Number-conserving solution for dynamical quantum backreaction in a Bose-Einstein condensate

Sang-Shin Baak [1], Caio C. Holanda Ribeiro [1,2] and Uwe R. Fischer [1]

[1] Department of Physics and Astronomy, Center for Theoretical Physics, Seoul National University, Seoul 08826, Korea

[2] Institute of Physics, University of Brasilia, 70919-970 Brasilia, Federal District, Brazil and International Centre of Physics, University of Brasilia, 70297-400 Brasilia, Federal District, Brazil

(Received 3 July 2022; accepted 27 October 2022; published 23 November 2022)

We provide a number-conserving approach to the backreaction problem of small quantum fluctuations onto a classical background for the exactly soluble dynamical evolution of a quasi-one-dimensional Bose-Einstein condensate, experimentally realizable in the ultracold gas laboratory. A force density exerted on the gas particles, which is of quantum origin, is uniquely identified as the deviation from the classical Eulerian force density. The backreaction equations are then explored for the specific example of a finite-size uniform density condensate initially at rest. By assuming that the condensate starts from a noninteracting regime and in its ground state, we fix a well-defined initial vacuum condition, which is driven out of equilibrium by instantaneously turning on the interactions. The assumption of this initial vacuum accounts for the ambiguity in choosing a vacuum state for interacting condensates, which is due to phase diffusion and the ensuing condensate collapse. As a major finding, we reveal that the time evolution of the condensate cloud leads to condensate density corrections that cannot in general be disentangled from the quantum depletion in measurements probing the power spectrum of the total density. Furthermore, while the condensate is initially at rest, quantum fluctuations give rise to a nontrivial condensate flux, from which we demonstrate that the quantum force density attenuates the classical Eulerian force. Finally, the knowledge of the particle density as a function of time for a condensate at rest determines, to order N^0 (where N is the total number of particles), the quantum force density, thus offering a viable route for obtaining experimentally accessible quantum backreaction effects.

DOI: 10.1103/PhysRevA.106.053319

I. INTRODUCTION

Field quantization in curved spacetimes leads to many intriguing phenomena [1,2]. The most illustrious example, with the largest impact in the physics community, is probably the Hawking radiation associated with the formation of event horizons around black holes. The emitted quantum particles, however, backreact onto the spacetime metric containing the horizon, causing it to fluctuate around its classical value. This backreaction problem was (partially) addressed by Hawking in [3] (where backreaction was stated to be a "difficult problem") and was entirely "ignored" in the original announcement that black holes radiate [4].

The difficulty (essentially, impossibility) in distinguishing the tiny Hawking radiation of real astrophysical black holes from the thermal background which surrounds it in the cosmos has led to the idea of analogs of Hawking radiation to be implemented in nonrelativistic parent systems [5–7], a development which culminated in the recent detection of this analog Hawking radiation in Bose-Einstein condensates (BECs) [8,9]. The BEC analog models are not limited to the simulation of kinematical phenomena on curved spacetimes such as the Hawking effect, and lend themselves to general studies of backreaction in a system with well-established microscopic behavior [10,11].

The backreaction problem of small quantum fluctuations onto the condensate in a dilute BEC at first glance might appear to be a straightforwardly soluble one within Bogoliubov theory. A careful inspection, however, reveals that, due to the nontrivial interchange of particles between the condensed and noncondensed (depleted) clouds, any proper notion of the separation of a "classical" condensate and quantum fluctuations on top of it can only be developed in a number-conserving theory, with the number of atoms N being fixed at all times. Therefore, the breakdown of particle-number conservation by the standard Bogoliubov expansion requires that expansions different from the latter should be employed in order to correctly account for backreaction effects, such as the number-conserving theories developed, e.g., in [12–19]. In [10] the authors considered backreaction by improving the standard Bogoliubov expansion via an expansion in powers of N, which is then capable of accounting for the interchange of particles between condensed and depleted clouds. We adopt in our work the same approach, because its predictions for backreaction can be straightforwardly interpreted in terms of possible experimental ramifications. Using the same approach, the backreaction effect on the analog background metric was investigated in [20].

Another important aspect of backreaction is related to using different field variables (e.g., switching to the quantum Madelung representation instead of using the scalar field operator). Different choices of variables produce conflicting results on the backreaction. This is illustrated, for example, by a comparison of a field-theory-inspired effective action approach to quantum backreaction [21] with an approach based on directly measurable quantities such as full density and current, as performed in [10] (also see [22]).

Thus, when backreaction is not axiomatically defined as for quantum field theory in curved spaces (cf., e.g., [1,2,23]), backreaction must be expressed in terms of measurable quantities. Following this principle, an unambiguous quantum backreaction definition was established in [10] as a quantum force density exerted on the gas particles which leads to an in principle measurable departure from classical Eulerian dynamics.

In the present paper we explore (and further extend) the quantum backreaction scheme defined in [10], by thoroughly analyzing a concretely soluble dynamical quasi-one-dimensional (quasi-1D) condensate model. We assume that our condensate initially has uniform density; such configurations have been experimentally implemented [24,25]. The system resides initially in the noninteracting ground state and is then suddenly quenched to an interacting regime, with finite coupling constant. The uniform density assumption leads to a set of field equations that can be exactly solved in the dominant order of condensate corrections.

An important phenomenon occurring in interacting BECs is that of condensate phase diffusion [15,26–30]. In particular, phase diffusion means that there is no stationary finite-size interacting condensate (note that this is distinct from the divergence of phase fluctuations in infinite one- and two-dimensional systems [31]). The absence of a stationary state then leads to the difficulty of properly specifying the initial configuration.

By assuming that the condensate is initially noninteracting, we provide a clear characterization of the initial configuration that, importantly, can be prepared in a controlled way in the laboratory. Indeed, this experimental procedure was adopted, for instance, in [32] in order to create stable condensates in well-characterized initial states. We show in this work that associated with the backreaction problem is a Cauchy problem, which requires the specification of the initial condensate state in order to produce meaningful solutions. For our quenched condensate, starting from a regime where the system is stationary, it is then possible to follow the system evolution in full detail from a well-defined initial state.

It should be stressed that the backreaction problem can be approached via a plethora of existing techniques. For example, within the context of analog gravity, in [33] the authors studied the problem of backreaction numerically in a black hole analog by implementing the truncated Wigner approximation (TWA), which amounts to evolving the system's Wigner functional by neglecting higher-order derivatives in its evolution [34,35]. Also, this approximation is suitable for numerically studying the evolution of analog Hawking radiation [36]. However, although the TWA is a numerical approximation to the full many-body problem and as such can be used to study even nonlinear regimes, its validity is limited by the number of quasiparticle modes included in the analysis. In contrast, the number-conserving Bogoliubov expansion we adopt here allows for exact solubility of relevant condensate configurations and it is limited only by the mean-field approximation.

It is also worth mentioning that different sources of backreaction on the condensate can exist besides the ones considered in [10]. Our findings reveal, as we will see, that when measuring, e.g., quantum depletion, one must necessarily take into account condensate backreaction coming from the atom interactions. In addition, the very measuring apparatus might imprint itself nontrivially on the condensate. Indeed, it was shown in [37] that Josephson-like oscillations between spatially separated condensates are modified by the nondestructive imaging of one of the condensates. Furthermore, in [38] the authors showed that the nondestructive imaging of a condensate also reduces atom-number fluctuations at the expense of increasing the phase spreading (squeezing).

From the number-conserving solution for the condensate evolution, our model enables the proper interpretation of measurable quantities such as density and current of the gas. When the interactions are turned on and quantum fluctuations emerge, we show that it is not possible to discern the quantum depletion from the corrections to the condensed cloud in density measurements, revealing a subtle but important aspect of measuring condensate depletion. Furthermore, we show that even though the condensate is initially at rest and there is no phonon flux, number conservation leads to a nonvanishing condensed particle flux, representing a bona fide quantum backreaction effect in our system. From the particle current, we then show that the total force density on the gas particles is given by a potential function, which, upon comparison with the Eulerian potential function, is used to show that the quantum force has an attenuating effect over the classical force.

Our work is organized as follows. We present in Sec. II the formal framework of the number-conserving approach to quantum backreaction which we employ. In Sec. III the condensate model is set up and the quantum fluctuations created by the coupling constant quench are studied. In Sec. IV we study quantum depletion, from which we verify the validity regime of our approach. In Sec. V we present the major results of our work, the corrections to the condensate resulting from backreaction of quantum fluctuations onto the condensate. We finish our discussion with a summary and final remarks in Sec. VI.

II. NUMBER-CONSERVING FORMULATION OF QUANTUM BACKREACTION

A. Expansion in powers of N

Let Ψ describe the bosonic field associated with the nonrelativistic dilute gas in 1+1 dimensions under study, whose evolution is here taken to be ruled in the s-wave approximation by the field equation

$$i\partial_t \Psi = \left(-\frac{\partial_x^2}{2m} + U + g|\Psi|^2\right)\Psi, \tag{1}$$

where we have set $\hbar = 1$ and U is the external (trapping) potential. We note that the theory described by (1) is invariant under global $U(1)$ transformations, which ensures the conservation of the system particle number N according to the Noether theorem. Specifically, the conservation law $\partial_t \rho + \partial_x J = 0$ holds, where $\rho = |\Psi|^2$, $J = (\Psi^* \partial_x \Psi - \Psi \partial_x \Psi^*)/2im$

are the system particle and current densities, respectively, and $N = \int dx \rho$.

The number-conserving theory we adopt in this work is obtained by expanding Ψ in powers of N as

$$\Psi = \phi_0 + \chi + \zeta + O(N^{-3/2}), \tag{2}$$

with the following set of scaling behaviors [10]:

$$N \to \infty \quad \text{with } Ng = \text{const}, \tag{3a}$$

$$\phi_0 = O(N^{1/2}), \quad \chi = O(N^0), \quad \zeta = O(N^{-1/2}). \tag{3b}$$

The above set of scalings leads, for the assumed contact interactions and at the respective orders of N, to the hierarchy of coupled equations (4)–(6) below [note that for long-range interactions, in general other scalings appear in the expansion (cf. [39])]. The first condition (3a) is necessary to control the potential $g|\Psi|^2$ in the field equation (1), for when $N \to \infty$ for a fixed coupling g, $\Psi \to 0$ is the only physical (normalizable) solution to the field equation. This condition is indeed required for a rigorous derivation of the Gross-Pitaevskii energy functional [40] and for complete Bose-Einstein condensation to occur in the limit $N \to \infty$ [41]. The second condition (3b) ensures that $|\zeta| \ll |\chi| \ll |\phi_0|$ and identifies the different magnitude scales in the system. The field χ is the fluctuating field and ζ encapsulates the backreaction effects from χ onto the given condensate configuration ϕ_0. Furthermore, our goal is to work with a number-conserving theory up to order N^0, i.e., the densities ρ and J should be expanded also up to order N^0. Thus, because both ρ and J are quadratic in Ψ, the expansion for Ψ must contain terms up to order $N^{-1/2}$, i.e., up to the order of ζ.

By plugging the expansion (2) into the field equation (1) and identifying terms according to their order in N (keeping in mind that $g \propto N^{-1}$), we obtain to leading order the Gross-Pitaevskii (GP) equation for ϕ_0,

$$i\partial_t \phi_0 = \left(-\frac{\partial_x^2}{2m} + U + g\rho_0\right)\phi_0, \tag{4}$$

with $\rho_0 = |\phi_0|^2$; in the next order in N the Bogoliubov–de Gennes (BdG) equation for χ,

$$i\partial_t \chi = \left(-\frac{\partial_x^2}{2m} + U + 2g\rho_0\right)\chi + g\phi_0^2\chi^*; \tag{5}$$

and finally a BdG-like equation with χ-dependent source terms for the final contribution to the expansion ζ,

$$i\partial_t \zeta = \left(-\frac{\partial_x^2}{2m} + U + 2g\rho_0\right)\zeta + g\phi_0^2\zeta^* + 2g|\chi|^2\phi_0 + g\chi^2\phi_0^*. \tag{6}$$

Alternatively, following [10], we can also define the field $\phi_c = \phi_0 + \zeta$ in such a way that both Eqs. (4) and (6) are written compactly in terms of the improved GP equation, which includes subleading terms

$$i\partial_t \phi_c = \left(-\frac{\partial_x^2}{2m} + U + g|\phi_c|^2 + 2g|\chi|^2\right)\phi_c + g\chi^2\phi_c^*. \tag{7}$$

It should be stressed, however, that within the expansion in powers of N, Eq. (7) represents *only* a compact way of writing Eqs. (4) and (6). Indeed, note that factors involving χ in Eq. (7) are of order N^{-1}, which should be compared with the order N^0 factors in the remaining terms. Thus, in order to keep the expansion consistent, the solution ϕ_c of Eq. (7) is the sum of a dominant order-$N^{1/2}$ term plus a subdominant correction of order $N^{-1/2}$. Using the fields ϕ_c and χ to describe the gas dynamics allows us to interpret the field ζ as modeling corrections to the condensate order parameter ϕ_0 due to the field χ. We will show in the following that this interpretation of the field ζ facilitates the proper formulation of backreaction of quantum fluctuations onto the classical background. Finally, quantization, within our approach, is achieved by promoting the classical field χ to the operator-valued distribution $\hat{\chi}$, taken to satisfy the equal-time bosonic commutation relation $[\hat{\chi}(t,x), \hat{\chi}^\dagger(t,x')] = \delta(x-x')$.[1]

In the following, working in the Heisenberg picture, we will consider only the quasiparticle vacuum state for the perturbations, for which $\langle\hat{\chi}\rangle := 0$. After quantization, the classical current and density become operator-valued distributions as well, and for the vacuum state under consideration we have

$$\rho := \langle\hat{\rho}\rangle = |\phi_c|^2 + \langle\hat{\chi}^\dagger\hat{\chi}\rangle + O(N^{-1/2}), \tag{8}$$

$$J := \langle\hat{J}\rangle = \frac{1}{m}\text{Im}[\phi_c^*\partial_x\phi_c + \langle\hat{\chi}^\dagger\partial_x\hat{\chi}\rangle] + O(N^{-1/2}). \tag{9}$$

Furthermore, the field ϕ_c is now given by the operator-valued version of Eq. (7), where it appears as a multiple of the identity operator. Thus the equation coincides with its vacuum expectation value and is given by

$$i\partial_t \phi_c = \left[-\frac{\partial_x^2}{2m} + U + g|\phi_c|^2 + 2g\langle\hat{\chi}^\dagger\hat{\chi}\rangle\right]\phi_c + g\langle\hat{\chi}^2\rangle\phi_c^*, \tag{10}$$

where the normal ordering prescription was taken.

It is instructive to define the averaged contributions to the densities stemming from the quantum fluctuations alone as $\rho_\chi := \langle\hat{\chi}^\dagger\hat{\chi}\rangle$ and $J_\chi := \text{Im}[\langle\hat{\chi}^\dagger\partial_x\hat{\chi}\rangle]/m$ in such a way that $\rho := \rho_c + \rho_\chi + O(N^{-1/2})$ and $J := J_c + J_\chi + O(N^{-1/2})$. It is then straightforward to show, from Eqs. (10) and (5), that

$$\partial_t\rho_c + \partial_x J_c = ig\left(\phi_c^2\langle\hat{\chi}^{\dagger 2}\rangle - \phi_c^{*2}\langle\hat{\chi}^2\rangle\right), \tag{11a}$$

$$\partial_t\rho_\chi + \partial_x J_\chi = -ig\left(\phi_c^2\langle\hat{\chi}^{\dagger 2}\rangle - \phi_c^{*2}\langle\hat{\chi}^2\rangle\right), \tag{11b}$$

thus ensuring that the theory is conserving ($\partial_t\rho + \partial_x J = 0$) up to our working ($N^0$) order in the densities and currents.

B. Corrections to the condensate background

We note that the right-hand sides of both Eqs. (11a) and (11b) are of order N^0, which is in accord with the left-hand side of Eq. (11b), but seems to fail for the left-hand side of Eq. (11a), which is of order N. The reason is that both ρ_c and J_c are determined by the field ϕ_c and thus they split into dominant $O(N)$ terms plus $O(N^0)$ corrections in such a way that the dominant terms on the left-hand side of

[1]Note that it is consistent, to leading Bogoliubov order, with the expansion (2) and the resulting evolution equations (4)–(6) to quantize the field χ only (and not also ζ) (cf. the discussion in [10]).

Eq. (11a) cancel exactly as they are built from a solution of the standard GP equation (4). Returning to the definitions of $\phi_c = \phi_0 + \zeta$ as well as ρ_c and J_c from Eqs. (8) and (9), we find that $\rho_c = \rho_0 + \rho_\zeta + O(N^{-1/2})$ and $J_c = J_0 + J_\zeta + O(N^{-1/2})$, where $J_0 = \mathrm{Im}[\phi_0^* \partial_x \phi_0]/m$,

$$\rho_\zeta = 2\,\mathrm{Re}[\phi_0^* \zeta], \tag{12}$$

$$J_\zeta = \frac{1}{m}\mathrm{Im}[\phi_0^* \partial_x \zeta + (\partial_x \phi_0)\zeta^*]. \tag{13}$$

It thus follows from Eq. (4) that $\partial_t \rho_c + \partial_x J_c = \partial_t \rho_\zeta + \partial_x J_\zeta = O(N^0)$, ensuring the consistency of Eq. (11a).

The density ρ_ζ and current density J_ζ are the corrections to the condensate contributions ρ_0 and J_0. Within the validity domain of Bogoliubov theory, i.e., when $\delta N := \int dx \rho_\chi \ll N = \int dx \rho_0$, the quantum fluctuations modeled by the field $\hat{\chi}$ remain small and independent of the condensate corrections ζ, which are in turn determined by ρ_χ and $\langle \hat{\chi}^2 \rangle$ through Eq. (6). In this regime, the dynamics of the field $\hat{\chi}$ is linear. Furthermore, the very condensate existence in the presence of interactions ($\delta N/N \ll 1$) leads to a nonvanishing quantum depletion ρ_χ and nonvanishing anomalous correlator $\langle \hat{\chi}^2 \rangle$, which in turn correct the condensate via the field ζ. This is the essence of the backreaction scheme we employ here.

C. Quantum force

We are now able to enunciate the definition of the backreaction scheme presented in [10]. The motivation for it comes from the fact that the gas separation into a condensate part modeled by ϕ_c and the one-particle quantum excitations $\hat{\chi}$ is an intricate concept, as the number of particles in each of these sectors is not conserved separately during the system development. This follows from Eqs. (11), which describe the local conservation law of particles in the condensate and depletion sectors.

Therefore, a consistent definition of the quantum backreaction must be formulated in terms of measurable quantities, which, within our nonrelativistic field theory, include the densities and current densities. By following the discussion in [10], in the absence of quantum fluctuations, the classical fluid described by the standard GP equation is ruled by the Euler equation $\partial_t J = f_{\mathrm{cl}}$ and the continuity equation $\partial_t \rho + \partial_x J = 0$, where the classical force density f_{cl} is

$$f_{\mathrm{cl}} = -\partial_x(\rho v^2) - \frac{\rho}{m}\partial_x\left(\frac{-\partial_x^2\sqrt{\rho}}{2m\sqrt{\rho}} + U + g\rho\right) \tag{14}$$

and $v = J/\rho$ is the average fluid velocity. However, when quantum fluctuations are taken into account, we should have $\partial_t J \neq f_{\mathrm{cl}}$, indicating a departure from the classical description induced by quantum effects, which unambiguously defines the quantum force $f_q := \partial_t J - f_{\mathrm{cl}}$. It then follows from the various definitions in the above that

$$\begin{aligned} f_q = {} & \partial_t J_\chi - v_0 \partial_t \rho_\chi + \partial_x\big(v_0 J_\chi - \rho_\chi v_0^2\big) + J_\chi \partial_x v_0 \\ & - \frac{\rho_0}{2m}\partial_x\left(\frac{g G^{(2)}}{\rho_0}\right) + \frac{\rho_\chi}{m}\partial_x\left(-\frac{\partial_x^2\sqrt{\rho_0}}{2m\sqrt{\rho_0}} + U + g\rho_0\right) \\ & - \frac{\rho_0}{4m^2}\partial_x\left[\frac{1}{\sqrt{\rho_0}}\partial_x^2\left(\frac{\rho_\chi}{\sqrt{\rho_0}}\right) - \frac{\rho_\chi}{\rho_0^{3/2}}\partial_x^2\sqrt{\rho_0}\right] \end{aligned} \tag{15}$$

holds up to order N^0. Here $v_0 = J_0/\rho_0$ is the zeroth-order condensate velocity and the (on-site, second-order) correlation $G^{(2)}$ is defined in terms of the quantum density operator $\hat{\rho} = (\phi_c^* + \hat{\chi}^\dagger)(\phi_c + \hat{\chi})$ as

$$G^{(2)} = \langle : (\hat{\rho} - \langle \hat{\rho} \rangle)^2 : \rangle, \tag{16}$$

where the colons indicate normal ordering, and is a contribution to the quantum force density originating in the local condensate density fluctuations. Measurements of the on-site correlation function $G^{(2)}$ in terms of the atom-number fluctuations are typically limited by the pixel-size resolution of the optical imaging [42], but more recently superresolution methods beating the diffraction limit have been developed, in principle allowing us to resolve the density fluctuations on scales of the order of the healing length [43,44].

D. Cauchy problem for backreaction analysis

As a final ingredient for the backreaction analysis under construction, in this section we show how the various equations presented in the above should be used to calculate relevant quantities. Specifically, we note that the unique identification of classical and quantum forces f_{cl} and f_q is not sufficient to determine the evolution of ρ and J completely through the system

$$\partial_t \rho + \partial_x J = 0, \tag{17a}$$

$$\partial_t J = f_{\mathrm{cl}} + f_q. \tag{17b}$$

Given the functional forms of both f_{cl} and f_q, the system of equations (17) has to be supplemented with initial conditions for the system densities ρ and J at some initial time, i.e., the gas configuration should be completely known initially in order for the backreaction problem to become a well-defined Cauchy problem, with a unique causal solution for ρ and J.

Furthermore, another subtle aspect of backreaction analysis in Bose gases is linked to the determination of the system initial configuration. Within our expansion scheme, a specification of ρ and J at a given time is characterized by ϕ_0, ζ, and a quantum state for the field operator $\hat{\chi}$ at that instant of time. Note that although this specification is simple from a mathematical point of view, it has a great impact on the applicability of the backreaction analysis to experiments devised to probe quantum effects. Indeed, in order to establish a concrete example, let us consider the problem of probing quantum depletion of a condensate through density measurements using, for instance, the technique of [45]. According to the definitions of Secs. II A and II B, the system's total density up to order N^0 reads $\rho = \rho_0 + \rho_\chi + \rho_\zeta + O(N^{-1/2})$, and measuring quantum depletion amounts to determining ρ_χ alone among the various density contributions. In any number-conserving analysis, ρ_χ can only be directly measured in experiments where it can be distinguished from ρ_ζ. We will return to this matter when we present our approach to backreaction in more detail below.

Also related to the notion of initial conditions is the phenomenon of condensate phase diffusion [15] when $g \neq 0$, the spontaneous destruction of the condensate off-diagonal long-range order. For our purposes, this phenomenon means that in the absence of coherent sources, no finite-size stationary condensate can exist. In particular, no finite-size condensate with $g \neq 0$ and $\partial_t \rho_\chi := 0$ is physically possible, even in the case where the field equations are stationary. Within the backreaction language, phase diffusion implies that there is no stationary instantaneous vacuum state that can be used as a quantum state for the initial system configuration, which contributes further to the difficulty in assessing condensate configurations at the laboratory. In the following sections we thoroughly explore a backreaction model with a well-defined initial state that is in principle experimentally accessible.

III. QUANTUM FLUCTUATIONS IN A UNIFORM DENSITY CONDENSATE

As discussed in Sec. II D, within our number-conserving theory, a condensate configuration is specified by the field ϕ_c or equivalently by ϕ_0 and ζ. We consider a uniform density condensate at rest inside a box trap of size ℓ and at zero temperature such that $\rho_0 = |\phi_0|^2$ is its density. The condensate is assumed to be in its ground state in a noninteracting regime for $t < 0$, i.e., $g = 0$ (see, e.g., Ref. [46] for an experimental realization of condensates in this regime). It thus follows from the absence of interactions that $\rho_\chi = \langle \hat{\chi}^2 \rangle := 0$ $(t < 0)$ for the system vacuum state and accordingly $\zeta := 0$ is a trivial solution to the backreaction equations which we take as the initial condensate configuration. In order to activate the quantum fluctuations and thus the backreaction, we assume that at $t \geqslant 0$ the atom interactions are turned on $(g = g_0 > 0)$, while the background condensate order parameter ϕ_0 remains unchanged. In this section we study in detail the field ϕ_0 and the quantization of χ for the resulting condensate configuration.

A. Background condensate

Condensates with an essentially uniform density profile can be prepared using current technology [24], and we assume for our purposes that the condensate under consideration is well approximated to be uniform. This might be realizable ever more accurately experimentally with new trapping techniques being developed (see [25] for an up-to-date review). A multitude of interesting applications of uniform density models has been described in [25] and we cite here in addition the recent application in the context of analog gravity and Hawking radiation [47].

The major benefit we gain by assuming a uniform condensate density is the exact solubility of Eqs. (5) and (6). Furthermore, we also assume that ϕ_0 has a simple harmonic time dependence, i.e., $\phi_0 = \exp(-i\mu t)\sqrt{\rho_0}$, where μ is the chemical potential and ρ_0 is constant. This configuration is a solution of Eq. (4) for the external potential

$$U = \mu + \frac{1}{2m}\partial_x[\delta(x - \ell/2) - \delta(x + \ell/2)]. \tag{18}$$

By plugging this external potential back into Eqs. (1) and (4)–(6) and integrating them around $\pm\ell/2$, we obtain that $\partial_x \Psi|_{x=\pm\ell/2} = 0$ and similarly for ϕ_0, χ, and ζ, i.e., this external potential translates to Neumann boundary conditions at the condensate walls. In what follows, we will omit the δ derivatives for the sake of notational convenience.

In order to activate the quantum fluctuations, we assume that at $t \geqslant 0$ the atom interactions are instantaneously turned on $(g = g_0 > 0)$, keeping ρ_0 constant. This is achieved by a time-dependent external potential given by

$$U = \mu - g\rho_0 \tag{19}$$

for $-\ell/2 < x < \ell/2$, and this represents all the input necessary to study backreaction in this system, as we will show.

Once the order-$N^{1/2}$ field ϕ_0 is determined, the condensate quantum fluctuations are given by the quantization of the field χ, and because of the uniform density assumption on ϕ_0, canonical quantization is straightforward. It consists in expanding χ in a complete set of eigenfunctions and imposing the canonical commutation relations, as we show in what follows. Also, in addition to $\hbar = 1$, we use units such that $m = 1$. We will render all equations dimensionless by scaling the quantities contained in them with powers of the healing length $\xi_0 = 1/\sqrt{\rho_0 g_0}$. For example, spatial coordinates are then expressed in units of ξ_0, time is expressed in units of ξ_0^2, and particle densities indicate the number of particles per healing length.

B. Canonical quantization of χ

Let $\chi = \exp(-i\mu t)\psi$, where from Eq. (5) ψ is solution of

$$i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi + \frac{g}{g_0}(\psi + \psi^*), \tag{20}$$

and we observe the boundary conditions $\partial_x \psi|_{x=\pm\ell/2} = 0$. Equation (20) can be cast in a spinorial form by defining the Nambu spinor $\Phi = (\psi, \psi^*)^\mathsf{T}$, where T stands for transpose. Thus, the field equation (20) implies

$$i\sigma_3 \partial_t \Phi = \left(-\frac{\partial_x^2}{2} + \frac{g}{g_0}\sigma_4\right)\Phi, \tag{21}$$

where σ_i $(i = 1, 2, 3)$ denote the usual Pauli matrices and $\sigma_4 = 1 + \sigma_1$. Equivalence between the two representations is recovered by requiring that the spinor Φ satisfies the reflection property $\Phi = \sigma_1 \Phi^*$ and upon quantization the commutation relations in terms of $\hat{\Phi}$ read

$$[\hat{\Phi}_a(t, x), \hat{\Phi}_b^\dagger(t, x')] = \sigma_{3,ab}\delta(x - x'). \tag{22}$$

Moreover, the Neumann boundary conditions (BCs) for ψ imply that the field Φ is subjected to the same conditions:

$$\partial_x \Phi|_{x=\pm\ell/2} = 0. \tag{23}$$

If Φ and Φ' are two solutions of Eq. (21) fulfilling the boundary conditions of Eq. (23), then

$$\langle \Phi, \Phi' \rangle = \int dx\, \Phi^\dagger(t, x)\sigma_3 \Phi'(t, x) \tag{24}$$

is a conserved (in time) quantity, which will be used as a scalar product on the space of classical solutions. Also, as the field modes have compact support, they have finite norms, which can be taken in general as

$$\langle \Phi, \Phi \rangle = \pm 1. \tag{25}$$

We stress that even though Eq. (21) may admit nonzero solutions with vanishing norm, we can *always* find an orthonormal basis as in Eq. (25). The plus and minus signs in Eq. (25) correspond to positive- and negative-norm modes, and we recall that for each solution Φ of Eq. (21), $\sigma_1 \Phi^*$ is also a solution of opposite norm sign. Thus there exists a one-to-one correspondence between positive- and negative-norm modes, which allows us to index the positive-norm solutions as Φ_n, $n = 0, 1, 2, \ldots$. With this, we can write the most general classical solution of Eq. (21) as

$$\Phi(t,x) = \sum_{n=0}^{\infty}[a_n\Phi_n(t,x) + b_n^*\sigma_1\Phi_n^*(t,x)], \tag{26}$$

and in view of the reflection property $\Phi = \sigma_1\Phi^*$, it follows that $b_n = a_n$. Now canonical quantization is defined by the promotion of Φ to the operator-valued distribution $\hat{\Phi}$ subjected to the condition (22), which corresponds to promoting each $a_n = \langle\Phi_n, \Phi\rangle$ to an operator $\hat{a}_n$ satisfying

$$[\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{n,n'}. \tag{27}$$

Concluding, a vacuum state $|0\rangle$ is defined by the kernel condition $\hat{a}_n|0\rangle = 0$, and from the identification $\Phi_n = (u_n, v_n)^{\mathrm{t}}$, we have, from Eq. (26),

$$\hat{\psi}(t,x) = \sum_{n=0}^{\infty}[\hat{a}_n u_n(t,x) + \hat{a}_n^\dagger v_n^*(t,x)]. \tag{28}$$

Therefore, in order to conclude the quantization procedure we need to find the set $\{\Phi_n\}_{n=0}^{\infty}$ of positive field modes.

C. Quantum field in the noninteracting regime

The positive-norm field modes $\Phi_n(t,x)$ $(n = 0, 1, 2, \ldots)$ at all times can be found by solving the field equation separately at $t < 0$ and $t > 0$, where the system dynamics is stationary. Let us focus on the noninteracting regime first. In this case, the field equation reads

$$i\sigma_3\partial_t\Phi = -\frac{\partial_x^2}{2}\Phi. \tag{29}$$

The solutions of Eq. (29) can be found as follows. As we are in a stationary regime, solutions of the form $\Phi(t,x) = \exp(-i\omega t)\Phi_\omega(x)$ exist for $\omega \geqslant 0$ such that

$$\omega\sigma_3\Phi_\omega = -\frac{\partial_x^2}{2}\Phi_\omega. \tag{30}$$

Now because Eq. (30) for $-\ell/2 < x < \ell/2$ is independent of x, we can find $\Phi_\omega(x) = \exp(ikx)\Phi_{\omega,k}$ with constant $\Phi_{\omega,k}$. This is possible only if

$$\omega = \pm\frac{k^2}{2}, \tag{31}$$

which has four distinct solutions $k_1 = \sqrt{2\omega} = -k_2 = -ik_3 = ik_4$, and $\Phi_{\omega,k_1} = \Phi_{\omega,k_2} = (1,0)^{\mathsf{T}}$ and $\Phi_{\omega,k_3} = \Phi_{\omega,k_4} = (0,1)^{\mathsf{T}}$. Thus a general solution of Eq. (30) must have the form

$$\Phi_\omega(x) = e^{ik_1x}\Phi_{\omega,k_1} + \sum_{i=2,3,4} S_{k_i}e^{ik_ix}\Phi_{\omega,k_i}. \tag{32}$$

By imposing Neumann BCs at $x = \pm\ell/2$ we obtain that $S_{k_3} = S_{k_4} = 0$, $k_1 := k_n = n\pi/\ell$ $(n = 0, 1, 2, 3, \ldots)$, $S_{k_2} = (-1)^n$, and $\omega := \Omega_n = n^2\pi^2/2l^2$, or

$$\Phi_n(t,x) = \frac{e^{-i\Omega_n t}[e^{ik_nx} + (-1)^n e^{-ik_nx}]}{\sqrt{2\ell(1+\delta_{0,n})}}(1,0)^{\mathsf{T}} \tag{33}$$

$(n = 0, 1, 2, 3, \ldots)$ are the positive-norm modes. The normalization constant is added to ensure that $\langle\Phi_n, \Phi_{n'}\rangle = \delta_{n,n'}$. Accordingly, the quantum field in the noninteracting regime assumes the form

$$\hat{\Phi}(t,x) = \sum_{n=0}^{\infty}[\hat{a}_n\Phi_n(t,x) + \hat{a}_n^\dagger\sigma_1\Phi_n^*(t,x)]. \tag{34}$$

We show in Appendix A that for $t < 0$ the commutation relation (22) for the expansion (34) is verified, which amounts to saying that the set of field modes just built is indeed complete.

D. Field modes in the interacting regime

In this section we extend the set $\{\Phi_n\}_{n=0}^{\infty}$ of positive-norm mode functions to the interacting regime $(t > 0)$. In order to obtain such an extension, we can proceed by solving the field equation for a complete set of mode functions in the interacting regime $(t > 0)$ and expand each Φ_n in terms of these functions. The field equation now reads

$$i\sigma_3\partial_t\Phi = \left(-\frac{\partial_x^2}{2} + \sigma_4\right)\Phi. \tag{35}$$

In contrast to the noninteracting regime, we call attention to the zero-norm-mode caveat of Eq. (35). Note that

$$\Phi = \Pi_0 := (1, -1)^{\mathsf{T}} \tag{36}$$

is a time-independent solution of Eq. (35), and because $\Pi_0^\dagger\sigma_3\Pi_0 = 0$, this nonzero solution has zero norm. Moreover, this is the only time-independent solution of Eq. (35) and clearly $\sigma_1\Pi_0^* = -\Pi_0$, which means that we cannot use the reflection property to build a second linearly independent (LI) solution. A procedure to find another LI solution was presented in [15], and in our case it is enough to see that

$$\tilde{\Pi}_0 = \tfrac{1}{2}(1,1)^{\mathsf{T}} - it\Pi_0 \tag{37}$$

is also an admissible field mode with zero norm. We note that $\tilde{\Pi}_0$ is not an eigenfunction of the time translation generator $i\partial_t$, $i\partial_t\tilde{\Pi}_0 = \Pi_0$, which is the mathematical expression for the breakdown of the system time translation symmetry exhibited by the field equations, implying that no interacting condensate free of external coherent sources exists in a steady state. In the language of Bose-Einstein condensates, the field mode Π_0 corresponds to a momentum operator, whereas $\tilde{\Pi}_0$ corresponds to an unbound phase operator, which gives rise to the notion of condensate phase diffusion (see for the definition of these operator notions [15,28]). Also, we observe in connection to analog gravity in BECs that finite-size black hole analogs present generic dynamical instabilities that also break the system time translation symmetry [48], and thus our quantum quench from a noninteracting regime offers a route for studying backreaction also in these systems.

All the other positive-norm mode functions of Eq. (35) can be found following the same procedure as before. We find that

$$\Pi_n = \frac{e^{-i\omega_n t}[e^{ik_nx} + (-1)^n e^{-ik_nx}]}{\sqrt{2\ell\left[1 - \left(\omega_n - k_n^2/2 - 1\right)^2\right]}}\left(1, \omega_n - k_n^2/2 - 1\right)^{\mathsf{T}} \tag{38}$$

for $n = 1, 2, 3, \ldots$ and $\omega_n = \sqrt{k_n^2(k_n^2/4 + 1)}$. Again, the normalization constant was chosen to guarantee $\langle\Pi_n, \Pi_{n'}\rangle = \delta_{n,n'}$, $n, n' \geqslant 1$. Therefore, the set $\{\Pi_0, \tilde{\Pi}_0, \Pi_n, \sigma_1\Pi_n^*\}$ is complete and thus we can write

$$\Phi_n = \alpha_{n,0}\Pi_0 + \beta_{n,0}\tilde{\Pi}_0 + \sum_{j=1}^{\infty}[\alpha_{n,j}\Pi_j - \beta_{n,j}\sigma_1\Pi_j^*] \tag{39}$$

for $t > 0$. The coefficients $\alpha_{n,j}$ and $\beta_{n,j}$ are then fixed by the field equation (21): The wave function Φ_n is a continuous function of t. In particular, we have $\Phi_n(0^+, x) = \Phi_n(0^-, x) := \Phi_n^{(-)}$. This condition applied to Eqs. (33) and (39) gives rise to a Fourier decomposition for the function $\Phi_n^{(-)}$, implying

$$\alpha_{n,0} = \frac{\langle\tilde{\Pi}_0, \Phi_n^{(-)}\rangle}{\langle\tilde{\Pi}_0, \Pi_0\rangle}, \quad \alpha_{n,j} = \langle\Pi_j, \Phi_n^{(-)}\rangle, \tag{40}$$

$$\beta_{n,0} = \frac{\langle\Pi_0, \Phi_n^{(-)}\rangle}{\langle\Pi_0, \tilde{\Pi}_0\rangle}, \quad \beta_{n,j} = \langle\sigma_1\Pi_j^*, \Phi_n^{(-)}\rangle, \tag{41}$$

where $j > 0$ and the functions Π_n are evaluated at $t = 0$. By performing the integrals we find

$$\alpha_{n,0} = \frac{\delta_{n,0}}{2\sqrt{\ell}}, \quad \alpha_{n,j} = \frac{\delta_{n,j}}{\sqrt{1-\left(\omega_n - k_n^2/2 - 1\right)^2}}, \tag{42a}$$

$$\beta_{n,0} = \frac{\delta_{n,0}}{\sqrt{\ell}}, \quad \beta_{n,j} = \frac{\left(\omega_n - k_n^2/2 - 1\right)(-1)^n\delta_{n,j}}{\sqrt{1-\left(\omega_n - k_n^2/2 - 1\right)^2}}, \tag{42b}$$

where $j > 0$. This concludes the determination of the complete set of positive-norm mode functions $\{\Phi_n\}_{n=0}^{\infty}$, and we show in Appendix A that the quantum field expansion of Eq. (34) satisfies the commutation relation of Eq. (22) for all t. In the next section we show how this quantization determines the evolution of the noncondensed cloud.

IV. CONDENSATE DEPLETION

In this section we focus on the evolution of the depleted cloud as the interactions are turned on. It should be stressed that the quantization developed in the preceding section and the results presented in this section are exactly the same as the ones obtained from the non-number-conserving Bogoliubov theory. Major differences between the number- and non-number-conserving approaches are found when we discuss connections to measurements in Sec. V.

The depleted cloud is characterized by the quantum depletion ρ_χ and the phonon flux J_χ, both defined in terms of $\hat{\chi}$ in the paragraph before Eqs. (11) as $\rho_\chi = \langle\hat{\chi}^\dagger\hat{\chi}\rangle$ and $J_\chi = \text{Im}[\langle\hat{\chi}^\dagger\partial_x\hat{\chi}\rangle]$. Referring back to the quantum field expansion in Eq. (28) and the definition $\hat{\chi} = \exp(-i\mu t)\hat{\psi}$, for our condensate configuration it follows that $J_\chi = 0$ for all t, meaning that the depleted cloud remains at rest with respect to the laboratory frame as long as the predictions of Bogoliubov theory are reliable. This is not true, however, for black hole analogs, in which cases the condensate necessarily flows. In such cases one has $J_\chi \neq 0$.

As for the quantum depletion, we find that $\rho_\chi := 0$ for $t < 0$ in the noninteracting regime by definition of the latter and

$$\rho_\chi = \frac{t^2}{\ell} + \frac{1}{2\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\omega_n^2}[(-1)^n + \cos(2k_n x)][1 - \cos(2\omega_n t)] \tag{43}$$

for $t \geqslant 0$ (interacting regime). Inspection of Eq. (43) reveals that the first contribution to the system depletion, namely, t^2/ℓ, comes from the zero mode in Eq. (37). It has a simple interpretation in terms of condensate phase diffusion [15]: As the condensate phase degrades, particles leave the condensed cloud. Moreover, we see that if $\ell \to \infty$ with ρ_0 kept constant, this contribution goes to zero. However, the second contribution diverges then, as no infinitely extended quasi-1D condensate exists [31].

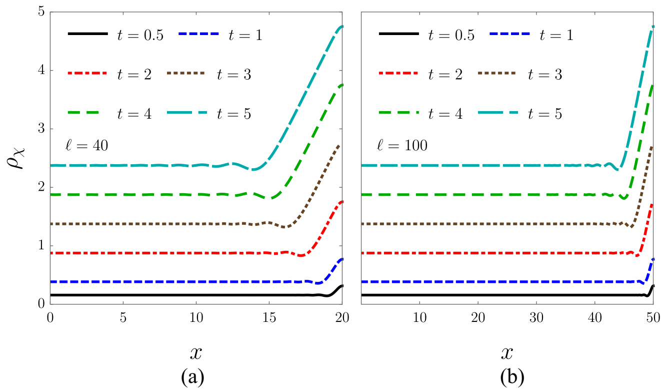

FIG. 1. (a) Evolution of the condensate depletion for a condensate of size $\ell = 40$. The curves are plotted for $x \geqslant 0$ only, using that ρ_χ is an even function of x [see Eq. (43)]. As time passes, we observe an overall depletion increase, initially more pronounced at the condensate wall at $x = \ell/2$. (b) Depletion profile evolution for a system of size $\ell = 100$. We note that the bulk depletion increase is insensitive to the existence of the condensate walls for the time periods considered in the plots. Here and in the following plots, units are chosen such that we have the scalings $x = x[\xi_0]$ and $t = t[\xi_0^2]$ for the particle densities $\rho_i = \rho_i[1/\xi_0]$, with $i = \chi, \zeta$, and for the current density $J_\zeta = J_\zeta[1/\xi_0^2]$.

In Fig. 1 we plot selected depletion profiles for two system sizes $\ell = 40$ and 100, from which we observe that when the interactions are turned on, depletion increases from zero following a pattern such that far from the condensate walls the system is insensitive to the existence of the (Neumann) boundary conditions, as can be inferred from the comparison between the plots in Figs. 1(a) and 1(b). Furthermore, as times passes, depletion increases faster closer to the condensate walls for both system sizes, giving rise to a wave with growing amplitude in the depleted cloud, which propagates towards the condensate bulk. Moreover, still from Figs. 1(a) and 1(b), in addition to the observed depletion insensitivity on the system size far from the walls, it also follows that the depletion profiles for both sizes are similar close to the walls, as shown in Fig. 2(a) for several system sizes at fixed time $t = 5$.

From our observation that far from the system walls the depletion growth rate is rather insensitive to the system size [cf. both Figs. 1(a) and 1(b)], it follows that the growth timescale depends only on the chemical potential $\rho_0 g_0$ (remember t is in units of $1/\rho_0 g_0$). Moreover, we recall that the quantum depletion of an infinitely extended homogeneous 3D condensate in its ground state is proportional to $\sqrt{\rho_{3D}g_{3D}^3}$ [45,49]. However, no analogous formula exists for the 1D condensate due to the infrared depletion divergence [31], whereas for (inhomogeneous) finite-size condensates the depletion dependence on

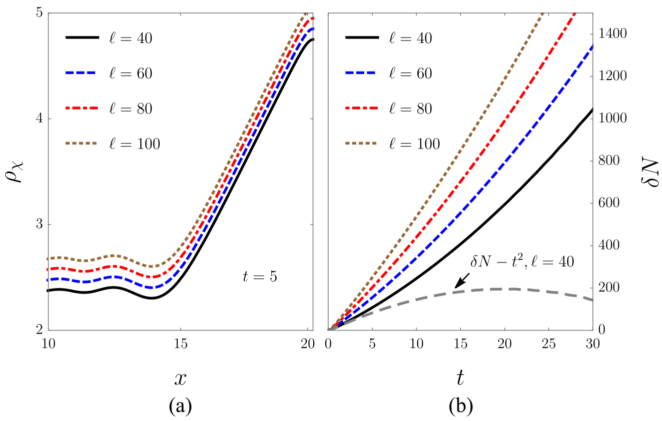

FIG. 2. (a) Depletion near-boundary behavior for several system sizes at $t = 5$. The profiles corresponding to larger condensates are translated to the left and slightly shifted to allow comparison with the smaller condensate profile. (b) Total number of particles in the depleted cloud as a function of time for several condensate sizes. Larger condensates correspond to faster growth of δN for fixed ρ_0. The long-dashed gray line depicts δN without the phase spreading contribution t^2 for $\ell = 40$ [cf. Eq. (44)], showing that δN is eventually dominated by the condensate phase degradation.

$\rho_0 g_0$ is necessarily model dependent. In our model, depletion is time dependent and we can obtain information regarding the depletion growth rate dependence on the condensate chemical potential $\rho_0 g_0$ far from the condensate walls for a fixed time duration. Indeed, Fig. 1(a) suggests that within the interval $1 \leqslant t \leqslant 5$ the depletion growth far from the condensate walls ($x \sim 0$) is fairly linear with t. Within our conventions, t is expressed in units of $\xi_0^2 = 1/\rho_0 g_0$ and thus to return to dimensionful time we must put $t \to \rho_0 g_0 t$. Hence, for a fixed parameter-independent time period, the quantum depletion in our model, after the interactions are turned on, is approximately linear in $\rho_0 g_0$, which should be compared with the 3D counterpart which is proportional to $\sqrt{\rho_{3D} g_{3D}^3}$.

Before solving the backreaction problem in the next section, we note an aspect of the quantum depletion which is of basic importance for our analysis, namely, the validity of the Bogoliubov expansion and consequently of the number-conserving expansion presented before Eqs. (3), expressed by the condition $\delta N/N \ll 1$, where δN is the number of depleted particles, which from Eq. (43) is found to be

$$\delta N = t^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}[1 - \cos(2\omega_n t)]. \tag{44}$$

We plot in Fig. 2(b) δN as a function of time for several system sizes. We recognize the first term in Eq. (44) to be the number of depleted particles due to the condensate phase spreading, which is independent of the condensate size. Also, we see from Fig. 2 that the depleted cloud is initially dominated by quasiparticle population and, as time passes, the condensate phase degradation becomes more relevant and eventually dominates the depleted cloud. We observe that for larger system sizes δN grows faster, which is expected as larger systems have more particles for a fixed ρ_0. Furthermore, for a system with size $\ell = 40$, we notice that for $t = 20$, $\delta N \sim 600$ (the timescale $\rho_0 g_0 t = 20$ can be arbitrarily large depending on the chemical potential $\rho_0 g_0$). By assuming a condensate with $N = 5000$ particles, we find $\delta N/N = 0.12$. For definiteness, in this work we assume $\delta N/N \lesssim 0.1$ as the validity regime for the field expansion in powers of N. Thus, for $\delta N/N = 0.12$ the results that follow from the expansion might accordingly not be reliable. We observe that for $N = 5000$, the results of Fig. 1 are within the expansion validity.

V. QUANTUM BACKREACTION

In this section we present the major results of our analysis, namely, the (up to the relevant order in N) exact solutions for the condensate corrections ρ_ζ and J_ζ in the interacting regime, i.e., for times $t > 0$. These quantities are determined by the coupled system of equations ($J_0 = J_\chi = 0$)

$$\partial_t \rho_\zeta + \partial_x J_\zeta = ig\big(\phi_0^2\langle\hat{\chi}^{\dagger 2}\rangle - \phi_0^{*2}\langle\hat{\chi}^2\rangle\big), \tag{45a}$$

$$\partial_t J_\zeta = f_{\rm cl} + f_{\rm q}, \tag{45b}$$

subject to the initial conditions $\rho_\zeta, J_\zeta = 0$ at $t = 0$. This initial-value problem then gives rise to the unique solution to the problem of how the condensate evolves during the unavoidable depleted cloud formation.

The system (45) plus initial conditions can be solved numerically with the aid of the correlations ρ_χ and $\langle\hat{\chi}^2\rangle$ calculated from the quantum field $\hat{\chi}$. Notwithstanding, for the particular condensate model adopted in our work, the exact solution for the backreaction problem can also be constructed by solving directly for the field ζ in Eq. (6) instead, from which the ρ_ζ and J_ζ can be determined. We describe in Appendix B the (rather cumbersome) construction of the solution for ζ: $\zeta = 0$ for $t < 0$ and the exact solution for $t \geqslant 0$ reads

$$e^{i\mu t}\sqrt{\rho_0}\zeta = -\frac{t^2}{2\ell} - \frac{1}{8\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\omega_n^2}\bigg\{2(-1)^n\bigg[1 - \cos 2\omega_n t + ik_n^2\bigg(\frac{\sin 2\omega_n t}{2\omega_n} - t\bigg)\bigg] + \cos 2k_n x\bigg[\frac{2 - k_n^2}{k_n^2 + 1} + 2\cos 2\omega_n t - \frac{4i\omega_n}{k_n^2}\sin 2\omega_n t - \frac{k_n^2 + 4}{k_n^2 + 1}\bigg(\cos\omega_{2n} t - \frac{i\omega_{2n}}{2k_n^2}\sin\omega_{2n} t\bigg)\bigg]\bigg\}, \tag{46}$$

which contains all the backreaction information within our number-conserving expansion to the relevant order. We now discuss separately the implications of ζ, as specified by Eq. (46).

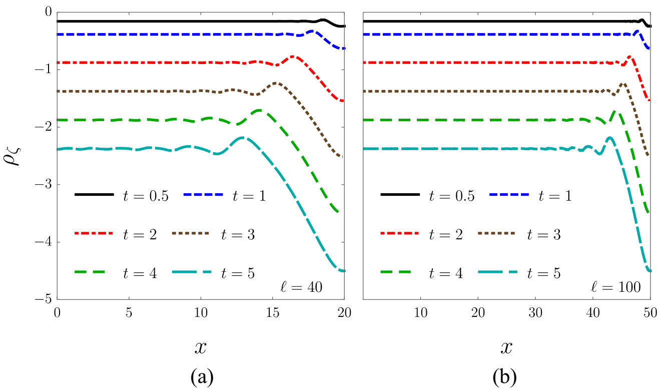

FIG. 3. (a) Evolution of the condensate correction ρ_ζ for a condensate of size $\ell = 40$. The curves are plotted for $x \geqslant 0$ as ρ_ζ is an even function of x [see Eq. (47)]. As time passes, we observe an overall depletion increase, initially more pronounced at the condensate wall at $x = \ell/2$. (b) Profile evolution of ρ_ζ for a system of size $\ell = 100$. We note that the condensate bulk corrections are insensitive to the existence of the condensate wall boundary region for the timescales considered in the plots.

A. Gas density

Let us start by studying the gas density $\rho = \rho_0 + \rho_\chi + \rho_\zeta + O(N^{-1/2})$. As the interactions are turned on, ρ_0 remains constant; ρ_χ, which models the evolution of the depleted gas cloud, was studied in the preceding section. Now, as a response to the evolution of ρ_χ dictated by the atom-number conservation, the condensate density $\rho_0 + \rho_\zeta$ is corrected by the function ρ_ζ, of the same order (N^0) as ρ_χ. We have from Eq. (12) that $\rho_\zeta = 2\,\mathrm{Re}[\exp(i\mu t)\sqrt{\rho_0}\zeta]$ and thus

$$\rho_\zeta = -\frac{t^2}{\ell} - \frac{1}{4\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\omega_n^2}\Bigg\{2(-1)^n[1-\cos(2\omega_n t)] + \cos(2k_n x)\Bigg[\frac{2-k_n^2}{1+k_n^2} + 2\cos(2\omega_n t) - \frac{k_n^2+4}{k_n^2+1}\cos(\omega_{2n} t)\Bigg]\Bigg\}. \tag{47}$$

We note first that ρ_ζ given by Eq. (47) is not proportional to ρ_χ and it is such that $\int dx \rho_\zeta = -\delta N$, where δN is the number of depleted particles given by Eq. (44). Therefore, we have $\int dx \rho = N$, and we verify that the total number of particles is indeed preserved up to order N^0.

We present in Fig. 3 selected plots for ρ_ζ. We notice that $\rho_\zeta < 0$ for all the profiles depicted in Fig. 3, indicating the decrease of the number of condensed atoms when depletion occurs. Furthermore, ρ_ζ shares some of the properties of ρ_χ discussed in Sec. IV, namely, the overall magnitude of ρ_ζ as a function of time for the timescales of Fig. 3 does not depend on the system size.

An important implication of our backreaction solution is linked to measurement processes in condensates. Let us consider, for instance, that an experiment is devised to determine quantum depletion in a condensate using the Bragg scattering technique of [45], where the authors explored the fact that for a particular condensate configuration the power spectrum of ρ_0 was exponentially suppressed in comparison to the power spectrum of ρ_χ. Thus a full density measurement is in principle capable of separating ρ_0 from ρ_χ. However, for a number-conserving analysis such as the one adopted in our model, density measurements can only detect the sum $\rho_\chi + \rho_\zeta$ [Fig. 4(a)] on top of the (in our model) constant background ρ_0. Therefore, if no feature exists in ρ_χ distinguishing it from ρ_ζ as it occurs for the system we consider (cf. Figs. 1 and 3 representing the buildup of ρ_χ and ρ_ζ, respectively), it is not possible to determine ρ_χ separately by the density power spectrum or in general via any measurement relying on an analysis of only the total density. As a corollary, our analysis also reveals that number conservation renders some measurement processes in condensates sensitive to the system initial condition, $\rho_\zeta = 0$ in our case. Indeed, a condensate determined at $t < 0$ by the uniform density order parameter ϕ_0 plus a nonzero ζ contribution of order $N^{-1/2}$, satisfying the field equations, necessarily presents a distinct ρ_ζ and *the same* ρ_χ after the interactions are turned on at $t = 0$.

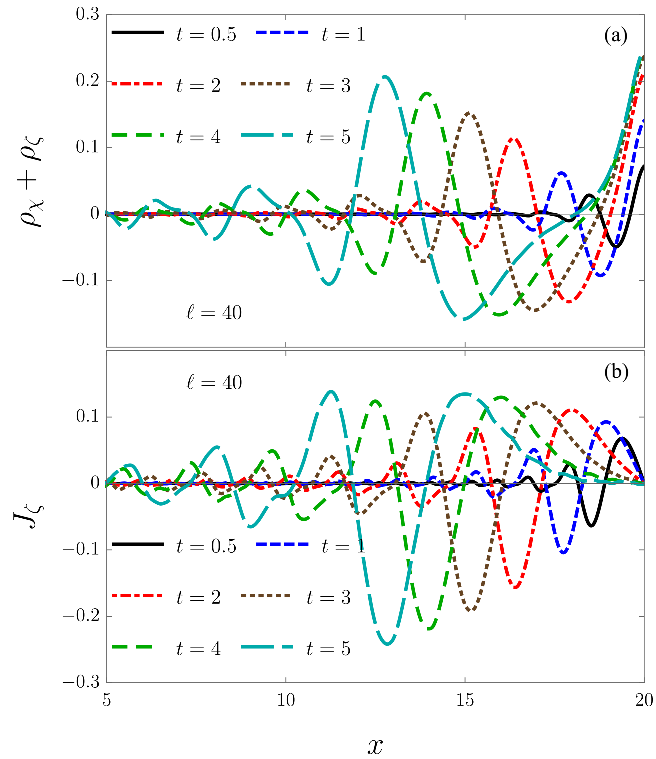

FIG. 4. (a) Evolution of the gas density on top of the condensate background ρ_0 for a system of size $\ell = 40$ and at several instants of time. These profiles represent the departure from a uniform density condensate profile as dictated by number-conserving backreaction effects. (b) Evolution of the condensate flux J_ζ for a condensate of size $\ell = 40$ and at several instants of time. The positive plot range $5 \leqslant x \leqslant 20$ is motivated by the fact that J_ζ is an odd function in view of Eq. (48). Note that the flux of particles vanishes at the condensate walls, reflecting the fact that the particles are indeed trapped inside the box.

In Fig. 4(a) we plot $\rho_\chi + \rho_\zeta$ for a condensate of size $\ell = 40$ for several instants of time. We observe that as time passes an oscillatory pattern emerges on top of the condensate density, which might be visible in the power spectrum of $\rho_\chi + \rho_\zeta$ and thus can be measured.

B. Induced condensate flow

In this section we discuss perhaps the most intriguing feature coming from the number conservation. As the interactions are turned on, even though there is no flux of depleted particles, the condensate particles undergo a nontrivial flow coming from the backreaction. We recall that the total flux of particles in the system is given by $J = J_0 + J_\chi + J_\zeta + O(N^{-1/2})$, and thus for our model, in which $J_0 = J_\chi = 0$, a condensate flow is modeled by a nonzero J_ζ. That J_ζ is necessarily nonzero follows directly from the number conservation of Eqs. (11) and $\rho_\chi + \rho_\zeta \neq 0$: $\partial_x J_\zeta = -\partial_t(\rho_\chi + \rho_\zeta) \neq 0$ and thus we must have $J_\zeta \neq 0$ during the condensate evolution. The particular form of J_ζ can be calculated with the aid of Eq. (13) and we find, using (46), that $J_\zeta = \mathrm{Im}[\phi_0^* \partial_x \zeta]$ is given by

$$J_\zeta = -\frac{2}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(2k_n x)}{k_n} \left[\frac{\sin(2\omega_n t)}{2\omega_n} - \frac{\sin(\omega_{2n} t)}{\omega_{2n}} \right]. \tag{48}$$

We present in Fig. 4(b) the evolution of the particle flux J_ζ for the condensate of size $\ell = 40$ presented in Fig. 4(a).

We note that although Eq. (48) was found from the exact solution of Eq. (46), as discussed above, the continuity equation, when $J_\chi = 0$, reads $\partial_x J_\zeta = -\partial_t(\rho_\chi + \rho_\zeta)$ and can be integrated to find J_ζ. Thus, number conservation implies that we can always find J_ζ in any model where $J_0 = J_\chi = 0$ and the full density is known, offering a route for determining the particle flux from density measurements. Note that for black hole analogs, for which $J_\chi \neq 0$, the determination of the condensate flux $J_0 + J_\zeta$, which, alongside the condensate density $\rho_0 + \rho_\zeta$, determines the effective metric, also requires the measurement of the phonon flux J_χ separately, showing how intricate it is to probe backreaction effects in such systems.

C. Quantum potential

In principle, the only quantized field in our condensate treatment is $\hat{\chi}$ and accordingly all the effects coming from $\hat{\chi}$ have a quantum origin. That includes ρ_ζ and J_ζ and justifies the backreaction nomenclature because in the absence of ρ_χ and $\langle \hat{\chi}^2 \rangle$, the field ζ vanishes; the noninteracting regime is an example of such a regime. Notwithstanding, as the depleted cloud evolves, the condensate response can be decomposed, according to the discussion of Sec. II C into a classical and a quantum part, which then enables us to identify the part of the force density which comes from quantum fluctuations, as defined in Eq. (15), and which we discuss now.

The total force density is defined by the derivative $\partial_t J = \partial_t J_\zeta$ (given that both J_0 and J_χ vanish) and we notice from Eq. (48) a mathematical issue with the backreaction analysis: The time derivative of J_ζ results in a slowly convergent series, preventing the numerical evaluation of the series. This means in particular that more quasiparticle modes are required in order to determine the quantum force. In order to circumvent this mathematical difficulty, we can explore the fact that the total force density $f_{\rm cl} + f_{\rm q}$ can be expressed as the gradient $\partial_t J_\zeta = -\partial_x V$, where

$$V = -\frac{1}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2k_n x)}{k_n^2} [\cos(2\omega_n t) - \cos(\omega_{2n} t)] \tag{49}$$

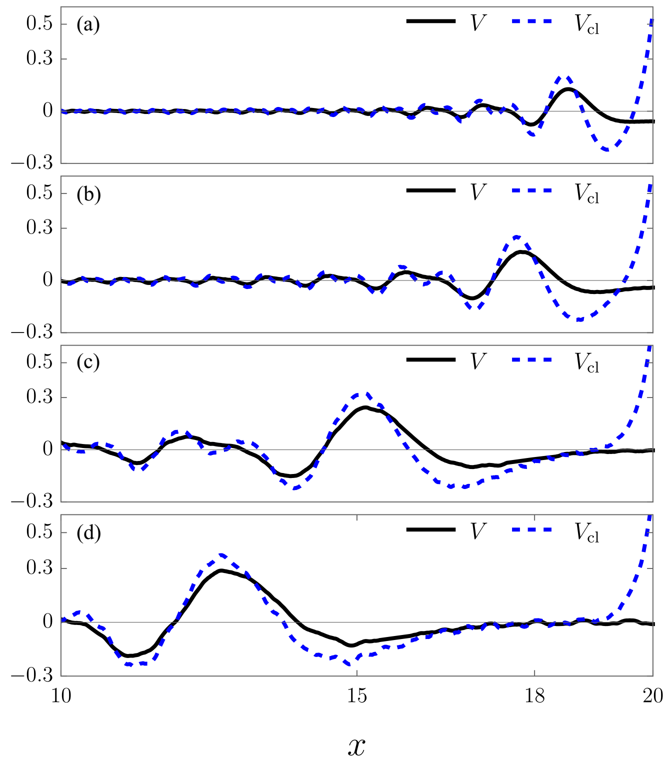

FIG. 5. Evolution of the total and classical potentials V and $V_{\rm cl}$, respectively, for a condensate of size $\ell = 40$ and at several instants of time: (a) $t = 0.5$, (b) $t = 1$, (c) $t = 3$, and (d) $t = 5$. The slopes of the curves represent the local force density exerted on the system particles. We note that for the considered time interval, the quantum force has the effect of attenuating the classical Eulerian force and that this attenuation is more pronounced near the condensate walls.

is, fortunately, a series with better convergence. Now it follows from Eq. (14) that for all condensates at rest the property $f_{\rm cl} = f_{\rm cl}(\rho, \partial_x\rho, \partial_x^2\rho, \partial_x^3\rho)$ holds, i.e., the classical force density depends solely on the particle density and its derivatives to our working order N^0, and here becomes

$$f_{\rm cl} = -\partial_x \left[\left(1 - \frac{\partial_x^2}{4} \right) (\rho_\chi + \rho_\zeta) \right]. \tag{50}$$

In particular, note that $f_{\rm cl}$ can be determined by (total) density measurements and it is a gradient $f_{\rm cl} = -\partial_x V_{\rm cl}$, where

$$V_{\rm cl} = \frac{1}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2k_n x)}{k_n^2} \times \left\{ \frac{\omega_{2n}^2}{4\omega_n^2} [1 - \cos(2\omega_n t)] - 1 + \cos(\omega_{2n} t) \right\}. \tag{51}$$

Hence, V represents the actual potential for the system particles, whereas $V_{\rm cl}$ is the expected potential if the particles were solely subjected to an Eulerian dynamics, therefore enabling the study of quantum force effects by comparing V and $V_{\rm cl}$, as presented in Fig. 5 for a condensate of size $\ell = 40$. We recall that the slopes in the plots of V and $V_{\rm cl}$ represent the total and classical force densities, respectively. Hence, from Fig. 5 we observe that the classical force is the dominant contribution on the condensate and the overall effect of the quantum force is the attenuation of $f_{\rm cl}$. In fact, this attenuation is even stronger near the condensate wall, where the observed force vanishes (no flux of particles) and the classical force is stronger, showing that near boundaries the quantum backreaction effects are more pronounced.

Concluding, because both J_ζ and the classical force are determined by the total density ρ for any condensate initially at rest, we anticipate the possibility of measuring the quantum force from density measurements alone via the route presented above.

VI. CONCLUSION

We have shown that the quantum force density deduced in [10], signaling a departure from Eulerian classical hydrodynamics due to quantum fluctuations, can be written solely in terms of quantities that have direct interpretation: the quantum depletion, the phonon flux, and the density fluctuations. Furthermore, we presented a discussion regarding the proper construction of the Cauchy problem for the backreaction analysis and its relation with condensate production at an experimental level, which revealed subtleties of the measurement of some theoretical predictions (e.g., the quantum depletion) imposed by the existence of condensates in states that meet the model assumptions, i.e., condensates in which ζ is initially known. Our model was then applied to a finite-size uniform density condensate, which represents an exactly soluble model and allows for a straightforward interpretation of the results.

As regards backreaction in BECs, the phenomenon of condensate phase diffusion plays a prominent role. Indeed, the existence of finite-size interacting quasi-1D condensates is associated with the continuous degradation of the system off-diagonal long-range order, which in particular implies that these quantum gases are not stationary systems. Accordingly, because the condensate spontaneously degrades, one has to add the initial configuration of the gas to the backreaction equations to study the system evolution. In order to circumvent this mathematical obstacle, we assumed that in our solved model the condensate was initially in a noninteracting regime, for which the system is indeed stationary and can be taken to reside in its ground state. By starting from this well-defined initial configuration, the required interacting regime can be accessed by driving the system out of equilibrium.

Among the consequences of the solutions of our model, we quote in particular that it highlights the problem of distinguishing condensate corrections from quantum depletion in measurements accessing properties of the total density (such as its power spectrum), as shown in Sec. V A. Furthermore, even though our uniform density condensate was initially at rest, backreaction from the quantum fluctuations gives rise to a condensate current dictated by number conservation, from which the total force density on the system was determined as a gradient of a potential function. Also, by explicitly computing the Eulerian force corresponding to the observed particle density $\rho = \rho_0 + \rho_\chi + \rho_\zeta$, it was possible to conclude that the quantum force density on the system particles attenuates the classical force, an effect more pronounced near the condensate walls.

We also call attention to the regime of validity of our number-conserving expansion: the smallness of the depleted cloud population, which imposes a timescale for the validity of the Bogoliubov expansion. The extension our analysis to longer times therefore requires the use of approximations that do not rely on the mean-field regime, the TWA being one possibility. In this sense, as our analysis provides an exact solution to the problem, it can be used to determine whether the TWA can be used to simulate the quantum force and other important quantities within the number-conserving backreaction scheme of [10]. Furthermore, an interesting continuation of our analysis can be obtained by exploring other sources of backreaction like the experiment-induce condensate phase spreading discussed in [38]. We believe that the determination of how the number-conserving backreaction scheme can be adapted to include measurement effects deserves a dedicated analysis.

An immediate application of the present approach is furnished by considering its consequences in analog gravity for, say, the backreaction of emitted Hawking radiation onto the condensate background [21,50]. Indeed, it was recently shown in [48] that the ramp-up of the Hawking radiation after an analog black hole is formed leads to a characteristic development of the depletion cloud outside the event horizon. This represents an important measurable quantity and indicates that nontrivial backreaction effects caused by the radiation on the condensate are expected to occur. Such an analysis in principle can be conducted via our procedure and can reveal novel radiation effects to directly measurable quantities. We also note in this connection that classical backreaction has already been studied experimentally in shallow water tanks (cf. Ref. [51]).

ACKNOWLEDGMENT

This work was supported by the National Research Foundation of Korea under Grants No. 2017R1A2A2A05001422 and No. 2020R1A2C2008103.

APPENDIX A: CANONICAL COMMUTATION RELATION

In this Appendix we show that the set $\{\Phi_n\}_{n=0}^\infty$ of field modes constructed in Sec. III D is complete, namely, the corresponding quantum field expansion satisfies Eq. (22) for $-\ell/2 < x, x' < \ell/2$. To this end, it is sufficient to show that $[\hat{\psi}(t,x), \hat{\psi}^\dagger(t,x')] = \delta(x-x')$ holds. By means of Eq. (28) we have

$$[\hat{\psi}(t,x), \hat{\psi}^\dagger(t,x')] = \sum_{n=0}^{\infty}[u_n(t,x)u_n^*(t,x') - v_n^*(t,x)v_n(t,x')]. \tag{A1}$$

Let us consider first the noninteracting regime $t < 0$. In this regime, the field modes are given by Eq. (33) and we find

$$[\hat{\psi}(t,x), \hat{\psi}^\dagger(t,x')] = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty}[e^{in\pi\Delta x/\ell} + (-1)^n e^{in\pi(x+x')/\ell}], \tag{A2}$$

where $\Delta x = x - x'$. We can write the equation above in terms of δ functions using Poisson's summation formula [52]

$$\frac{1}{2\ell}\sum_{n=-\infty}^{\infty} e^{in\pi y/\ell} = \sum_{n=-\infty}^{\infty}\delta(y-2\ell n), \tag{A3}$$

where $\delta(x) = (1/2\pi)\int dk \exp(ikx)$, from which we obtain

$$[\hat{\psi}(t,x), \hat{\psi}^\dagger(t,x')] = \sum_{n=-\infty}^{\infty}[\delta(\Delta x - 2\ell n) + \delta(x + x' - 2\ell n - \ell)]. \tag{A4}$$

By inspecting the right-hand side of the equation above we conclude that for $-\ell/2 < x$ and $x' < \ell/2$ all the δ functions give zero contribution except $\delta(\Delta x)$, thus verifying that the field is canonically quantized in the noninteracting regime.

In the interacting regime, the field modes assume the form in Eqs. (39) and (42) and straightforward manipulations lead, for $[\hat{\psi}(t,x), \hat{\psi}^\dagger(t,x')]$, to the identical equation (A2).

APPENDIX B: BUILDING THE CONDENSATE CORRECTIONS

In this Appendix we show how to derive the main result of our work, the solution of Eq. (6) in the interacting regime, presented in Eq. (46). We start by making the ansatz

$$\zeta(t,x) = \exp(-i\mu t) f(t,x)/\sqrt{\rho_0}, \tag{B1}$$

where f is a solution of

$$i\partial_t f = -\tfrac{1}{2}\partial_x^2 f + (f + f^*) + 2\rho_\chi + \langle\hat{\psi}^2\rangle, \tag{B2}$$

with $f = 0$ at $t = 0$ and $\partial_x f = 0$ at $x = \pm\ell/2$ for all times. Inspired by how the solutions of Eq. (20) pertaining to the BdG equation were built, we define the spinor $F = (f, f^*)^\mathsf{T}$, which is a solution of

$$i\partial_t\sigma_3 F = -\frac{1}{2}\partial_x^2 F + \sigma_4 F + \begin{pmatrix} 2\rho_\chi + \langle\hat{\psi}^2\rangle \\ 2\rho_\chi + \langle\hat{\psi}^{\dagger 2}\rangle\end{pmatrix}. \tag{B3}$$

Thus our goal is to solve Eq. (B3) subject to $F = 0$ at $t = 0$ and with the Neumann boundary conditions imposed on F.

It follows from the quantum field expansion of Eq. (28) for $t > 0$ that

$$2\rho_\chi + \langle\hat{\psi}^2\rangle = \frac{t^2 - it}{\ell} + \frac{1}{\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{k_n^2(k_n^2+4)} \times [(-1)^n + \cos(2k_n x)] \times \left\{\left(2 - k_n^2\right)[1 - \cos(2\omega_n t)] - 2i\omega_n\sin(2\omega_n t)\right\}. \tag{B4}$$

Note that from Eq. (B4) the quantity $2\rho_\chi + \langle\hat{\psi}^2\rangle$ is a sum over the various field mode contributions, and because Eq. (B3) is linear, F can be constructed by summing the solutions of Eq. (B3) for each term in Eq. (B4) (indexed by n) subjected to the initial and spatial boundary conditions. Accordingly, we write $F(t,x) = \sum_{n=0}^{\infty} F_n(t,x)$ and solve for each F_n such that $F_n(0,x) = 0$ and F_n satisfies Neumann boundary conditions at $x = \pm\ell/2$.

We now determine each F_n. For $n = 0$ the result is straightforwardly obtained, $F_0 = -(t^2/2\ell)(1,1)^\mathsf{T}$. For $n > 0$ we need to solve

$$\left(i\partial_t\sigma_3 + \frac{1}{2}\partial_x^2 - \sigma_4\right)F_n = \frac{(-1)^n[(-1)^n + \cos(2k_n x)]}{\ell k_n^2\left(k_n^2+4\right)}\left[\left(2 - k_n^2\right)\begin{pmatrix}1\\1\end{pmatrix} - e^{2i\omega_n t}\begin{pmatrix}\omega_n - k_n^2/2 + 1\\ -\omega_n - k_n^2/2 + 1\end{pmatrix} - e^{-2i\omega_n t}\begin{pmatrix}-\omega_n - k_n^2/2 + 1\\ \omega_n - k_n^2/2 + 1\end{pmatrix}\right]. \tag{B5}$$

Suppose that $\tilde{F}_n$ is a particular solution for the equation above. Then the required F_n must be of the form

$$F_n = a_0\Pi_0 + b_0\tilde{\Pi}_0 + \sum_{j=1}^{\infty}[a_j\Pi_j - b_j\sigma_1\Pi_j^*] + \tilde{F}_n, \tag{B6}$$

using that the set of field modes $\{\Pi_0, \tilde{\Pi}_0, \Pi_n, \sigma_1\Pi_n^*\}$ is complete as discussed in Sec. III D. The coefficients a_j and b_j are uniquely defined by the initial condition $F_n(0,x) = 0$, which leads to a Fourier expansion of $-\tilde{F}_n(0,x)$, in the exactly same fashion as presented in Sec. III D, namely,

$$a_0\Pi_0 + b_0\tilde{\Pi}_0 + \sum_{j=1}^{\infty}[a_j\Pi_j - b_j\sigma_1\Pi_j^*] = -\tilde{F}_n, \tag{B7}$$

where

$$a_0 = -\frac{\langle\tilde{\Pi}_0, \tilde{F}_n\rangle}{\langle\tilde{\Pi}_0, \Pi_0\rangle}, \quad a_j = -\langle\Pi_j, \tilde{F}_n\rangle,$$
$$b_0 = -\frac{\langle\Pi_0, \tilde{F}_n\rangle}{\langle\Pi_0, \tilde{\Pi}_0\rangle}, \quad b_j = -\langle\sigma_1\Pi_j^*, \tilde{F}_n\rangle, \tag{B8}$$

and all the functions are evaluated at $t = 0$. Moreover, because $\tilde{F}_n = \sigma_1\tilde{F}_n^*$, we should have $b_j = -\langle\sigma_1\Pi_j^*, \tilde{F}_n\rangle = b_j = \langle\Pi_j^*, \sigma_1\tilde{F}_n\rangle = \langle\Pi_j, \sigma_1\tilde{F}_n^*\rangle^* = \langle\Pi_j, \tilde{F}_n\rangle^* = -a_j^*$.

Therefore, the problem is reduced to the determination of the particular solution $\tilde{F}_n$, which can be obtained as follows. We first note that the right-hand side of Eq. (B5) is the sum of constant terms that depend on x or t only and terms that depend on both x and t. Thus one strategy for finding $\tilde{F}_n$ is to use the field equation's linearity and solve for each term separately, which can be done in a straightforward manner. We find that

$$\tilde{F}_n = -\frac{(-1)^n}{2\ell k_n^2\left(k_n^2+4\right)}\left\{(-1)^n\left[\left(2 - k_n^2\right)\begin{pmatrix}1\\1\end{pmatrix} - \frac{e^{2i\omega_n t}}{\omega_n}\begin{pmatrix}\omega_n - k_n^2/2\\ \omega_n + k_n^2/2\end{pmatrix} - \frac{e^{-2i\omega_n t}}{\omega_n}\begin{pmatrix}\omega_n + k_n^2/2\\ \omega_n - k_n^2/2\end{pmatrix}\right] + 2\cos(2k_n x)\left[\frac{2-k_n^2}{1+k_n^2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{e^{2i\omega_n t}}{k_n^2}\begin{pmatrix}-\omega_n + k_n^2/2\\ \omega_n + k_n^2/2\end{pmatrix} + \frac{e^{-2i\omega_n t}}{k_n^2}\begin{pmatrix}\omega_n + k_n^2/2\\ -\omega_n + k_n^2/2\end{pmatrix}\right]\right\} \tag{B9}$$

is a particular solution to Eq. (B5). Using this particular solution and the modes given by Eqs. (36)–(38), we find the set of coefficients (B8) in the expansion (B6):

$$a_0 = 0, \quad b_0 = -\frac{1}{\ell\left(k_n^2 + 4\right)}, \tag{B10}$$

$$a_j = -\delta_{j,2n}\frac{2(-1)^n\left(2 - \omega_{2n} + k_{2n}^2/2\right)}{\sqrt{2\ell\left[1 - \left(\omega_{2n} - k_{2n}^2/2 - 1\right)^2\right]\omega_{2n}^2}}, \tag{B11}$$

which give rise to the solution presented in Eq. (46).

[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[2] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1995).

[3] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).

[4] S. W. Hawking, Black hole explosions? Nature (London) **248**, 30 (1974).

[5] W. G. Unruh, Experimental Black-Hole Evaporation? Phys. Rev. Lett. **46**, 1351 (1981).

[6] M. Visser, Hawking Radiation without Black Hole Entropy, Phys. Rev. Lett. **80**, 3436 (1998).

[7] C. Barceló, S. Liberati, and M. Visser, Analogue gravity, Living Rev. Relativ. **14**, 3 (2011).

[8] J. R. Muñoz de Nova, K. Golubkov, V. I. Kolobov, and J. Steinhauer, Observation of thermal Hawking radiation and its temperature in an analogue black hole, Nature (London) **569**, 688 (2019).

[9] V. I. Kolobov, K. Golubkov, J. R. Muñoz de Nova, and J. Steinhauer, Observation of stationary spontaneous Hawking radiation and the time evolution of an analogue black hole, Nat. Phys. **17**, 362 (2021).

[10] R. Schützhold, M. Uhlmann, Y. Xu, and U. R. Fischer, Quantum backreaction in dilute Bose-Einstein condensates, Phys. Rev. D **72**, 105005 (2005).

[11] U. R. Fischer, in *Quantum Analogues: From Phase Transitions to Black Holes and Cosmology*, edited by W. G. Unruh and R. Schützhold (Springer, Berlin, 2007), pp. 93–113.

[12] M. Girardeau and R. Arnowitt, Theory of many-boson systems: Pair theory, Phys. Rev. **113**, 755 (1959).

[13] M. D. Girardeau, Comment on "Particle-number-conserving Bogoliubov method which demonstrates the validity of the time-dependent Gross-Pitaevskii equation for a highly condensed Bose gas", Phys. Rev. A **58**, 775 (1998).

[14] C. W. Gardiner, Particle-number-conserving Bogoliubov method which demonstrates the validity of the time-dependent Gross-Pitaevskii equation for a highly condensed Bose gas, Phys. Rev. A **56**, 1414 (1997).

[15] Y. Castin and R. Dum, Low-temperature Bose-Einstein condensates in time-dependent traps: Beyond the $U(1)$ symmetry-breaking approach, Phys. Rev. A **57**, 3008 (1998).

[16] S. A. Gardiner and S. A. Morgan, Number-conserving approach to a minimal self-consistent treatment of condensate and noncondensate dynamics in a degenerate Bose gas, Phys. Rev. A **75**, 043621 (2007).

[17] T. P. Billam and S. A. Gardiner, Coherence and instability in a driven Bose–Einstein condensate: A fully dynamical number-conserving approach, New J. Phys. **14**, 013038 (2012).

[18] S. Ripka, J. Blaizot, and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, Cambridge, 1986).

[19] T. Kita, Self-consistent perturbation expansion for Bose-Einstein condensates satisfying Goldstone's theorem and conservation laws, Phys. Rev. B **80**, 214502 (2009).

[20] Y. Kurita, M. Kobayashi, H. Ishihara, and M. Tsubota, Particle creation in Bose-Einstein condensates: Theoretical formulation based on conserving gapless mean-field theory, Phys. Rev. A **82**, 053602 (2010).

[21] R. Balbinot, S. Fagnocchi, and A. Fabbri, Quantum effects in acoustic black holes: The backreaction, Phys. Rev. D **71**, 064019 (2005).

[22] R. Schützhold, Quantum back-reaction problems, in *Proceedings of From Quantum to Emergent Gravity: Theory and Phenomenology — PoS(QG-Ph)*, Vol. 043 (2008) p. 036.

[23] R. M. Wald, Axiomatic renormalization of the stress tensor of a conformally invariant field in conformally flat spacetimes, Ann. Phys. (NY) **110**, 472 (1978).

[24] A. L. Gaunt, T. F. Schmidutz, I. Gotlibovych, R. P. Smith, and Z. Hadzibabic, Bose-Einstein Condensation of Atoms in a Uniform Potential, Phys. Rev. Lett. **110**, 200406 (2013).

[25] N. Navon, R. P. Smith, and Z. Hadzibabic, Quantum gases in optical boxes, Nat. Phys. **17**, 1334 (2021).

[26] A. J. Leggett and F. Sols, On the concept of spontaneously broken gauge symmetry in condensed matter physics, Found. Phys. **21**, 353 (1991).

[27] A. Imamoğlu, M. Lewenstein, and L. You, Inhibition of Coherence in Trapped Bose-Einstein Condensates, Phys. Rev. Lett. **78**, 2511 (1997).

[28] M. Lewenstein and L. You, Quantum Phase Diffusion of a Bose-Einstein Condensate, Phys. Rev. Lett. **77**, 3489 (1996).

[29] J. Javanainen and M. Wilkens, Phase and Phase Diffusion of a Split Bose-Einstein Condensate, Phys. Rev. Lett. **78**, 4675 (1997).

[30] A. J. Leggett and F. Sols, Comment on "Phase and Phase Diffusion of a Split Bose-Einstein Condensate", Phys. Rev. Lett. **81**, 1344 (1998).

[31] P. C. Hohenberg, Existence of long-range order in one and two dimensions, Phys. Rev. **158**, 383 (1967).

[32] E. A. Donley, N. R. Claussen, S. L. Cornish, J. L. Roberts, E. A. Cornell, and C. E. Wieman, Dynamics of collapsing and exploding Bose–Einstein condensates, Nature (London) **412**, 295 (2001).

[33] Y.-H. Wang, T. Jacobson, M. Edwards, and C. W. Clark, Induced density correlations in a sonic black hole condensate, SciPost Phys. **3**, 022 (2017).

[34] A. Sinatra, C. Lobo, and Y. Castin, Classical-Field Method for Time Dependent Bose-Einstein Condensed Gases, Phys. Rev. Lett. **87**, 210404 (2001).

[35] A. Sinatra, C. Lobo, and Y. Castin, The truncated Wigner method for Bose-condensed gases: limits of validity and applications, J. Phys. B **35**, 3599 (2002).

[36] I. Carusotto, S. Fagnocchi, A. Recati, R. Balbinot, and A. Fabbri, Numerical observation of Hawking radiation from acoustic black holes in atomic Bose–Einstein condensates, New J. Phys. **10**, 103001 (2008).

[37] J. F. Corney and G. J. Milburn, Homodyne measurements on a Bose-Einstein condensate, Phys. Rev. A **58**, 2399 (1998).

[38] D. A. R. Dalvit, J. Dziarmaga, and R. Onofrio, Measurement-induced squeezing of a Bose-Einstein condensate, Phys. Rev. A **65**, 033620 (2002).

[39] R. Schützhold, M. Uhlmann, Y. Xu, and U. R. Fischer, Mean-field expansion in Bose-Einstein condensates with finite-range interactions, Int. J. Mod. Phys. B **20**, 3555 (2006).

[40] E. H. Lieb, R. Seiringer, and J. Yngvason, Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional, Phys. Rev. A **61**, 043602 (2000).

[41] E. H. Lieb and R. Seiringer, Proof of Bose-Einstein Condensation for Dilute Trapped Gases, Phys. Rev. Lett. **88**, 170409 (2002).

[42] J. Esteve, J.-B. Trebbia, T. Schumm, A. Aspect, C. I. Westbrook, and I. Bouchoule, Observations of Density Fluctuations in an Elongated Bose Gas: Ideal Gas and Quasicondensate Regimes, Phys. Rev. Lett. **96**, 130403 (2006).

[43] M. McDonald, J. Trisnadi, K.-X. Yao, and C. Chin, Super-resolution Microscopy of Cold Atoms in an Optical Lattice, Phys. Rev. X **9**, 021001 (2019).

[44] S. Subhankar, Y. Wang, T.-C. Tsui, S. L. Rolston, and J. V. Porto, Nanoscale Atomic Density Microscopy, Phys. Rev. X **9**, 021002 (2019).

[45] R. Lopes, C. Eigen, N. Navon, D. Clément, R. P. Smith, and Z. Hadzibabic, Quantum Depletion of a Homogeneous Bose-Einstein Condensate, Phys. Rev. Lett. **119**, 190404 (2017).

[46] Y. Chen, M. Horikoshi, M. Kuwata-Gonokami, and K. Yoshioka, Creating a noninteracting Bose gas in equilibrium at finite temperature, Phys. Rev. A **105**, 043312 (2022).

[47] J. Curtis, G. Refael, and V. Galitski, Evanescent modes and step-like acoustic black holes, Ann. Phys. (NY) **407**, 148 (2019).

[48] C. C. Holanda Ribeiro, S.-S. Baak, and U. R. Fischer, Existence of steady-state black hole analogs in finite quasi-one-dimensional Bose-Einstein condensates, Phys. Rev. D **105**, 124066 (2022).

[49] T. D. Lee, K. Huang, and C. N. Yang, Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties, Phys. Rev. **106**, 1135 (1957).

[50] S. Liberati, G. Tricella, and A. Trombettoni, Back-reaction in canonical analogue black holes, Appl. Sci. **10**, 8868 (2020).

[51] S. Patrick, H. Goodhew, C. Gooding, and S. Weinfurtner, Back-reaction in an Analogue Black Hole Experiment, Phys. Rev. Lett. **126**, 041105 (2021).

[52] V. A. De Lorenci and E. S. Moreira, Renormalized scalar propagator around a dispiration, Phys. Rev. D **67**, 124002 (2003).