynamics of the GDSs associated to these matrices are different because each one corresponds to different Lindblad operators, which are not associated with the symmetry in (83). Nonetheless, all these GDSs have the same thermal equilibrium state $\hat{\sigma}^{\text{th}}$. This is checked in Appendix E through the symplectic diagonalization of \mathbb{V}^{th} in (E6), where the matrix $\frac{1}{2} \mathbb{A}_i \mathbb{A}_i^{-1} = \mathbb{K}^{\text{th}} \oplus \mathbb{K}^{\text{th}}$ is the direct sum of the symplectic spectrum of \mathbb{V}^{th} that does not depend on $|s|$.

V. MASTER EQUATION OF A GDS SATISFYING A $\hat{\sigma}^{\text{th}}$ -DBC

Here, we prove that the master equation of a GDS that satisfies a $\hat{\sigma}^{\text{th}}$ -DBC, i.e., satisfying all theorems in last section, has the form of the quantum optical master equation (QOME) [see Eq. (5.4.14) in [30]]:

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} &= -\frac{\iota}{\hbar} [\hat{H}_{\text{OME}}, \hat{\rho}_t] \\ &+ \sum_k \frac{\bar{\gamma}_k}{2} (\bar{n}_k + 1) (2\hat{X}_k^- \hat{\rho}_t \hat{X}_k^+ - \{\hat{X}_k^+ \hat{X}_k^-, \hat{\rho}_t\}) \\ &+ \sum_k \frac{\bar{\gamma}_k}{2} \bar{n}_k (2\hat{X}_k^+ \hat{\rho}_t \hat{X}_k^- - \{\hat{X}_k^- \hat{X}_k^+, \hat{\rho}_t\}), \end{aligned} \quad (88)$$

where $\bar{\gamma}_k$ are the coupling constants between the bath and the system, $\bar{n}_k = (e^{\hbar \beta \bar{\omega}_k} - 1)^{-1}$ is the Planck factor with $\bar{\omega}_k > 0$, and the operators \hat{X}_k^\pm are eigenoperators of \hat{H}_{OME} , i.e.,

$$[\hat{H}_{\text{OME}}, \hat{X}_k^\pm] = \pm \hbar \bar{\omega}_k \hat{X}_k^\pm, \quad (89)$$

so $(\hat{X}_k^-)^\dagger = \hat{X}_k^+$.

In order to rewrite the master equation of a GDS satisfying a $\hat{\sigma}^{\text{th}}$ -DBC, first we rewrite the Lindblad operators

in (8b) as

$$\hat{L}_j = |s_j\rangle \hat{X}_j^-, \quad (90a)$$

$$\hat{L}_{n+j} = e^{-\frac{1}{2}\hbar\beta\omega_j} \hat{L}_j^\dagger = |s_j\rangle e^{-\frac{1}{2}\hbar\beta\omega_j} \hat{X}_j^+ \quad (90b)$$

with

$$\hat{X}_j^- = \bar{\mathbf{I}}_j^\dagger \mathbf{J} \hat{\mathbf{x}} \quad \text{and} \quad \hat{X}_j^+ = (\hat{X}_j^-)^\dagger = (\bar{\mathbf{I}}_j^*)^\dagger \mathbf{J} \hat{\mathbf{x}}, \quad (91)$$

when employing Eq. (87) for $j = 1, \dots, n$. Therefore, using these expressions for the Lindblad operators, the master equation in (1) becomes

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} &= -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}_t] \\ &+ \sum_{j=1}^n \frac{|s_j|^2}{2\hbar} (2\hat{X}_j^- \hat{\rho}_t \hat{X}_j^+ - \{\hat{X}_j^+ \hat{X}_j^-, \hat{\rho}_t\}) \\ &+ \sum_{j=1}^n \frac{|s_j|^2}{2\hbar} e^{-\hbar\beta\omega_j} (2\hat{X}_j^+ \hat{\rho}_t \hat{X}_j^- - \{\hat{X}_j^- \hat{X}_j^+, \hat{\rho}_t\}). \end{aligned} \quad (92a)$$

$$(92b)$$

In Appendix F, we prove that \hat{X}_j^\pm are eigenoperators of \hat{H} in (42), i.e.,

$$[\hat{H}, \hat{X}_j^\pm] = \pm \hbar\omega_j \hat{X}_j^\pm \quad (j = 1, \dots, n), \quad (93)$$

while in Appendix G there is a proof for

$$|s_j|^2 = \hbar\bar{\gamma}_j \bar{n}_j e^{\hbar\beta\omega_j} = \hbar\bar{\gamma}_j (\bar{n}_j + 1), \quad (94)$$

where now the Planck factor is

$$\bar{n}_k = (e^{\hbar\beta\omega_k} - 1)^{-1}. \quad (95)$$

With all these, we show that the master equation (92) has the structure of a QOME like the one in (88).

Using Eq. (51) and writing $\hat{U}_{\mathbf{S}^{\text{th}}}$ for the metaplectic operator associated with the symplectic matrix \mathbf{S}^{th} , i.e., $\hat{U}_{\mathbf{S}^{\text{th}}}^\dagger \hat{\mathbf{x}} \hat{U}_{\mathbf{S}^{\text{th}}} = \mathbf{S}^{\text{th}} \hat{\mathbf{x}}$, one can find

$$\hat{H} = \frac{1}{2} \hat{\mathbf{x}}^\dagger \mathbb{B} \hat{\mathbf{x}} = \hat{U}_{\mathbf{S}^{\text{th}}}^\dagger \hat{H}_{\text{ho}} \hat{U}_{(\mathbf{S}^{\text{th}})}, \quad (96)$$

where

$$\hat{H}_{\text{ho}} := \frac{1}{2} \hat{\mathbf{x}}^\dagger (\mathbb{w} \oplus \mathbb{w}) \hat{\mathbf{x}} = \sum_{j=1}^n \frac{\omega_j}{2} (\hat{p}_j^2 + \hat{q}_j^2)$$

is the Hamiltonian of a multimode harmonic oscillator. The eigenequation for \hat{H}_{ho} is

$$\hat{H}_{\text{ho}} |\mathbf{n}\rangle = E_{\mathbf{n}} |\mathbf{n}\rangle, \quad E_{\mathbf{n}} = \sum_{l=1}^n \hbar\omega_l \left(n_l + \frac{1}{2} \right), \quad (97)$$

where $|\mathbf{n}\rangle = |n_1\rangle \otimes \dots \otimes |n_n\rangle$, $\mathbf{n} = (n_1, \dots, n_n)$ with $n_l = 0, \dots, +\infty$. Therefore, due to the similarity relation between \hat{H}_{ho} and \hat{H} , the eigenequation of \hat{H} is $\hat{H} |\phi_{\mathbf{n}}\rangle = E_{\mathbf{n}} |\phi_{\mathbf{n}}\rangle$, with

$$|\phi_{\mathbf{n}}\rangle = \hat{U}_{\mathbf{S}^{\text{th}}}^\dagger |\mathbf{n}\rangle. \quad (98)$$

Now, from (93), it is straightforward to verify that \hat{X}_j^+ and \hat{X}_j^- are the creation and annihilation operators of a quantum $\hbar\omega_j$ associated to the states $|\phi_{\mathbf{n}}\rangle$; that is, $\hat{X}_j^\pm |\phi_{\mathbf{n}}\rangle = \alpha_{\mathbf{n}^\pm} |\phi_{\mathbf{n}^\pm}\rangle$ is an eigenvector of \hat{H} with energy $E_{\mathbf{n}^\pm}$ such that $\mathbf{n}_j^\pm = (n_1, \dots, n_j \pm 1, \dots, n_n)$. Since \hat{X}_j^\pm satisfies the commutation

relation (93), the operator $\hat{X}_j^+ \hat{X}_j^-$ is the number operator in the eigenbasis $\{|\phi_{\mathbf{n}}\rangle\}$, i.e., $\hat{X}_j^+ \hat{X}_j^- |\phi_{\mathbf{n}}\rangle = n_j |\phi_{\mathbf{n}}\rangle$, therefore, $\bar{\alpha}_{\mathbf{n}_j^+} = \sqrt{n_j + 1}$ and $\bar{\alpha}_{\mathbf{n}_j^-} = \sqrt{n_j}$.¹⁰

It is worth to note that we consider the same mode structure for \hat{H}_{ho} and \hat{H} ; what changes is the nature of the stationary states, i.e., while $|\mathbf{n}\rangle$ are separable states, $|\phi_{\mathbf{n}}\rangle$ could be entangled with respect to the considered mode structure. In an analogous way, we can rewrite the thermal equilibrium state in (41) as

$$\begin{aligned} \hat{\sigma}^{\text{th}} &= \frac{1}{\mathcal{Z}^{\text{th}}} e^{-\frac{\beta}{2} \hat{\mathbf{x}}^\dagger (\mathbf{S}^{\text{th}})^\dagger (\mathbb{w} \oplus \mathbb{w}) \mathbf{S}^{\text{th}} \hat{\mathbf{x}}} \\ &= \hat{U}_{\mathbf{S}^{\text{th}}}^\dagger \hat{\sigma}_{\text{ho}}^{\text{th}} \hat{U}_{\mathbf{S}^{\text{th}}}, \end{aligned} \quad (99)$$

where we recognize in

$$\hat{\sigma}_{\text{ho}}^{\text{th}} = \frac{1}{\mathcal{Z}^{\text{th}}} e^{-\frac{\beta}{2} \hat{\mathbf{x}}^\dagger (\mathbb{w} \oplus \mathbb{w}) \hat{\mathbf{x}}} \quad (100)$$

the Gibbs's state of the multimode harmonic oscillator with Hamiltonian \hat{H}_{ho} in (96). The state $\hat{\sigma}_{\text{ho}}^{\text{th}}$ is manifestly separable, so the possible entanglement of the state $\hat{\sigma}^{\text{th}}$ is due to the action of the unitary operation $\hat{U}_{(\mathbf{S}^{\text{th}})}$, where \mathbf{S}^{th} is the matrix of the symplectic diagonalization of the Hessian matrix \mathbb{B} , as shown in Eq. (51).

Noteworthy that usually in the derivation of the QOME, it is assumed that the commutation relation (89) is valid [30]. Here, the quantum detailed balance condition for a GDS shows that commutation relation (93) must be valid instead. When \mathbf{S}^{th} is the identity and $\hat{H}_{\text{eff}} = \hat{H} = \hat{H}_{\text{ho}}$, we have that $\hat{X}_j^- = \hat{a}_j$ and $\hat{X}_j^+ = \hat{a}_j^\dagger$ are the usual creation and annihilation operators of the multimode quantum harmonic oscillator. Then, in this case the QOME (92) is the well-known master equation of a multimode damped harmonic oscillator [43].

VI. ADDITIONAL PROPERTIES OF GDSs SATISFYING A $\hat{\sigma}^{\text{th}}$ -DBC

A. Temperature dependence of \mathbb{D} and \mathbb{C}

The final form for the diffusion and dissipation matrices of GDSs that satisfy a $\hat{\sigma}^{\text{th}}$ -DBC is obtained replacing (94) into (86), so

$$\mathbb{D} = \hbar^2 (\mathbf{S}^{\text{th}})^{-1} (\bar{\mathbb{g}} \mathbb{k}^{\text{th}} \oplus \bar{\mathbb{g}} \mathbb{k}^{\text{th}}) (\mathbf{S}^{\text{th}})^{-\dagger}, \quad (101a)$$

$$\mathbb{C} = \frac{\hbar}{2} (\mathbf{S}^{\text{th}})^{-1} \mathbf{J}^\dagger (\bar{\mathbb{g}} \oplus \bar{\mathbb{g}}) (\mathbf{S}^{\text{th}})^{-\dagger}, \quad (101b)$$

where \mathbb{k}^{th} is given in (53) and $\bar{\mathbb{g}}$ is the diagonal matrix containing the coupling constants, i.e.,

$$\bar{\mathbb{g}} = \text{diag}(\bar{\gamma}_1, \dots, \bar{\gamma}_n). \quad (102)$$

It is worth to remember here that \mathbf{S}^{th} and \mathbb{k}^{th} are ultimately determined by \mathbb{B} , the Hessian matrix of \hat{H} that defines the thermal equilibrium state. However, the coupling constants can take different values, thus defining different pair of matrices, \mathbb{D} and \mathbb{C} , all having the same equilibrium state. These are the multiples solutions of Eq. (62).

¹⁰This can be checked through a lengthy, but not difficult, calculation using the definition of \hat{X}_j^\pm in (91) and $|\phi_{\mathbf{n}}\rangle$ in (98).

Note that here we consider that the coupling constants do not depend on temperature. This is consistent with the fact that the Fokker-Planck equation (34) for Gaussian states corresponds to an Ornstein-Uhlenbeck process [38], where the drift term $\frac{\partial \mathcal{J}\mathcal{C}\mathbf{x}}{\partial \mathbf{x}^\top} = \text{tr}(\mathcal{J}\mathcal{C})$ corresponds to a drift force $-\mathcal{J}\mathcal{C}\mathbf{x}$ which does not depend on the temperature [11]. From Eq. (66), we can write $\mathbb{D} = \hbar(\mathbf{S}^{\text{th}})^{-1}(\mathfrak{d} \oplus \mathfrak{d})(\mathbf{S}^{\text{th}})^{-\top}$, where we use that \mathbf{S}^{th} is symplectic and that $\mathbb{Q}^\dagger(\iota\mathfrak{d} \oplus (-\iota\mathfrak{d}))\mathbb{Q} = \hbar\mathcal{J}(\mathfrak{d} \oplus \mathfrak{d})$. Analogously, from Eq. (67), we can write $\mathbb{C} = (\mathbf{S}^{\text{th}})^{-1}\mathcal{J}^\top(\mathfrak{j}_\mathcal{C} \oplus \mathfrak{j}_\mathcal{C})(\mathbf{S}^{\text{th}})^{-\top}$, where $\mathbb{Q}^\dagger(\mathfrak{j}_\mathcal{C} \oplus \mathfrak{j}_\mathcal{C})\mathbb{Q} = \mathfrak{j}_\mathcal{C} \oplus \mathfrak{j}_\mathcal{C}$. Comparing with Eqs. (101), we have $\mathfrak{d} = \hbar\bar{\mathfrak{g}}\mathbb{k}^{\text{th}}$ and $\mathfrak{j}_\mathcal{C} = \frac{\hbar}{2}\bar{\mathfrak{g}}$, so $\mathfrak{d} = 2\mathfrak{j}_\mathcal{C}\mathbb{k}^{\text{th}}$. Therefore, the diffusion and dissipation matrices of a GDS satisfying a QDBC verify the condition (68); this condition guarantees that the GDS has a thermal equilibrium state.

For the validity of the reciprocal implication (every GDS that attains thermal equilibrium must satisfy a QDBC), we need to allow an arbitrary temperature dependence for the coupling constants in the matrix $\bar{\mathfrak{g}}$. This arbitrariness is clear taking into account (68), where $\mathfrak{d} = 2\mathfrak{j}_\mathcal{C}\mathbb{k}^{\text{th}}$; now $\mathfrak{j}_\mathcal{C}$ can depend on temperature still matching (101b) with $\mathfrak{j}_\mathcal{C} = \frac{\hbar}{2}\bar{\mathfrak{g}}$ for an arbitrary dependence of $\bar{\mathfrak{g}}$ on temperature.

B. High- and low-temperature limits

The high-temperature limit is obtained when considering $\hbar\beta\|\omega\| \ll 1$ in (101a), where $\|\omega\| = \max\{\omega_1, \dots, \omega_n\}$, so one can write $\mathbb{k}^{\text{th}} \approx (\hbar\beta\omega)^{-1} + \frac{1}{12}\hbar\beta\omega + O(\hbar\beta\omega)^3$ and neglecting higher-order terms, we write

$$\mathbb{D} \approx \hbar\beta^{-1}(\mathbf{S}^{\text{th}})^{-1}(\bar{\mathfrak{g}}\omega^{-1} \oplus \bar{\mathfrak{g}}\omega^{-1})(\mathbf{S}^{\text{th}})^{-\top}. \quad (103)$$

Inserting this into Eq. (62) with \mathbb{C} in (101b), using the symplectic condition for \mathbf{S}^{th} , and the symplectic diagonalization in (51), one obtains

$$\mathbb{V} \approx (\hbar\beta\mathbb{B})^{-1}. \quad (104)$$

This limit is also called classical limit and the Wigner function in (37) for this covariance matrix is the classical Boltzmann factor $\exp[-\beta H]$ for the classical version (Wigner symbol) of Hamiltonian (42) [36].

The low-temperature limit corresponds to $\hbar\beta\|\omega\| \gg 1$ and in this case $\mathbb{k}^{\text{th}} = \frac{1}{2}\mathbb{1} + O(e^{-\hbar\beta\omega})$. From (101),

$$\mathbb{D} \approx \hbar\frac{1}{2}(\mathbf{S}^{\text{th}})^{-1}(\mathbf{S}^{\text{th}})^{-\top}\mathcal{J}\mathbb{C}. \quad (105)$$

Consequently,

$$\mathbb{V}^{\text{th}} \approx \frac{1}{2}(\mathbf{S}^{\text{th}})^{-1}(\mathbf{S}^{\text{th}})^{-\top}, \quad (106)$$

and the thermal equilibrium state corresponds to a pure Gaussian state $\hat{\sigma}^{\text{th}} \approx |\phi_{\mathbf{n}=0}\rangle\langle\phi_{\mathbf{n}=0}|$, where $|\phi_{\mathbf{n}=0}\rangle$ is the ground state of the Hamiltonian \hat{H} in (42).

C. Pure diffusive regime

This regime corresponds to the limit $\bar{n}_j \rightarrow +\infty$ (i.e., $\beta \rightarrow 0$) together with $\bar{\gamma}_j \rightarrow 0$, such that $\bar{n}_j\bar{\gamma}_j = \bar{c}_j$ are constant values for $j = 1, \dots, n$. In such limit, the coefficients in (92a) become equal to the ones in (92b) for the master equation of the GDS [see also Eq. (88)]. For the diffusion and dissipation

matrices, respectively, we have

$$\mathbb{D} = \hbar^2(\mathbf{S}^{\text{th}})^{-1}\bar{\mathfrak{c}} \oplus \bar{\mathfrak{c}}(\mathbf{S}^{\text{th}})^{-\top}, \quad (107a)$$

$$\mathbb{C} = 0, \quad (107b)$$

where $\bar{\mathfrak{c}} = \text{diag}(\bar{c}_1, \dots, \bar{c}_n)$. In this case the GDS has no thermal equilibrium solution because \mathbb{D} and \mathbb{C} do not satisfy Eq. (62).

It is worth to note that the master equation in (92) has no pure dissipative regime, i.e., $\mathbb{D} = 0$, since Eq. (13) is violated. However, if \hbar represents an effective Planck constant in the master equation of the GDS in (92), the semiclassical limit $\hbar \ll 1$ and the low-temperature condition $\hbar\beta\|\omega\| \gg 1$ guarantee the validity of (105), thus, the contributions from \mathbb{D} can be effectively neglected when compared to those coming from \mathbb{C} ; in this case the evolution is thus dominated by dissipation [44].

VII. ROLE OF \hat{H}_{eff} IN THERMALIZATION

So far, we completely answer the question of which kind of environments, characterized by the diffusion and dissipation matrices \mathbb{D} and \mathbb{C} , allow the existence of a given thermal equilibrium state in a GDS. This state is given by a Gaussian Gibbs state $\hat{\sigma}^{\text{th}} = e^{-\beta\hat{H}}/\mathcal{Z}^{\text{th}}$, corresponding to a quadratic Hamiltonian \hat{H} of positive-elliptic type, which is completely characterized by its covariance matrix \mathbb{V}^{th} such that $\mathcal{J}\mathbb{V}^{\text{th}}$ belongs to the commuting set $\{\mathcal{J}\mathbb{V}^{\text{th}}, \mathcal{J}\mathbb{D}, \mathcal{J}\mathbb{C}\}$.

However, it is interesting to analyze this result from the perspective of a fixed GDS that governs the system to thermal equilibrium. In this case, the commuting matrices $\mathcal{J}\mathbb{D}$ and $\mathcal{J}\mathbb{C}$ determine $\mathcal{J}\mathbb{B}$ through the relations (62) and (47). Then, it is clear that the effective action of the environment, through the nonunitary part of the evolution in (9), is to confine the system in phase space (with coordinates \mathbf{x}) around the equilibrium point $\langle\hat{\mathbf{x}}\rangle^{\text{th}} = 0$. The energetic balance between the environment and the system is described by the quantum positive-elliptic Hamiltonian \hat{H} , which is dynamically stable (with spectrum necessarily discrete with bounded eigenstates). If we admit that every GDS that leads to thermal equilibrium satisfies a QDBC, the form of the master equation in (92) together with (93) and (94) helps to understand in detail the energetic balance of the thermal equilibrium process. Indeed, this process is determined by the action of the eigenoperators \hat{X}_j^\pm of \hat{H} , which promote transitions between the energy levels of this Hamiltonian as a consequence of the interaction with the environment. This contrasts with the usual derivation [30] of the QOME in (88), where $\{\hat{X}_j^\pm\}$ is considered as a basis of eigenoperators of \hat{H}_{OME} , which expands the algebra of operators acting on the system [30].

The induced environmental confinement process of the system in thermal equilibrium is not influenced by the unitary dynamics generated by \hat{H}_{eff} since the commuting set of matrices $\{\mathcal{J}\mathbb{V}^{\text{th}}, \mathcal{J}\mathbb{D}, \mathcal{J}\mathbb{C}\}$, which ultimately determine the thermal equilibrium through (62), do not depend on the Hessian matrix \mathbb{B}' of \hat{H}_{eff} . However, there are some of these Hamiltonians that allow the process of thermalization to occur; these are determined by the condition $[\hat{H}, \hat{H}_{\text{eff}}] = 0$. Using Eq. (43) and the results in Appendix H, all these Hamiltonians are written

as

$$\hat{H}_{\text{eff}} = \frac{1}{2} \hat{\mathbf{x}}^\top \mathbb{B}' \hat{\mathbf{x}} = \hat{U}_{\mathbf{S}^{\text{th}}}^\dagger \hat{H}'_{\text{ho}} \hat{U}_{(\mathbf{S}^{\text{th}})}, \quad (108)$$

where we employed the same reasoning performed in Eq. (96) with $\mathbb{B}' = (\mathbf{S}^{\text{th}})^\top (\mathbb{B} \oplus \mathbb{B}) \mathbf{S}^{\text{th}}$. The Hamiltonian \hat{H}'_{ho} is the one of a multimode harmonic oscillator, i.e.,

$$\hat{H}'_{\text{ho}} := \frac{1}{2} \hat{\mathbf{x}}^\top (\mathbb{B} \oplus \mathbb{B}) \hat{\mathbf{x}} = \sum_{j=1}^n \frac{\lambda_j}{2} (\hat{p}_j^2 + \hat{q}_j^2)$$

with $\mathbb{B} = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $\lambda_j \in \mathbb{R}$, $j = 1, \dots, n$.

Likewise \hat{H} , the Hamiltonian \hat{H}_{eff} in (108) is also elliptic, or dynamically stable. However, contrary to the multimode harmonic oscillator \hat{H}_{ho} in the expression (96), the arbitrary frequencies λ_j of \hat{H}'_{ho} can be positive, negative, or even null. Consequently, \hat{H}_{eff} can be the null Hamiltonian and the system still thermalizes to the same state $\hat{\sigma}^{\text{th}}$. Note, however, that taking a null Hamiltonian is quite different of considering the interaction representation of the QOME in (88), where the density operator turns to $\hat{\rho}_t^I = e^{\frac{i}{\hbar} \hat{H}_{\text{OME}} t} \hat{\rho}_i e^{-\frac{i}{\hbar} \hat{H}_{\text{OME}} t}$ since the frequencies $\tilde{\omega}_k$ still correspond to \hat{H}_{OME} through the Planck factor \tilde{n}_k .

It is worth to note that the operators \hat{X}_k^\pm are eigenoperators of \hat{H} when the master equation (92) of a GDS thermalizes [see Eq. (93)], but they are also eigenoperators of \hat{H}_{eff} , i.e., $[\hat{H}, \hat{X}_j^\pm] = \pm \hbar \lambda_j \hat{X}_j^\pm$ with $j = 1, \dots, n$. Nevertheless, the frequencies ω_k in the master equation (92) are associated with \hat{H} . Thus, the energetic balance of the thermalization process is associated with transitions in the Hamiltonian \hat{H} associated with the confinement of the system.

VIII. CONCLUSIONS

When a GDS has a thermal equilibrium state $\hat{\sigma}^{\text{th}}$, every initial condition of the evolution ends up on this state. Without loss of generality, we can choose $(\hat{\mathbf{x}})^{\text{th}} = 0$, so the thermal state $\hat{\sigma}^{\text{th}}$ must have the form in (41) with \hat{H} in (42), and the Hamiltonian of the unitary part of the evolution for the GDS given by \hat{H}_{eff} , of the form in (44) such that $\hat{H} = \hat{H}_{\text{eff}}$ or $[\hat{H}, \hat{H}_{\text{eff}}] = 0$. Another prerequisite for $\hat{\sigma}^{\text{th}}$ came from the normalization of a density operator: in order to have unity trace, the Hessian matrix \mathbb{B} of \hat{H} must be positive definite [36]. This means that \hat{H} is elliptic, i.e., its classical counterpart in (54) generates a Hamiltonian flow given by the symplectic matrix $\hat{\mathbf{S}}_t$ in (55) around the elliptic fixed point $\mathbf{x} = 0$. Therefore, \hat{H} is dynamically stable having a discrete energy spectrum corresponding to bound states.

The nonunitary part of the evolution of a GDS is determined by the diffusion and dissipation matrices \mathbb{D} and \mathbb{C} , respectively. These are the real and imaginary parts, respectively, of the decoherence matrix \mathbb{F} [see Eq. (11)]. In (62), we show the relation between the covariance matrix \mathbb{V}^{th} of $\hat{\sigma}^{\text{th}}$ and the matrices \mathbb{D} and \mathbb{C} for every GDS with a thermal equilibrium state, which has appeared before in [11] in a narrower context. Further, we show that \mathbb{V}^{th} is completely determined by the symplectic matrix \mathbf{S}^{th} in (49), which diagonalizes symplectically the Hessian matrix \mathbb{B} as shown in (51), and by the symplectic spectrum of \mathbb{B} , which contains the eigenfrequencies $\omega_j > 0$ of the system Hamiltonian.

However, Eq. (62) does not establish neither the temperature dependence of \mathbb{D} and \mathbb{C} nor characterize all the different

GDSs that have the same thermal equilibrium state $\hat{\sigma}^{\text{th}}$. This deficiency can be fixed extending Alicki's QDBC (developed for systems described by discrete Hilbert spaces) to n -mode bosonic systems. We have shown how to implement this extension and the result is that all GDSs verifying a QDBC have a thermal equilibrium state whose dynamics is characterized by symplectically invariant diffusion and dissipation matrices [see Eq. (72)]. This condition allows the characterization of the Lindblad operators of a GDS satisfying a QDBC in Theorem 5.

The corresponding master equation has the structure of a quantum optical master equation (QOME) [see Eq. (92) together with Eqs. (93) and (94)]. As a consequence, the matrices \mathbb{D} and \mathbb{C} have the specific structure in (101), which sets explicitly their dependence on temperature. These expressions also show that different GDSs with the same thermal equilibrium state differ from each other only on the value of the coupling constants, namely, $\tilde{\gamma}_j$ in (94), because \mathbf{S}^{th} is determined by the symplectic diagonalization of \mathbb{B} , i.e., the Hessian matrix of \hat{H} . Usually, the coupling constants $\tilde{\gamma}_j$ do not depend on temperature, however, we show that if we allow these to have an arbitrary temperature dependence, then it is possible to say that every GDS with thermal equilibrium satisfies a QDBC.

We also have shown that it is possible to define a pure diffusive regime for any GDS satisfying the QDBC, where the GDS lost its stationary solution. This is a specific example of the necessity to balance diffusion and dissipation to achieve a stationary solution in any quantum dynamical semigroup [11].

Finally, we also show that the contribution of \hat{H}_{eff} to the dynamics of a GDS has no influence in the thermalization process. However, by determining all \hat{H}_{eff} such that $[\hat{H}, \hat{H}_{\text{eff}}] = 0$, we have determined all Hamiltonians that allow thermalization. These Hamiltonians are also dynamically stable as \hat{H} , however, with arbitrary eigenenergy frequencies. Therefore, the energetic balance of the thermalization process, determined by \hat{H} , is completely independent of \hat{H}_{eff} . This marks a fundamental difference between the QOME for a GDS satisfying a QDBC and the usual QOME [30].

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APPENDIX A: PROOF OF LEMMA 1

Let us prove first that $\mathbb{D} > 0$ under the hypothesis of the Lemma. Since \mathbb{C} is antisymmetric ($\mathbb{C} = -\mathbb{C}^\top$) and invertible, the spectral theorem [45] guarantees that there is a unitary diagonalizing matrix \mathbb{U} , such that $\mathbb{U} \mathbb{C} \mathbb{U}^\dagger = (-i\mathbb{c}) \oplus (i\mathbb{c}) = \mathbb{C}'$, where $\mathbb{c} = \text{diag}(c_1, \dots, c_n)$, with $c_i > 0$, for $i = 1, \dots, n$. Thus, using (13) we have $\mathbb{U} \hbar \mathbb{F} \mathbb{U}^\dagger = \mathbb{D}' + \mathbb{C}' \geq 0$, where $\mathbb{D}' = \mathbb{U} \mathbb{D} \mathbb{U}^\dagger$ and $\mathbb{C}' = (\hbar \mathbb{c}) \oplus (-\hbar \mathbb{c})$. Now consider the matrix \mathbb{F}^\top and apply the same unitary transformation, reaching $\mathbb{U} \mathbb{F}^\top \mathbb{U}^\dagger = \mathbb{D}' - \mathbb{C}' \geq 0$, which is positive semidefinite,

since \mathbb{F}^\top is Hermitian and has the same eigenvalues of \mathbb{F} . For a complex $2n$ -dimensional vector \mathbf{z} , the above matrix inequalities are equivalent to $\mathbf{z}^\dagger \mathbb{D}' \mathbf{z} \pm \mathbf{z}^\dagger \mathbb{C}' \mathbf{z} \geq 0$, for all vectors \mathbf{z} , which implies $\mathbf{z}^\dagger \mathbb{D}' \mathbf{z} > 0$, thus $\mathbb{D} > 0$.

To prove that $\mathbb{F} > 0$, it is enough to use that $\mathbf{z}^\dagger \mathbb{C} \mathbf{z} = 0$ for all vectors \mathbf{z} since \mathbb{C} is antisymmetric. Consequently, $\hbar \mathbf{z}^\dagger \mathbb{F} \mathbf{z} = \mathbf{z}^\dagger \mathbb{D} \mathbf{z}$ since $\mathbb{D} > 0$ under the hypothesis of the Lemma 1 in Sec. II, thus $\mathbb{F} > 0$. This finishes the proof.

Observing the particular case where $\mathbb{C} = 0$, one has $\mathbb{F} > 0$ iff $\mathbb{D} > 0$, which shows that the converse of the Lemma is not true.

APPENDIX B: PROOF OF THEOREM 2

Since $\bar{\Lambda}_t$ is a GDS and satisfies a $\hat{\sigma}^{\text{th}}$ -DBC, we have

$$\langle \bar{\Lambda}_t[\hat{A}], \hat{B} \rangle_{\text{GNS}} = \langle \hat{A}, \bar{\Lambda}_t[\hat{B}] \rangle_{\text{GNS}}. \quad (\text{B1})$$

Evaluating for $\hat{A} = \hat{1}$, we have $\bar{\Lambda}_t[\hat{1}] = \hat{1}$ and thus

$$\langle \hat{1}, \hat{B} \rangle_{\text{GNS}} = \langle \hat{1}, \bar{\Lambda}_t[\hat{B}] \rangle_{\text{GNS}}. \quad (\text{B2})$$

However, $\langle \hat{1}, \hat{B} \rangle_{\text{GNS}} = \langle \hat{\sigma}, \hat{B} \rangle$ and $\langle \hat{1}, \bar{\Lambda}_t[\hat{B}] \rangle_{\text{GNS}} = \langle \hat{\sigma}, \bar{\Lambda}_t[\hat{B}] \rangle = \langle \Lambda_t[\hat{\sigma}], \hat{B} \rangle$, where we use that Λ_t is the adjoint of $\bar{\Lambda}_t$ with respect to the Hilbert-Schmidt scalar product $\langle \cdot, \cdot \rangle$. Therefore, we arrive to

$$\langle \hat{\sigma}^{\text{th}}, \hat{B} \rangle = \langle \Lambda_t[\hat{\sigma}^{\text{th}}], \hat{B} \rangle, \quad (\text{B3})$$

valid for all operators \hat{B} such the Hilbert-Schmidt scalar product on both sides is finite. In particular, we can use $\hat{B} = \hat{R}_x$, a reflection operator defined in Sec. II B, therefore, (B3) is just the equality between the Wigner symbols of $\hat{\sigma}^{\text{th}}$ and $\Lambda_t[\hat{\sigma}^{\text{th}}]$, which implies that $\Lambda_t[\hat{\sigma}^{\text{th}}] = \hat{\sigma}^{\text{th}}$.

APPENDIX C: PROOF OF THEOREM 3

Let us start writing

$$\begin{aligned} \langle \bar{\Lambda}_t[\Xi_\tau[\hat{A}]], \hat{B} \rangle_{\text{GNS}} &= \text{Tr}(\hat{\sigma}^{\text{th}}(\bar{\Lambda}_t[\Xi_\tau[\hat{A}]])^\dagger \hat{B}) \\ &= \text{Tr}(\hat{\sigma}^{\text{th}}(\Xi_\tau[\hat{A}])^\dagger \bar{\Lambda}_t[\hat{B}]) = \text{Tr}(\hat{\sigma}^{\text{th}} \Xi_{\tau^*}[\hat{A}^\dagger] \bar{\Lambda}_t[\hat{B}]) \\ &= \text{Tr}(\hat{\sigma}^{\text{th}} e^{\frac{i}{\hbar} \hat{H} \tau^*} \hat{A}^\dagger e^{-\frac{i}{\hbar} \hat{H} \tau^*} \bar{\Lambda}_t[\hat{B}]) \end{aligned}$$

$$= \text{Tr}(e^{-\frac{i}{\hbar} \hat{H} \tau^*} \bar{\Lambda}_t[\hat{B}^\dagger]^\dagger e^{\frac{i}{\hbar} \hat{H} \tau^*} \hat{\sigma}^{\text{th}} \hat{A}^\dagger), \quad (\text{C1})$$

where in the first line we used that $\bar{\Lambda}_t$ satisfies a $\hat{\sigma}^{\text{th}}$ -DBC and the notation τ^* means complex conjugation of τ ; in the third line, we used that $[e^{\pm \frac{i}{\hbar} \hat{H} \tau}, \hat{\sigma}^{\text{th}}] = 0$ and that $\bar{\Lambda}_t$ is a real superoperator, i.e., $\bar{\Lambda}_t[\hat{B}^\dagger]^\dagger = \bar{\Lambda}_t[\hat{B}]$.

Following [19], we specialize (C1) for $\tau^* = -i\hbar\beta$, therefore, $e^{\frac{i}{\hbar} \hat{H}(-i\hbar\beta)} = (\hat{\sigma}^{\text{th}})^{-1}$, then

$$\begin{aligned} \langle \bar{\Lambda}_t[\Xi_{i\hbar\beta}[\hat{A}]], \hat{B} \rangle_{\text{GNS}} &= \text{Tr}(\hat{\sigma}^{\text{th}} \bar{\Lambda}_t[\hat{B}^\dagger]^\dagger \hat{A}^\dagger) \\ &= \text{Tr}(\hat{\sigma}^{\text{th}} (\hat{B}^\dagger)^\dagger \bar{\Lambda}_t[\hat{A}^\dagger]) = \text{Tr}(\hat{\sigma}^{\text{th}} \hat{B} \bar{\Lambda}_t[\hat{A}^\dagger]) \\ &= \text{Tr}(\bar{\Lambda}_t[\hat{A}^\dagger] \hat{\sigma}^{\text{th}} \hat{B}) = \text{Tr}(\hat{\sigma}^{\text{th}} (\hat{\sigma}^{\text{th}})^{-1} \bar{\Lambda}_t[\hat{A}^\dagger] \hat{\sigma}^{\text{th}} \hat{B}) \\ &= \text{Tr}(\hat{\sigma}^{\text{th}} (\hat{\sigma}^{\text{th}} \bar{\Lambda}_t[\hat{A}]) (\hat{\sigma}^{\text{th}})^{-1} \hat{B}) \\ &= \text{Tr}(\hat{\sigma}^{\text{th}} (\Xi_{i\hbar\beta}[\bar{\Lambda}_t[\hat{A}]]))^\dagger \hat{B}) \\ &= \langle \Xi_{i\hbar\beta}[\bar{\Lambda}_t[\hat{A}]], \hat{B} \rangle_{\text{GNS}}. \end{aligned} \quad (\text{C2})$$

Since this identity is valid for any operator \hat{B} , we can set it equal to the reflection operator, i.e., $\hat{B} = \hat{R}_x$. Then, in this case, Eq. (C2) just restates that the Wigner symbols of the operator $\hat{\sigma}^{\text{th}} \bar{\Lambda}_t[\Xi_{i\hbar\beta}[\hat{A}]]$ and of $\hat{\sigma}^{\text{th}} \Xi_{i\hbar\beta}[\bar{\Lambda}_t[\hat{A}]]$ are identical, therefore also the operators themselves. For any finite value β , the operator $\hat{\sigma}^{\text{th}}$ has an inverse, then it must be true that

$$\bar{\Lambda}_t[\Xi_{i\hbar\beta}[\hat{A}]] = \Xi_{i\hbar\beta}[\bar{\Lambda}_t[\hat{A}]], \quad (\text{C3})$$

for all \hat{A} , which means that $\bar{\Lambda}_t$ commutes with $\Xi_{i\hbar\beta}$ for all t .

Noting that $(\hat{\sigma}^{\text{th}})^{\pm \frac{i}{\hbar\beta}} = (e^{-\beta \hat{H}})^{\pm \frac{i}{\hbar\beta}} = e^{\mp \frac{i}{\hbar} \hat{H}}$, therefore, we can define $(\Xi_{i\hbar\beta})^{-\frac{i}{\hbar\beta}}[\cdot] = (\hat{\sigma}^{\text{th}})^{-\frac{i}{\hbar\beta}} \cdot ((\hat{\sigma}^{\text{th}})^{-1})^{-\frac{i}{\hbar\beta}} = e^{\frac{i}{\hbar} \hat{H} t} \cdot e^{-\frac{i}{\hbar} \hat{H} t} = \Xi_t[\cdot]$. But, according to (C3), $\bar{\Lambda}_t$ commutes with $\Xi_{i\hbar\beta}$ and thus it must commute with any function of $\Xi_{i\hbar\beta}$, then with $(\Xi_{i\hbar\beta})^{\frac{i}{\hbar} t} = \Xi_t$ for all $t \in \mathbb{C}$. Taking the time derivative on both sides of Eq. (C3) and using $d\bar{\Lambda}_t[\cdot]/dt = \tilde{\mathcal{L}}[\bar{\Lambda}_t[\cdot]]$, we get $\tilde{\mathcal{L}}[\bar{\Lambda}_t[\Xi_{i\hbar\beta}[\hat{A}]]] = \Xi_{i\hbar\beta}[\tilde{\mathcal{L}}\bar{\Lambda}_t[\hat{A}]]$, which shows that $\tilde{\mathcal{L}}$ commutes with $\Xi_{i\hbar\beta}$ when $t = 0$. Following the same reasoning as before, $\tilde{\mathcal{L}}$ commutes with Ξ_t .

APPENDIX D: DEMONSTRATION OF EQ. (74)

Let us start from the left-hand side of (74):

$$\begin{aligned} \Xi_{-t} \left[\frac{\partial}{\partial \hat{\mathbf{x}}^\top} [\cdot] \right] &= \Xi_{-t} \left[\frac{i}{\hbar} [(\mathbf{J}\hat{\mathbf{x}})^\top, \cdot] \right] = \frac{i}{\hbar} \left(\Xi_{-t} [(\mathbf{J}\hat{\mathbf{x}})^\top \cdot] - \frac{i}{\hbar} \Xi_{-t} [\cdot (\mathbf{J}\hat{\mathbf{x}})^\top] \right) = \frac{i}{\hbar} \left(\Xi_{-t} [(\mathbf{J}\hat{\mathbf{x}})^\top] \Xi_{-t} [\cdot] - \frac{i}{\hbar} \Xi_{-t} [\cdot] \Xi_{-t} [(\mathbf{J}\hat{\mathbf{x}})^\top] \right) \\ &= \frac{i}{\hbar} \left(-\Xi_{-t} [\hat{\mathbf{x}}^\top] \mathbf{J} \Xi_{-t} [\cdot] + \frac{i}{\hbar} \Xi_{-t} [\cdot] \Xi_{-t} [\hat{\mathbf{x}}^\top] \mathbf{J} \right) = \frac{i}{\hbar} \left(-\hat{\mathbf{x}}^\top \tilde{\mathbf{S}}_t^{-\top} \mathbf{J} \Xi_{-t} [\cdot] + \frac{i}{\hbar} \Xi_{-t} [\cdot] \hat{\mathbf{x}}^\top \tilde{\mathbf{S}}_t^{-\top} \mathbf{J} \right) \\ &= \frac{i}{\hbar} \left(-\hat{\mathbf{x}}^\top \mathbf{J} \tilde{\mathbf{S}}_t \Xi_{-t} [\cdot] + \frac{i}{\hbar} \Xi_{-t} [\cdot] \hat{\mathbf{x}}^\top \mathbf{J} \tilde{\mathbf{S}}_t \right) = \frac{i}{\hbar} \left((\mathbf{J}\hat{\mathbf{x}})^\top \Xi_{-t} [\cdot] - \frac{i}{\hbar} \Xi_{-t} [\cdot] (\mathbf{J}\hat{\mathbf{x}})^\top \right) \tilde{\mathbf{S}}_t = \frac{\partial}{\partial \hat{\mathbf{x}}^\top} [\Xi_{-t} [\cdot]] \tilde{\mathbf{S}}_t, \end{aligned} \quad (\text{D1})$$

where we used that $\Xi_{-t}[\hat{\mathbf{x}}^\top] = \hat{\mathbf{x}} \tilde{\mathbf{S}}_t^{-\top}$ and $\tilde{\mathbf{S}}_t^{-\top} \mathbf{J} = \mathbf{J} \tilde{\mathbf{S}}_t$, because $\tilde{\mathbf{S}}_t$ is a symplectic matrix.

APPENDIX E: DEMONSTRATION OF EQ. (78)

In order to prove Eq. (78), note that the decoherence matrix \mathbb{F} , that stems from Theorem 4, must be invariant under a congruence with $\tilde{\mathbf{S}}_t$ in (55). This follows using (72) into the definition of the matrices \mathbb{D} and \mathbb{C} in (10), and the fact that $\tilde{\mathbf{S}}_t$ is a real matrix.

Now, using Eq. (55) we can write $\tilde{\mathbf{S}}_t(\mathbb{Q}\mathbf{S}^{\text{th}})^{-1} = (\mathbb{Q}\mathbf{S}^{\text{th}})^{-1} e^{(\mathbb{J}\mathbb{B})_d t}$, so the column vectors $(\mathbb{Q}\mathbf{S}^{\text{th}})^{-1}|_k = \bar{\mathbf{I}}_k$ for

$k = 1, \dots, 2n$ are the complex eigenvectors of $\tilde{\mathbf{S}}_t$. Respecting the block order in the matrix in (52), we can write

$$\tilde{\mathbf{S}}_t \bar{\mathbf{I}}_j = e^{i\omega_j t} \bar{\mathbf{I}}_j, \quad (\text{E1a})$$

$$\tilde{\mathbf{S}}_t \bar{\mathbf{I}}_{n+j} = \tilde{\mathbf{S}}_t \bar{\mathbf{I}}_j^* = e^{-i\omega_j t} \bar{\mathbf{I}}_j^*, \quad (\text{E1b})$$

where $j = 1, \dots, n$ and $\omega_j > 0$ are the frequencies corresponding to the symplectic spectra of \mathbb{B} in (51). According to (E1), it is clear that the canonical form of \mathbb{F} in (78) is

manifestly invariant under a congruence through $\tilde{\mathbf{S}}_r$ and is positive definite, as it must be.

Let us now prove that the matrices \mathfrak{s} and \mathfrak{r} in (78) satisfy the relation in Eq. (80). We first note that the expression of \mathbb{F} in (78) can be rewritten as in (84), which allows us to show that the matrix $(\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger}$ diagonalizes simultaneously the Hamiltonian matrix $\mathbb{J}\mathbb{D}$ and the skew Hamiltonian matrix $\mathbb{J}\mathbb{C}$, therefore, we must have $[\mathbb{J}\mathbb{D}, \mathbb{J}\mathbb{C}] = 0$. Indeed, from (84), we have

$$\begin{aligned}\mathbb{J}\mathbb{D} &= \hbar\mathbb{J}(\mathbf{S}^{\text{th}})^{-1}\mathbb{A}_r(\mathbf{S}^{\text{th}})^{-\text{T}} = \hbar(\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{J}\mathbb{A}_r(\mathbf{S}^{\text{th}})^{-\text{T}} \\ &= \hbar(\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger}\mathbb{Q}\mathbb{J}\mathbb{A}_r\mathbb{Q}^{\dagger}\mathbb{Q}(\mathbf{S}^{\text{th}})^{-\text{T}} \\ &= (\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger}(\mathbb{J}\mathbb{D})_{\text{d}}((\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger})^{-1},\end{aligned}\quad (\text{E2})$$

where \mathbb{Q} is the unitary matrix defined in (26). In the above steps, the symplectic condition $\mathbb{J}(\mathbf{S}^{\text{th}})^{-1} = (\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{J}$ and the fact that $\hbar\mathbb{Q}\mathbb{J}\mathbb{A}_r\mathbb{Q}^{\dagger}$ is a diagonal matrix were employed. Since Eq. (E2) is a similarity transformation the matrix $\hbar\mathbb{Q}\mathbb{J}\mathbb{A}_r\mathbb{Q}^{\dagger}$ has the eigenvalues of $\mathbb{J}\mathbb{D}$ in its diagonal. Therefore, we can write

$$(\mathbb{J}\mathbb{D})_{\text{d}} = \frac{i\hbar}{2}[(|\mathfrak{s}|^2 + |\mathfrak{r}|^2) \oplus (-|\mathfrak{s}|^2 - |\mathfrak{r}|^2)]. \quad (\text{E3})$$

It is worth to note that, according to Lemma 2, the diagonal matrix $\mathfrak{d} = \frac{1}{2}(|\mathfrak{s}|^2 + |\mathfrak{r}|^2)$ contains the symplectic spectrum of \mathbb{D} . In an analogous way, we have

$$\begin{aligned}\mathbb{J}\mathbb{C} &= \mathbb{J}(\mathbf{S}^{\text{th}})^{-1}(-\mathbb{J}\mathbb{A}_i)(\mathbf{S}^{\text{th}})^{-\text{T}} = (\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{A}_i(\mathbf{S}^{\text{th}})^{-\text{T}} \\ &= (\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger}\mathbb{A}_i\mathbb{Q}(\mathbf{S}^{\text{th}})^{-\text{T}} \\ &= (\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger}(\mathbb{J}\mathbb{C})_{\text{d}}((\mathbf{S}^{\text{th}})^{\text{T}}\mathbb{Q}^{\dagger})^{-1},\end{aligned}\quad (\text{E4})$$

where the eigenvalue matrix corresponding to $\mathbb{J}\mathbb{C}$ is

$$(\mathbb{J}\mathbb{C})_{\text{d}} = \mathbb{Q}^{\dagger}\mathbb{A}_i\mathbb{Q} = \mathbb{A}_i = \mathfrak{j}\mathfrak{c}. \quad (\text{E5})$$

From (E2) and (E4), we immediately realize that $\mathbb{J}\mathbb{D}$ commutes with $\mathbb{J}\mathbb{C}$, as we wanted to prove.

The condition $[\mathbb{J}\mathbb{D}, \mathbb{J}\mathbb{C}] = 0$ in Theorem 1 (proved in the main text) determines univocally the covariance matrix \mathbb{V}^{th} of a thermal equilibrium state $\hat{\sigma}^{\text{th}}$. Therefore, using the expressions (62) and (86), we can write

$$\mathbb{V}^{\text{th}} = (\mathbf{S}^{\text{th}})^{-1}\frac{1}{2}\mathbb{A}_r\mathbb{A}_i^{-1}(\mathbf{S}^{\text{th}})^{-\text{T}}, \quad (\text{E6})$$

where we used the symplectic condition $\mathbb{J}(\mathbf{S}^{\text{th}}) = (\mathbf{S}^{\text{th}})^{-\text{T}}\mathbb{J}$. Comparing with Eq. (49), we immediately recognize in Eq. (E6) the symplectic spectrum of \mathbb{V}^{th} , i.e.,

$$\frac{1}{2}\mathbb{A}_r\mathbb{A}_i^{-1} = \mathfrak{k}^{\text{th}} \oplus \mathfrak{l}^{\text{th}}. \quad (\text{E7})$$

Equating the matrix elements on both sides of (E7), we obtain $\frac{1}{2}\frac{1+a_j}{1-a_j} = \kappa_j^{\text{th}}$ with $a_j = \frac{|r_j|^2}{|s_j|^2}$ and $j = 1, \dots, n$. Inverting these equations and rewriting in a matrix structure, we get

$$\frac{|\mathfrak{r}|^2}{|\mathfrak{s}|^2} = \frac{2\mathfrak{k}^{\text{th}} - \mathbb{1}}{2\mathfrak{k}^{\text{th}} + \mathbb{1}}. \quad (\text{E8})$$

Now using (31), (45), and (51), we arrive to the equality $g(\mathfrak{k}^{\text{th}}) = \hbar\beta\omega$. Then, employing the identity $\exp[-2 \coth^{-1}(2x)] = \frac{2x-1}{2x+1}$ for $x \geq \frac{1}{2}$, we can rewrite

Eq. (E8) as

$$\frac{|\mathfrak{r}|^2}{|\mathfrak{s}|^2} = e^{-\hbar\beta\omega}, \quad (\text{E9})$$

which finally implies the relation in (80).

APPENDIX F: PROOF OF EQ. (93)

Let us prove an equivalent statement of Eq. (93), i.e., in a GDS satisfying a $\hat{\sigma}^{\text{th}}$ -DBC we have

$$[\hat{H}, \hat{L}_j] = -\hbar\omega_j\hat{L}_j \Rightarrow [\hat{H}, \hat{L}_j^{\dagger}] = \hbar\omega_j\hat{L}_j^{\dagger}, \quad (\text{F1})$$

for $j = 1, \dots, n$, \hat{H} given in (42), and the Lindblad operators \hat{L}_j in (90).

For a generic symmetric matrix \mathbb{X} and a generic complex vector \mathbf{l} , it is straightforward to show that

$$\left[\frac{1}{2}\hat{\mathbf{x}}^{\text{T}}\mathbb{X}\hat{\mathbf{x}}, \mathbf{l}^{\text{T}}\mathbb{J}\hat{\mathbf{x}} \right] = i\hbar\mathbf{l}^{\text{T}}\mathbb{X}\hat{\mathbf{x}} = i\hbar\mathbf{l}^{\text{T}}\mathbb{X}\mathbb{J}^{\text{T}}\mathbb{J}\hat{\mathbf{x}}, \quad (\text{F2})$$

which is the commutation relation for a generic quadratic Hamiltonian with a linear Lindblad operator. But, if the GDS satisfies a $\hat{\sigma}^{\text{th}}$ -DBC according to Eq. (55), $\bar{\mathbf{l}}_j$ appearing in (91) is an eigenvector of the matrix $\mathbb{J}\mathbb{B}$, i.e., $\mathbb{J}\mathbb{B}\bar{\mathbf{l}}_j = i\omega_j\bar{\mathbf{l}}_j$, which is the same as say that $\bar{\mathbf{l}}_j = (\mathbb{Q}\mathbf{S}^{\text{th}})^{-1}\bar{\mathbf{l}}_j$ (see Appendix E). Therefore, using (F2) with $\mathbf{l}_j^{\text{T}}\mathbb{B}\mathbf{l}_j = i\omega_j\bar{\mathbf{l}}_j^{\text{T}}$ we arrive to (F1).

APPENDIX G: PROOF OF EQ. (94)

Let us start noting that for the thermal state $\hat{\sigma}^{\text{th}}$ in (41), or equivalently in (99), the eigenenergy distribution is the usual (multimode) Planck distribution:

$$P(\mathbf{n}) = \langle \phi_{\mathbf{n}} | \hat{\sigma}^{\text{th}} | \phi_{\mathbf{n}} \rangle = \langle \mathbf{n} | \hat{\sigma}_{\text{ho}}^{\text{th}} | \mathbf{n} \rangle = \prod_{j=1}^n P_{n_j}(\bar{n}_j), \quad (\text{G1})$$

which is nothing but a consequence of Eqs. (96) and (98). In the above equation,

$$P_{n_j}(\bar{n}_j) = \frac{1}{\bar{n}_j + 1} \left(\frac{\bar{n}_j}{\bar{n}_j + 1} \right)^{n_j}, \quad (\text{G2})$$

where \bar{n}_j is in (95). Note that $\bar{n}_j(\bar{n}_j + 1)^{-1} = e^{-\hbar\beta\omega_j}$.

This distribution can be recovered considering the stationary regime $\frac{d\hat{\rho}_i}{dt} = 0$ in (92), subsequently taking the diagonal matrix elements in the eigenbasis $\{|\phi_{\mathbf{n}}\rangle\}$ and using the commutation relation $[\hat{X}_j^-, \hat{X}_j^+] = 1$, indeed

$$\begin{aligned}& \sum_{j=1}^n \frac{|s_j|^2}{\hbar} [(n_j + 1)P_{n_j+1} - n_j P_{n_j}] \\ & + \sum_{j=1}^n \frac{|s_j|^2}{\hbar} e^{-\hbar\beta\omega_j} [n_j P_{n_j-1} - (n_j + 1)P_{n_j}] = 0,\end{aligned}$$

where $P_{n_j} = \langle \phi_{\mathbf{n}} | \hat{\sigma}^{\text{th}} | \phi_{\mathbf{n}} \rangle$ and $P_{n_j \pm 1} = \langle \phi_{\mathbf{n}^{\pm}} | \hat{\sigma}^{\text{th}} | \phi_{\mathbf{n}^{\pm}} \rangle$. As can be directly checked, the solution is

$$P_{n_j} = \frac{\hbar\bar{\gamma}_j}{|s_j|^2} e^{-\hbar\beta\omega_j n_j}. \quad (\text{G3})$$

The constant $\bar{\gamma}_j$ is included in order to $\hbar\bar{\gamma}_j/|s_j|^2$ be dimensionless. Comparing Eqs. (G3) and (G2) we arrive to (94).

APPENDIX H: ALGEBRA OF COMMUTING ELLIPTIC HAMILTONIAN MATRICES

In this Appendix we will prove the following theorem:

Theorem 6. Consider a positive-definite symmetric matrix \mathbb{B} and a symmetric matrix \mathbb{B}' such that $[\mathbb{J}\mathbb{B}, \mathbb{J}\mathbb{B}'] = 0$, then

$$\mathbb{J}\mathbb{B}' = (\mathbb{Q}\mathbb{S})^{-1} \iota x \oplus (-\iota x) (\mathbb{Q}\mathbb{S}), \quad (\text{H1})$$

where \mathbb{Q} is defined in (26), $x := \text{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$ for $i = 1, \dots, n$, and \mathbb{S} is a symplectic matrix that diagonalizes symplectically \mathbb{B} , i.e.,

$$\mathbb{S}^{-\text{T}}\mathbb{B}\mathbb{S} = \omega \oplus \omega, \quad \omega = \text{diag}(\omega_1, \dots, \omega_n) \quad (\text{H2})$$

for $\omega_i > 0$ and $i = 1, \dots, n$. Further, Eq. (H1) is equivalent to a ‘‘symplectic diagonalization’’ of \mathbb{B}' :

$$\mathbb{S}^{-\text{T}}\mathbb{B}'\mathbb{S}^{-1} = x \oplus x. \quad (\text{H3})$$

Let $\tilde{\mathbb{S}}$ be a symplectic matrix such that $\tilde{\mathbb{S}}^{-\text{T}}\mathbb{B}\tilde{\mathbb{S}} = \omega \oplus \omega$, then

$$\begin{aligned} [\mathbb{J}\mathbb{B}, \mathbb{J}\mathbb{B}'] &= [\mathbb{J}\tilde{\mathbb{S}}^{\text{T}}(\omega \oplus \omega)\tilde{\mathbb{S}}, \mathbb{J}\mathbb{B}'] \\ &= \mathbb{J}\tilde{\mathbb{S}}^{\text{T}}[\mathbb{J}(\omega \oplus \omega), \tilde{\mathbb{S}}^{-\text{T}}\mathbb{B}'\tilde{\mathbb{S}}^{-1}]\tilde{\mathbb{S}}, \end{aligned} \quad (\text{H4})$$

where we used the symplectic condition $\tilde{\mathbb{S}}\mathbb{J} = \mathbb{J}\tilde{\mathbb{S}}^{-\text{T}}$ and $\mathbb{J}(\omega \oplus \omega) = (\omega \oplus \omega)\mathbb{J}$. Writing

$$\tilde{\mathbb{B}} := \tilde{\mathbb{S}}^{-\text{T}}\mathbb{B}'\tilde{\mathbb{S}}^{-1} = \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{y}^{\text{T}} & \tilde{z} \end{pmatrix},$$

where \tilde{x} , \tilde{y} , and \tilde{z} are $n \times n$ real matrices such that $\tilde{x} = \tilde{x}^{\text{T}}$ and $\tilde{z} = \tilde{z}^{\text{T}}$, and using the commutation relation $[\mathbb{J}\mathbb{B}, \mathbb{J}\mathbb{B}'] = 0$ in (H4) one attains

$$\tilde{y}^{\text{T}} = -\omega^{-1}\tilde{y}\omega = -\omega\tilde{y}\omega^{-1} \quad \text{and} \quad \tilde{z} = \omega^{-1}\tilde{z}\omega = \omega\tilde{z}\omega^{-1};$$

consequently, $[\tilde{y}, \omega] = [\tilde{z}, \omega] = 0$, thus $\tilde{x} = \tilde{z}$ and \tilde{y} is skew symmetric $\tilde{y}^{\text{T}} = -\tilde{y}$. All of these relations enable us to write

$$\tilde{\mathbb{B}} := \tilde{\mathbb{S}}^{-\text{T}}\mathbb{B}'\tilde{\mathbb{S}}^{-1} = \begin{pmatrix} \tilde{x} & \tilde{y} \\ -\tilde{y} & \tilde{x} \end{pmatrix}. \quad (\text{H5})$$

Multiplying by \mathbb{J} from left, considering the symplecticity of $\tilde{\mathbb{S}}$, and using \mathbb{Q} , the last equation is equivalently rewritten as

$$(\mathbb{Q}\tilde{\mathbb{S}})\mathbb{J}\mathbb{B}'(\mathbb{Q}\tilde{\mathbb{S}})^{-1} = \mathbb{Q}\mathbb{J}\tilde{\mathbb{B}}\mathbb{Q}^{\dagger} = (\iota\tilde{x} - \tilde{y}) \oplus (-\iota\tilde{x} - \tilde{y}). \quad (\text{H6})$$

The above particular block structure is a consequence of the degenerated structure of the diagonal matrix $\omega \oplus \omega$, where each diagonal element is at least doubly degenerated.

The two blocks in the matrix of rightmost equality in (H6) are skew Hermitian and, moreover, they are complex conjugates of each other. Recalling that a skew-Hermitian matrix has pure imaginary (possibly null) eigenvalues and is unitarily diagonalizable [46], then there exists a unitary matrix u such that $u(\iota\tilde{x} - \tilde{y})u^{\dagger} = \iota x$, where $x = \text{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix containing the eigenvalues $\lambda_j \in \mathbb{R}$ of the matrix $\iota\tilde{x} - \tilde{y}$.

Defining $\mathbb{S} = \mathbb{R}\tilde{\mathbb{S}}$ with

$$\mathbb{R} = \mathbb{Q} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \mathbb{Q}^{\dagger} \in \text{Sp}(2n, \mathbb{R}) \cap \text{O}(2n) \quad (\text{H7})$$

a real symplectic and orthogonal matrix, from (H6) we can write

$$\mathbb{Q}\mathbb{S}\mathbb{J}\mathbb{B}'(\mathbb{Q}\mathbb{S})^{-1} = \iota x \oplus (-\iota x), \quad (\text{H8})$$

which is Eq. (H1). Since \mathbb{S} is symplectic and using that $\mathbb{Q}^{\dagger}(\iota x \oplus (-\iota x))\mathbb{Q} = \mathbb{J}(x \oplus x)$, we immediately recover Eq. (H3) from (H8).

However, it remains to prove that \mathbb{S} satisfies (H2). If the symplectic spectrum in ω is nondegenerate, conditions $[\tilde{y}, \omega] = [\tilde{z}, \omega] = 0$, $\tilde{x}^{\text{T}} = \tilde{x}$, and $\tilde{y}^{\text{T}} = -\tilde{y}$ imply $\tilde{x} = x = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $\tilde{y} = 0$, so one can choose $u = \mathbb{1}$ such that $\mathbb{R} = \mathbb{1}$ and $\mathbb{S} = \tilde{\mathbb{S}}$; consequently, \mathbb{S} satisfies (H2), as claimed. When the symplectic spectrum in ω is degenerate, \tilde{x} is diagonal outside the degenerate subspaces of ω , while \tilde{y} is null outside the same subspaces. Therefore, the unitary matrix u has to diagonalize $\iota\tilde{x} - \tilde{y}$ only inside the degenerate subspaces of ω . This is possible choosing u block diagonal such $u\iota\tilde{x} - \tilde{y}u^{\dagger} = \iota x$. In this case we also have that $\mathbb{R}^{\text{T}}\omega \oplus \omega\mathbb{R} = \omega \oplus \omega$, and therefore $\mathbb{S} = \mathbb{R}\tilde{\mathbb{S}}$ also diagonalize symplectically \mathbb{B} , i.e.,

$$\mathbb{B} = \mathbb{S}^{\text{T}}\omega \oplus \omega\mathbb{S} = \tilde{\mathbb{S}}^{\text{T}}\mathbb{R}^{\text{T}}\omega \oplus \omega\mathbb{R}\tilde{\mathbb{S}} = \tilde{\mathbb{S}}^{\text{T}}\omega \oplus \omega\tilde{\mathbb{S}}, \quad (\text{H9})$$

which shows that \mathbb{S} satisfies (H2) for the degenerate case; with this we finish the proof of the theorem.

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