

Comparative study for two-terminal transport through a lossy one-dimensional quantum wireShun Uchino *Advanced Science Research Center, Japan Atomic Energy Agency, Tokai 319-1195, Japan*

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Motivated by the realization of a dissipative quantum point contact in ultracold atomic gases, we investigate a two-terminal mesoscopic transport system in which a single-particle loss is locally present in a one-dimensional chain. By means of the Dyson equation approach in the Keldysh formalism that can incorporate dissipative effects, we reveal analytic structures of the particle and energy currents whose formal expressions correspond to ones in certain three-terminal systems where particle loss is absent. The obtained formulas are also consistent with non-Hermitian and three-terminal Landauer-Büttiker analyses. The universality of the current expressions holds regardless of the quantum statistics, and may be useful for understanding lossy two-terminal transport in terms of three-terminal transport and vice versa.

DOI: [10.1103/PhysRevA.106.053320](https://doi.org/10.1103/PhysRevA.106.053320)**I. INTRODUCTION**

Along with the intensive momentum of quantum technology, research on quantum simulation that extracts the essential features of complex quantum many-body problems is developing at a rapid pace [1]. Since Feynman's proposal [2], several promising quantum simulators such as ultracold atomic gases [3], trapped ions [4], photonic systems [5], and superconducting qubits [6] have been introduced and a broad spectrum of quantum many-body problems ranging from condensed matter physics to astrophysics, and high-energy physics such as particle and nuclear physics [7] have been investigated.

Among them, ultracold atomic gases have reached a high level of maturity owing to the controllability and flexibility of experiments. As is often said, one of the advantages in ultracold atomic gases is that the systems can normally be treated as closed ones in that the effects of an uncontrollable external environment are negligible. In fact, thermalization and nonthermalization problems occurring in closed quantum many-body systems have successfully been verified with atomic gases [8]. More recently, controllable atomic gases enable the exploration of open quantum many-body systems as well. By applying various schemes to perturb the systems [9–15], one can engineer one-body, two-body, and three-body losses and analyze dissipation effects in quantum many-body systems.

How dissipation affects quantum transport is an important theme in quantum technology [16,17] and is now diagnosed in ultracold atomic gases. The epochal realization is lossy Josephson junction arrays of Bose-Einstein condensates, where negative differential conductance [18] and bistability between superfluid and resistive states [19,20] have been found. Another striking realization, which is the subject of this paper, is two-terminal transport via a dissipative quantum point contact [14], where a local single-particle loss is introduced inside a short one-dimensional (1D) wire and leads to

a conductance plateau of less than $1/h$. Although the dissipation effects discussed in ultracold atomic gases can usually be explained in terms of the Lindblad master equation [21], unlike Bose-Einstein condensates where the quantum jump term being difficult to handle is greatly simplified in the Gross-Pitaevskii analysis, a proper prescription in a quantum point contact system has yet to be fully understood. In order to explain the experimental data of the dissipative quantum point contact, Ref. [14] introduced the Landauer-Büttiker formalism with a complex potential in the belief that the quantum jump term does not play an essential role in transport. While an analysis with a complex potential has indeed been utilized in the context of inelastic scattering problems [22,23], it may also be important to understand dissipative quantum point contact transport on the basis of the Lindblad formalism including the quantum jump term.

In this paper, we investigate two-terminal current flows through the lossy one-dimensional chain that is a model of a dissipative quantum point contact. We note that a similar model has recently been addressed in Refs. [24,25]. Compared to these previous works, here we focus on the analytic structures of the energy current as well as the particle current, and a connection to other approaches including the non-Hermitian Landauer-Büttiker formalism. To this end, we adopt an analysis based on Dyson equations in the Keldysh functional integral approach extended to open quantum systems [26], and reveal a relationship between such a dissipative system and a similar system in which dissipation is absent but an additional reservoir is attached. By taking an appropriate limit shown below, we show that regardless of fermions or bosons, the particle and energy currents in these two systems coincide with each other. In addition, the formal expressions of the currents are found to be consistent with the ones based on the non-Hermitian analysis obtained in Ref. [14] and in certain three-terminal Landauer-Büttiker analyses. What is remarkable is that formal current expressions of lossy two-terminal systems can be reproduced with three-terminal systems

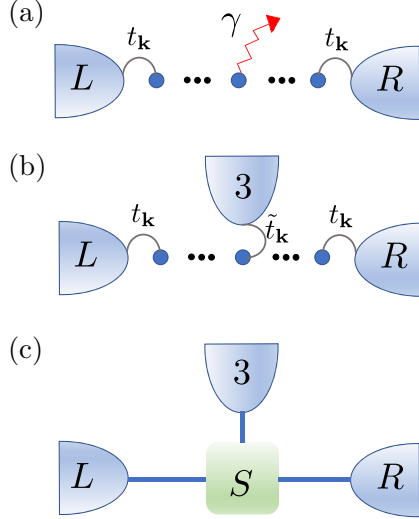


FIG. 1. Mesoscopic transport systems discussed in this paper. (a) The lossy two-terminal transport system where the single-particle loss occurs at the center site in the 1D chain. (b) Three-terminal transport system where the center site in the 1D chain couples to the third reservoir. (c) Three-terminal scattering problem where the sample region in the Landauer-Büttiker formalism is characterized by an S matrix.

consisting of nonempty reservoirs, which paves the way to a simulation of lossy two-terminal transport realized in ultracold atomic gases in terms of lossless three-terminal transport realized in condensed matter systems and vice versa.

The paper is organized as follows. In Sec. II we perform a formulation for the systems shown in Figs. 1(a) and 1(b). In Secs. III and IV we apply our formalism to single-site and multisite cases, respectively. In Sec. V we perform a three-terminal Landauer-Büttiker analysis for the system of Fig. 1(c). In Sec. VI we summarize and provide an outlook on this work. In the Appendix, we give a derivation of the functional integral form of the partition function whose density matrix obeys the Lindblad master equation.

II. FORMULATION OF THE PROBLEM

We consider a two-terminal mesoscopic transport system of noninteracting particles that can be either bosons or fermions. In this work, we focus on a situation where a one-dimensional tight-binding chain is attached between two macroscopic reservoirs. By adopting units of $\hbar = k_B = 1$, the corresponding Hamiltonian is given by

$$\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_{1D} + \hat{H}_T, \quad (1)$$

$$\hat{H}_{1D} = \sum_{i=-N}^N \epsilon_i \hat{d}_i^\dagger \hat{d}_i - \sum_{i=-N}^{N-1} t_i (\hat{d}_i^\dagger \hat{d}_{i+1} + \hat{d}_{i+1}^\dagger \hat{d}_i), \quad (2)$$

$$\hat{H}_T = - \sum_{\mathbf{k}} [t_{\mathbf{k}} \hat{\psi}_{L,\mathbf{k}}^\dagger \hat{d}_{-N} + t_{\mathbf{k}} \hat{\psi}_{R,\mathbf{k}}^\dagger \hat{d}_N] + \text{H.c.} \quad (3)$$

Here, $\hat{H}_{L(R)} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu_{L(R)}) \hat{\psi}_{L(R),\mathbf{k}}^\dagger \hat{\psi}_{L(R),\mathbf{k}}$, where $\epsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$ with particle mass m , is the Hamiltonian of the left (right) reservoir measured from the chemical potential $\mu_{L(R)}$, \hat{H}_{1D} is the Hamiltonian of the one-dimensional chain with nearest-

neighbor hopping t_i , and \hat{H}_T is the coupling between the reservoirs and the one-dimensional chain that is modeled as the tunneling term with tunneling amplitude $t_{\mathbf{k}}$. For a symmetric reason, we assume that the on-site potential energy ϵ_i has the property $\epsilon_i = \epsilon_{-i}$, the hopping amplitude satisfies $t_{-i} = t_{i-1}$ for $i = 0, 1, \dots, N$, and the site number of the 1D chain L is set to be $L = 2N + 1$. In addition, $\hat{\psi}_{L(R)}$ and \hat{d} are respectively the annihilation field operators in the reservoir $L(R)$ and one-dimensional chain.

For comparative purposes, we also introduce a three-terminal system described by the following Hamiltonian:

$$\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_{1D} + \hat{H}_T + \hat{H}_3, \quad (4)$$

$$\hat{H}_T = \hat{H}_T - \sum_{\mathbf{k}} \tilde{t}_{\mathbf{k}} \hat{\psi}_{3,\mathbf{k}}^\dagger \hat{d}_0 + \text{H.c.} \quad (5)$$

Here, \hat{H}_3 is an additional reservoir Hamiltonian that is similar to \hat{H}_L and \hat{H}_R and is expressed with the field operator $\hat{\psi}_{3,\mathbf{k}}$. In addition, we assume that a coupling to the third reservoir with tunnel amplitude $\tilde{t}_{\mathbf{k}}$ occurs at the center site in the 1D chain.

We now come back to Eq. (1) and introduce dissipation such that a single-particle loss is present at the center site in the chain. Inspired by the dissipative quantum point contact experiment [14], we assume that the system obeys the following Lindblad master equation [27],

$$\partial_\tau \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \gamma \left(\hat{d}_0 \hat{\rho} \hat{d}_0^\dagger - \frac{\{\hat{d}_0^\dagger \hat{d}_0, \hat{\rho}\}}{2} \right), \quad (6)$$

where $\hat{\rho}$ is the density matrix operator and γ is the dissipation rate. For $\gamma = 0$, this equation is reduced to the von Neumann equation describing unitary time evolutions. In the presence of nonzero γ , additional terms describing dissipation effects come out. It is also convenient to introduce the adjoint equation. For a Heisenberg operator \hat{A} , the corresponding adjoint equation is written as [27]

$$\partial_\tau \hat{A} = i[\hat{H}, \hat{A}] + \gamma \left[\hat{d}_0^\dagger \hat{A} \hat{d}_0 - \frac{\{\hat{d}_0^\dagger \hat{d}_0, \hat{A}\}}{2} \right]. \quad (7)$$

By using the above, the operator of the particle number growth rate in each reservoir is given by

$$\partial_\tau \hat{N}_{L(R)} = i \sum_{\mathbf{k}} t_{\mathbf{k}} \hat{\psi}_{L(R),\mathbf{k}}^\dagger \hat{d}_{-N(N)} + \text{H.c.}, \quad (8)$$

where we use

$$[\hat{N}_{L(R)}, \hat{d}_0] = [\hat{N}_{L(R)}, \hat{d}_0^\dagger] = 0. \quad (9)$$

Similarly, the operator of the energy absorption rate in each reservoir is given by

$$\partial_\tau \hat{H}_{L(R)} + \mu_{L(R)} \partial_\tau \hat{N}_{L(R)} = i \sum_{\mathbf{k}} t_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{\psi}_{L(R),\mathbf{k}}^\dagger \hat{d}_{-N(N)} + \text{H.c.}, \quad (10)$$

where we note that $\mu_{L(R)} \partial_\tau \hat{N}_{L(R)}$ is added in order to describe the energy flow. By using these operators, the particle and energy current operators flowing between reservoirs are respectively expressed as

$$\hat{I} = \frac{\partial_\tau (-\hat{N}_L + \hat{N}_R)}{2}, \quad (11)$$

$$\hat{I}_E = \frac{\partial_\tau (-\hat{H}_L - \mu_L \hat{N}_L + \hat{H}_R + \mu_R \hat{N}_R)}{2}, \quad (12)$$

where of course $\mu_{L(R)}$ is assumed to be a constant in time [28].

In order to evaluate the averages of the operators defined above, we harness the functional integral formulation of the Keldysh formalism [29]. There, we can consider the following partition function [26] (see also the Appendix),

$$Z = \int \mathcal{D}[\bar{\psi}, \psi, \bar{d}, d] e^{iS_L + iS_R + iS_{ID} + iS_T + iS_{\text{loss}}}, \quad (13)$$

where

$$S_{L(R)} = \int_{-\infty}^{\infty} d\tau [\bar{\psi}_{L(R)}^+ i\partial_\tau \psi_{L(R)}^+ - \bar{\psi}_{L(R)}^- i\partial_\tau \psi_{L(R)}^- - H_{L(R)}^+ + H_{L(R)}^-], \quad (14)$$

$$S_{ID} = \int_{-\infty}^{\infty} d\tau \left[\sum_{i=-N}^N \{ \bar{d}_i^+ (i\partial_\tau - \epsilon_i) d_i^+ - \bar{d}_i^- (i\partial_\tau - \epsilon_i) d_i^- \} + \sum_{i=-N}^{N-1} \{ \bar{d}_{i+1}^+ d_{i+1}^+ + \bar{d}_{i+1}^- d_{i+1}^- - \bar{d}_i^- d_{i+1}^- - \bar{d}_{i+1}^- d_i^- \} \right], \quad (15)$$

$$S_T = \int_{-\infty}^{\infty} d\tau \sum_{\mathbf{k}} [t_{\mathbf{k}} \bar{\psi}_{L,\mathbf{k}}^+ d_{-N}^+ + t_{\mathbf{k}}^* \bar{d}_{-N}^+ \psi_{L,\mathbf{k}}^+ + t_{\mathbf{k}} \bar{\psi}_{R,\mathbf{k}}^+ d_N^+ + t_{\mathbf{k}}^* \bar{d}_N^+ \psi_{R,\mathbf{k}}^+ - t_{\mathbf{k}} \bar{\psi}_{L,\mathbf{k}}^- d_{-N}^- - t_{\mathbf{k}}^* \bar{d}_{-N}^- \psi_{L,\mathbf{k}}^- - t_{\mathbf{k}} \bar{\psi}_{R,\mathbf{k}}^- d_N^- - t_{\mathbf{k}}^* \bar{d}_N^- \psi_{R,\mathbf{k}}^-], \quad (16)$$

$$S_{\text{loss}} = \int_{-\infty}^{\infty} d\tau i\gamma \left[\frac{\bar{d}_0^+ d_0^+ + \bar{d}_0^- d_0^-}{2} - \bar{d}_0^- d_0^+ \right]. \quad (17)$$

Here, the superscripts $+$ and $-$ represent forward and backward contours, respectively, and $H_{L(R)}^\pm = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu_{L(R)}) \bar{\psi}_{L(R),\mathbf{k}}^\pm \psi_{L(R),\mathbf{k}}^\pm$. Although the expressions above are available regardless of bosons or fermions, we note that the fields are expressed with complex numbers for bosons and Grassmann numbers for fermions. As can be seen from the expressions above, an advantage of this formalism is that the seemingly complicated dissipator terms that consist of non-Hermitian and quantum jump terms can be replaced by the quadratic action term S_{loss} .

In place of the representation based on the forward and backward counters, we convert to the following rotated representation [29]:

$$\begin{pmatrix} \psi^{cl} \\ \psi^q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \quad \begin{pmatrix} d^{cl} \\ d^q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} d^+ \\ d^- \end{pmatrix}, \quad (18)$$

$$\begin{pmatrix} \bar{\psi}^{cl} \\ \bar{\psi}^q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{\psi}^+ \\ \bar{\psi}^- \end{pmatrix}, \quad \begin{pmatrix} \bar{d}^{cl} \\ \bar{d}^q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{d}^+ \\ \bar{d}^- \end{pmatrix}. \quad (19)$$

Under this transformation, the reservoir action is expressed as

$$S_{L(R)} = \sum_{\mathbf{k}} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \begin{pmatrix} \bar{\psi}_{L(R)}^{cl} & \bar{\psi}_{L(R)}^q \end{pmatrix}_\tau \begin{pmatrix} 0 & [g_{L(R)}^{-1}(\mathbf{k}, \tau - \tau')]^A \\ [g_{L(R)}^{-1}(\mathbf{k}, \tau - \tau')]^R & [g_{L(R)}^{-1}(\mathbf{k}, \tau - \tau')]^K \end{pmatrix} \begin{pmatrix} \psi_{L(R)}^{cl} \\ \psi_{L(R)}^q \end{pmatrix}_{\tau'}, \quad (20)$$

where $[g^{-1}]^{R(A)}$ and $[g^{-1}]^K$ are the retarded (advanced) and Keldysh components in inverse Green's function, respectively. In addition, S_{loss} is transformed into

$$S_{\text{loss}} = \int_{-\infty}^{\infty} d\tau \begin{pmatrix} \bar{d}_0^{cl} & \bar{d}_0^q \end{pmatrix} \begin{pmatrix} 0 & -\frac{i\gamma}{2} \\ \frac{i\gamma}{2} & i\gamma \end{pmatrix} \begin{pmatrix} d_0^{cl} \\ d_0^q \end{pmatrix}. \quad (21)$$

Thus, noninfinitesimal imaginary parts appear in the inverse Green's function at the center site. It is also instructive to note that the conventional representation for fermions is the so-called Larkin-Ovchinnikov one where ψ and d fields transform in the same way as Eq. (18), while $\bar{\psi}$ and \bar{d} fields transform in the following way [29]: $(\bar{\psi}^{\pm}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} (\bar{\psi}^{\pm})$, and $(\bar{d}^{\pm}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} (\bar{d}^{\pm})$. In this paper, however, we stick to using Eq. (19), which allows us to discuss both bosonic and fermionic transport in a unified manner.

We are interested in steady transport in which average currents do not depend on time. In this case, it is convenient to move onto the frequency space. By taking into account the fact that the origins of the frequencies between reservoirs are shifted by a chemical potential difference $\Delta\mu \equiv \mu_L - \mu_R$, the reservoir action in the frequency space is given by

$$S_{L/R} = \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \begin{pmatrix} \bar{\psi}_{L/R}^{cl} & \bar{\psi}_{L/R}^q \end{pmatrix} \begin{pmatrix} 0 & [g_{L/R}^{-1}(\mathbf{k}, \omega \mp \Delta\mu/2)]^A \\ [g_{L/R}^{-1}(\mathbf{k}, \omega \mp \Delta\mu/2)]^R & [g_{L/R}^{-1}(\mathbf{k}, \omega \mp \Delta\mu/2)]^K \end{pmatrix} \begin{pmatrix} \psi_{L/R}^{cl} \\ \psi_{L/R}^q \end{pmatrix}. \quad (22)$$

Here, the retarded and advanced components are determined as

$$[g^{-1}(\mathbf{k}, \omega)]^{R/A} = \omega - \xi_{\mathbf{k}} \pm i0^+. \quad (23)$$

Since we deal with the two-terminal transport system in which two macroscopic reservoirs are in thermal equilibrium where the fluctuation-dissipation relation holds in each reservoir [29], $[g^{-1}]^K$ obeys

$$[g^{-1}(\mathbf{k}, \omega)]^K = ([g^{-1}(\mathbf{k}, \omega)]^R - [g^{-1}(\mathbf{k}, \omega)]^A) [1 \pm 2n(\omega)], \quad (24)$$

where $n(\omega) = \frac{1}{e^{\omega/T} \mp 1}$ with temperature T is the distribution function and the upper and lower signs are for bosons and fermions, respectively.

On the other hand, the partition function in the three-terminal system described by Eq. (4) is straightforwardly obtained as

$$\tilde{Z} = \int \mathcal{D}[\bar{\psi}, \psi, \bar{d}, d] e^{iS_L + iS_R + iS_{1D} + i\tilde{S}_T + S_3}, \quad (25)$$

where

$$S_3 = \int_{-\infty}^{\infty} d\tau [\bar{\psi}_3^+ i\partial_\tau \psi_3^+ - \bar{\psi}_3^- i\partial_\tau \psi_3^- - H_3^+ + H_3^-], \quad (26)$$

$$\tilde{S}_T = S_T + \int_{-\infty}^{\infty} d\tau \sum_{\mathbf{k}} [\tilde{t}_{\mathbf{k}} \bar{\psi}_{3,\mathbf{k}}^+ d_0^+ + \tilde{t}_{\mathbf{k}}^* \bar{d}_0^+ \psi_{3,\mathbf{k}}^+ - \tilde{t}_{\mathbf{k}} \bar{\psi}_{3,\mathbf{k}}^- d_0^- - \tilde{t}_{\mathbf{k}}^* \bar{d}_0^- \psi_{3,\mathbf{k}}^-]. \quad (27)$$

For a calculation of physical quantities, one needs to evaluate the Green's functions. In fact, it is easy to confirm that the particle and energy currents are respectively expressed as

$$\begin{aligned} I(\tau) &= \frac{\sum_{\mathbf{k}} \text{Re}[\pm t_{\mathbf{k}} G_{dL}^K(\mathbf{k}, \tau, \tau) \mp t_{\mathbf{k}} G_{dR}^K(\mathbf{k}, \tau, \tau)]}{2} \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sum_{\mathbf{k}} \text{Re}[\pm t_{\mathbf{k}} G_{dL}^K(\mathbf{k}, \omega) \mp t_{\mathbf{k}} G_{dR}^K(\mathbf{k}, \omega)], \end{aligned} \quad (28)$$

$$\begin{aligned} I_E(\tau) &= \frac{\sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \text{Re}[\pm t_{\mathbf{k}} G_{dL}^K(\mathbf{k}, \tau, \tau) \mp t_{\mathbf{k}} G_{dR}^K(\mathbf{k}, \tau, \tau)]}{2} \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \text{Re}[\pm t_{\mathbf{k}} G_{dL}^K(\mathbf{k}, \omega) \mp t_{\mathbf{k}} G_{dR}^K(\mathbf{k}, \omega)]. \end{aligned} \quad (29)$$

Here,

$$G_{dL}^K(\mathbf{k}, \tau, \tau') = -i \langle [\hat{d}_{-N}(\tau), \hat{\psi}_{L,\mathbf{k}}^\dagger(\tau')]_{\pm} \rangle, \quad (30)$$

$$G_{dR}^K(\mathbf{k}, \tau, \tau') = -i \langle [\hat{d}_N(\tau), \hat{\psi}_{R,\mathbf{k}}^\dagger(\tau')]_{\pm} \rangle, \quad (31)$$

and the upper and lower signs are again for bosons and fermions, respectively. A straightforward way to calculate $G_{dL(R)}^K$ is to invert the matrix of the action defined in (13), which becomes an efficient algorithm for numerical analyses of the average currents [24,30–33]. With this method, however, it is not so easy to establish analytic structures of the currents. To overcome this, we therefore adopt an approach based on Dyson equations [34], where the action is split into nonperturbative and perturbative parts. For the dissipative (three-terminal) situation, one can treat $S_{L(R)}$, the on-site energy term in S_{1D} , and $S_{\text{loss}}(S_3)$ as the former part, and the hopping term in S_{1D} and $S_T(\tilde{S}_T)$ as the latter part. Under this setting, a calculation in the presence of dissipation can be done similarly with one in the absence of dissipation. We note that the analysis with Dyson equations becomes exact by treating the perturbative part at all order levels, in which case the result corresponds to the one with the matrix inversion of the action.

III. SINGLE-SITE CASE

In order to see the essential structure of the average particle and energy currents both in dissipative two-terminal and three-terminal systems, we start to analyze the $L = 1$ case,

i.e., the noninteracting quantum dot system. For the application of Dyson equations in the Keldysh formalism in our system, it is convenient to use the following relation [35],

$$\begin{aligned} \sum_{\mathbf{k}} t_{\mathbf{k}} G_{dL(R)}^K(\mathbf{k}, \omega) &= - \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 [G_d^R(\omega) g_{L(R)}^K(\mathbf{k}, \omega) \\ &\quad + G_d^K(\omega) g_{L(R)}^A(\mathbf{k}, \omega)], \end{aligned} \quad (32)$$

where g is the uncoupled Green's function that is obtained from the inverse of the matrix (22), and G_d is the full Green's function of the 1D chain that contains the effects of the perturbative part. Notice also that in order to obtain the above, we applied the Langreth rule in the Keldysh formalism [34]. Owing to this deformation, the remaining task is to calculate $G_d^{R/K}$.

A. Dissipative situation

We first look at the dissipative situation corresponding to the situation of Fig. 1(a). There, the retarded and advanced Green's function in the single site obey the following Dyson equation [34]:

$$[G_d^{R(A)}(\omega)]^{-1} = [g_d^{R(A)}(\omega)]^{-1} - \Sigma^{R(A)}(\omega). \quad (33)$$

We note that the retarded or advanced component is decoupled and therefore satisfies $[G^{-1}]^{R(A)} = [G^{R(A)}]^{-1}$. We also introduce the retarded and advanced components of the self-energy,

$$\begin{aligned} \Sigma^{R(A)}(\omega) &= \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 [g_L^{R(A)}(\mathbf{k}, \omega - \Delta\mu/2) \\ &\quad + g_R^{R(A)}(\mathbf{k}, \omega + \Delta\mu/2)]. \end{aligned} \quad (34)$$

By introducing

$$\begin{aligned} R(\omega) &\equiv 2 \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 \text{Re}[g_L^R(\mathbf{k}, \omega - \Delta\mu/2)] \\ &= 2 \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 \text{Re}[g_R^R(\mathbf{k}, \omega + \Delta\mu/2)], \end{aligned} \quad (35)$$

$$\begin{aligned} \Gamma(\omega) &\equiv -2 \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 \text{Im}[g_L^R(\mathbf{k}, \omega - \Delta\mu/2)] \\ &= -2 \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 \text{Im}[g_R^R(\mathbf{k}, \omega + \Delta\mu/2)], \end{aligned} \quad (36)$$

the full retarded and advanced Green's functions are obtained as

$$G_d^{R/A}(\omega) = \frac{1}{\omega - \epsilon_0 - R(\omega) \pm i[\frac{\gamma}{2} + \Gamma(\omega)]}. \quad (37)$$

In addition, the Keldysh component obeys the following Dyson equation:

$$G_d^K(\omega) = [1 + G_d^R(\omega)\Sigma^R(\omega)]g_d^K(\omega)[1 + \Sigma^A(\omega)G_d^A(\omega)] + G_d^R(\omega)\Sigma^K(\omega)G_d^A(\omega). \quad (38)$$

$$G_d^K(\omega) = \frac{-2i[\frac{\gamma}{2} + \Gamma(\omega)] \mp 2i\Gamma(\omega)[n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2)]}{[\omega - \epsilon_0 - R(\omega)]^2 + [\frac{\gamma}{2} + \Gamma(\omega)]^2}. \quad (41)$$

In total, the average current is determined as

$$I = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\mathcal{T}(\omega) + \frac{\mathcal{L}(\omega)}{2} \right] \times [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)], \quad (42)$$

where we introduce

$$\mathcal{T}(\omega) = \frac{[\Gamma(\omega)]^2}{[\omega - \epsilon_0 - R(\omega)]^2 + [\frac{\gamma}{2} + \Gamma(\omega)]^2}, \quad (43)$$

$$\mathcal{L}(\omega) = \frac{\Gamma(\omega)\gamma}{[\omega - \epsilon_0 - R(\omega)]^2 + [\frac{\gamma}{2} + \Gamma(\omega)]^2}. \quad (44)$$

Physically, $\mathcal{T}(\omega)$ and $\mathcal{L}(\omega)$ represent the transmittance and loss probability, respectively. When $\gamma = 0$, $\mathcal{T}(\omega)$ is indeed reduced to transmittance in the noninteracting quantum dot system [35,36]. In addition, when $\gamma \neq 0$, $\mathcal{T}(\omega)$ is reduced to one in a phenomenological model with inelastic widths in which $\mathcal{L}(\omega)$ is absent [37]. On the other hand, the presence of $\mathcal{L}(\omega)$ is peculiar to the lossy system. The physical meaning of $\mathcal{L}(\omega)$ becomes clearer by noting that the average of the particle loss rate is expressed as

$$-\dot{N} = -\dot{N}_L - \dot{N}_R = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{L}(\omega)[n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2)]. \quad (45)$$

It is also instructive to point out the relation with Refs. [24,25]. When the frequency dependence of $\Gamma(\omega)$ is neglected and $R(\omega) = 0$, the particle current obtained in this work is identical to the one in Refs. [24,25]. The first condition is called the wide-band approximation and is reasonable for low-energy transport such that the frequency dependence in Γ is negligible is concerned. In addition, the second condition means that the energy shift due to the couplings with the reservoirs is negligible, and corresponds to the situation where the Lamb shift is neglected in the context of quantum optics. For instance, these conditions are simultaneously satisfied if $t_{\mathbf{k}}$ is

By using

$$\begin{aligned} \Sigma^K(\omega) &= \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 [g_L^K(\mathbf{k}, \omega - \Delta\mu/2) + g_R^K(\mathbf{k}, \omega + \Delta\mu/2)] \\ &= -2i\Gamma(\omega)[1 \pm n_L(\omega - \Delta\mu/2) \pm n_R(\omega + \Delta\mu/2)], \end{aligned} \quad (39)$$

and

$$\begin{aligned} &[1 + G_d^R(\omega)\Sigma^R(\omega)]g_d^K(\omega)[1 + \Sigma^A(\omega)G_d^A(\omega)] \\ &= G_d^R(\omega)[g_d^R(\omega)]^{-1}g_d^K(\omega)[g_d^A(\omega)]^{-1}G_d^A(\omega) \\ &= -G_d^R(\omega)[g_d^{-1}(\omega)]^K G_d^A(\omega), \end{aligned} \quad (40)$$

the Dyson equation for the Keldysh component can be solved as

momentum independent such as the point contact tunneling [38,39] and the constant density-of-states approximation in the reservoir that is often used in fermionic systems is reasonable. In contrast, our formulation is available even if such conditions are not valid, which contains superfluid bosonic systems [40–42].

With the use of the above results, we can also obtain the energy current. To this end, we note

$$\begin{aligned} &\sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 \epsilon_{\mathbf{k}} \text{Im}[g_{L/R}^R(\mathbf{k}, \omega \mp \Delta\mu/2)] \\ &= \sum_{\mathbf{k}} |t_{\mathbf{k}}|^2 (\omega + \mu) \text{Im}[g_{L/R}^R(\mathbf{k}, \omega \mp \Delta\mu/2)], \end{aligned} \quad (46)$$

where we use $\text{Im}[g_{L/R}^R] \propto \delta(\omega + \mu - \frac{k^2}{2m})$ with the average chemical potential between reservoirs $\mu = \frac{\mu_L + \mu_R}{2}$. By using this relation, the average energy current is shown to be expressed as

$$I_E = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega + \mu) \left[\mathcal{T}(\omega) + \frac{\mathcal{L}(\omega)}{2} \right] \times [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)]. \quad (47)$$

In order to compare with the result in Ref. [14], we bear in mind that while ω adopted in Ref. [14] corresponds to the absolute energy, $\omega = 0$ in our formulation corresponds to the energy at the average chemical potential. By taking into account this energy shift between two formulations, it turns out that the formal expressions of Eqs. (42), (45), and (47) are consistent with ones found in the non-Hermitian Landauer-Büttiker analysis [14].

B. Three-terminal situation

We now consider the situation where atom loss is absent but instead the noninteracting dot is attached to three terminals [Fig. 1(b)]. In this case, the retarded and advanced compo-

nents of the self-energy are given by

$$\begin{aligned} \Sigma^{R(A)}(\omega) = & \sum_{\mathbf{k}} [|\tilde{t}_{\mathbf{k}}|^2 g_L^{R(A)}(\mathbf{k}, \omega - \Delta\mu/2) \\ & + |\tilde{t}_{\mathbf{k}}|^2 g_R^{R(A)}(\mathbf{k}, \omega + \Delta\mu/2) + |\tilde{t}_{\mathbf{k}}|^2 g_3^{R(A)}(\mathbf{k}, \omega + V)], \end{aligned} \quad (48)$$

where $V \equiv \mu - \mu_3$ with the chemical potential of the third reservoir μ_3 . As in the case of the previous section, we introduce

$$R_3(\omega) \equiv 2 \sum_{\mathbf{k}} |\tilde{t}_{\mathbf{k}}|^2 \text{Re}[g_3^R(\mathbf{k}, \omega + V)], \quad (49)$$

$$\Gamma_3(\omega) \equiv -2 \sum_{\mathbf{k}} |\tilde{t}_{\mathbf{k}}|^2 \text{Im}[g_3^R(\mathbf{k}, \omega + V)]. \quad (50)$$

Then, the full retarded and advanced Green's functions are obtained as

$$G_d^{R/A}(\omega) = \frac{1}{\omega - \epsilon_0 - R(\omega) - \frac{R_3(\omega)}{2} \pm i[\Gamma(\omega) + \frac{\Gamma_3(\omega)}{2}]}. \quad (51)$$

In addition, the Keldysh component of the self-energy is

$$\begin{aligned} \Sigma^K(\omega) = & -2i\Gamma(\omega)[1 \pm n_L(\omega - \Delta\mu/2) \pm n_R(\omega + \Delta\mu/2)] \\ & - i\Gamma_3(\omega)[1 \pm 2n_3(\omega + V)], \end{aligned} \quad (52)$$

where the upper and lower signs are for bosons and fermions, respectively, and n_3 is the distribution function of the third reservoir. Since $[g^{-1}]^K = 0$ in the absence of single-particle loss, we obtain

$$\begin{aligned} G_d^K(\omega) = & \frac{-i\Gamma_3(\omega)[1 \pm 2n_3(\omega + V)]}{[\omega - \epsilon_0 - R(\omega) + \frac{R_3(\omega)}{2}]^2 + [\frac{\Gamma_3(\omega)}{2} - \Gamma(\omega)]^2} \\ & - \frac{2i\Gamma(\omega)[1 \pm n_L(\omega - \Delta\mu/2) \pm n_R(\omega + \Delta\mu/2)]}{[\omega - \epsilon_0 - R(\omega) - \frac{R_3(\omega)}{2}]^2 + [\frac{\Gamma_3(\omega)}{2} + \Gamma(\omega)]^2}. \end{aligned} \quad (53)$$

Now that the full Green's functions are obtained, we turn to compare Eqs. (37) and (41) and Eqs. (51) and (53). Then, it turns out that the full Green's functions in the three-terminal system coincide with the ones in the dissipative two-terminal system when the following conditions are satisfied:

$$\Gamma_3(\omega) = \gamma, \quad (54)$$

$$R_3(\omega) = 0, \quad (55)$$

$$n_3(\omega + V) = 0. \quad (56)$$

Equations (54) and (55) represent the wide-band approximation and the disregard of the Lamb shift, respectively. In addition, Eq. (56) means that the third reservoir is empty. The last condition is physically sound, since the dissipator in Eq. (6) (terms proportional to γ) does not contain the gain effect [27]. In total, when Eqs. (54)–(56) are satisfied, it follows that the transport quantities calculated in the previous section such as the particle and heat current and particle loss rate also coincide with those in the lossy two-terminal system.

Although the conditions of (54)–(56) are necessary for exact correspondence including the frequency dependences of

\mathcal{T} and \mathcal{L} , one can relax them as far as a formal correspondence is concerned. In particular, it is straightforward to confirm that without (54)–(56), the average particle and energy currents from left to right are expressed as

$$\begin{aligned} I = & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\tilde{\mathcal{T}}(\omega) + \frac{\tilde{\mathcal{L}}(\omega)}{2} \right] \\ & \times [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)], \end{aligned} \quad (57)$$

$$\begin{aligned} I_E = & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega + \mu) \left[\tilde{\mathcal{T}}(\omega) + \frac{\tilde{\mathcal{L}}(\omega)}{2} \right] \\ & \times [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)], \end{aligned} \quad (58)$$

where

$$\tilde{\mathcal{T}}(\omega) = \frac{[\Gamma(\omega)]^2}{(\omega - \epsilon_0 - R(\omega) - \frac{R_3(\omega)}{2})^2 + [\frac{\Gamma_3(\omega)}{2} + \Gamma(\omega)]^2}, \quad (59)$$

$$\tilde{\mathcal{L}}(\omega) = \frac{\Gamma(\omega)\Gamma_3(\omega)}{(\omega - \epsilon_0 - R(\omega) - \frac{R_3(\omega)}{2})^2 + [\frac{\Gamma_3(\omega)}{2} + \Gamma(\omega)]^2}. \quad (60)$$

We note that Eqs. (57) and (58) are formally equivalent to Eqs. (42) and (47) in that the currents are expressed with transmittance and loss probability in a similar manner. The only difference from the dissipative situation is that the additional frequency dependences originating from the frequency dependence of g_3 and momentum dependence of $\tilde{t}_{\mathbf{k}}$ come out in $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{L}}$. What is especially remarkable is that the average currents do not depend on the distribution of the third reservoir.

It should also be noted that all the average quantities in the three-terminal situation are not independent of n_3 . In fact, the average particle loss rate in the three-terminal situation, which is merely the particle gain rate of the third terminal, is expressed as

$$\begin{aligned} -\dot{N} = & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\tilde{\mathcal{L}}(\omega)\{n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2)\} \\ & - 2\tilde{\mathcal{L}}(\omega)n_3(\omega + V)]. \end{aligned} \quad (61)$$

The expression above is not formally consistent with Eq. (45) due to the presence of the last term proportional to n_3 . Thus, for generic $\tilde{\mathcal{L}}$, the condition (56) is essential to the consistency of the particle loss rate between the dissipative and three-terminal situations. At the same time, notice that

$$\int d\omega \tilde{\mathcal{L}}(\omega)n_3(\omega + V) = 0 \quad (62)$$

can also be achieved for a nonempty third reservoir. Indeed, the above equality is satisfied, if $\tilde{\mathcal{L}}(\omega) = 0$ for ω 's such that $n_3(\omega + V) \neq 0$. Moreover, $\mu_3 < \mu_L, \mu_R$ is also necessary to obtain nontrivial effects of $\tilde{\mathcal{L}}$ in transport. In this case, the gain effect from the third reservoir is neglected and lossy two-terminal transport corresponds to the lossless three-terminal one.

IV. MULTISITE CASE

Although the essential structure of the correspondence between lossy two-terminal and lossless three-terminal systems already appears in the single-site problem, in order to convince ourselves of it, we now look at the multisite case.

To this end, we point out the following relations,

$$\sum_{\mathbf{k}} t_{\mathbf{k}} G_{dL}^K(\mathbf{k}, \omega) = - \sum_{\mathbf{k}} t_{\mathbf{k}}^2 [G_{d,11}^R(\omega) g_L^K(\mathbf{k}, \omega - \Delta\mu/2) + G_{d,11}^K(\omega) g_L^A(\mathbf{k}, \omega - \Delta\mu/2)], \quad (63)$$

$$\sum_{\mathbf{k}} t_{\mathbf{k}} G_{dR}^K(\mathbf{k}, \omega) = - \sum_{\mathbf{k}} t_{\mathbf{k}}^2 [G_{d,LL}^R(\omega) g_R^K(\mathbf{k}, \omega + \Delta\mu/2) + G_{d,LL}^K(\omega) g_R^A(\mathbf{k}, \omega + \Delta\mu/2)], \quad (64)$$

which is the multisite generalization of Eq. (32). Here, we introduce the following $L \times L$ matrix of a Green's function

$$[\mathbf{g}_d^{R/A}]^{-1} = \begin{pmatrix} \omega - \epsilon_N \pm i0^+ & 0 & \dots & 0 \\ 0 & \omega - \epsilon_{N-1} \pm i0^+ & 0 & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & \omega - \epsilon_0 \pm i\frac{\gamma}{2} & \dots \\ 0 & \dots & \dots & \omega - \epsilon_N \pm i0^+ \end{pmatrix}, \quad (67)$$

and

$$\Sigma^{R/A} = \begin{pmatrix} \frac{R(\omega) \mp i\Gamma(\omega)}{2} & -t_{N-1} & 0 & \dots & 0 \\ -t_{N-1} & 0 & -t_{N-2} & & \\ 0 & -t_{N-2} & 0 & & \\ \vdots & & & & \\ 0 & \dots & 0 & -t_{N-1} & \frac{R(\omega) \mp i\Gamma(\omega)}{2} \end{pmatrix}, \quad (68)$$

we obtain

$$\mathbf{G}_d^{R/A}(\omega) = \begin{pmatrix} \omega - \epsilon_N - \frac{R(\omega)}{2} \pm i\frac{\Gamma(\omega)}{2} & t_{N-1} & 0 & \dots & 0 \\ t_{N-1} & \omega - \epsilon_{N-1} \pm i0^+ & t_{N-2} & & \\ 0 & & \ddots & & \\ \vdots & & & \omega - \epsilon_0 \pm i\frac{\gamma}{2} & \dots \\ 0 & & & \dots & t_{N-1} & \omega - \epsilon_N - \frac{R(\omega)}{2} \pm i\frac{\Gamma(\omega)}{2} \end{pmatrix}^{-1}. \quad (69)$$

Thus, it turns out that the inverse of the retarded or advanced Green's function is expressed with the so-called tridiagonal matrix. In general, a nonsingular tridiagonal matrix,

$$T = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & \ddots & 0 \\ 0 & \ddots & \ddots & b_{L-1} \\ 0 & 0 & c_{L-1} & a_L \end{pmatrix}, \quad (70)$$

is known to be inverted as [43–46]

$$T_{i,j}^{-1} = \begin{cases} (-1)^{i+j} b_i \dots b_{j-1} \theta_{i-1} \phi_{j+1} / \theta_L, & i < j, \\ \theta_{i-1} \phi_{j+1} / \theta_L, & i = j, \\ (-1)^{i+j} c_j \dots c_{i-1} \theta_{j-1} \phi_{i+1} / \theta_L, & i > j. \end{cases} \quad (71)$$

in the 1D chain:

$$\mathbf{G}_d = \begin{pmatrix} G_{d,11} & G_{d,12} & \dots & G_{d,1L} \\ G_{d,21} & G_{d,22} & & \vdots \\ \vdots & & \ddots & \vdots \\ G_{d,L1} & G_{d,L2} & \dots & G_{d,LL} \end{pmatrix}. \quad (65)$$

For example, $G_{d,11}^R(\tau, \tau') = -i\theta(\tau - \tau') \langle [d_{-N(N)}(\tau), d_{-N(N)}^\dagger(\tau')]_{\pm} \rangle$. As in the case of the single site, what we need to do for the determination of the current formulas is to evaluate the retarded, advanced, and Keldysh components of \mathbf{G}_d .

A. Dissipative situation

In the multisite case, the Dyson equation for retarded and advanced components is expressed as

$$[\mathbf{G}_d^{R/A}(\omega)]^{-1} = [\mathbf{g}_d^{R/A}(\omega)]^{-1} - \Sigma^{R/A}(\omega). \quad (66)$$

Since

Here, θ_i and ϕ_i satisfy the following recurrence relations,

$$\theta_i = a_i\theta_{i-1} - b_{i-1}c_{i-1}\theta_{i-2}, \quad i = 2, 3, \dots, L, \quad (72)$$

$$\phi_i = a_i\phi_{i+1} - b_i c_i \phi_{i+2}, \quad i = L-1, \dots, 1, \quad (73)$$

with initial conditions $\theta_0 = 1$, $\theta_1 = a_1$, $\phi_{L+1} = 1$, and $\phi_L = a_L$. Thus, by applying the above formula, $\mathbf{G}_d^{R/A}$ can be exactly obtained.

On the other hand, since

$$[\mathbf{g}_d^{-1}]^K = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \\ & & i\gamma & \\ 0 & \cdots & & \ddots & \\ & & & & 0 \end{pmatrix}, \quad (74)$$

and

$$\Sigma^K = \begin{pmatrix} -i\Gamma(\omega)[1 \pm 2n_L(\omega - \Delta\mu/2)] & 0 & \cdots & & 0 \\ 0 & 0 & & & \\ \vdots & & \ddots & & \\ 0 & \cdots & & -i\Gamma(\omega)[1 \pm 2n_R(\omega + \Delta\mu/2)] & \end{pmatrix}, \quad (75)$$

the Keldysh component is expressed as

$$\begin{aligned} \mathbf{G}_d^K(\omega) &= -\mathbf{G}_d^R(\omega)[\mathbf{g}_d^{-1}(\omega)]^K \mathbf{G}_d^A(\omega) + \mathbf{G}_d^R(\omega)\Sigma^K(\omega)\mathbf{G}_d^A(\omega) \\ &= \mathbf{G}_d^R(\omega) \begin{pmatrix} -i\Gamma(\omega)[1 \pm 2n_L(\omega - \Delta\mu/2)] & 0 & \cdots & & 0 \\ 0 & 0 & & & \\ \vdots & & \ddots & & \\ & & & -i\gamma & \\ 0 & \cdots & & & \ddots & \\ & & & & & -i\Gamma(\omega)[1 \pm 2n_R(\omega + \Delta\mu/2)] \end{pmatrix} \mathbf{G}_d^A(\omega). \end{aligned} \quad (76)$$

Based on the results obtained above, we obtain the average particle current and particle loss rate. By using symmetries such as $G_{d11}^{R/A} = G_{dLL}^{R/A}$ and $G_{d,L1}^{R/A} = G_{d,L1}^{R/A}$, the particle current is expressed as

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Gamma(\omega) \left[-\text{Im}[G_{d,11}^R(\omega)] - \frac{\Gamma(\omega)\{|G_{d,11}^R(\omega)|^2 - |G_{d,1L}^R(\omega)|^2\}}{2} \right] \\ &\quad \times [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)]. \end{aligned} \quad (77)$$

In addition, the particle loss rate is expressed as

$$\begin{aligned} -\dot{N} &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\mp \Gamma(\omega)\{2 \text{Im}[G_{d,11}^R(\omega)] + \Gamma(\omega)\{|G_{d,11}^R|^2 + |G_{d,1L}^R|^2\} + \gamma |G_{d,1}^{L\pm}(\omega)|^2\} \right. \\ &\quad \left. + \Gamma(\omega)\{-\Gamma(\omega)\{|G_{d,11}^R|^2 + |G_{d,1L}^R|^2\} - 2 \text{Im}[G_{d,11}^R(\omega)]\}[n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2)]\right]. \end{aligned} \quad (78)$$

In order to obtain convenient forms, we consider the following identity:

$$\mathbf{G}_d^R - \mathbf{G}_d^A = \mathbf{G}_d^R \{(\mathbf{G}_d^A)^{-1} - (\mathbf{G}_d^R)^{-1}\} \mathbf{G}_d^A. \quad (79)$$

This leads to

$$2 \text{Im}[G_{d,11}^R(\omega)] = -\Gamma(\omega)\{|G_{d,11}^R|^2 + |G_{d,1L}^R|^2\} - \gamma |G_{d,1}^{L\pm}(\omega)|^2. \quad (80)$$

Thus, the particle current and particle loss rate are obtained as

$$I = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\mathcal{T}^L(\omega) + \frac{\mathcal{L}^L(\omega)}{2} \right] [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)], \quad (81)$$

$$-\dot{N} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{L}^L(\omega) [n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2)], \quad (82)$$

where

$$\mathcal{T}^L(\omega) = [\Gamma(\omega)]^2 |G_{d,1L}^R(\omega)|^2, \tag{83}$$

$$\mathcal{L}^L(\omega) = \Gamma(\omega) \left[-\Gamma(\omega) \{ |G_{d,11}^R|^2 + |G_{d,1L}^R|^2 \} - 2 \text{Im}[G_{d,11}^R(\omega)] \right], \tag{84}$$

are the transmittance and loss probability, respectively. It is also straightforward to show that the energy current is obtained as

$$I_E = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega + \mu) \left[\mathcal{T}^L(\omega) + \frac{\mathcal{L}^L(\omega)}{2} \right] [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)]. \tag{85}$$

Equations (81), (82), and (85) are multisite generalization of Eqs. (42), (45), and (47).

In the case of $L = 1$, it is easy to check that the transmittance [Eq. (83)] and loss probability [Eq. (84)] coincide with those that appeared in the single-site case. For multisites, explicit forms of Eqs. (83) and (84) in a generic 1D chain are lengthy. If $\epsilon \equiv \epsilon_0 = \epsilon_1 = \dots \epsilon_N$ and $t \equiv t_0 = t_1 = \dots t_{N-1}$, and $R(\omega) = 0$, however, one can obtain simpler expressions. For the $L = 3$ case, we obtain

$$\mathcal{T}^{L=3}(\omega) = \frac{[\Gamma(\omega)]^2 t^4}{(\omega - \epsilon + [\Gamma(\omega)]^2/4)[(\omega - \epsilon)^4 + (\omega - \epsilon)^2([\Gamma(\omega)]^2/4 - 4t^2 + \gamma^2/4) + (2t^2 + \Gamma\gamma/4)^2]}, \tag{86}$$

$$\mathcal{L}^{L=3}(\omega) = \frac{\Gamma(\omega)t^2\gamma}{(\omega - \epsilon)^4 + (\omega - \epsilon)^2([\Gamma(\omega)]^2/4 - 4t^2 + \gamma^2/4) + [2t^2 + \Gamma(\omega)\gamma/4]^2}, \tag{87}$$

and for the $L = 5$ case, we obtain

$$\begin{aligned} \mathcal{T}^{L=5}(\omega) &= [\Gamma(\omega)]^2 t^8 / [(\omega - \epsilon)^2 [\Gamma(\omega)]^2/4 + [(\omega - \epsilon)^2 - t^2]^2] \\ &\times [(\omega - \epsilon)^6 + t^4 [\Gamma(\omega) + \gamma/2]^2 + (\omega - \epsilon)^4 ([\Gamma(\omega)]^2/4 - 6t^2 + \gamma^2/4) + (\omega - \epsilon)^2 \{t^2(9t^2 - \gamma^2/2) \\ &+ (-4t^2 + \gamma^2/4)[\Gamma(\omega)]^2/4\}], \end{aligned} \tag{88}$$

$$\begin{aligned} \mathcal{L}^{L=5}(\omega) &= \Gamma(\omega)t^4\gamma / [(\omega - \epsilon)^6 + t^4 [\Gamma(\omega) + \gamma/2]^2 + (\omega - \epsilon)^4 ([\Gamma(\omega)]^2/4 - 6t^2 + \gamma^2/4) \\ &+ (\omega - \epsilon)^2 \{t^2(9t^2 - \gamma^2/2) + (-4t^2 + \gamma^2/4)[\Gamma(\omega)]^2/4\}]. \end{aligned} \tag{89}$$

In this way, one can obtain the explicit forms of $\mathcal{T}^L(\omega)$ and $\mathcal{L}^L(\omega)$, provided that the number of the sites is fixed and the corresponding $G_{d,11}^R$ and $G_{d,1L}^R$ are determined.

B. Three-terminal situation

In the case of three terminals, the retarded or advanced component of the self-energy is given by

$$\Sigma^{R/A} = \begin{pmatrix} \frac{R(\omega) \mp i\Gamma(\omega)}{2} & -t_{N-1} & 0 & \dots & 0 \\ -t_{N-1} & 0 & -t_{N-2} & & \\ 0 & & \ddots & & \\ \vdots & & & \frac{R_3(\omega) \mp i\Gamma_3(\omega)}{2} & \\ 0 & \dots & & \ddots & t_{N-1} \\ & & & -t_{N-1} & \frac{R(\omega) \mp i\Gamma(\omega)}{2} \end{pmatrix}, \tag{90}$$

and we obtain

$$\mathbf{G}_d^{R/A} = \begin{pmatrix} \omega - \epsilon_N - \frac{R(\omega)}{2} \pm i\frac{\Gamma(\omega)}{2} & t_{N-1} & 0 & \dots & 0 \\ t_{N-1} & \omega - \epsilon_{N-1} \pm i0^+ & t_{N-2} & & \\ 0 & & \ddots & & \\ \vdots & & & \omega - \epsilon_0 - \frac{R_3(\omega)}{2} \pm i\frac{\Gamma_3(\omega)}{2} & \\ 0 & \dots & & \ddots & t_{N-1} \\ & & & t_{N-1} & \omega - \epsilon_N - \frac{R(\omega)}{2} \pm i\frac{\Gamma(\omega)}{2} \end{pmatrix}^{-1}. \tag{91}$$

In addition, the absence of particle loss means $[\mathbf{g}_d^{-1}(\omega)]^K = \mathbf{0}$, and therefore the Keldysh component of the self-energy is given by

$$\Sigma^K = \begin{pmatrix} -i\Gamma(\omega)[1 \pm 2n_L(\omega - \Delta\mu/2)] & 0 & \cdots & & 0 \\ 0 & 0 & & & \\ \vdots & & \ddots & & \\ & & & -i\Gamma_3(\omega)[1 \pm 2n_3(\omega + V)] & \\ 0 & \cdots & & \ddots & -i\Gamma(\omega)[1 \pm 2n_R(\omega + \Delta\mu/2)] \end{pmatrix}. \quad (92)$$

Thus, the Keldysh component of the full Green's function is obtained as

$$\begin{aligned} \mathbf{G}_d^K(\omega) &= -\mathbf{G}_d^R(\omega)[\mathbf{g}_d^{-1}(\omega)]^K \mathbf{G}_d^A(\omega) + \mathbf{G}_d^R(\omega)\Sigma^K(\omega)\mathbf{G}_d^A(\omega) \\ &= \mathbf{G}_d^R(\omega) \begin{pmatrix} -i\Gamma(\omega)[1 \pm 2n_L(\omega - \Delta\mu/2)] & 0 & \cdots & & 0 \\ 0 & 0 & & & \\ \vdots & & \ddots & & \\ & & & -i\Gamma_3(\omega)[1 \pm 2n_3(\omega + V)] & \\ 0 & \cdots & & \ddots & -i\Gamma(\omega)[1 \pm 2n_R(\omega + \Delta\mu/2)] \end{pmatrix} \\ &\quad \times \mathbf{G}_d^A(\omega). \end{aligned} \quad (93)$$

By comparing Eqs. (91) and (93) with Eqs. (69) and (76), the full Green's functions in the three-terminal system correspond to the ones in the lossy two-terminal system when Eqs. (54)–(56) are satisfied. In this case, it follows that the current formulas of the three-terminal system obey Eqs. (81), (82), and (85).

When it comes to the formal correspondence between lossy and lossless transport, we can relax those conditions as in the case of the single site. Indeed, the average particle and energy currents are generally shown to be reduced to Eqs. (81) and (85), respectively. In addition, by using the following relation,

$$2 \operatorname{Im}[G_{d,11}^R(\omega)] = -\Gamma(\omega)\{|G_{d,11}^R(\omega)|^2 + |G_{d,1L}^R(\omega)|^2\} - \Gamma_3(\omega)|G_{d,1\frac{L+1}{2}}^R(\omega)|^2, \quad (94)$$

the particle loss rate is obtained as

$$-\dot{N} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{L}^L(\omega)[n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2) - 2n_3(\omega + V)], \quad (95)$$

which is consistent with the single-site case.

V. THREE-TERMINAL LANDAUER-BÜTTIKER ANALYSIS

So far, we have examined the correspondence between the lossy two-terminal and lossless three-terminal systems. When this correspondence holds, the current formulas are given by Eqs. (81), (82), and (85) whose formal expressions are consistent with the non-Hermitian Landauer-Büttiker analysis. Such a coincidence anticipates that a similar result may be obtained with the three-terminal Landauer-Büttiker analysis [Fig. 1(c)].

To see this, we point out that in the Landauer-Büttiker formalism, the particle current operator in each reservoir is given by

$$\hat{I}_j(\tau) = \int \frac{d\omega}{2\pi} \int d\omega' e^{i(\omega-\omega')\tau} [\hat{a}_j^\dagger(\omega)\hat{a}_j(\omega') - \hat{b}_j^\dagger(\omega)\hat{b}_j(\omega')], \quad (96)$$

where $j = L, R, 3$, and \hat{a} and \hat{b} are annihilation operators for the incoming and outgoing states, respectively. We note that the positive current direction is from each reservoir to the channel. The operators \hat{a} and \hat{b} are connected through the

following symmetric S matrix:

$$\begin{pmatrix} b_L \\ b_R \\ b_3 \end{pmatrix} = \begin{pmatrix} r & t & t_L \\ t' & r' & t_R \\ t'_L & t'_R & r_3 \end{pmatrix} \begin{pmatrix} a_L \\ a_R \\ a_3 \end{pmatrix}. \quad (97)$$

In the Landauer-Büttiker, it is assumed that the incoming operators (\hat{a}_j and \hat{a}_j^\dagger) obey

$$\langle \hat{a}_L^\dagger(\omega)\hat{a}_L(\omega') \rangle = \delta(\omega - \omega')n_L(\omega - \Delta\mu/2), \quad (98)$$

$$\langle \hat{a}_R^\dagger(\omega)\hat{a}_R(\omega') \rangle = \delta(\omega - \omega')n_R(\omega + \Delta\mu/2), \quad (99)$$

$$\langle \hat{a}_3^\dagger(\omega)\hat{a}_3(\omega') \rangle = \delta(\omega - \omega')n_3(\omega + V), \quad (100)$$

$$\langle \hat{a}_i^\dagger(\omega)\hat{a}_j(\omega') \rangle = 0, \quad i \neq j. \quad (101)$$

In addition, the S matrix obeys the unitary condition

$$SS^\dagger = 1, \quad (102)$$

which leads to

$$|r|^2 + |t|^2 + |t_L|^2 = 1, \quad (103)$$

$$|r'|^2 + |t'|^2 + |t_R|^2 = 1, \quad (104)$$

$$|r_3|^2 + |t'_L|^2 + |t'_R|^2 = 1. \quad (105)$$

Thus, each average current is obtained as

$$I_L = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [|t(\omega)|^2 [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)] + |t_L(\omega)|^2 [n_L(\omega - \Delta\mu/2) - n_3(\omega + V)]], \quad (106)$$

$$I_R = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [|t'(\omega)|^2 [n_R(\omega + \Delta\mu/2) - n_L(\omega - \Delta\mu/2)] + |t_R(\omega)|^2 [n_R(\omega + \Delta\mu/2) - n_3(\omega + V)]], \quad (107)$$

$$I_3 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [|t'_L(\omega)|^2 [n_3(\omega + V) - n_L(\omega - \Delta\mu/2)] + |t'_R(\omega)|^2 [n_3(\omega + V) - n_R(\omega + \Delta\mu/2)]]. \quad (108)$$

Up to now, we were dedicated to the generic three-terminal Landauer-Büttiker analysis. In order to obtain a connection to the dissipative two-terminal system discussed in the previous sections, we consider

$$|t|^2 = |t'|^2, \quad (109)$$

$$|t_L|^2 = |t_R|^2. \quad (110)$$

The first condition above represents the symmetry of the conduction channel such that the transmittance from the left to right reservoirs is identical to the one from the right to left reservoirs. In addition, the second condition means that the transmittance into the third reservoir is common between the left and right reservoirs. For example, an S -matrix introduced in Ref. [47] satisfies these conditions.

When the above conditions hold, the average particle current from left to right is reduced to

$$I = \frac{I_L - I_R}{2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[|t(\omega)|^2 + \frac{|t_L(\omega)|^2}{2} \right] \times [n_L(\omega - \Delta\mu/2) - n_R(\omega + \Delta\mu/2)]. \quad (111)$$

In addition, the particle loss rate is

$$\begin{aligned} -\dot{N} &= I_L + I_R \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |t_L(\omega)|^2 [n_L(\omega - \Delta\mu/2) + n_R(\omega + \Delta\mu/2) \\ &\quad - 2n_3(\omega + V)]. \end{aligned} \quad (112)$$

Thus, by treating

$$|t_L(\omega)|^2 = \mathcal{L}(\omega), \quad (113)$$

$$|t(\omega)|^2 = \mathcal{T}(\omega), \quad (114)$$

the expressions above are consistent with the ones obtained in the previous sections, and

$$\int d\omega \mathcal{L}(\omega) n_3(\omega) = 0 \quad (115)$$

is necessary to neglect the gain effect from the third reservoir. Moreover, we can obtain the following relation,

$$\mathcal{T}(\omega) + \mathcal{R}(\omega) + \mathcal{L}(\omega) = 1, \quad (116)$$

with reflectance $\mathcal{R} = |r|^2 = |r'|^2$. The relation above is exactly the one obtained in Refs. [14,48,49]. We also note that

by using the following energy current operator,

$$\begin{aligned} \hat{I}_{E,j}(\tau) &= \int \frac{d\omega}{2\pi} \int d\omega' e^{i(\omega-\omega')\tau} \left[\frac{\omega + \omega'}{2} + \mu \right] \\ &\quad \times [\hat{a}_j^\dagger(\omega) \hat{a}_j(\omega') - \hat{b}_j^\dagger(\omega) \hat{b}_j(\omega)], \end{aligned} \quad (117)$$

it is straightforward to see that a similar correspondence occurs in the energy current [50].

Although one may think that the non-Hermitian Landauer-Büttiker analysis may contain a deficiency on the treatment of the quantum jump term, it is indeed consistent with the three-terminal analysis that does not drop any term. The reason is interpreted as follows. In the Landauer-Büttiker formalism, the scattering region and leads (reservoirs) play separate roles and especially what happens in the scattering region is assumed not to affect the thermodynamics of the leads. A modification of the distribution in a system occurs by the quantum jump term, yet it is irrelevant to the present Landauer-Büttiker analysis, since the dissipation takes place in the scattering region and the distribution function appearing in the Landauer-Büttiker formula is entirely determined by the leads.

VI. SUMMARY AND OUTLOOK

In this work, we have investigated the analytic structure of the two-terminal flows through the lossy one-dimensional chain. By using the analysis in the basis of the Dyson equations, we have succeeded in obtaining the generic expressions for the particle current (81), the loss rate (82), and the energy current (85), which are identical to the ones in a certain three-terminal system without particle loss. Although the theoretical backgrounds are not exactly the same between analyses with and without the Lindblad formulation, the formal correspondence occurs if the gain effect of the third reservoir is negligible.

It is also remarkable that the formal expressions on the currents are consistent with the non-Hermitian Landauer-Büttiker analysis [14] and with a certain three-terminal Landauer-Büttiker analysis. In the context of mesoscopic transport, the incoherent scattering of electrons has been explained with the introduction of additional reservoirs [51]. Our analyses gave a prescription and a theoretical basis of lossy mesoscopic systems with the Landauer-Büttiker formalism, and paves the way to simulate lossy two-terminal transport with three-terminal transport and vice versa.

The universality discussed in this work may be applied to other mesoscopic systems. In particular, applications both for bosonic [40–42] and fermionic [32,33,52] superfluid reservoirs are relevant to cold-atom experiments. It is also interesting to look into the dissipation effects of mesoscopic spin transport in which interactions neglected in this work may play important roles [53–59]. In addition, a microscopic understanding of the dissipation effects for internal mesoscopic transport systems [60] could also be an interesting future work.

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APPENDIX: DERIVATION OF THE PARTITION FUNCTION

In order to make this paper self-contained, here we give a derivation of the functional integral form of the partition function whose density matrix obeys the Lindblad master equation. Our derivation relies on Ref. [26], where the bosonic action has explicitly been derived.

For this purpose, it is useful to consider the coherent state obeying

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle, \quad (\text{A1})$$

$$\langle\psi|\hat{\psi}^\dagger = \langle\psi|\bar{\psi}, \quad (\text{A2})$$

where $\hat{\psi}$ and $\hat{\psi}^\dagger$ are the field annihilation and creation operators, respectively, and ψ and $\bar{\psi}$ are complex numbers for bosons and Grassmann numbers for fermions. The coherent state satisfies the following overcomplete relation,

$$\hat{1} = \int d[\bar{\psi}, \psi] e^{-\bar{\psi}\psi} |\psi\rangle\langle\psi|, \quad (\text{A3})$$

where $d[\bar{\psi}, \psi] = \frac{d\bar{\psi}d\psi}{\pi}$ for bosons and $d[\bar{\psi}, \psi] = d\bar{\psi}d\psi$ for fermions. In addition, the overlap of two coherent states is

By using the coherent state, the density matrix operator is expressed as

$$\hat{\rho}_n = \int d[\bar{\psi}_n^+, \psi_n^+, \bar{\psi}_n^-, \psi_n^-] e^{-\bar{\psi}_n^+ \psi_n^+ - \bar{\psi}_n^- \psi_n^-} |\pm\psi_n^+\rangle\langle\pm\psi_n^+| \hat{\rho}_n |\psi_n^-\rangle\langle\psi_n^-|, \quad (\text{A9})$$

$$\hat{\rho}_{n+1} = \int d[\bar{\psi}_{n+1}^+, \psi_{n+1}^+, \bar{\psi}_{n+1}^-, \psi_{n+1}^-] e^{-\bar{\psi}_{n+1}^+ \psi_{n+1}^+ - \bar{\psi}_{n+1}^- \psi_{n+1}^-} |\pm\psi_{n+1}^+\rangle\langle\pm\psi_{n+1}^+| \hat{\rho}_{n+1} |\psi_{n+1}^-\rangle\langle\psi_{n+1}^-|. \quad (\text{A10})$$

Since

$$\langle\pm\psi_{n+1}^+|\hat{\rho}_{n+1}|\psi_{n+1}^-\rangle = \langle\pm\psi_{n+1}^+|\hat{\rho}_n|\psi_{n+1}^-\rangle + \delta_\tau \langle\pm\psi_{n+1}^+|\mathcal{L}\hat{\rho}_n|\psi_{n+1}^-\rangle, \quad (\text{A11})$$

we have

$$\langle\pm\psi_{n+1}^+|\hat{\rho}_{n+1}|\psi_{n+1}^-\rangle = \int d[\bar{\psi}_n^+, \psi_n^+, \bar{\psi}_n^-, \psi_n^-] e^{(\bar{\psi}_{n+1}^+ - \bar{\psi}_n^+) \psi_n^+ + \bar{\psi}_n^- (\psi_{n+1}^- - \psi_n^-)} \langle\pm\psi_n^+|\hat{\rho}_n|\psi_n^-\rangle, \quad (\text{A12})$$

$$\begin{aligned} \langle\pm\psi_{n+1}^+|\mathcal{L}\hat{\rho}_n|\psi_{n+1}^-\rangle &= -i\langle\pm\psi_{n+1}^+|[\hat{H}, \hat{\rho}_n]|\psi_{n+1}^-\rangle + \gamma\langle\pm\psi_{n+1}^+|\left(\hat{L}\hat{\rho}_n\hat{L}^\dagger - \frac{\{\hat{L}^\dagger\hat{L}, \hat{\rho}_n\}}{2}\right)|\psi_{n+1}^-\rangle \\ &= \int d[\bar{\psi}_n^+, \psi_n^+, \bar{\psi}_n^-, \psi_n^-] e^{(\bar{\psi}_{n+1}^+ - \bar{\psi}_n^+) \psi_n^+ + \bar{\psi}_n^- (\psi_{n+1}^- - \psi_n^-)} \\ &\quad \times \left[-iH_n^+ + iH_n^- + \gamma\left(L_n^+ \bar{L}_n^- - \frac{\bar{L}_n^+ L_n^+ + \bar{L}_n^- L_n^-}{2}\right) \right] \langle\pm\psi_n^+|\hat{\rho}_n|\psi_n^-\rangle. \end{aligned} \quad (\text{A13})$$

Here, $H_n^{+(-)} = H(\pm\bar{\psi}_{n+1}^{+(-)}, \pm\psi_{n+1}^{+(-)})$ consists of fields on the $+(-)$ contour only, and the same holds true for L_n^\pm . By using

$$\partial_\tau \psi_n^\pm = \frac{\psi_{n+1}^\pm - \psi_n^\pm}{\delta_\tau}, \quad (\text{A14})$$

given by

$$\langle\psi|\psi'\rangle = e^{\bar{\psi}\psi'}. \quad (\text{A4})$$

By means of the coherent state, the partition function $Z = \text{Tr} \rho(\tau)$ can be expressed as

$$Z = \int d[\bar{\psi}, \psi] e^{-\bar{\psi}\psi} \langle\pm\psi|\hat{\rho}(\tau)|\psi\rangle. \quad (\text{A5})$$

Here, the upper and lower signs are respectively for bosons and fermions, and especially the minus sign for fermions originates from the anticommutation property of Grassmann numbers [29].

We next look at the density matrix operator whose dynamics obeys the following Lindblad master equation,

$$\begin{aligned} \partial_\tau \hat{\rho} &= \mathcal{L}\hat{\rho} \\ &= -i[\hat{H}, \hat{\rho}] + \gamma\left[\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{\{\hat{L}^\dagger\hat{L}, \hat{\rho}\}}{2}\right], \end{aligned} \quad (\text{A6})$$

where \mathcal{L} and L are Liouvillian and Lindblad operators, respectively. The solution of the above equation is formally expressed as

$$\begin{aligned} \hat{\rho}(\tau_f) &= e^{(\tau_f - \tau_0)\mathcal{L}} \hat{\rho}(\tau_0) \\ &= \lim_{N \rightarrow \infty} (1 + \delta_\tau \mathcal{L})^N \hat{\rho}(\tau_0). \end{aligned} \quad (\text{A7})$$

We now decompose the time evolution from τ_0 to τ_f into a sequence of small steps of duration $\delta_\tau = (\tau_f - \tau_0)/N$, and denote the density matrix operator after the n th step at $\tau_n = \tau_0 + \delta_\tau n$ by $\hat{\rho}_n = \hat{\rho}(\tau_n)$. We then have

$$\hat{\rho}_{n+1} = (1 + \delta_\tau \mathcal{L})\hat{\rho}_n + O(\delta_\tau^2). \quad (\text{A8})$$

we obtain

$$e^{(\bar{\psi}_{n+1}^+ - \bar{\psi}_n^+) \psi_n^+ + \bar{\psi}_n^- (\psi_{n+1}^- - \psi_n^-)} = e^{\delta_\tau [\partial_\tau \bar{\psi}_n^+ \psi_n^+ + \bar{\psi}_n^- \partial_\tau \psi_n^-]}. \quad (\text{A15})$$

In addition, we note

$$1 + \delta_\tau \left[-iH_n^+ + iH_n^- + \gamma \left(L_n^+ \bar{L}_n^- - \frac{\bar{L}_n^+ L_n^+ + \bar{L}_n^- L_n^-}{2} \right) \right] \approx e^{\delta_\tau [-iH_n^+ + iH_n^- + \gamma (L_n^+ \bar{L}_n^- - \frac{\bar{L}_n^+ L_n^+ + \bar{L}_n^- L_n^-}{2})]}. \quad (\text{A16})$$

Therefore, we obtain

$$\langle \pm \psi_{n+1}^+ | \hat{\rho}_{n+1} | \psi_{n+1}^- \rangle = \int d[\bar{\psi}_n^+, \psi_n^+, \bar{\psi}_n^-, \psi_n^-] e^{\delta S_n} \langle \pm \psi_n^+ | \hat{\rho}_n | \psi_n^- \rangle, \quad (\text{A17})$$

where

$$\delta S_n = \delta_\tau \left[\bar{\psi}_n^+ i \partial_\tau \psi_n^+ - \bar{\psi}_n^- i \partial_\tau \psi_n^- - H_n^+ + H_n^- - i\gamma \left(L_n^+ \bar{L}_n^- - \frac{\bar{L}_n^+ L_n^+ + \bar{L}_n^- L_n^-}{2} \right) \right]. \quad (\text{A18})$$

Thus, the partition function is obtained as

$$\begin{aligned} Z &= \text{Tr} e^{(\tau_f - \tau_0) \mathcal{L}} \hat{\rho}(\tau_0) \\ &= \int \mathcal{D}[\bar{\psi}^+, \psi^+, \bar{\psi}^-, \psi^-] e^{iS} \langle \pm \psi^+(\tau_0) | \hat{\rho}(\tau_0) | \psi^-(\tau_0) \rangle. \end{aligned} \quad (\text{A19})$$

When we are interested in a steady state, we take $\tau_0 \rightarrow -\infty$ and $\tau_f \rightarrow \infty$. There, we can make an assumption that the initial state in the infinite past does not affect the steady state, and we can ignore the boundary term, $\langle \pm \psi^+(\tau_0) | \hat{\rho}(\tau_0) | \psi^-(\tau_0) \rangle$.

In this case, the partition function is reduced to

$$Z = \int \mathcal{D}[\bar{\psi}^+, \psi^+, \bar{\psi}^-, \psi^-] e^{iS}. \quad (\text{A20})$$

Finally, we apply the above formula into our system, where $\hat{L} = \hat{d}_0$. There, we should be aware of the sign of the quantum jump term in the action $-i\gamma L^+ \bar{L}^-$. Especially, for fermions where the antiperiodic boundary condition is adopted (A5), the jump term is expressed as $i\gamma d_0 \bar{d}_0$, which has the opposite sign to bosons. This is in contrast to systems with two-particle loss where the sign difference between fermions and bosons does not matter, since the Lindblad operator is bosonic [61]. By using the anticommutation relation for Grassmann numbers, however, we obtain the partition function (13) whose formal expression is independent of quantum statistics.

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