Integrodifferential equation in the multimode Jaynes-Cummings model

Katsuya Hasebe
Aichi University, Nagoya, 453-8777, Japan

Akihiro Nakayama ,* Toshiyuki Okubo, and Motoomi Yamanoi
Faculty of Science and Technology, Meijo University, Nagoya 468-8502, Japan

(Received 26 September 2022; accepted 18 November 2022; published 1 December 2022)

We investigate the multimode Jaynes-Cummings model. The population of the excited state of an atom is expressed by an integrodifferential equation. The exact solution is found in an idealized case by taking the infinite limit of the number of modes. The solution shows a mathematical reason why the kinklike behavior of the excited-state population appears. We also confirm that numerical calculations agree with the exact solution.

DOI: 10.1103/PhysRevA.106.062201

I. INTRODUCTION

The Jaynes-Cummings model (JC model) was proposed to investigate a system which consists of an atom and the electromagnetic field in a cavity [1]. The atom is assumed to be a two-level system (TLS), and the electromagnetic field is assumed to have a single mode whose energy is close to the resonance energy of the atom. The interaction consists of a process in which the excited state transfers to the ground state with an emission of photon and its inverse process. This restriction of interaction is nowadays called the rotating-wave approximation.

The JC model has been extended to apply various quantum systems [2,3]. The multimode extension is introduced in a study of spontaneous emission of an atom in a cavity [4,5]. The atom interacts with many modes of the electromagnetic field. In these works, the model is solved by diagonalizing the Hamiltonian in a few cases where the number of modes is small, and the population of the excited state of the atom is obtained. When the number of modes is large, the population is estimated numerically. A TLS in a one-dimensional electromagnetic field is studied in another approach [6], and analytical expression of the population is shown in the case where the number of modes is infinite. However, its explicit form is not given and numerical results are shown in the case of a finite number of modes.

In a subsequent work of the spontaneous emission in a cavity, a kinklike behavior of the population of the excited state is emphasized [7]. The kinks appear at every round-trip time of the photon in the cavity (see also Figs. 1 and 2 in this paper). A photon packet emitted by the atom is reflected from the boundary of the cavity and is absorbed by the atom again. The kinks originate in this reflected photon packet. The profile of the motion of the photon packet is clearly depicted in studies of the multimode JC model [8] and the Rabi model [9]. The interpretation is intuitively clear, but these results are obtained by numerical calculations.

Another approach is often adopted to obtain the excited-state population. In the case of a large number of modes, the diagonalization of the Hamiltonian is difficult, and Volterra's integrodifferential equations are used to describe the population of the excited state. A heat bath model, which is a model for dissipation, is studied by this method [10]. In this model, the system is one harmonic oscillator and the reservoir is many harmonic oscillators. The decay of the system is expressed by integrodifferential equations. A TLS in dissipative cavities is also investigated by a similar method [11,12]. These systems are described by a kind of multimode JC model including the environment. It is shown that the revival, where the excited-state population takes almost the same value as the initial state, appears at every round-trip time. The appearance of revival is clear also in a three-dimensional ellipsoidal cavity, where the round-trip time takes a single value for all directions [13]. In these works, authors investigate the case where the number of modes is infinite, but they do not show the explicit form of the solution.

In this paper the multimode JC model is investigated analytically. We concentrate on the time evolution of the excited-state population of the TLS. The amplitude of the excited state satisfies an integrodifferential equation. The original form of this equation cannot be exactly solved. However, by taking an infinite limit of the number of modes, we can find the exact solution with a recursive formula. In order to validate the limit, we compare the solution with numerical results which are obtained without taking the limit [7,10–12]. The solution supports the interpretation of the kinks due to the boundary effect and gives an analytical expression for the kinks.

The paper is organized as follows: In Sec. II, we present the model and derive an integrodifferential equation. In Sec. III, we solve the equation in the case where coupling constants have the same value for all modes. The exact solution is obtained in the infinite limit of the number of modes and is given by a recursive formula. We also confirm the validity of

*spock@meijo-u.ac.jp

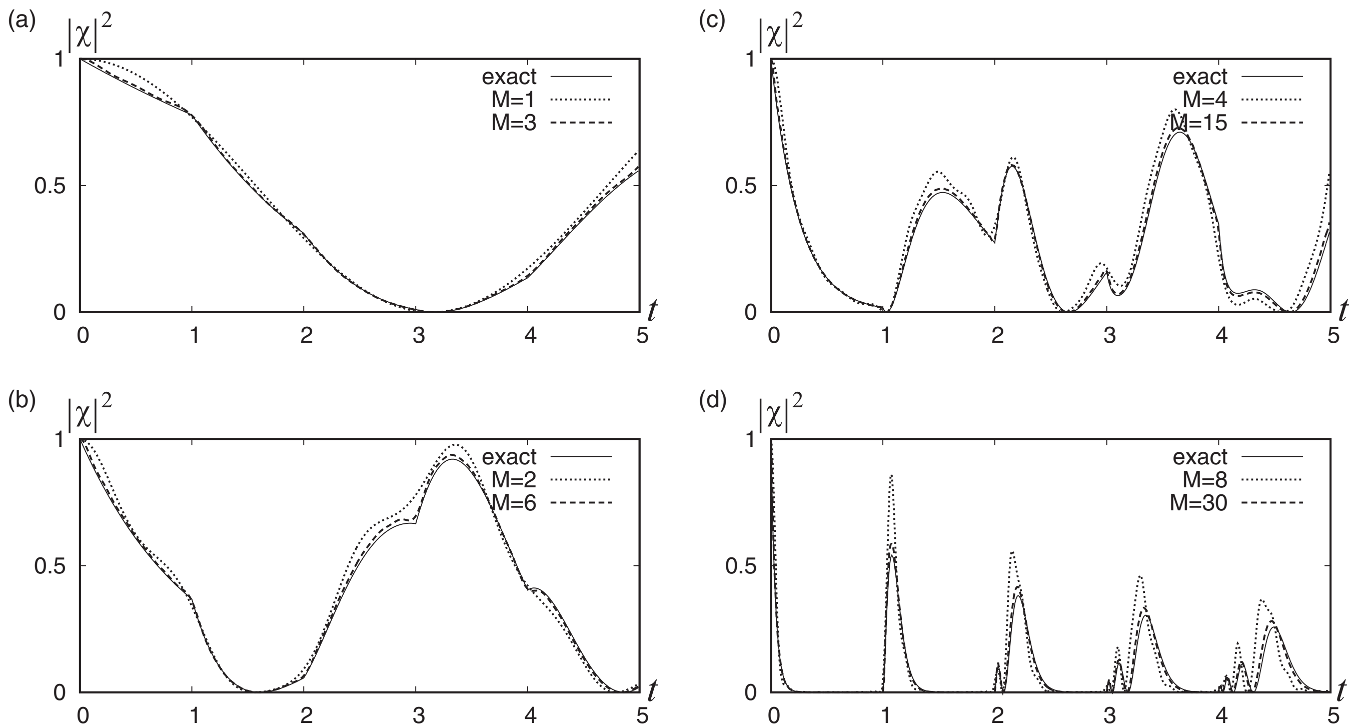

FIG. 1. $|\chi|^2$ given by the exact solution and numerical calculations in the case of $\alpha = 0$. Parameters are chosen as (a) $g = 0.5$ and $M = 1$ and 3, (b) $g = 1.0$ and $M = 2$ and 6, (c) $g = 2.0$ and $M = 4$ and 15, and (d) $g = 5.0$ and $M = 8$ and 30. The unit of time t is the round-trip time L/c.

the solution numerically. Section IV is devoted to summary and discussion.

II. JC MODEL AND VOLTERRA EQUATION

We investigate the time dependence of the excited-state population of a TLS in a one-dimensional cavity. The TLS is supposed to exist at the center of the cavity for simplicity. The energy level of the photon in the cavity can be indicated by an integer n. The energy of the nth mode is $\omega_n = n\omega_1$, where ω_1 is the lowest energy. It is well-known that the TLS at the center of the cavity interacts with the modes with the odd number index $n = 1, 3, 5, \ldots$ Then we reassign the index n to the $(2n-1)$th mode. The energy of the nth mode is given by $\omega_n = (2n-1)\omega_1$.

The multimode JC model is described by the Hamiltonian

$$\mathcal{H} = \omega_0 \frac{\sigma_z + 1}{2} + \sum_{n=1}^{N} \omega_n a_n^\dagger a_n + \sum_{n=1}^{N} g_n(\sigma^- a_n^\dagger + \sigma^+ a_n), \quad (1)$$

where ω_0 and ω_n are energy levels for the TLS and the photon, respectively. Because the TLS is similar to a spin, the Hamiltonian is written by Pauli matrices. The eigenstates of σ_z with eigenvalues -1 and 1 represent the ground state $|g\rangle$ and the excited state $|e\rangle$, respectively. $\sigma^\pm$ are defined by $\sigma^+ = |e\rangle\langle g|$ and $\sigma^- = |g\rangle\langle e|$. $a_n^\dagger$ and a_n are creation and annihilation operators for the nth mode of the photon. The number of modes N should be infinite, but is tentatively considered to be finite. The coupling constants g_n are taken to be real without loss of generality.

To simplify the expression, we choose the lowest energy of the photon as the unit of energy. Moreover, we make it dimensionless by setting $\omega_1 = \pi$. The excitation energy of the TLS is rewritten by use of this unit:

$$\omega_0 = \pi(2J-1), \quad \omega_n = \pi(2n-1), \quad (2)$$

where $n = 1, 2, 3, \ldots$, and J is not generally an integer. The dimension for each variable is lost, but it can be restored in final results if necessary.

We suppose that the TLS is initially in the excited state $|e\rangle$ and there are no photons in the cavity, and we investigate the probability of the excited state. The state vector at the time t can be expanded as

$$|\psi\rangle = e^{-i\omega_0 t}\chi(t)|e\rangle \otimes |0\rangle + \sum_{n=1}^{N} e^{-i\omega_n t}\eta_n(t)|g\rangle \otimes |1_n\rangle, \quad (3)$$

where $|0\rangle$ is the no-photon state and $|1_n\rangle \equiv a_n^\dagger|0\rangle$ is the one-photon state of the nth mode. $\chi(t)$ and $\eta_n(t)$ are coefficients for these Fock states. The phase factors $e^{-i\omega_0 t}$ and $e^{-i\omega_n t}$ are introduced to simplify equations. Then the Schrödinger equation $i|\dot{\psi}\rangle = \mathcal{H}|\psi\rangle$ reduces to

$$i\dot{\chi} = \sum_{n=1}^{N} g_n e^{-2\pi i(n-J)t}\eta_n, \quad (4)$$

$$i\dot{\eta}_n = g_n e^{2\pi i(n-J)t}\chi. \quad (5)$$

Our purpose is to find the solution $\chi(t)$ under the initial conditions $\chi(0) = 1$ and $\eta_n(0) = 0$. Equations (4) and (5) are formally solved as

$$\eta_n(t) = -ig_n \int_0^t e^{2\pi i(n-J)s}\chi(s)ds \qquad (6)$$

and

$$\begin{aligned}\dot{\chi}(t) &= -\sum_{n=1}^{N} g_n^2 \int_0^t e^{-2\pi i(n-J)(t-s)}\chi(s)ds \\ &= -\sum_{n=1}^{N} g_n^2 \int_0^t e^{-2\pi i(n-J)\tau}\chi(t-\tau)d\tau. \qquad (7)\end{aligned}$$

When the parameter J is not an integer, the excited energy of the TLS is not equal to any energy level of photon. The difference between the excited energy of the TLS and the nearest energy level of the photon is called detuning. We rewrite J as

$$J = J_0 + \alpha \quad (0 \leqslant \alpha < 1), \qquad (8)$$

where J_0 is the integer part of J and α is the detuning. Then Eq. (7) becomes

$$\dot{\chi}(t) = -\varepsilon^{\tau} \sum_{n=1}^{N} g_n^2 \int_0^t e^{-2\pi i(n-J_0)\tau}\chi(t-\tau)d\tau, \qquad (9)$$

where $\varepsilon = e^{2\pi i\alpha}$.

Changing the order of the sum and the integral, we obtain an integrodifferential equation:

$$\dot{\chi}(t) = \int_0^t K(\tau)\chi(t-\tau)d\tau, \qquad (10)$$

where

$$K(\tau) = -\varepsilon^{\tau} \sum_{n=1}^{N} g_n^2 e^{-2\pi i(n-J_0)\tau}. \qquad (11)$$

III. SOLUTION OF THE INTEGRODIFFERENTIAL EQUATION

The kernel (11) for Eq. (10) is explicitly obtained for small N, but is obtained only by numerical calculation for large N. In this section we find the exact solution analytically in an idealized case. The exact solution clarifies the reason why the kinklike behavior of $|\chi|^2$ appears in numerical results for large N.

A. Analytical solution

We investigate an idealized case $g_n = g$ and $N = 2J_0 - 1$, which means that the excited energy of the TLS exists in the middle of the energy levels of the photon except for the detuning. Then the kernel (11) is

$$\begin{aligned}K(\tau) &= -g^2\varepsilon^{\tau} \sum_{m=-M}^{M} e^{-2\pi i m\tau} \\ &= -g^2\varepsilon^{\tau} D(\tau), \qquad (12)\end{aligned}$$

where $M = J_0 - 1$, $m = n - J_0$, and $D(\tau)$ is the Dirichlet kernel $D(\tau) = \sin[(2M+1)\pi\tau]/\sin\pi\tau$. This expression is shown in Ref. [7] for example.

Here we consider the large-M limit. The number of modes becomes infinite but the excited energy of the TLS is always in the middle of those levels. In such a case the Dirichlet kernel approaches the Dirac comb:

$$\sum_{m=-M}^{M} e^{-2\pi i m\tau} \to \sum_{k=-\infty}^{+\infty} \delta(\tau - k), \qquad (13)$$

where k is an integer. The same expression is obtained in Ref. [12], but the analytical solution of Eq. (10) is not given.

We note the meaning of the integer k. When the length of the cavity is L, the wave number of the lowest mode is π/L and the round-trip time of a photon packet in the cavity is L/c. Because we have set the unit of energy as $\omega_1 = \pi$, the round-trip time becomes $L/c = 1$. $\tau = k$ indicates the kth return of a photon packet emitted at $\tau = 0$.

We find the exact solution of Eq. (10) with Eqs. (12) and (13) analytically. For this purpose, we divide the whole range of time into the intervals $0 \leqslant t < 1$, $1 \leqslant t < 2$, and so on. If t is in the jth interval ($j \leqslant t < j+1$), where j is a positive integer, Eq. (10) is rewritten as

$$\begin{aligned}\dot{\chi}(t) &= -g^2 \sum_{k=0}^{j} \int_0^t \varepsilon^k \delta(\tau - k)\chi(t-\tau)d\tau \\ &= \gamma_0\chi(t) + \gamma_1\chi(t-1) + \cdots + \gamma_j\chi(t-j), \quad (14)\end{aligned}$$

where

$$\gamma_0 = -\tfrac{1}{2}g^2, \quad \gamma_j = -\varepsilon^j g^2. \qquad (15)$$

In Eq. (15), we used $\int_0^{0^+} \delta(\tau)d\tau = 1/2$.

The solution $\chi(t)$ can be recursively found from the 0th interval $0 \leqslant t < 1$. When $0 \leqslant t < 1$, Eq. (14) reduces to

$$\dot{\chi}(t) = \gamma_0\chi(t), \qquad (16)$$

and the solution is

$$\chi = e^{\gamma_0 t}. \qquad (17)$$

In this interval the photon reflected from the boundary does not arrive at the TLS. The behavior of the TLS is the same as that in free space and shows the exponential decay.

When $1 \leqslant t < 2$, the right-hand side of Eq. (14) contains the second term $\chi(t-1)$, but this term is equal to Eq. (17). $\chi(t-1)$ is interpreted as an effect of the photon reflected from the boundary in the 0th interval.

To find a general expression, we consider the jth interval ($j \leqslant t < j+1$) and suppose that $\chi(t)$ is already obtained until the $(j-1)$th interval. We introduce a new function, $\chi_k(\tau)$, with a new variable, $\tau = t - k$ $(0 \leqslant \tau < 1)$, for each k $(k \leqslant j)$. Then τ becomes common variable for all intervals and all terms $\chi(t-k)$ can be rewritten by τ as $\chi(t) = \chi_j(\tau)$, $\chi(t-1) = \chi_{j-1}(\tau)$, ..., $\chi(t-j) = \chi_0(\tau)$. Equation (14) becomes

$$\begin{aligned}\dot{\chi}_j(\tau) &= \gamma_0\chi_j(\tau) + \gamma_1\chi_{j-1}(\tau) + \gamma_2\chi_{j-2}(\tau) + \cdots + \gamma_n\chi_0(\tau) \\ &= \sum_{k=0}^{j} \gamma_k\chi_{j-k}(\tau). \qquad (18)\end{aligned}$$

By the redefinition $\chi_k(\tau) = e^{\gamma_0\tau}\zeta_k(\tau)(0 \leqslant k \leqslant j)$, we find

$$\dot{\zeta}_j(\tau) = \sum_{k=1}^{j} \gamma_k \zeta_{j-k}(\tau), \tag{19}$$

where $\zeta_0(\tau) = 1$ from Eq. (17). The general solution of Eq. (19) is given by a polynomial up to the jth order of τ,

$$\zeta_j(\tau) = \sum_{m=0}^{j} A_{jm}\tau^m, \tag{20}$$

where coefficients A_{jk} are recursively obtained as follows.

Substituting Eq. (20) into Eq. (19), we find

$$\begin{aligned}\sum_{m=1}^{j} mA_{jm}\tau^{m-1} &= \sum_{k=1}^{j} \gamma_k \sum_{m=0}^{j-k} A_{j-k\,m}\tau^m \\ &= \sum_{m=1}^{j} \sum_{k=1}^{j-m+1} \gamma_k A_{j-k\,m-1}\tau^{m-1},\end{aligned} \tag{21}$$

where we changed the order of summations and replaced the index m by $m-1$. By comparing each coefficients of τ^{m-1} in Eq. (21) we obtain

$$mA_{jm} = \sum_{k=1}^{j-m+1} \gamma_k A_{j-k\,m-1}. \tag{22}$$

Here A_{j0} are determined by the continuity of the wave function $\chi_j(0) = \chi_{j-1}(1)$, that is, $\zeta_j(0) = e^{\gamma_0}\zeta_{j-1}(1)$:

$$A_{j0} = e^{\gamma_0} \sum_{m=0}^{j-1} A_{j-1\,m}, \tag{23}$$

where $A_{00} = 1$ from Eq. (17).

We show the exact solution for a few intervals (see Appendix):

$$\begin{aligned}\chi_0(\tau) &= e^{\gamma_0\tau} = e^{-g^2\tau/2}, \\ \chi_1(\tau) &= e^{-g^2\tau/2}(A_{10} + A_{11}\tau), \\ \chi_2(\tau) &= e^{-g^2\tau/2}(A_{20} + A_{21}\tau + A_{22}\tau^2),\end{aligned} \tag{24}$$

where coefficients A_{jm} are

$$\begin{aligned}A_{10} &= e^{\gamma_0}A_{00} = e^{-g^2/2}, \\ A_{11} &= \gamma_1 A_{00} = -\varepsilon g^2, \\ A_{20} &= e^{\gamma_0}(A_{10} + A_{11}) = e^{-g^2/2}(e^{-g^2/2} - \varepsilon g^2), \\ A_{21} &= \gamma_1 A_{10} + \gamma_2 A_{00} = -\varepsilon g^2 e^{-g^2/2} - \varepsilon^2 g^2, \\ A_{22} &= \tfrac{1}{2}\gamma_1 A_{11} = \tfrac{1}{2}\varepsilon^2 g^4.\end{aligned} \tag{25}$$

This result shows that the amplitudes of the excited state are continuous, $\chi_0(1) = \chi_1(0)$ and $\chi_1(1) = \chi_2(0)$, but their derivatives with respect to τ are discontinuous. The exact solution reveals the origin of kinklike behavior of the excited-state population.

B. Comparison of analytical and numerical solutions

In this subsection, we compare the exact solution with numerical calculations. The exact solution is obtained in the limit $M \to \infty$. Numerical calculations can be carried out only for finite M, where the number of modes is $2M+1$. If we take $N = \infty$ in Eq. (1), the range of summation in Eq. (12) becomes $-M \leqslant m \leqslant \infty$. In the case of finite M, the contribution from the modes $M < m$ is discarded. In the case of infinite M, the extra contribution from the modes $m < -M$ is included. If they agree with each other, both methods are considered to be appropriate. Because we concentrate on the kinklike behavior of the population in this work, we confirm the validity of the analysis by the agreement of the behavior for various g.

Figure 1 shows the exact solution and numerical results for several values of M in the case of $\alpha = 0$ (zero detuning). When the coupling constant is small ($g = 0.5$), the numerical result shows a good agreement with the exact solution even for $M = 3$, i.e., seven modes. When the coupling constant g is large, the numerical calculation requires larger M to reproduce the exact solution. In the case of $g = 5$, M should be larger than 30 at least.

Inversely, we can use the limit $M \to \infty$ to solve the multimode JC model in cases where the number of modes is appropriately large. The limit $M \to \infty$ means that there is no lower bound of the Hamiltonian. However, when the energy level of the TLS is appropriately larger than the lowest energy of the photon, this limit is a good approximation for a finite number of modes.

Figure 2 shows the exact solution and numerical results for several values of M in the case of $\alpha = 1/2$ (maximum detuning). The dependencies on the number of modes are quite similar to those in Fig. 1, though the profiles of $|\chi|^2$ are rather different from those in Fig. 1. We can find the effect of detuning in the oscillation period of $|\chi|^2$ for small g. The kinklike behaviors are seen in both figures. Our analysis clarifies the mathematical reason why such kinks appear.

IV. SUMMARY AND DISCUSSION

In this paper we investigated the multimode JC model and found the exact solution for the population of the excited state of a TLS in an idealized case. The population is described by an integrodifferential equation, which has already been obtained in other works [7,10,12]. By supposing the number of modes is infinite, the integrodifferential equation is converted to a series of equations for time intervals $0 \leqslant t < 1$, $1 \leqslant t < 2$, and so on. The exact solution is recursively obtained for each time interval, and its derivative with respect to time t has discontinuities. This is the reason why the kinklike behavior appears in the profile of the excited-state population [7].

We compare the exact solution with numerical results for a finite number of modes. Numerical results agree with the exact solution if the number of modes is appropriately large depending on the coupling constant. When the coupling constant is small, it is sufficient to take into account several modes. On the other hand, several tens of modes are necessary when the coupling constant is large. The infinite limit of the number of modes is a good approximation when the energy

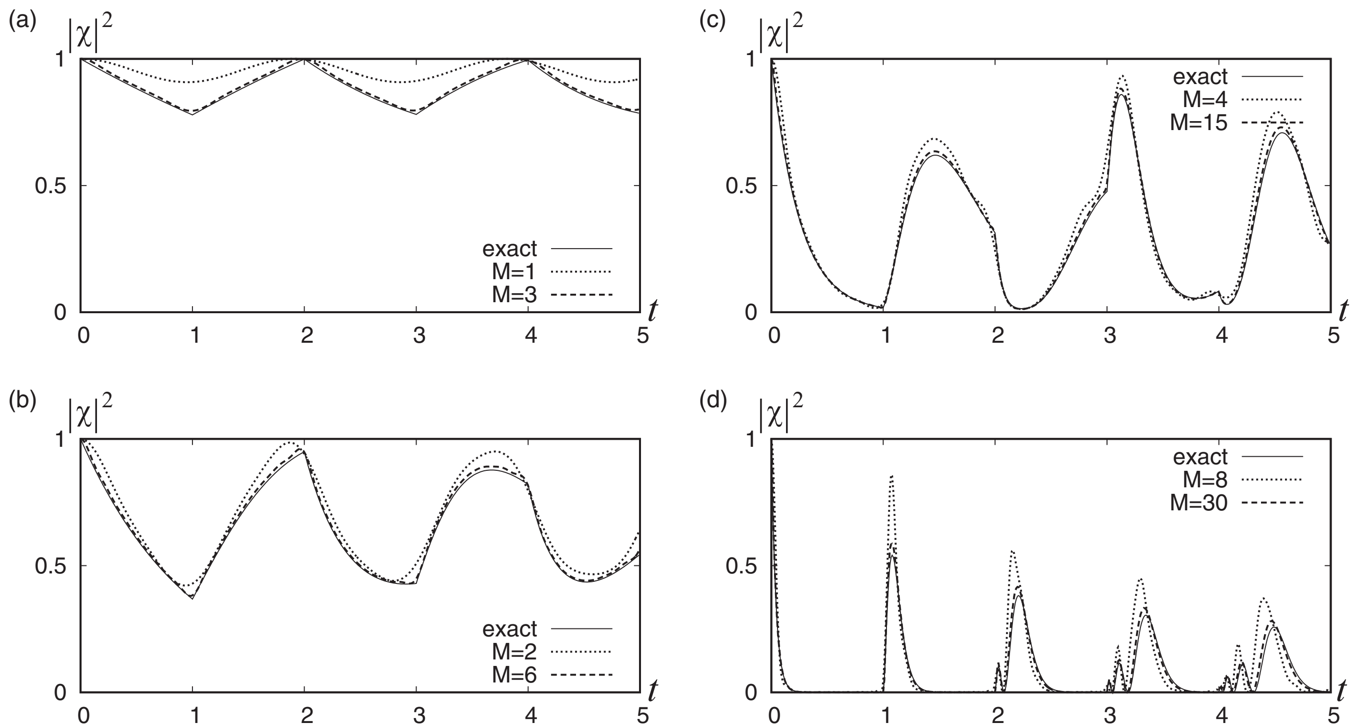

FIG. 2. $|\chi|^2$ given by the exact solution and numerical calculations in the case of $\alpha = 1/2$. Parameters are chosen as (a) $g = 0.5$ and $M = 1$ and 3, (b) $g = 1.0$ and $M = 2$ and 6, (c) $g = 2.0$ and $M = 4$ and 15, and (d) $g = 5.0$ and $M = 8$ and 30. The unit of time t is the round-trip time L/c.

level of the TLS is large compared to the lowest energy of the photon.

The exact solution is obtained under the condition that the coupling constants g_n take the same value for all modes. This condition is essential to obtain the Dirichlet kernel and the Dirac Comb in Eqs. (12) and (13). Therefore, we have little freedom to make a change of the coupling constants. For example, we cannot arbitrarily change the position of the atom, which existed at the center of the cavity in this work. It induces an involved n dependence of g_n. However, it may be possible to apply the method to a model with the dipole coupling $g_n \propto \sqrt{n}$, which is the standard interaction between an atom and the electromagnetic field. In this case, the present method requires some modifications. Such a case will be investigated in the future.

On the other hand, our method is not applicable to the Rabi model. The method relies on the feature of the multimode JC model, where the Hilbert space is restricted to $|e\rangle \otimes |0\rangle$ and $|g\rangle \otimes |1_n\rangle$ under the rotating-wave approximation. Because such a restriction does not exist in the Rabi model, it is difficult to find an integrodifferential equation similar to Eq. (10).

APPENDIX: EXACT SOLUTION

We show the exact solution until the 7th interval:

$$\chi_j(\tau) = e^{-g^2\tau/2}\sum_{m=0}^{j} A_{jm}\tau^m, \tag{A1}$$

where $j = 0, 1, \dots, 6$. To simplify expressions, we define new parameters as $q \equiv -g^2$, $\mu \equiv e^{-g^2/2}$, and $\varepsilon = e^{2\pi i\alpha}$. Then the coefficients A_{jm} are

$$\begin{aligned} A_{00} &= 1,\\ A_{10} &= \mu,\\ A_{11} &= \varepsilon q,\\ A_{20} &= \varepsilon\mu q + \mu^2,\\ A_{21} &= \varepsilon^2 q + \varepsilon\mu q,\\ A_{22} &= \tfrac{1}{2}\varepsilon^2 q^2, \end{aligned} \tag{A2}$$

$$\begin{aligned} A_{30} &= \tfrac{1}{2}\{\varepsilon^2\mu(q^2+2q) + 4\varepsilon\mu^2 q + 2\mu^3\},\\ A_{31} &= \varepsilon^3 q + \varepsilon^2\mu(q^2+q) + \varepsilon\mu^2 q,\\ A_{32} &= \tfrac{1}{2}\{2\varepsilon^3 q^2 + \varepsilon^2\mu q^2\},\\ A_{33} &= \tfrac{1}{6}\varepsilon^3 q^3, \end{aligned} \tag{A3}$$

$$\begin{aligned} A_{40} &= \tfrac{1}{6}\{\varepsilon^3\mu(q^3+6q^2+6q)\\ &\quad + 12\varepsilon^2\mu^2(q^2+q) + 18\varepsilon\mu^3 q + 6\mu^4\},\\ A_{41} &= \tfrac{1}{2}\{2\varepsilon^4 q + \varepsilon^3\mu(q^3+4q^2+2q)\\ &\quad + 2\varepsilon^2\mu^2(2q^2+q) + 2\varepsilon\mu^3 q\},\\ A_{42} &= \tfrac{1}{2}\{3\varepsilon^4 q^2 + \varepsilon^3\mu(q^3+2q^2) + \varepsilon^2\mu^2 q^2\},\\ A_{43} &= \tfrac{1}{6}\{3\varepsilon^4 q^3 + \varepsilon^3\mu q^3\},\\ A_{44} &= \tfrac{1}{24}\varepsilon^4 q^4, \end{aligned} \tag{A4}$$

$$
\begin{aligned}
A_{50} &= \tfrac{1}{24}\{\varepsilon^4\mu(q^4+12q^3+36q^2+24q)\\
&\quad+\varepsilon^3\mu^2(32q^3+96q^2+48q)\\
&\quad+\varepsilon^2\mu^3(108q^2+72q)+96\varepsilon\mu^4q+24\mu^5\},\\
A_{51} &= \tfrac{1}{6}\{6\varepsilon^5q+\varepsilon^4\mu(q^4+9q^3+18q^2+6q)\\
&\quad+\varepsilon^3\mu^2(12q^3+24q^2+6q)\\
&\quad+\varepsilon^2\mu^3(18q^2+6q)+6\varepsilon\mu^4q\},\\
A_{52} &= \tfrac{1}{4}\{8\varepsilon^5q^2+\varepsilon^4\mu(q^4+6q^3+6q^2)\\
&\quad+4\varepsilon^3\mu^2(q^3+q^2)+2\varepsilon^2\mu^3q^2\},\\
A_{53} &= \tfrac{1}{6}\{6\varepsilon^5q^3+\varepsilon^4\mu(q^4+3q^3)+\varepsilon^3\mu^2q^3\},\\
A_{54} &= \tfrac{1}{24}(4\varepsilon^5+\varepsilon^4\mu)q^4,\\
A_{55} &= \tfrac{1}{120}\varepsilon^5q^5,
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
A_{60} &= \tfrac{1}{120}\{\varepsilon^5\mu(q^5+20q^4+120q^3+240q^2+120q)\\
&\quad+\varepsilon^4\mu^2(80q^4+480q^3+720q^2+240q)\\
&\quad+\varepsilon^3\mu^3(540q^3+1080q^2+360q)\\
&\quad+\varepsilon^2\mu^4(960q^2+480q)+600\varepsilon\mu^5q+120\mu^6\},\\
A_{61} &= \tfrac{1}{24}\{24\varepsilon^6q\\
&\quad+\varepsilon^5\mu(q^5+16q^4+72q^3+96q^2+24q)\\
&\quad+\varepsilon^4\mu^2(32q^4+144q^3+144q^2+24q)\\
&\quad+\varepsilon^3\mu^3(108q^3+144q^2+24q)\\
&\quad+\varepsilon^2\mu^4(96q^2+24q)+24\varepsilon\mu^5q\},\\
A_{62} &= \tfrac{1}{12}\{30\varepsilon^6q^2+\varepsilon^5\mu(q^5+12q^4+36q^3+24q^2)\\
&\quad+\varepsilon^4\mu^2(12q^4+36q^3+18q^2)\\
&\quad+\varepsilon^3\mu^3(18q^3+12q^2)+6\varepsilon^2\mu^4q^2\},\\
A_{63} &= \tfrac{1}{12}\{20\varepsilon^6q^3+\varepsilon^5\mu(q^5+8q^4+12q^3)\\
&\quad+\varepsilon^4\mu^2(4q^4+6q^3)+2\varepsilon^3\mu^3q^3\},\\
A_{64} &= \tfrac{1}{24}\{10\varepsilon^6q^4+\varepsilon^5\mu(q^5+4q^4)+\varepsilon^4\mu^2q^4\},\\
A_{65} &= \tfrac{1}{120}\{5\varepsilon^6q^5+\varepsilon^5\mu q^5\},\\
A_{66} &= \tfrac{1}{720}\varepsilon^6q^6.
\end{aligned}
\tag{A6}
$$

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[2] B. W. Shore and P. L. Knight, J. Mod. Opt. **40**, 1195 (1993).
[3] J. Larson and T. Mavrogordatos, *The Jaynes–Cummings Model and Its Descendants* (IOP, Bristol, 2021).
[4] S. Swain, J. Phys. A: Gen. Phys. **5**, 1587 (1972).
[5] S. Swain, J. Phys. A: Gen. Phys. **5**, 1601 (1972).
[6] R. Davidson and J. J. Kozak, J. Math. Phys. **12**, 903 (1971).
[7] J. Parker and C. R. Stroud, Jr., Phys. Rev. A **35**, 4226 (1987).
[8] V. Bužek, G. Drobny, M. S. Kim, G. Adam, and P. L. Knight, Phys. Rev. A **56**, 2352 (1997).
[9] C. Sánchez Muñoz, F. Nori, and S. De Liberato, Nat. Commun. **9**, 1924 (2018).
[10] M. Bauer and D. Bernard, J. Stat. Mech.: Theory Exp. (2011) P04016.
[11] D. O. Krimer, M. Liertzer, S. Rotter, and H. E. Türeci, Phys. Rev. A **89**, 033820 (2014).
[12] N. Német, A. Carmele, S. Parkins, and A. Knorr, Phys. Rev. A **100**, 023805 (2019).
[13] G. Alber and N. Trautmann, J. Phys.: Conf. Ser., **672**, 012004 (2016).