Effects of gravity in extra dimensions in atomic phenomena

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We use the difference between theory and experiment for energy intervals in simple atomic systems (hydrogen, muonium, positronium, and deuteron) to find limits on the size of extra space dimensions in the Arkani-Hamed–Dimopoulos–Dvali model for gravitational potential on short distances. As an additional experimental fact we use the absence of the small size gravitational bound states of elementary particles. We demonstrate that the perturbation theory approach does not work and more reliable results are obtained by solving the Dirac equations for an electron in Coulomb and gravitational fields. These results probe smaller distances than the distance between nuclei in molecules and the limits are significantly stronger than the limits on the size of extra dimensions obtained using the spectra of hydrogen molecules.

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I. INTRODUCTION

Observations indicate that our universe has three noncompact spatial dimensions, with all forces and particles operating inside these dimensions. However, there are popular theoretical models with extra spatial dimensions such as string theories—see, e.g., Ref. [1]. These models have been motivated by the search for a theory unifying all interactions and producing finite results which do not require hiding infinities using the renormalization procedures.

In this paper, we examine the Arkani-Hamed–Dimopoulos–Dvali (ADD) model [2] which aimed at solving the hierarchy problem by proposing that gravity can propagate through n extra spatial compact dimensions as well as the regular three noncompact dimensions. An ordinary Newtonian gravitational interaction between elementary particles is many orders of magnitude smaller than other interactions. In the ADD model the observed Newton gravitational law in our three dimensions will be significantly strengthened at a distance smaller than the size of the extra dimensions R. The model also introduces a higher-dimensional Planck mass M that is related to R and the observed three-dimensional Planck mass $M_{\rm Pl}$. The Newtonian gravitational potential will change from $1/r$ to a more singular $1/r^{n+1}$ dependence at distances smaller than the size of the extra dimensions, R. This may be easily explained by the Gauss integrated flux formula for the gravitational force since the size of the "surface" in $(2+n)$ dimensions [$(2+n)$D] is proportional to r^{2+n} [2] (see also the Randall-Sundrum multidimensional models [3,4] which have a similar small-distance gravity). The search for macroscopic effects has not found any deviation from Newton law. The gravity force has been observed to obey the inverse-square law down to the μm scale [5] (see also Refs. [6–10]). LHC searches for extra dimensions look for signs of graviton emission, with Refs. [11,12] obtaining constraints on the higher-dimensional Planck mass from $M_2 > 10$ TeV to $M_6 > 5$ TeV, where M_n is the Planck mass for n extra dimensions.

References [13,14] proposed using atomic spectroscopy to search for extra dimensions and proceeded to calculate atomic energy shifts by treating the gravitational potential as a small perturbation. While the works did not place constraints, they had chosen the proton radius as a cutoff parameter to obtain a finite energy shift produced by the gravitational potential g/r^{n+1} for $n > 1$. With this cutoff, the interaction between the electron and quark inside the proton has been excluded without justification. Reference [15] used Bethe's nonrelativistic treatment to calculate the radiative correction to the energy shift produced by the gravitational potential, with the proton radius cutoff for hydrogen and a 10^{-17} m cutoff for muonium; constraints obtained from $1s$-$2s$ spectroscopy range from $R_3 \sim 10\ \mu$m to $R_6 \sim 0.01$ nm, where R_n is the radius of n extra dimensions. Reference [16] attempted to solve the proton radius puzzle with extra dimensions and used perturbation theory for a range of cutoff parameters; these constraints were comparable to Ref. [15]. However, all these results strongly depend on an unknown (and practically arbitrary) cutoff parameter. Note that there are several other works, for example, Refs. [17,18], that investigate gravitational effects with additional modifications of the model such as the brane thickness.

The cutoff problem has been avoided in Ref. [19] where the authors studied the effects of the potential g/r^{n+1} between nuclei in H_2, D_2, and HD^+ molecules. Since nuclei in molecules are separated by distances exceeding the Bohr radius a_B, there is no need for a cutoff parameter here. However, the effects of the potential g/r^{n+1} are much bigger for subatomic distances which give the main contribution to the energy shift in atoms.

In this paper we want to solve the cutoff problem using the absence of small size bound states between two elementary particles which could be produced by the singular potential g/r^{n+1} if the cutoff parameter is too small. Indeed, the highly singular nature of the ADD gravitational potential introduces a fall-to-center problem, where there must exist some new physics mechanism that cuts off the gravitational potential below some cutoff radius r_c. To solve the cutoff problem empirically, we will use simultaneously two experimental facts: The absence of the collapsed gravitational bound states of elementary particles and the difference between the measured and calculated (using QED) transition energy. In this way we can get conservative limits on the size of the extra dimensions by excluding the area of unknown physics at distances smaller than the electroweak scale.

II. OVERVIEW OF THE ADD MODEL

The ADD model was first proposed in Ref. [2] and introduced n extra spatial compactified dimensions of size R. There is also a change to the Planck mass, where a higher-dimensional Planck mass M is defined with respect to R and replaces the observed three-dimensional Planck mass M_{Pl},

$$\left(\frac{cM_{\mathrm{Pl}}}{\hbar}\right)^2 = R^n\left(\frac{cM}{\hbar}\right)^{n+2}. \quad (1)$$

This means that Gauss's law can be applied to the gravitational potential for two masses m_1 and m_2 separated by a distance r,

$$V(r) = -\frac{Gm_1m_2}{r} \quad \text{for } r \gg R,$$
$$-\frac{Gm_1m_2R^n}{r^{n+1}} = -\frac{\hbar^{n+1}}{c^{n-1}M^{n+2}}\frac{m_1m_2}{r^{n+1}} \quad \text{for } r \ll R, \quad (2)$$

where $G = 6.674 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the regular gravitational constant. The long-range component of the potential matches the regular Newtonian gravitational potential that is known to be accurate from μm scales to astronomical scales. However, the short-range component has a more singular form, therefore this is the region where the gravitational potential can become significantly stronger with extra dimensions.

For atomic calculations it is convenient to present a relation between the Planck mass M for $4 + n$ dimensions and the radius of extra dimensions R in the following form:

$$Mc^2 = (1.22 \times 10^{19})^{\frac{2}{n+2}}10^{-\frac{6n}{n+2}}\left(\frac{3.73a_B}{R}\right)^{\frac{n}{n+2}}\ \mathrm{GeV}. \quad (3)$$

We introduce the dimensionless variable S to describe the ratio of the three-dimensional gravitational and Coulomb potential

$$S \equiv \frac{Gm_1m_2}{e^2} = \frac{m_1m_2}{\alpha M_{\mathrm{Pl}}^2}, \quad (4)$$

and therefore the short-range ADD potential can be written as

$$V(r) = -\frac{\hbar c\alpha SR^n}{r^{n+1}}. \quad (5)$$

We can also consider the gravity-Coulomb boundary

$$\frac{\hbar c\alpha SR^n}{r_{gc}^{n+1}} = \frac{e^2}{r_{gc}}, \quad (6)$$

which provides the cutoff radius $r_{gc} = S^{1/n}R$ for the applicability of the perturbation theory for $n > 1$. Indeed, for $n > 2$ the matrix elements of $V(r) = g/r^{n+1}$ are dominated by the small distance integral near the cutoff radius r_c, $\int_b d^3r/r^{n+1} \sim 1/r_c^{n-2}$. For $n = 2$ the divergence is logarithmic. We can present the correction to energy exactly as

$$\delta E = \langle\psi|H|\psi\rangle - \langle\psi_0|H_0|\psi_0\rangle$$
$$= \langle\psi_0|H - H_0|\psi_0\rangle + \langle\psi|H|\psi\rangle - \langle\psi_0|H|\psi_0\rangle, \quad (7)$$

where $H - H_0 = V(r)$ is the gravitational potential. Since in the area $r < r_{gc}$ the correction to the wave function is not small, $\psi - \psi_0 \gg \psi_0$ for $r \ll r_{gc}$, the first-order perturbation theory result is incorrect even if δE is small. Indeed, in this case the last two terms in Eq. (7) are bigger than the first term presenting the first-order correction to the energy. The correction to energy δE is small since the volume occupied by the perturbation $V(r)$ is small. However, inside this volume the perturbation is much bigger than the Coulomb potential and the correction to the wave function is very large.

One may say that the distance r_{gc} may be associated with the scale where all interactions have a comparable strength.

III. THE PERTURBATION APPROACH

As mentioned, we expect that perturbation theory for the potential $V(r) = g/r^{n+1}$ is applicable for $n = 1$ and may be not applicable for $n > 1$. To make a link to ADD theory, we should find estimates for the size of the extra dimensions R_n, which determines the area of the highly singular gravitational potential, $V(r) = g/r^{n+1}$ for $r < R_n$. We will do these first estimates using experimental data from muonium, postronium, and hydrogen spectra and perturbation theory.

For $n = 1$, we use the known expectation values of $1/r^2$ potentials [20] and obtain the energy shift produced by the potential Eq. (5),

$$\delta E_1 = \frac{\hbar c\alpha SRZ^2}{n_p^3\left(l + \frac{1}{2}\right)a^2}, \quad (8)$$

where a is the reduced atomic Bohr radius, Z is the nuclear charge (in units of proton electric charge e), n_p is the principal quantum number, and l is the angular momentum quantum number.

For $n = 2$ and $n \geqslant 3$, we estimate leading terms enhanced by the small cutoff parameter r_c in the first-order perturbation energy shift for s orbitals:

$$\delta E_2 = \frac{4\hbar c\alpha SR^2Z^3}{n_p^3a^3}\ln\left(\frac{a}{r_c}\right), \quad (9)$$

$$\delta E_{n\geqslant 3} = \frac{4\hbar c\alpha SR^nZ^3}{(n-2)n_p^3a^3r_c^{n-2}}. \quad (10)$$

Note that for p orbitals, the energy shift for $n > 1$ is negligible compared to the s-wave shift. We consider two different choices for the cutoff parameter r_c. First, we consider the point r_{gc} where the gravitational and Coulomb potentials are equal. This choice corresponds to the boundary of applicability of the perturbation theory. Defining the variable $\lambda \equiv \frac{1}{n_i^3} - \frac{1}{n_j^3}$ as the inverse cubed principal quantum number difference of the two s-wave states, we determine the size for any extra dimensions R_n for a given energy shift from gravity for a transition between s orbitals $n_i \rightarrow n_f$; for the $2s$-$2p_{1/2}$ transition we should

TABLE I. Input parameters for simple systems. All systems have an atomic charge $Z = 1$ and reduced Bohr radius a_B (with the exception of positronium with $2a_B$).

System	$\Delta E = E_{\rm expt} - E_{\rm theor}$	Max. ΔE (eV)	Ref.
H 1s-2s	−0.9 (5.4) kHz	2.2×10^{-11}	[25]
H 1s-3s	1.2 (4.1) kHz	2.2×10^{-11}	[26]
Mu 1s-2s	5.6 (9.9) MHz	6.4×10^{-8}	[27]
Mu 2s-2$p_{1/2}$	0.084 (2.500) MHz	1.08×10^{-8}	[28]
Ps 1s-2s	−5.78 (3.5) MHz	1.3×10^{-8}	[29–31]

take $\lambda = 1/8$. Assuming a cutoff radius $r_c = r_{gc} = S^{1/n}R$ from equality of the gravitational and Coulomb potentials, we obtain

$$R_1 = \frac{a^2\delta E}{2\hbar c\alpha S Z^2\lambda}, \tag{11}$$

$$R_2 = \sqrt{\frac{a^3\delta E}{4\hbar c\alpha S Z^3\lambda\ln(a/S^{1/n}R_2)}}, \tag{12}$$

$$R_{n\geqslant 3} = \sqrt{\frac{a^3(n-2)\delta E}{4\hbar c\alpha S^{2/n}Z^3\lambda}}. \tag{13}$$

Note for $n = 2$, R_2 must be obtained using iterations; all other R_n can be trivially calculated.

To have inputs for δE, we consider the maximal energy shift between experimental and theoretical energy levels in simple systems. The systems were chosen due to their small deviation between theoretical and experimental results, which is presented in Table I. Spectroscopy results for the hydrogen and muonium Lamb shifts are new, while muonium 1s-2s and positronium measurements are older; see Refs. [21,22] for recent reviews in muonium and positronium spectroscopy. Significant improvements for muonium and positronium 1s-2s spectroscopy are expected in the near future [23,24].

In our calculations, we use the S values for each system: $S_{ee} = 2.4 \times 10^{-43}$ for positronium and $S_{e\mu} = 4.9 \times 10^{-41}$ for muonium. For hydrogen, stable gravitational bound states could be formed between an electron and pointlike quark. However, when we calculate the gravitational energy shift, the total proton mass density defines the strength of the gravitational potential. For $n > 2$, the quark-electron potential $V(r) = g/r^{n+1}$ with a cutoff r_c, which is very small on an atomic scale, is practically equivalent to the contact potential $V(r) = C\delta(r)$, where $C = \int V(r)d^3r$. For a finite-size proton or nucleus this approximation gives a potential proportional to the mass density $m\rho(r)$, where $\int \rho(r)d^3r = 1$. In light atoms, the s-wave electron wave function tends to be constant at $r \to 0$ and the finite size of the nucleus has no effect, i.e., we may take $V(r) = C\delta(r)$ for $n > 2$.

For $n = 1$ the effect of the potential $V(r) = g/r^2$ is not sensitive to both the cutoff radius and nuclear size. For $n = 2$ there is only a very weak logarithmic sensitivity to these parameters. Thus we may conclude that the formulas presented above may be used to estimate the energy shift for a hydrogen atom with $S_{ep} = 4.4 \times 10^{-40}$. Use of the proton mass (instead of the quark mass) for the estimate of the cutoff parameter overestimates r_c and makes the limits on R weaker but more reliable.

A smaller value of the cutoff distance r_c comes from the condition of the absence of the gravitational bound states of elementary particles (since they have not been observed experimentally). The estimate follows from the equality of the relativistic kinetic energy $\hbar c/r$ and gravitational energy Eq. (2) for weakly bound states. It leads to

$$(r_c/R)^n \sim m_1 m_2/M_{\rm Pl}^2, \tag{14}$$

which gives

$$r_c \sim \alpha^{1/n} r_{gc}. \tag{15}$$

This choice leads to stronger constraints on the size of the extra dimensions (see Table IV). Note that we still use the nonrelativistic expression for the gravitational interaction (2). The estimation of the role of the relativistic effects is presented in the Appendix. Relativistic corrections increase the gravitational interaction and make the constraint on the radius of extra dimensions R stronger.

A. Deuteron binding energy

The wave function of the deuteron may be found using the short-range character of the strong interaction and the relatively small binding energy of the deuteron. Outside the interaction range, we use the solution to the Schrödinger equation for zero potential. Within the interaction range $r_0 = 1.2$ fm, the wave function has a constant value for the s orbital,

$$\psi(r) = \begin{cases} \frac{Be^{-\kappa r}}{r} & \text{for } r > r_0, \\ \frac{B\,J(0)}{r_0} & \text{for } r < r_0, \end{cases} \tag{16}$$

where the normalization constant B is given by $4\pi B^2 = 2\kappa$ for $\kappa = \sqrt{2m|E|} = 4.56 \times 10^7$ eV (reduced mass $m = m_p/2$ and binding energy $|E| = 2.22$ MeV). The Jastrow factor, $J(0) = 0.4$ [32], is included to account for the nucleon repulsion at a short distance. For $n = 1$ we obtain

$$\delta E_1 = -\frac{2\kappa R_1 \alpha S_{pn}}{r_0}, \tag{17}$$

where $S_{pn} = 0.81 \times 10^{-36}$, $\alpha S_{pn} = 0.59 \times 10^{-38}$. Following Ref. [33] we take the difference between experimental [34] and theoretical [35] results as $E_{\rm expt} - E_{\rm theor} = -13.7$ eV. This gives $R_1 < 3 \times 10^{16}$ m.

For $n > 1$ the quark-quark gravitational potential $V(r) = -g/r^{n+1}$ is more singular than the strong interaction, so the latter may be neglected in the area $r \sim r_c$. The cutoff radius r_c is very small on a nuclear scale, so the gravitational interaction may be approximated by the contact potential $V(r) = -C\delta(r)$, where $C = -\int V(r)d^3r$. This contact interaction is similar to the weak interaction mediated by the Z boson. From the results of the weak effects calculation we know that the finite size of the nucleons produces a suppression of the effects described by the Jastrow factor $J(0)$ [32]. After accounting for this factor we may assume that nucleons are pointlike particles with an interaction $V(r) = -C\delta(r)$.

TABLE II. Parameters of extra dimensions obtained from the deuteron data. n is the number of extra dimensions, R is the size of extra dimensions, r_c is the cutoff parameter, and M is the Planck mass for extra dimensions (3). Numbers in square brackets mean powers of ten.

n	R (m)	r_c (m)	Mc^2 (GeV)
2	5.1	4.5[−19]	22
3	4.8[−06]	9.6[−19]	25
4	3.8[−09]	1.1[−18]	32
5	5.2[−11]	1.2[−18]	38
6	3.0[−12]	1.3[−18]	43
7	4.0[−13]	1.4[−18]	47

Using the wave function in Eq. (16) we obtain the following estimate for the energy shift in the case of $n > 1$,

$$\delta E_n = \langle\psi| - C_n\delta(\mathbf{r})|\psi\rangle = -\frac{C_n\kappa J(0)^2}{2\pi r_0^2}$$
$$= -C_n 0.8 \times 10^{36}\,\frac{\text{eV}}{\text{m}^2}, \quad (18)$$

where for $n = 2$,

$$C_2 = 4\pi\alpha S_{pn} R_2^2 \ln\frac{r_0}{r_c}, \quad (19)$$

and for $n > 2$,

$$C_n = \frac{4\pi\alpha S_{pn} R_n^2}{(n-2)}\left(\frac{R_n}{r_c}\right)^{n-2}. \quad (20)$$

We can estimate the cutoff radius r_c from the condition of the absence of the small size gravitational bound states Eq. (14). Note that we want to exclude the gravitational bound state for two quarks while all quarks contribute to the deuteron binding energy. To have a conservative estimate of the limits on R and M, we use a constituent quark mass $m_q = 300$ MeV in Eq. (14) (mass $m_q \sim 5$ MeV would give a smaller r_c, bigger energy shifts, and stronger limits). Using $E_{\text{expt}} - E_{\text{theor}} = -13.7$ eV we obtain estimates for R_n and M_n presented in Table II, and Figs. 1 and 2. Formally, these estimates appear to be the strongest constraints among two-body systems. However, deuteron is a system with a strong interaction and these constraints are probably less reliable than the constraints from hydrogen, muonium, and positronium. In any case, constraints from deuteron should be treated as order-of-magnitude estimates.

IV. NUMERICAL CALCULATIONS

We performed the numerical calculations of the energy shifts caused by the gravitation potential in order to find limits on the size of extra dimensions R. We do this in two ways. First, we use the perturbation theory (PT) approach described in the previous section. Second, we solve Dirac equations for the $1s$, $2s$, $2p_{1/2}$, $3s$ states of hydrogen, muonium, and positronium. The energy shift is found as the difference of energies calculated in the Coulomb potential and in the potential with an added gravitational contribution. The size of extra dimensions R is used in this approach as a fitting parameter to fit the input energy shifts from Table I (we call these shifts "experimental" shifts and use the notation ΔE_{expt} for them). The calculations with Dirac equations are done to check whether PT actually works. Note that a small value of the correction to the energy does not necessarily mean that PT is applicable. The correction is small because it comes from a very small region in the vicinity of the cutoff radius. However, in this region the change of the potential is not small, leading to a large change in the wave function and the breaking of PT.

We perform the calculations using two ways of defining the cutoff parameter r_c [$V_g(r) = V_g(r_c)$ for $r < r_c$]. Using $r_c = r_{gc}$ [see Eq. (6)] leads to conservative estimates of the size of extra dimensions which are in agreement with the perturbation theory calculations. The corresponding results are presented in Table III. In another approach we use a much smaller value of r_c which we find from the condition of the absence of the small size gravitational bound states of elementary particles.

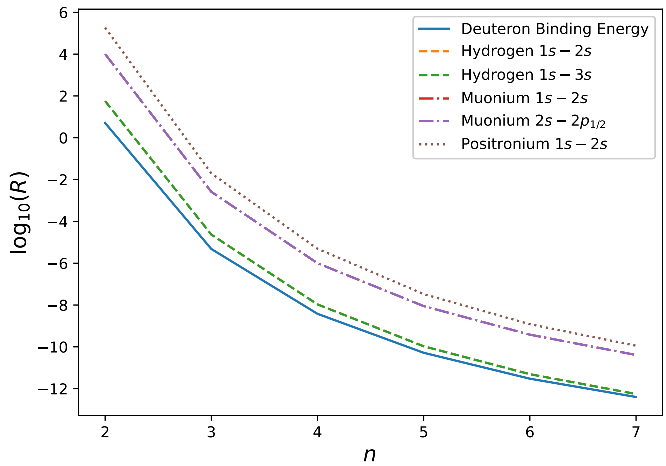

FIG. 1. Upper limits on the size of extra dimension R for varying number of extra dimensions n based on Tables II and IV. Transitions $1s$-$2s$ and $1s$-$3s$ in hydrogen as well as $1s$-$2s$ and $2p_{1/2}$-$2s$ in muonium give practically the same curves.

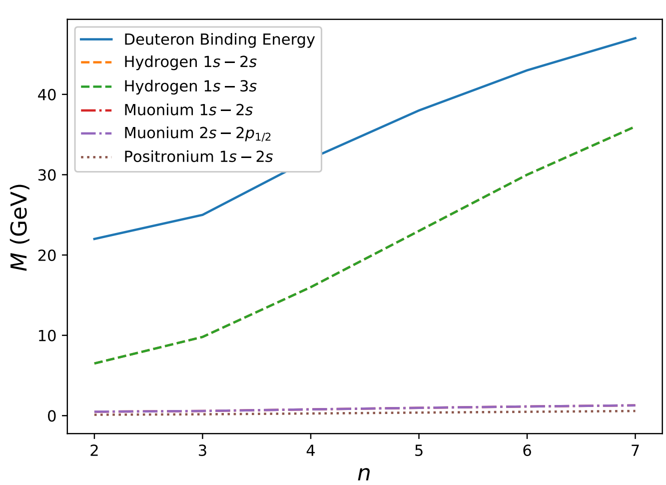

FIG. 2. Lower limits on the Planck mass M for varying number of extra dimensions n based on Tables II and IV. Transitions $1s$-$2s$ and $1s$-$3s$ in hydrogen as well as $1s$-$2s$ and $2p_{1/2}$-$2s$ in muonium give practically the same curves.

TABLE III. Parameters of extra dimensions obtained from comparing experimental and theoretical energy shifts. Experimental shifts $\Delta E_{\rm expt}$ are taken from Table I. Parameter n is the number of extra dimensions; ΔE_a is the energy shift given by Eqs. (8)–(10); $\Delta E_{\rm PT0} = \langle\psi_{ns}^{(0)}|V|\psi_{ns}^{(0)}\rangle - \langle\psi_{1s}^{(0)}|V|\psi_{1s}^{(0)}\rangle$, where $\psi^{(0)}$ are unperturbed wave functions; $\Delta E_{\rm PT} = \langle\psi_{ns}|V|\psi_{ns}\rangle - \langle\psi_{1s}|V|\psi_{1s}\rangle$, where ψ are wave functions obtained by solving Dirac equations including Coulomb and gravitational potentials. Energy shifts ΔE_D, obtained using Dirac equations, are fitted to $\Delta E_{\rm expt}$ by varying size of extra dimensions R, i.e., $\Delta E_D = \Delta E_{\rm expt}$. The values of R_a are given by Eqs. (11)–(13); R_D is chosen to fit the experimental energy shift by solving Dirac equations; $r_{gca} = S^{1/n}R_a$, $r_{gc} = S^{1/n}R_D$. M is the Planck mass for extra dimensions (3). The calculations were done at the cutoff parameter $r_c = r_{gc}$. Due to the selection of the cutoff parameter, there is no area where the gravitational potential (the perturbation) is bigger than the Coulomb potential. Equality $\Delta E_{\rm PT0} = \Delta E_{\rm PT} = \Delta E_D$ indicates that perturbation theory works for such a cutoff parameter. Numbers in square brackets mean powers of ten.

n	ΔE_a (eV)	$\Delta E_{\rm PT0}$ (eV)	$\Delta E_{\rm PT}$ (eV)	R_a (m)	r_{gca} (m)	R_D (m)	r_{gc} (m)	Mc^2 (GeV)
				Hydrogen, $1s$-$2s$, $\Delta E_D = \Delta E_{\rm expt} = 2.23 \times 10^{-11}$ eV				
1	2.23[−11]	2.23[−11]	2.23[−11]	5.63[+16]	2.48[−23]	5.63[+16]	2.48[−23]	80.5
2	2.41[−11]	2.27[−11]	2.27[−11]	3.06[+02]	6.42[−18]	3.18[+02]	6.68[−18]	2.75
3	1.69[−11]	2.26[−11]	2.26[−11]	3.37[−04]	2.56[−17]	2.93[−04]	2.23[−17]	2.15
4	1.29[−11]	2.15[−11]	2.15[−11]	2.50[−07]	3.62[−17]	1.90[−07]	2.75[−17]	2.36
5	1.09[−11]	2.19[−11]	2.19[−11]	3.30[−09]	4.44[−17]	2.31[−09]	3.11[−17]	2.54
6	9.44[−12]	2.20[−11]	2.20[−11]	1.86[−10]	5.13[−17]	1.21[−10]	3.33[−17]	2.70
7	8.04[−12]	2.14[−11]	2.14[−11]	2.40[−11]	5.73[−17]	1.44[−11]	3.44[−17]	2.88
				Hydrogen, $1s$-$3s$, $\Delta E_D = \Delta E_{\rm expt} = 2.19 \times 10^{-11}$ eV				
1	2.22[−11]	2.22[−11]	2.22[−11]	5.53[+16]	2.44[−23]	5.08[+16]	2.24[−23]	83.3
2	2.41[−11]	2.27[−11]	2.27[−11]	3.03[+02]	6.36[−18]	3.03[+02]	6.36[−18]	2.82
3	1.62[−11]	2.16[−11]	2.16[−11]	3.34[−04]	2.54[−17]	2.74[−04]	2.08[−17]	2.24
4	1.32[−11]	2.20[−11]	2.20[−11]	2.48[−07]	3.59[−17]	1.83[−07]	2.66[−17]	2.42
5	1.12[−11]	2.23[−11]	2.23[−11]	3.27[−09]	4.40[−17]	2.22[−09]	2.99[−17]	2.61
6	9.58[−12]	2.24[−11]	2.24[−11]	1.84[−10]	5.08[−17]	1.16[−10]	3.20[−17]	2.79
7	8.54[−12]	2.28[−11]	2.28[−11]	2.38[−11]	5.68[−17]	1.42[−11]	3.38[−17]	2.92
				Muonium, $1s$-$2s$, $\Delta E_D = \Delta E_{\rm expt} = 6.41 \times 10^{-8}$ eV				
1	6.41[−08]	6.41[−08]	6.41[−08]	1.43[+21]	7.12[−20]	1.43[+21]	7.12[−20]	2.7
2	7.04[−08]	6.48[−08]	6.48[−08]	5.67[+04]	4.00[−16]	5.95[+04]	4.20[−16]	0.20
3	4.74[−08]	6.32[−08]	6.32[−08]	3.73[−02]	1.37[−15]	3.21[−02]	1.18[−15]	0.13
4	3.90[−08]	6.50[−08]	6.50[−08]	2.31[−05]	1.94[−15]	1.80[−05]	1.51[−15]	0.11
5	3.32[−08]	6.65[−08]	6.65[−08]	2.73[−07]	2.38[−15]	1.97[−07]	1.71[−15]	0.11
6	2.71[−08]	6.32[−08]	6.32[−08]	1.43[−08]	2.75[−15]	9.31[−09]	1.78[−15]	0.10
7	2.54[−08]	6.78[−08]	6.79[−08]	1.76[−09]	3.07[−15]	1.11[−09]	1.93[−15]	0.098
				Muonium, $2s$-$2p_{1/2}$, $\Delta E_D = \Delta E_{\rm expt} = 1.08 \times 10^{-8}$ eV				
1	1.00[−8]	1.02[−8]	1.02[−8]	2.55[+21]	1.25[−19]	2.40[+21]	1.19[−19]	2.3
2	1.16[−8]	1.07[−8]	1.07[−8]	6.18[+04]	4.33[−16]	6.48[+04]	4.57[−16]	0.19
3	8.01[−9]	1.08[−8]	1.08[−8]	4.05[−02]	1.48[−15]	3.52[−02]	1.29[−15]	0.12
4	6.26[−9]	1.07[−8]	1.07[−8]	2.50[−05]	2.09[−15]	1.93[−05]	1.62[−15]	0.11
5	5.18[−9]	1.06[−8]	1.06[−8]	2.96[−07]	2.56[−15]	2.08[−07]	1.81[−15]	0.10
6	4.53[−9]	1.06[−8]	1.06[−8]	1.55[−08]	2.96[−15]	1.01[−08]	1.93[−15]	0.098
7	3.98[−9]	1.06[−8]	1.06[−8]	1.90[−09]	3.31[−15]	1.16[−09]	2.02[−15]	0.095
				Ps, $1s$-$2s$, $\Delta E_D = \Delta E_{\rm expt} = 1.34 \times 10^{-8}$ eV				
1	1.34[−8]	1.34[−8]	1.34[−8]	2.49[+23]	5.97[−20]	2.49[+23]	5.97[−20]	0.49
2	1.50[−8]	1.26[−8]	1.26[−8]	1.04[+06]	5.08[−16]	1.10[+06]	5.39[−16]	0.047
3	9.71[−9]	9.71[−9]	9.71[−9]	2.86[−01]	1.78[−15]	2.43[−01]	1.51[−15]	0.038
4	7.97[−9]	7.97[−9]	7.97[−9]	1.14[−04]	2.51[−15]	8.75[−05]	1.94[−15]	0.040
5	6.58[−9]	6.59[−9]	6.59[−9]	1.03[−06]	3.08[−15]	7.20[−07]	2.16[−15]	0.042
6	5.68[−9]	5.68[−9]	5.68[−9]	4.51[−08]	3.56[−15]	2.93[−08]	2.31[−15]	0.044
7	5.16[−9]	5.17[−9]	5.17[−9]	4.87[−09]	3.97[−15]	3.02[−09]	2.46[−15]	0.045

In practice this means that the wave function, found from solving the Dirac equation, does not oscillate at small distances. We found that in all cases $r_c = r_{gca}/100$, where $r_{gca} = S^{1/n}R_a$, in agreement with an estimate from Eq. (14). The results are presented in Table IV and Figs. 1 and 2.

In Tables III and IV we present energy shifts and corresponding values of r_c, R, and M obtained in several different ways. ΔE_a is calculated using formulas (8)–(10) but with the radius of extra dimensions found from the fitting of the energy shift ΔE_D obtained by solving the Dirac equations. $\Delta E_{\rm PT0} = \langle\psi_{1s}^{(0)}|V|\psi_{ns}^{(0)}\rangle$, where $\psi^{(0)}$ are unperturbed wave functions.

To avoid a misunderstanding, we should note that we used a slightly different method of cutoff in the analytical approach and in the numerical solution of the Dirac equation. $\Delta E_{\rm PT0} = \Delta E_a$ if integration for $\Delta E_{\rm PT0}$ is done from r_c to infinity, as in the analytical approach. We did such calculations as a test

TABLE IV. The same as Table III but the calculations were done at the smaller values of the cutoff parameter r_c, chosen from the condition of the absence of the gravitational bound states of elementary particles [this r_c also agrees with estimate (14)]; the values of R_a are given by Eqs. (11)–(13). Energy shifts ΔE_D, obtained using Dirac equations, are fitted to $\Delta E_{\rm expt}$ by varying size of extra dimensions R_D, and M is the Planck mass for extra dimensions (3). ΔE_a is the energy shift given by Eqs. (8)–(10) for $R = R_D$; $\Delta E_{\rm PT0} = \langle\psi^{(0)}_{ns}|V|\psi^{(0)}_{ns}\rangle - \langle\psi^{(0)}_{1s}|V|\psi^{(0)}_{1s}\rangle$, where $\psi^{(0)}$ are unperturbed wave functions; $\Delta E_{\rm PT} = \langle\psi_{ns}|V|\psi_{ns}\rangle - \langle\psi_{1s}|V|\psi_{1s}\rangle$, where ψ are wave functions obtained by solving Dirac equations including Coulomb and gravitational potentials. The main contribution to ΔE comes from the area where the gravitational potential is much bigger than the Coulomb potential. Large differences between the values of $\Delta E_{\rm PT0}$, $\Delta E_{\rm PT}$, and ΔE_D indicate that perturbation theory is not applicable for this small value of the cutoff parameter r_c. Therefore, all constraints are obtained using ΔE_D.

n	ΔE_a (eV)	$\Delta E_{\rm PT0}$ (eV)	$\Delta E_{\rm PT}$ (eV)	R_a (m)	r_c (m)	R_D (m)	Mc^2 (GeV)
			Hydrogen, $1s$-$2s$, $\Delta E_D = \Delta E_{\rm expt} = 2.23 \times 10^{-11}$ eV				
2	1.01[−12]	9.49[−13]	1.26[−08]	3.06[+02]	6.42[−20]	5.73[+01]	6.5
3	7.58[−13]	7.59[−13]	8.41[−10]	3.37[−04]	2.56[−19]	2.35[−05]	9.8
4	7.85[−13]	7.86[−13]	3.23[−10]	2.50[−07]	3.62[−19]	1.08[−08]	16
5	8.04[−13]	8.05[−13]	1.96[−10]	3.30[−09]	4.44[−19]	1.07[−10]	23
6	8.17[−13]	8.17[−13]	1.36[−10]	1.86[−10]	5.13[−19]	4.97[−12]	30
7	8.25[−13]	8.26[−13]	9.73[−11]	2.40[−11]	5.73[−19]	5.59[−13]	36
			Hydrogen, $1s$-$3s$, $\Delta E_D = \Delta E_{\rm expt} = 2.19 \times 10^{-11}$ eV				
2	1.10[−12]	1.03[−12]	1.33[−08]	3.03[+02]	6.36[−20]	5.69[+01]	6.5
3	8.18[−13]	8.19[−13]	7.21[−10]	3.34[−04]	2.54[−19]	2.33[−05]	9.8
4	8.48[−13]	8.49[−13]	3.49[−10]	2.48[−07]	3.59[−19]	1.07[−08]	16
5	8.61[−13]	8.62[−13]	1.09[−10]	3.27[−09]	4.40[−19]	1.06[−10]	23
6	8.74[−13]	8.75[−13]	7.92[−11]	1.84[−10]	5.08[−19]	4.91[−12]	30
7	8.85[−13]	8.86[−13]	7.33[−11]	2.38[−11]	5.68[−19]	5.53[−13]	36
			Muonium, $1s$-$2s$, $\Delta E_D = \Delta E_{\rm expt} = 6.41 \times 10^{-8}$ eV				
2	2.84[−09]	2.62[−09]	2.24[−05]	5.67[+04]	4.00[−18]	1.01[+04]	0.49
3	2.17[−09]	2.17[−09]	1.55[−06]	3.73[−02]	1.37[−17]	2.60[−03]	0.58
4	2.25[−09]	2.25[−09]	9.50[−07]	2.31[−05]	1.94[−17]	1.00[−06]	0.78
5	2.32[−09]	2.33[−09]	1.60[−06]	2.73[−07]	2.38[−17]	8.89[−09]	0.97
6	2.35[−09]	2.35[−09]	4.77[−07]	1.43[−08]	2.75[−17]	3.83[−10]	1.14
7	2.37[−09]	2.37[−09]	2.82[−07]	1.76[−09]	3.07[−17]	4.09[−11]	1.28
			Muonium, $2s$-$2p_{1/2}$, $\Delta E_D = \Delta E_{\rm expt} = 1.08 \times 10^{-8}$ eV				
2	4.06[−10]	3.81[−10]	1.65[−5]	6.18[+4]	4.00[−18]	1.01[+04]	0.49
3	3.11[−10]	4.16[−10]	3.31[−6]	4.05[−2]	1.37[−17]	2.61[−03]	0.58
4	3.23[−10]	5.37[−10]	6.60[−7]	2.50[−5]	1.94[−17]	1.00[−06]	0.78
5	3.27[−10]	6.54[−10]	2.38[−7]	2.56[−7]	2.38[−17]	8.87[−09]	0.97
6	3.32[−10]	7.75[−10]	1.98[−7]	1.55[−8]	2.75[−17]	3.83[−10]	1.1
7	3.41[−10]	9.11[−10]	6.35[−7]	1.90[−9]	3.07[−17]	4.09[−11]	1.3
			Ps, $1s$-$2s$, $\Delta E_D = \Delta E_{\rm expt} = 1.34 \times 10^{-8}$ eV				
2	5.89[−10]	5.21[−10]	6.02[−06]	1.04[+06]	5.08[−18]	1.85[+05]	0.11
3	4.55[−10]	4.55[−10]	3.27[−07]	2.86[−01]	1.78[−17]	1.99[−02]	0.17
4	4.72[−10]	4.73[−10]	2.00[−07]	1.14[−04]	2.51[−17]	4.92[−06]	0.27
5	4.81[−10]	4.82[−10]	7.83[−08]	1.03[−06]	3.08[−17]	3.34[−08]	0.38
6	4.91[−10]	4.92[−10]	8.26[−08]	4.51[−08]	3.56[−17]	1.21[−09]	0.48
7	4.93[−10]	4.94[−10]	4.12[−08]	4.87[−09]	3.97[−17]	1.13[−10]	0.58

and found good agreement between the two. However, in the tables we present values of $\Delta E_{\rm PT0}$ and $\Delta E_{\rm PT}$ found by integrating from zero to infinity. This is done to make a meaningful comparison with the energy shift obtained from solving the Dirac equations. The same gravitational potential is used to calculate $\Delta E_{\rm PT0}$, $\Delta E_{\rm PT}$, and ΔE_D in the Dirac equations. If $\Delta E_{\rm PT0} = \Delta E_{\rm PT} = \Delta E_{\rm expt}$ (we have $\Delta E_D = \Delta E_{\rm expt}$), then one could say that PT works. We see that this is the case only for $r_c = r_{gca}$ (Table III).

The values of R_a and r_{gca} are given by Eqs. (11)–(13) and (6). The values of R_D and r_{gc} are found from solving the Dirac equations. We believe that the results obtained with the use of Dirac equations are more reliable since they are not based on the perturbation theory. The Planck mass for extra dimensions M was found using the Dirac equation values of R (R_D).

V. CONCLUSION

In the present paper we investigated possibilities to probe the ADD gravitational potential Eq. (2) at subatomic distances. This potential does not include relativistic corrections. However, as it is demonstrated in the Appendix, relativistic corrections increase the gravitational interaction and lead to much stronger constraints. Therefore, our estimates of the limits on the size of extra dimensions, based on the potential Eq. (2), are conservative. By imposing the condition that there are no gravitational bound states between the considered elementary particles (which have not been observed), we cut off an area of unknown physics at distances smaller than the electroweak scale. Constraints on the size of extra dimensions R and Planck mass M, obtained using the difference between the experimental and calculated values of the deuteron binding energy, have been presented in Table II. However, deuteron is a system with a strong interaction and these constraints are probably less reliable than the constraints from hydrogen, muonium, and positronium presented in Table IV. The perturbation theory is not applicable if the main contribution to the energy shift comes from the area where the gravitational potential exceeds the Coulomb potential. Therefore, we obtain the energy shift using a solution of the Dirac equation including both Coulomb and gravitational potentials. The corresponding limit on the size of extra dimensions is denoted by R_D.

We investigate the gravitational potential at subatomic distances and obtain stronger constraints than that obtained from the spectra of hydrogen molecules where the distance between the nuclei exceeds the Bohr radius—see Ref. [19]. Our constraints are also stronger than that presented in Ref. [15] which are based on the estimate of the radiative correction.

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APPENDIX: RELATIVISTIC ESTIMATES

As known, all types of energy contribute to the gravitational potential. For example, the kinetic energy of nearly massless quarks ($m_u \approx 3$ MeV, $m_d \approx 5$ MeV) makes a significant contribution to the nucleon mass. The kinetic energy of an ultrarelativistic particle may be estimated as $E_k = pc \sim \hbar c/r$, so mass m in the gravitational potential should be replaced by $\hbar/cr$, and the corresponding gravitational interaction energy of two particles is

$$U \sim -\frac{G}{r}\left(\frac{\hbar}{cr}\right)^2\left(\frac{R_n}{r}\right)^n. \qquad \text{(A1)}$$

For weakly bound states the kinetic energy and potential energy compensate each other, $E_k = |U|$, and this equality defines the cutoff radius excluding the formation of the bound states, $r_c = \hbar/M_n c$, where M_n is the Planck mass for extra n dimensions—see Eq. (1). Note that one can find similar estimates in Ref. [36].

Perturbation theory gives the following estimate for the energy shift produced by the potential Eq. (A1) with the cutoff $r_c = \hbar/M_n c$ (in natural units $\hbar = c = 1$):

$$\delta E \sim \frac{4\pi\psi(0)^2}{nM_n^2}. \qquad \text{(A2)}$$

For hydrogen $\delta E < 2.2 \times 10^{-11}$ eV and we obtain the limit $M_n > 100$ GeV$/n^{1/2}$, which is much stronger than the limit obtained using the nonrelativistic gravitational potential. For deuterium the limit is even stronger, $M_n > 170$ GeV$/n^{1/2}$.

Thus, we conclude that the use of the nonrelativistic gravitational potential underestimates the energy shift δE and gives conservative limits on the Planck mass M_n and radius of extra dimensions R_n.

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