Quantifying coherence in terms of Fisher information

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In quantum metrology, the parameter estimation accuracy is bounded by quantum Fisher information. In this paper we present coherence measures in terms of (quantum) Fisher information by directly considering the postselective nonunitary parametrization process. This coherence measure demonstrates the apparent operational meaning by the exact connection between coherence and parameter estimation accuracy. We also discuss the distinction between our coherence measure and the quantum Fisher information subject to unitary parametrization. The analytic coherence measure is given for qubit states.

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I. INTRODUCTION

Quantum coherence, as a fundamental feature in quantum physics, has attracted a great deal of attention in recent years. Many works have investigated the role of coherence in quantum optics [1–4], quantum thermodynamics [5–7], quantum phase transitions [8], quantum biology [9,10], and quantum information science [11–18]. These works have promoted not only the development of related applications but also the development of the resource theory of coherence [19,20], where coherence is treated as a physical resource under some limited conditions. Benefiting from an operational view and axiomatic approach, one can quantify coherence in a rigorous manner, study the transformation of coherence, and reveal the connection between coherence with other fundamental quantum features [21–32]. In particular, some coherence measures contain obvious operational meanings, which provide us with a way to understand (interpret) coherence from the viewpoint of quantum information processes (QIPs) and find the potential relation between coherence with some characteristics in QIPs [33–39].

It has been shown that the coherence of the probing state in many quantum metrology processes is often a key ingredient [11–13]. For instance, in the usual phase estimation for the parameter θ with unitary parametrization $\mathcal{U}_\theta(\cdot) = e^{-i\theta H}(\cdot)e^{i\theta H}$ [40–44], coherence with regard to the eigenvectors of the Hermitian operator H is necessary. Furthermore, the optimal estimation accuracy of an unknown parameter could be obtained by the state with maximal coherence in the sequential protocol [13]. The estimation accuracy is bounded by quantum Fisher information (QFI), a crucial ingredient in quantum metrology [45–48]. A simple calculation can show that QFI subject to unitary parametrization $\mathcal{U}_\theta(\cdot)$ in the qubit case [44] is monotonic with some coherence measures (such as l_1-norm coherence). Many works have investigated the relation between quantum coherence with Fisher information (FI) and QFI [49–58]. Coherence within some *particular settings* could be understood by QFI (or FI) [51,52,57]. Significantly, QFI in unitary parametrization is closely connected with unspeakable coherence [52,59], a special case of resource theory of asymmetry [60–62]. In addition, based on QFI concerning the dephasing parameter, coherence measure has been given in the sense of strictly incoherent operations as free operations [51]. However, up to now, the estimation accuracy and FI (or QFI) has not been used to directly quantify quantum coherence in *general* scenarios. An intuitive challenge is that QFI with unitary parametrization $\mathcal{U}_\theta(\cdot)$ in the usual sense is not a coherence measure in the general resource theory of coherence [20]. For example, two-dimensional maximally coherent states (MCSs) could be obtained under incoherent operations from three-dimensional MCSs [33,53,63], but the QFI of the former is strictly larger than the latter, which directly violates the monotonicity of a good measure. Therefore, it is significant to find an appropriate parametrization process for establishing coherence measures and further investigating the role of coherence in quantum metrology.

In this paper we successfully establish several equivalent coherence measures in the general resource theory of coherence by the FI (and QFI) subject to a type of nonunitary parametrization. Since the optimal estimation accuracy is bounded by FI which is asymptotically attained with maximum likelihood estimators [45–47], our measure naturally inherits the operational meaning of FI through the optimal estimation accuracy with non-unitary parametrization. We also show that in the qubit case, our coherence measure can be equivalently understood through unitary parametrization and the analytic expression can be obtained. Our coherence measure not only builds a direct relation between coherence and parameter estimation accuracy (or FI) but also sheds new light on the roles of the nonunitary parametrization process. The remainder of this paper is organized as follows. In Sec. II we first introduce the fundamental concepts of resource theory of coherence and our parametrization process and then present several main theorems to build the coherence measure based on FI. In Sec. III we give the analytic result of the coherence

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measure in the qubit case and discuss the difference from the QFI with unitary parametrization. We summarize in Sec. IV.

II. COHERENCE IN TERMS OF QFI

In this section we first introduce the resource theory of coherence established mainly based on the incoherent (free) operations and incoherent (free) states [20]. Considering the preferred basis $\{|n\rangle\}$, the incoherent state is defined by $\varrho = \sum_n q_n|n\rangle\langle n|$, with $\mathcal{I}$ denoting the set of incoherent states, and the incoherent operations (IOs) with the Kraus representation $\{K_n : \sum_l K_l^\dagger K_l = \mathbb{I}\}$ is a special type of completely positive and trace-preserving map defined by $\frac{K_l\varrho K_l^\dagger}{\mathrm{tr}(K_l\varrho K_l^\dagger)} \in \mathcal{I}$ for $\varrho \in \mathcal{I}$. In this sense, a good coherence measure $C(\rho)$ for any state ρ should satisfy the following conditions.

(i) Non-negativity. $C(\rho) \geqslant 0$ is saturated if and only if $\rho \in \mathcal{I}$.

(ii) Monotonicity. $C(\mathcal{E}(\rho)) \leqslant C(\rho)$ for any incoherent operation $\mathcal{E}(\cdot)$.

(iii) Strong monotonicity. $\sum_n p_n C(K_n\rho K_n^\dagger/p_n) \leqslant C(\rho)$ for any IO $\{K_n\}$, with $p_n = \mathrm{Tr}(K_n\rho K_n^\dagger)$.

(iv) Convexity. $C(\rho) \leqslant \sum_i p_i C(\rho_i)$ for any $\rho = \sum_i p_i\rho_i$.

To present a valid coherence measure, we begin with the following parametrization process. Considering a state ρ undergoing quantum channel $\mathcal{E}_\theta$ depending on parameter θ, the unknown parameter could be estimated from measurements on $\mathcal{E}_\theta(\rho)$. Here we are interested in the free parametrization processes $\mathcal{E}_\theta = \{E_x(\theta)\}$,

$$E_x(\theta) = \sum_n b_n^x(\theta)|g_x(n)\rangle\langle n|, \quad \sum_x E_x(\theta)^\dagger E_x(\theta) = \mathbb{I}, \quad (1)$$

where $\{|n\rangle\}$ is the preferred incoherent basis and $g_x(\cdot)$ is a map from one integer to another.

In order to focus on the role of coherence, we desire that within the parametrization process, the incoherent probe cannot affect parameter estimation. That is, the measurement outcomes $\mathcal{E}_\theta(\varrho)$ and $\{\varrho_x, p_x\}$ obtained from an incoherent probe ($\varrho \in \mathcal{I}$) do not depend on parameter θ, where $p_x = \mathrm{tr}[E_x(\theta)\varrho E_x(\theta)^\dagger]$ and $\varrho_x = E_x(\theta)\varrho E_x(\theta)^\dagger/p_x$. Thus $|b_n^x(\theta)|$ does not depend on the parameter θ and $E_x(\theta)$ can be rewritten as

$$E_x(\theta) = \sum_n c_n^x e^{ih_n^x(\theta)}|g_x(n)\rangle\langle n|, \quad (2)$$

where c_n^x is parameter independent and h_n^x is a real function. In fact, it is very similar to the case of the usual phase estimation $\mathcal{U}_\theta(\cdot) = e^{-i\theta H}(\cdot)e^{i\theta H}$ mentioned in the Introduction. One can find that the measurement outcomes of an incoherent probe in the phase estimation do not depend on the parameter θ either. In addition, $\mathcal{U}_\theta$ can be expressed based on $e^{-iH\theta} = \sum_n e^{-ih_n\theta}|n\rangle\langle n|$ (h_n is eigenvalue of H), which is analogous to Eq. (2). In this sense, the parametrization process $\mathcal{E}_\theta$ can be understood as a generalization of unitary phase estimation to the nonunitary case.

In addition, we could restrict $\partial_\theta h_n^x(\theta) \in [0, 1]$ and the conclusion in a more general case could be derived from this case (a detailed discussion is given in Appendix A). Based on the Stinespring dilation theorem [64], the operations could be implemented by a controlled unitary operator [65–67] and an operation swapping specified states. The details are given in Appendix B. All the operations of interest (operations in Eq. (2) with $\partial_\theta h_n^x(\theta) \in [0, 1]$) comprise a set denoted by G. We note that an IO satisfying $\mathrm{rank}[E_x(\theta)^\dagger E_x(\theta)] = 1$ is of particular interest in the paper, so we use G_1 to represent the IO set with this particular property.

If the postselection is allowed, the IO $\mathcal{E}_\theta$ performed on a quantum state ρ will directly lead to the probability distribution

$$P^{\mathcal{E}}(x|\theta) = \mathrm{tr}[E_x(\theta)\rho E_x(\theta)^\dagger]. \quad (3)$$

If the postselection is not allowed, the state after the IO will become $\mathcal{E}_\theta(\rho)$. We can operate a positive-operator-valued measure (POVM) $\mathcal{M} = \{M_x\}$ on the state $\mathcal{E}_\theta(\rho)$ and obtain the probability distribution family as

$$P_{\mathcal{M}}^{\mathcal{E}}(x|\theta) = \mathrm{tr}[M_x\mathcal{E}_\theta(\rho)], \quad (4)$$

where the subscript $\mathcal{M}$ denotes the general POVM.

The FI of the distribution $P(x|\theta)$ is given by

$$F(P, \theta_0) = \sum_x P(x|\theta_0)\left[\frac{\partial \ln P(x|\theta)}{\partial\theta}\bigg|_{\theta_0}\right]^2 \quad (5)$$

and the QFI of $P_{\mathcal{M}}^{\mathcal{E}}(x|\theta)$ for any given θ_0 can be written as

$$F_Q(\rho, \mathcal{E}, \theta_0) = \max_{\mathcal{M}} F\left(P_{\mathcal{M}}^{\mathcal{E}}, \theta_0\right). \quad (6)$$

Based on above the FI and QFI, we can establish two coherence measures, respectively, which will be given by the following two theorems.

Theorem 1. The coherence of a state ρ can be quantified by the maximal FI for a given parameter θ_0 as

$$C^{\theta_0}(\rho) = \max_{\mathcal{E}\in G} F(P^{\mathcal{E}}, \theta_0), \quad (7)$$

where $F(P^{\mathcal{E}}, \theta_0)$ is the FI of the distribution in Eq. (3).

Proof. We need to prove $C^{\theta_0}(\rho)$ by satisfying conditions (i)–(iv).

(i) *Non-negativity*. If ρ is incoherent, for any $\mathcal{E}$ and x we have

$$\begin{aligned} &E_x(\theta)\rho E_x(\theta)^\dagger \\ &= \sum_n b_n^x(\theta)|g_x(n)\rangle\langle n|\rho\sum_m b_m^{x*}(\theta)|m\rangle\langle g_x(m)| \\ &= \sum_{nm} b_n^x(\theta)b_m^{x*}(\theta)\rho_{nm}|g_x(n)\rangle\langle g_x(m)| \\ &= \sum_n \left|b_n^x(\theta)\right|^2\rho_{nn}|g_x(n)\rangle\langle g_x(n)|, \end{aligned} \quad (8)$$

which does not depend on θ due to $b_n^x(\theta) = c_n^x e^{ih_n^x(\theta)}$. Thus $P^{\mathcal{E}}(x|\theta)$ does not depend on θ either, which means that

$$F(P^{\mathcal{E}}, \theta_0) = \sum_x\left[\frac{\partial P^{\mathcal{E}}(x|\theta)}{\partial\theta}\bigg|_{\theta_0}\right]^2\frac{1}{P^{\mathcal{E}}(x|\theta_0)} = 0. \quad (9)$$

Equation (9) leads to $C^{\theta_0}(\rho) = 0$.

Conversely, if a d-dimensional ρ has nonzero off-diagonal entries, without loss of generality, we can set $\rho_{12} = |\rho_{12}|e^{i\alpha}$.

There exists an IO $\{E_i\} \in G$,

$$E_1(\theta) = \frac{\sqrt{2}}{2} e^{i(\theta+\gamma)}|1\rangle\langle 1| + \frac{\sqrt{2}}{2}|1\rangle\langle 2|,$$
$$E_2(\theta) = -\frac{\sqrt{2}}{2} e^{i(\theta+\gamma)}|2\rangle\langle 1| + \frac{\sqrt{2}}{2}|2\rangle\langle 2|,$$
$$E_3(\theta) = \sum_{n=3}^{d} |n\rangle\langle n|, \tag{10}$$

with $\alpha + \theta_0 + \gamma \in [-\pi/2, 0) \bigcup (0, \pi/2]$, such that

$$P^{\mathcal{E}}(1|\theta_0) = \mathrm{tr}[E_1(\theta_0)\rho E_1(\theta_0)^\dagger] \neq 0,$$
$$\partial_\theta \, \mathrm{tr}[E_1(\theta)\rho E_1(\theta)^\dagger]|_{\theta_0} \neq 0, \tag{11}$$

which obviously shows that $C^{\theta_0}(\rho) \neq 0$ and $C^{\theta_0}(\rho) > 0$.

(iii) *Strong monotonicity.* Suppose ρ undergoes an arbitrary IO

$$K_l = \sum_n a_n^l |f_l(n)\rangle\langle n|. \tag{12}$$

The postmeasurement ensemble $\{t_l, \rho_l\}$ reads

$$t_l = \mathrm{tr}(K_l \rho K_l^\dagger), \quad \rho_l = \frac{K_l \rho K_l^\dagger}{t_l}. \tag{13}$$

Let $\mathcal{E}^{(l)} = \{E_x^l(\theta)\}_x$ be the optimal IO for ρ_l such that

$$C^{\theta_0}(\rho_l) = F(P_l, \theta_0), \tag{14}$$

where $E_x^l(\theta) = \sum_n b_n^{lx}(\theta)|g_{lx}(n)\rangle\langle n|$ and

$$\begin{aligned} P_l(x|\theta) &= \mathrm{tr}\left[E_x^l(\theta)\rho_l E_x^l(\theta)^\dagger\right] \\ &= \frac{\mathrm{tr}\left[E_x^l(\theta) K_l \rho K_l^\dagger E_x^l(\theta)^\dagger\right]}{t_l} \\ &= \frac{P(x, l|\theta)}{t_l}. \end{aligned} \tag{15}$$

Here $P(x, l|\theta)$ represents the probability distribution from $\mathcal{E}' = \{E'_{xl}(\theta)\}_{xl}$ with

$$E'_{xl}(\theta) = E_x^l(\theta) K_l = \sum_n a_n^l b_{f_l(n)}^{lx}(\theta)|g_{lx}[f_l(n)]\rangle\langle n|, \tag{16}$$

which implies $\mathcal{E}' \in G$. Therefore, we arrive at

$$\begin{aligned} \sum_l t_l C^{\theta_0}(\rho_l) &= \sum_l t_l F(P_l, \theta_0) \\ &= \sum_l t_l \sum_{x \in S_l} \left[\frac{\partial P_l(x|\theta)}{\partial \theta}\bigg|_{\theta_0} \right]^2 \frac{1}{P_l(x|\theta_0)} \\ &= \sum_l t_l \sum_{x \in S_l} \left[\frac{\partial P(l, x|\theta)}{\partial \theta}\bigg|_{\theta_0} \right]^2 \frac{1}{P(l, x|\theta_0) t_l} \\ &= \sum_l \sum_{x \in S_l} \left[\frac{\partial P(l, x|\theta)}{\partial \theta}\bigg|_{\theta_0} \right]^2 \frac{1}{P(l, x|\theta_0)} \\ &= F(P, \theta_0) \leqslant C^{\theta_0}(\rho), \end{aligned} \tag{17}$$

where S_l indicates the region of x in P_l and the last inequality is because $\mathcal{E}'$ may not be the optimal one for ρ.

(iv) *Convexity.* For any ensemble $\{t_i, \sigma_i\}$ with the corresponding mixed state $\rho = \sum_i t_i \sigma_i$, let $\mathcal{E} = \{E_x(\theta)\}$ be the optimal IO for ρ in the sense of $C^{\theta_0}(\rho) = F(P, \theta_0)$, with $P(x|\theta) = \mathrm{tr}[E_x(\theta)\rho E_x^\dagger(\theta)]$. For the state σ_i, we define

$$P_i(x|\theta) = \mathrm{tr}[E_x(\theta)\sigma_i E_x^\dagger(\theta)]; \tag{18}$$

then

$$\begin{aligned} \sum_i t_i P_i(x|\theta) &= \sum_i t_i \, \mathrm{tr}[E_x(\theta)\sigma_i E_x^\dagger(\theta)] \\ &= \mathrm{tr}[E_x(\theta)\rho E_x^\dagger(\theta)] \\ &= P(x|\theta). \end{aligned} \tag{19}$$

However, $\mathcal{E}$ may not be optimal for σ_i, which implies that

$$C^{\theta_0}(\sigma_i) \geqslant F(P_i, \theta_0), \tag{20}$$

so we can immediately get

$$\begin{aligned} \sum_i t_i C^{\theta_0}(\sigma_i) &\geqslant \sum_i t_i F(P_i, \theta_0) \\ &\geqslant F\left(\sum_i t_i P_i, \theta_0\right) = F(P, \theta_0) \\ &= C^{\theta_0}(\rho), \end{aligned} \tag{21}$$

where the second inequality is due to the convexity of the FI.

Since (iii) and (iv) hold, it is natural that (ii) is satisfied. The proof is completed. ■

From Theorem 1, coherence could be quantified by the FI of the probability distribution in Eq. (3). In some sense, this implies the connection between coherence and estimation accuracy for incoherent nonunitary parametrization. In fact, G in the definition (7) could be replaced by its subset G_1 from the following lemma.

Lemma 1. For any $\mathcal{E} = \{E_x(\theta)\} \in G$ there always exists another $\mathcal{E}' = \{\tilde{E}_x(\theta)\} \in G_1$ such that

$$F(P^{\mathcal{E}}, \theta_0) \leqslant F(P^{\mathcal{E}'}, \theta_0), \tag{22}$$

where $F(P^{\mathcal{E}}, \theta_0)$ and $F(P^{\mathcal{E}'}, \theta_0)$ are the FI of $P^{\mathcal{E}}(x|\theta)$ and $P^{\mathcal{E}'}(x|\theta)$, respectively.

Proof. Letting $\mathcal{E} = \{E_x(\theta)\} \in G$, we can rewrite $\{E_x(\theta)\}$ as

$$\begin{aligned} E_x(\theta) &= \sum_n c_n^x e^{ih_n^x(\theta)}|g_x(n)\rangle\langle n| \\ &= \sum_n c_n^x |g_x(n)\rangle\langle n| \sum_m e^{ih_m^x(\theta)}|m\rangle\langle m| \\ &= A_x U_x(\theta), \end{aligned} \tag{23}$$

where $A_x = \sum_n c_n^x |g_x(n)\rangle\langle n|$ and $U_x(\theta) = \sum_m e^{ih_m^x(\theta)} |m\rangle\langle m|$. Thus we have

$$\begin{aligned} E_x(\theta)^\dagger E_x(\theta) &= U_x(\theta)^\dagger A_x^\dagger A_x U_x(\theta) \\ &= U_x(\theta)^\dagger \left(\sum_i |\psi_i^x\rangle\langle\psi_i^x|\right) U_x(\theta) \\ &= \sum_i U_x(\theta)^\dagger |\psi_i^x\rangle\langle\psi_i^x| U_x(\theta) \\ &= \sum_i |\phi_i^x(\theta)\rangle\langle\phi_i^x(\theta)| = \sum_i \tilde{E}_{x,i}(\theta)^\dagger \tilde{E}_{x,i}(\theta), \end{aligned} \tag{24}$$

where $\sum_i |\psi_i^x\rangle\langle\psi_i^x|$ denotes the eigendecomposition of $A_x^\dagger A_x$ (the eigenvalue is absorbed in $|\psi_i^x\rangle$), $|\phi_i^x(\theta)\rangle = U_x(\theta)^\dagger|\psi_i^x\rangle$, and $\tilde{E}_{x,i}(\theta) = |i\rangle\langle\phi_i^x(\theta)|$. It is obvious that $\mathcal{E}' = \{\tilde{E}_{x,i}(\theta)\}_{xi} \in G_1$. From the Cauchy-Schwarz inequality $[|\langle v|w\rangle|^2 \leqslant \langle v|v\rangle\langle w|w\rangle]$ [68], we can obtain

$$[\partial_\theta P(x|\theta)|_{\theta_0}]^2 \leqslant \sum_i \frac{[\partial_\theta P_i(x|\theta)|_{\theta_0}]^2}{P_i(x|\theta_0)} \sum_i P_i(x|\theta_0), \tag{25}$$

where $P(x|\theta) = \mathrm{tr}(\rho E_x^\dagger E_x)$ and $P_i(x|\theta) = \mathrm{tr}(\rho \tilde{E}_{x,i}^\dagger \tilde{E}_{x,i})$, and thus

$$\frac{[\partial_\theta P(x|\theta)|_{\theta_0}]^2}{P(x|\theta_0)} \leqslant \sum_i \frac{[\partial_\theta P_i(x|\theta)|_{\theta_0}]^2}{P_i(x|\theta_0)}. \tag{26}$$

The inequality holds for every x, which implies that $F(P^{\mathcal{E}}, \theta_0) \leqslant F(P^{\mathcal{E}'}, \theta_0)$. ■

From the lemma, maximizing the FI over the set G can be realized by the optimization over the set G_1, which effectively reduces the range of the optimized IO.

Theorem 1 mainly focuses on the FI with the related probability distribution generated via the postselective IO on a state. Next we would build another coherence measure defined by the QFI with respect to parametrization in G,

$$C_Q^{\theta_0}(\rho) = \max_{\mathcal{E}\in G} F_Q(\rho, \mathcal{E}, \theta_0). \tag{27}$$

To do this, we first give another lemma.

Lemma 2. The maximal QFI subject to parametrization in G is upper bounded by the FI directly induced by the optimal postselective IO parametrization process, namely,

$$\max_{\mathcal{E}\in G} F_Q(\rho, \mathcal{E}, \theta_0) \leqslant \max_{\mathcal{E}\in G} F(P^{\mathcal{E}}, \theta_0), \tag{28}$$

where $P^{\mathcal{E}}$ is the distribution in Eq. (3).

Proof. Suppose $\tilde{\mathcal{E}}$ and $\mathcal{M}$ are the optimal parametrization and measurement for the optimal F_Q, respectively. From Eq. (4) we have $P_{\mathcal{M}}^{\tilde{\mathcal{E}}}(x|\theta) = \mathrm{tr}[\sum_i |\psi_i^x\rangle\langle\psi_i^x|\tilde{\mathcal{E}}_\theta(\rho)] = \sum_i P_i(x|\theta)$, where $\sum_i |\psi_i^x\rangle\langle\psi_i^x|$ represents the eigendecomposition of M_x. In particular, $P_i(x|\theta) = \langle\psi_i^x|\tilde{\mathcal{E}}_\theta(\rho)|\psi_i^x\rangle$, which can be rewritten as

$$\begin{aligned}
P_i(x|\theta) &= \mathrm{tr}\left[|i\rangle\langle\psi_i^x|\tilde{\mathcal{E}}_\theta(\rho)|\psi_i^x\rangle\langle i|\right] \\
&= \sum_{ynn'} \mathrm{tr}\left[|i\rangle\langle\psi_i^x|b_n^y(\theta)|g_y(n)\rangle\langle n|\rho|n'\rangle\right. \\
&\quad \left.\times \langle g_y(n')|b_{n'}^{y*}(\theta)|\psi_i^x\rangle\langle i|\right] \\
&= \sum_{ynn'} \mathrm{tr}\left[b_n^{ixy}(\theta)|i\rangle\langle n|\rho|n'\rangle\langle i|b_{n'}^{ixy*}(\theta)\right] \\
&= \sum_y \mathrm{tr}[E_{ixy}(\theta)\rho E_{ixy}(\theta)^\dagger] \\
&= \sum_y P(ixy|\theta),
\end{aligned} \tag{29}$$

where $b_n^{ixy}(\theta) = \langle\psi_i^x|b_n^y(\theta)|g_y(n)\rangle$ and $E_{ixy}(\theta) = \sum_n b_n^{ixy}(\theta)|i\rangle\langle n|$. It is obvious that $\mathcal{E}_\theta' = \{E_{ixy}(\theta)\} \in G$. Then

$$\begin{aligned}
\max_{\mathcal{E}} F_Q(\rho, \mathcal{E}, \theta_0) &= F\left(P_{\mathcal{M}}^{\tilde{\mathcal{E}}}, \theta_0\right) \\
&= \sum_x \frac{\left[\partial_\theta P_{\mathcal{M}}^{\tilde{\mathcal{E}}}(x|\theta)|_{\theta_0}\right]^2}{P_{\mathcal{M}}^{\tilde{\mathcal{E}}}(x|\theta_0)} \\
&= \sum_x \frac{\left[\sum_i \partial_\theta P_i(x|\theta)|_{\theta_0}\right]^2}{\sum_i P_i(x|\theta_0)} \\
&\leqslant \sum_{ix} \frac{[\partial_\theta P_i(x|\theta)|_{\theta_0}]^2}{P_i(x|\theta_0)} \\
&= \sum_{ix} \frac{\left[\sum_y \partial_\theta P(ixy|\theta)|_{\theta_0}\right]^2}{\sum_y P(ixy|\theta_0)} \\
&\leqslant \sum_{ixy} \frac{\left[\partial_\theta P(ixy|\theta)|_{\theta_0}\right]^2}{P(ixy|\theta_0)} \\
&= F(P, \theta_0) \leqslant \max_{\mathcal{E}} F(P^{\mathcal{E}}, \theta_0),
\end{aligned} \tag{30}$$

where P is the distribution from $\mathcal{E}_\theta'$, the first two inequalities could be derived based on the Cauchy-Schwarz inequality, and the derivation process is similar to that in Eqs. (25) and (26), namely, from

$$\left(\sum_i \partial_\theta P_i(x|\theta)|_{\theta_0}\right)^2 \leqslant \sum_i \frac{[\partial_\theta P_i(x|\theta)|_{\theta_0}]^2}{P_i(x|\theta_0)} \sum_i P_i(x|\theta_0)$$

we could obtain the first inequality and from

$$\left(\sum_y \partial_\theta P(ixy|\theta)|_{\theta_0}\right)^2 \leqslant \sum_y \frac{[\partial_\theta P(ixy|\theta)|_{\theta_0}]^2}{P(ixy|\theta_0)} \sum_y P(ixy|\theta_0)$$

we could reach the second inequality. Thus we complete the proof. ■

Next we show that $C_Q^{\theta_0}(\rho)$ in Eq. (27) is equivalent to $C^{\theta_0}(\rho)$ and can also quantify the quantum coherence of ρ.

Theorem 2. For a given density matrix ρ,

$$C_Q^{\theta_0}(\rho) = C^{\theta_0}(\rho). \tag{31}$$

Proof. From Lemma 1, $C^{\theta_0}(\rho)$ could be written as

$$C^{\theta_0}(\rho) = \max_{\mathcal{E}\in G_1} F(P^{\mathcal{E}}, \theta_0). \tag{32}$$

Suppose $\mathcal{E} = \{E_z(\theta)\}$ is the optimal operation in G_1 such that

$$C^{\theta_0}(\rho) = F(P^{\mathcal{E}}, \theta_0), \tag{33}$$

where

$$P^{\mathcal{E}}(z|\theta) = \mathrm{tr}[E_z(\theta)\rho E_z(\theta)^\dagger] \tag{34}$$

and $\mathrm{rank}[E_z(\theta)^\dagger E_z(\theta)] = 1$. Without loss of generality, $E_z(\theta)$ could be written as

$$E_z(\theta) = |z\rangle\langle\phi_z(\theta)|. \tag{35}$$

Define

$$P_{\mathcal{P}}^{\mathcal{E}}(z|\theta) = \mathrm{tr}[|z\rangle\langle z|\mathcal{E}_\theta(\rho)|z\rangle\langle z|], \tag{36}$$

where $\mathcal{P}$ indicates the projective measurements on the parametrized state. Note that

$$\begin{aligned}P^{\mathcal{E}}(z|\theta) &= \mathrm{tr}(E_z\rho E_z^\dagger)\\ &= \mathrm{tr}\left[|z\rangle\langle z|\left(\sum_{z'}E_{z'}\rho E_{z'}^\dagger\right)|z\rangle\langle z|\right]\\ &= \mathrm{tr}[|z\rangle\langle z|\mathcal{E}_\theta(\rho)|z\rangle\langle z|] = P_{\mathcal{P}}^{\mathcal{E}}(z|\theta)\end{aligned}\tag{37}$$

and thus

$$\begin{aligned}C^{\theta_0}(\rho) &= F(P^{\mathcal{E}},\theta_0) = F\left(P_{\mathcal{P}}^{\mathcal{E}},\theta_0\right) \leqslant \max_{\mathcal{M}} F\left(P_{\mathcal{M}}^{\mathcal{E}},\theta_0\right)\\ &= F_Q(\rho,\mathcal{E},\theta_0) \leqslant \max_{\mathcal{E}\in G} F_Q(\rho,\mathcal{E},\theta_0) = C_Q^{\theta_0}(\rho).\end{aligned}\tag{38}$$

Conversely, from Lemma 2 we can immediately arrive at

$$C^{\theta_0}(\rho) \geqslant C_Q^{\theta_0}(\rho),\tag{39}$$

and thus we can get $C_Q^{\theta_0}(\rho) = C^{\theta_0}(\rho)$, which finishes the proof. ■

We have shown that the coherence measures based on the QFI and FI subject to the postselective parametrization are equivalent to each other. The most distinct advantage of this type of coherence measure is that it can be straightforwardly connected with the parameter estimation process in terms of the Cramér-Rao bound [46–48,69].

Let us consider an incoherent nonunitary parametrization $\mathcal{E} = \{E_x(\theta)\} \in G$ on ρ as introduced previously. Then we will obtain a probability distribution $P_{\mathcal{M}}^{\mathcal{E}}(x|\theta)$ through a POVM on ρ_θ or obtain $P^{\mathcal{E}}(x|\theta)$ directly through postselection of $\mathcal{E}$. With maximum likelihood estimators $\hat{\theta}_{\mathcal{M}}$ with respect to $P_{\mathcal{M}}^{\mathcal{E}}$ or $\hat{\theta}$ with respect to $P^{\mathcal{E}}$, the Cramér-Rao bound can be asymptotically attained. That is, the mean square error $(\delta\hat{\theta}_{\mathcal{M}})^2 = E[(\hat{\theta}_{\mathcal{M}}-\theta)^2]$ and $(\delta\hat{\theta})^2 = E[(\hat{\theta}-\theta)^2]$ approach $\frac{1}{nF}$ in the asymptotic sense, where E indicates the expectation value, θ is the true value, and n denotes the runs of detection. Thus, in the asymptotic limit, the estimation accuracy $\frac{1}{n(\delta\hat{\theta}_{\mathcal{M}})^2}$ approaches $F(P_{\mathcal{M}}^{\mathcal{E}},\theta)$, which is naturally bounded by $C_Q^{\theta}(\rho)$ based on Eq. (26). In particular, the bound $C_Q^{\theta}(\rho)$ can be asymptotically achieved with the optimal parametrization process and optimal POVM. Similarly, $\frac{1}{n(\delta\hat{\theta})^2}$ approaches $F(P^{\mathcal{E}},\theta)$ in the asymptotic scenario and simultaneously reaches $C^{\theta}(\rho)$ in an asymptotic sense with an optimal parametrization process. Note that the two measures are equivalent; therefore, our coherence measure can be understood as the optimal accuracy through two different estimation processes as well as the corresponding incoherent nonunitary parametrization.

In fact, the optimized $\mathcal{M}$ in $C_Q^{\theta_0}$ [Eq. (27)] can be replaced by $\mathcal{P}$, the projective measurement on the preferred basis. In this sense, the above two coherence measures have an equivalent expression $C_{\mathcal{P}}^{\theta_0}(\rho) = \max_{\mathcal{E}\in G} F(P_{\mathcal{P}}^{\mathcal{E}},\theta_0)$. This can be understood as follows. We first have $C^{\theta_0}(\rho) \leqslant C_{\mathcal{P}}^{\theta_0}(\rho)$ from the second equality in Eq. (38). Note that $\mathcal{M}$ in $C_Q^{\theta_0}(\rho)$ contains a projective measurement, which implies that $C_Q^{\theta_0}(\rho) \geqslant C_{\mathcal{P}}^{\theta_0}(\rho)$. Combining the above two inequalities with Theorem 2, we obtain $C_{\mathcal{P}}^{\theta_0}(\rho) = C^{\theta_0}(\rho) = C_Q^{\theta_0}(\rho)$. Although they are identical in value, they imply different details of operational meanings and give us different ways to understand coherence.

III. CONNECTION WITH QFI BASED ON UNITARY PARAMETRIZATION

Although the coherence measure has obvious operational meaning based on quantum metrology, an analytically computable expression seems not to be easy. Next we will show that for a two-dimensional quantum state, the analytic result could be obtained and the coherence measure can be realized by FI with unitary parametrization. However, our measure is not equivalent to that based on unitary parametrization in high-dimensional cases, which is proved later.

Theorem 3. For a two-dimensional state ρ, the coherence based on Theorem 1 can be given as

$$C^{\theta_0}(\rho) = F_Q(\rho,U_\theta,\theta_0),\tag{40}$$

where F_Q is the QFI of ρ subject to unitary parametrization $U_\theta = e^{i\theta}|1\rangle\langle 1| + |2\rangle\langle 2|$.

Proof. For qubit states ρ, let the IO $\{E_x\} \in G$ read

$$E_x(\theta) = a_1'^x e^{ih_1'^x(\theta)}|f_x(1)\rangle\langle 1| + a_2'^x e^{ih_2'^x(\theta)}|f_x(2)\rangle\langle 2|,\tag{41}$$

where $a_1'^x$ or $a_2'^x$ may be zero. The Kraus operator could be written as

$$E_x(\theta) = a_1^x e^{ih_1^x(\theta)}|f_x(1)\rangle\langle 1| + a_2^x e^{ih_2^x(\theta)}|f_x(2)\rangle\langle 2|,\tag{42}$$

where $a_j^x = a_j'^x e^{ih_j'^x(\theta_0)}$ and $h_j^x(\theta) = h_j'^x(\theta) - h_j'^x(\theta_0)$ for $j = 1, 2$. According to Lemma 1 and its proof, the optimal IO can be rank-1 with the form $\{|i\rangle\langle\psi_i^x(\theta)|\}$, which means $f_x(1) = f_x(2)$ for any x. Then we have

$$\begin{aligned}P(x|\theta) = \mathrm{tr}\,\big[&|a_1^x|^2\rho_{11}|f_x(1)\rangle\langle f_x(1)| + |a_2^x|^2\rho_{22}|f_x(2)\rangle\langle f_x(2)|\\ &+ \rho_{12}a_1^x a_2^{x*}e^{i[h_1^x(\theta)-h_2^x(\theta)]}|f_x(1)\rangle\langle f_x(2)|\\ &+ \rho_{21}a_1^{x*}a_2^x e^{-i[h_1^x(\theta)-h_2^x(\theta)]}|f_x(2)\rangle\langle f_x(1)|\big]\\ = &|a_1^x|^2\rho_{11} + |a_2^x|^2\rho_{22} + \rho_{12}a_1^x a_2^{x*}e^{i[h_1^x(\theta)-h_2^x(\theta)]}\\ &+ \rho_{21}a_1^{x*}a_2^x e^{-i[h_1^x(\theta)-h_2^x(\theta)]},\end{aligned}\tag{43}$$

and thus

$$\begin{aligned}F(P,\theta_0) &= \sum_x \frac{\left[2\,\mathrm{Im}\left(\rho_{12}a_1^x a_2^{x*}\right)\right]^2\left[\partial_\theta h_1^x(\theta)\big|_{\theta_0} - \partial_\theta h_2^x(\theta)\big|_{\theta_0}\right]^2}{|a_1^x|^2\rho_{11} + |a_2^x|^2\rho_{22} + 2\mathrm{Re}\left(\rho_{12}a_1^x a_2^{x*}\right)}\\ &\leqslant \sum_x \frac{\left[2\,\mathrm{Im}\left(\rho_{12}a_1^x a_2^{x*}\right)\right]^2}{|a_1^x|^2\rho_{11} + |a_2^x|^2\rho_{22} + 2\,\mathrm{Re}\left(\rho_{12}a_1^x a_2^{x*}\right)},\end{aligned}\tag{44}$$

where the inequality could be saturated by the function taken as $h_1^x(\theta) = \theta$ and $h_2^x(\theta) = 0$ and the corresponding IO reads $E_x(\theta) = K_x U_\theta$, with

$$\begin{aligned}K_x &= a_1^x|f_x(1)\rangle\langle 1| + a_2^x|f_x(2)\rangle\langle 2|,\\ U_\theta &= e^{i\theta}|1\rangle\langle 1| + |2\rangle\langle 2|,\end{aligned}\tag{45}$$

where $f_x(1) = f_x(2)$ and $\{K_x\} \in G_1$. In this sense, the probability distribution can be rewritten as

$$\begin{aligned}P^{\mathcal{E}}(x|\theta) &= \mathrm{tr}[E_x(\theta)\rho E_x(\theta)^\dagger] = \mathrm{tr}(K_x U_\theta\rho U_\theta^\dagger K_x^\dagger)\\ &= \mathrm{tr}(U_\theta\rho U_\theta^\dagger K_x^\dagger K_x) = P_{\mathcal{M}}(x|\theta),\end{aligned}\tag{46}$$

where $P_{\mathcal{M}}$ can be understood as a distribution generated by a unitary parametrization U_θ followed by a rank-1 POVM $\mathcal{M} =$

$\{K_x^\dagger K_x\}$. Considering the above optimal IO, we arrive at

$$C^{\theta_0}(\rho) = \max_{\mathcal{E}\in G_1} F(P^{\mathcal{E}}, \theta_0) = \max_{\mathcal{M}} F(P_{\mathcal{M}}, \theta_0) = F_Q(\rho, U_\theta, \theta_0), \tag{47}$$

which finishes the proof. ∎

In fact, in the general high-dimensional case, C^{θ_0} is distinct from the FI with unitary parametrization. To demonstrate the difference, we will give a concrete example. Consider a state with maximal coherence

$$|\phi\rangle = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T \tag{48}$$

and the parametrization $\mathcal{E} = \{E_x(\theta)\}$ expressed as

$$E_x(\theta) = a_1^x e^{ih_1^x\theta}|f_x(1)\rangle\langle 1| + a_2^x e^{ih_2^x\theta}|f_x(1)\rangle\langle 2| + a_3^x e^{ih_3^x\theta}|f_x(1)\rangle\langle 3|, \tag{49}$$

with a_n^x and h_n^x $(x = 1, \ldots, 9)$ to be given at the end. Define $\rho = |\phi\rangle\langle\phi|$. The probability distribution is

$$\begin{aligned}P(x|0) &= \mathrm{tr}[E_x(\theta)|\phi\rangle\langle\phi|E_x(\theta)^\dagger] \\ &= \rho_{11}|a_1^x|^2 + \rho_{22}|a_2^x|^2 + \rho_{33}|a_3^x|^2 \\ &\quad + 2\,\mathrm{Re}\left(\rho_{12}a_1^x a_2^{x*} + \rho_{12}a_2^x a_3^{x*} + \rho_{31}a_3^x a_1^{x*}\right)\end{aligned} \tag{50}$$

and

$$\begin{aligned}\partial_\theta P(x|\theta)|_0 &= 2\,\mathrm{Im}\left[\rho_{12}a_1^x a_2^{x*}\left(h_1^x - h_2^x\right) + \rho_{23}a_2^x a_3^{x*}\left(h_2^x - h_3^x\right)\right. \\ &\quad \left. + \rho_{31}a_3^x a_1^{x*}\left(h_3^x - h_1^x\right)\right].\end{aligned} \tag{51}$$

Therefore, the corresponding FI reads

$$F(P^{\mathcal{E}}, 0) = \sum_x \frac{[\partial_\theta P(x|\theta)|_0]^2}{P(x|0)} = 0.9410. \tag{52}$$

From the definition, we have $C^0(\rho) \geqslant F(P^{\mathcal{E}}, 0)$.

To compare our measure with the QFI subject to the optimal unitary parametrization in G, we calculate $\max_{U_\theta\in G} F_Q(|\phi\rangle, U_\theta, 0)$, where U_θ is the unitary operator expressed as

$$U_\theta = \sum_n e^{ih_n(\theta)}|n\rangle\langle n|, \tag{53}$$

with $\partial_\theta h_n(\theta) \in [0, 1]$ [based on Appendix A, other cases with different range of $\partial_\theta h_n(\theta)$ lead to the same conclusion]. When eigenvalues of the parametrized state $U_\theta\rho U_\theta^\dagger$ are parameter independent, the QFI could be calculated from the equation [44,70]

$$F_Q(\rho, U_\theta, \theta_0) = \sum_{ij} \frac{2(P_i - P_j)^2}{P_i + P_j}|\langle\varphi_i|\partial_\theta\varphi_j\rangle|^2, \tag{54}$$

where $\{P_i\}$ and $\{|\varphi_i\rangle\}$ denote the eigenvalues and eigenvectors of $U_\theta\rho U_\theta^\dagger$, respectively, and we use $|\partial_\theta\varphi_j\rangle$ to briefly express the partial derivative $\frac{\partial|\varphi_j\rangle}{\partial\theta}|_{\theta_0}$. In addition, the terms with $P_i = P_j = 0$ are not included in the summation. Further, for a pure state $\rho = |\psi\rangle\langle\psi|$, let $\{|\psi_i\rangle\}$ be the basis vectors satisfying $|\psi\rangle = |\psi_1\rangle$; then the corresponding $P_1 = 1$ and the residual eigenvalues P_i $(i \neq 1)$ are zero. Then the eigenvectors of $U_\theta\rho U_\theta^\dagger$ are $\{U_\theta|\psi_i\rangle\}$. Defining

$$H_\theta = \sum_n \partial_\theta h_n(\theta)|n\rangle\langle n|, \tag{55}$$

we have

$$\begin{aligned}&F_Q(|\psi\rangle, U_\theta, \theta_0) \\ &= \sum_i \frac{2(1 - P_i)^2}{1 + P_i}\langle\psi|U_{\theta_0}^\dagger U_{\theta_0}H_{\theta_0}|\psi_i\rangle\langle\psi_i|H_{\theta_0}U_{\theta_0}^\dagger U_{\theta_0}|\psi\rangle \\ &\quad + \sum_i \frac{2(P_i - 1)^2}{P_i + 1}\langle\psi_i|U_{\theta_0}^\dagger U_{\theta_0}H_{\theta_0}|\psi\rangle\langle\psi|H_{\theta_0}U_{\theta_0}^\dagger U_{\theta_0}|\psi_i\rangle \\ &= 4\langle\psi|H_{\theta_0}\sum_i|\psi_i\rangle\langle\psi_i|H_{\theta_0}|\psi\rangle - 4\langle\psi|H_{\theta_0}|\psi\rangle\langle\psi|H_{\theta_0}|\psi\rangle \\ &= 4\langle\psi|H_{\theta_0}^2|\psi\rangle - 4\langle\psi|H_{\theta_0}|\psi\rangle^2.\end{aligned} \tag{56}$$

This result does not depend on the choice of $|\psi_i\rangle$ as long as $|\psi\rangle = |\psi_1\rangle$, and the optimal QFI $\max_{U_\theta\in G} F_Q(|\phi\rangle, U_\theta, 0)$ can be calculated as

$$\max_{U_\theta\in G} F_Q(|\phi\rangle, U_\theta, 0) = \max_{H\in S} 4\langle\phi|H^2|\phi\rangle - 4\langle\phi|H|\phi\rangle^2 = \tfrac{8}{9}, \tag{57}$$

where $|\phi\rangle$ is the three-dimensional MCS in Eq. (48) and S is the set of operators $H = h_1|1\rangle\langle 1| + h_2|2\rangle\langle 2| + h_3|3\rangle\langle 3|$ $(h_i \in [0, 1])$. Thus $C^0(\rho) > \max_{U_\theta\in G} F(|\phi\rangle, U_\theta, 0)$, which indicates that C^{θ_0} is different from the FI with unitary parametrization.

Finally, we present all the coefficients of E_x in the above calculation by defining $A^x = [a_1^x, a_2^x, a_3^x]$, where

$$\begin{aligned}A^1 &= \frac{[0, \sqrt{0.4}, \sqrt{0.6}]}{\sqrt{3}}, \\ A^2 &= \frac{[0, \sqrt{0.4}e^{-i2\pi/3}, \sqrt{0.6}e^{i2\pi/3}]}{\sqrt{3}}, \\ A^3 &= \frac{[0, \sqrt{0.4}e^{-i4\pi/3}, \sqrt{0.6}e^{i4\pi/3}]}{\sqrt{3}}, \\ A^4 &= \frac{[\sqrt{0.4}, \sqrt{0.6}, 0]}{\sqrt{3}}, \\ A^5 &= \frac{[\sqrt{0.4}, \sqrt{0.6}e^{i2\pi/3}, 0]}{\sqrt{3}}, \\ A^6 &= \frac{[\sqrt{0.4}, \sqrt{0.6}e^{i4\pi/3}, 0]}{\sqrt{3}}, \\ A^7 &= \frac{[\sqrt{0.6}, 0, \sqrt{0.4}]}{\sqrt{3}}, \\ A^8 &= \frac{[\sqrt{0.6}, 0, \sqrt{0.4}e^{i2\pi/3}]}{\sqrt{3}}, \\ A^9 &= \frac{[\sqrt{0.6}, 0, \sqrt{0.4}e^{i4\pi/3}]}{\sqrt{3}}.\end{aligned} \tag{58}$$

In addition,

$$\begin{aligned}&h_1^x = 0, \quad h_2^x = 1, \quad h_3^x = 0, \quad x = 1, 2, 3 \\ &h_1^x = 1, \quad h_2^x = 0, \quad h_3^x = 0, \quad x = 4, \ldots, 9.\end{aligned} \tag{59}$$

IV. CONCLUSION

In this paper we have established coherence measures based on FI subject to the incoherent nonunitary parametrization process. The coherence measure could be defined by two forms based on FI or QFI, which both imply the direct operational meaning by the connection with the parameter estimation accuracy. In addition, we compared our measure with QFI in unitary parametrization and found that in the qubit case, our coherence measure can be equivalently understood through unitary parametrization and can be analytically calculated. Our coherence also sheds new light on the roles of the nonunitary parametrization process.

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APPENDIX A: REGION OF $\partial_\theta h(\theta)$

In the main text $C_Q^{\theta_0}$ and C^{θ_0} were defined under a certain condition $\frac{\partial h_n^x(\theta)}{\partial\theta} \in [0, 1]$. In fact, measures defined under other conditions can be transformed to the original C^{θ_0}.

We first consider the case that $\frac{\partial h_n^x(\theta)}{\partial\theta}$ is finite and suppose $\max_{n,x} |\frac{\partial h_n^x(\theta)}{\partial\theta}| \leqslant k$ (k is finite). Define $\tilde{C}_k^{\theta_0}$ as the function defined in a similar way to C^{θ_0} (in Theorem 1) but with the different condition $\max_{n,x} |\frac{\partial h_n^x(\theta)}{\partial\theta}| \leqslant k$ and $G^{(k)}$ as the set of the corresponding channels, namely,

$$\tilde{C}_k^{\theta_0}(\rho) = \max_{\mathcal{E}\in G^{(k)}} F(\tilde{P}^{\mathcal{E}}, \theta_0), \tag{A1}$$

where

$$\tilde{P}^{\mathcal{E}}(x|\theta) = \mathrm{tr}[\tilde{E}_x(\theta)\rho\tilde{E}_x(\theta)^\dagger],$$
$$\tilde{E}_x(\theta) = \sum_n a_n^x e^{ih_n^x(\theta)}|f_x(n)\rangle\langle n|, \tag{A2}$$

and $\max\limits_{n,x} |\frac{\partial h_n^x(\theta)}{\partial\theta}| \leqslant k$. We find that $\tilde{C}_k^{\theta_0}$ has a connection with the previous coherence measure.

Lemma 3. The function $\tilde{C}_k^{\theta_0}$ satisfies that

$$C^{\gamma_0}(\rho) = \frac{1}{4k^2}\tilde{C}_k^{\theta_0}(\rho), \tag{A3}$$

where $\gamma_0 = 2k\theta_0$.

In this sense, investigation under the condition $\frac{\partial h_n^x(\theta)}{\partial\theta} \in [0, 1]$ could cover all other situations where k is finite. Next we give a brief proof.

Proof. Suppose $\{E_x\}$ are Kraus operators of the channel in $G^{(1/2)}$, namely,

$$E_x(\gamma) = \sum_n a_n^x e^{iu_n^x(\gamma)}|f_x(n)\rangle\langle n|, \tag{A4}$$

where $|\partial_\gamma u_n^x| \leqslant \frac{1}{2}$. Define

$$\hat{E}_x(\gamma) = e^{i\gamma/2}E_x(\gamma) = \sum_n a_n^x e^{iv_n^x(\gamma)}|f_x(n)\rangle\langle n|, \tag{A5}$$

where $v_n^x(\gamma) = u_n^x(\gamma) + \gamma/2$ and thus $\partial_\gamma v_n^x \in [0, 1]$. The two channels lead to identical effects, that is,

$$\begin{aligned}\hat{E}_x(\gamma)\rho\hat{E}_x(\gamma)^\dagger &= e^{i\gamma/2}E_x(\gamma)\rho e^{-i\gamma/2}E_x(\gamma)^\dagger\\ &= E_x(\gamma)\rho E_x(\gamma)^\dagger,\end{aligned} \tag{A6}$$

from which we have

$$C^{\gamma_0}(\rho) = \tilde{C}_{1/2}^{\gamma_0}(\rho). \tag{A7}$$

Consider $\tilde{E}_x$ in Eq. (A2). Defining S_l^x as the set satisfying $f_x(n) = l$ when $n \in S_l^x$, then

$$\tilde{P}(x|\theta) = \sum_l \sum_{n,n'\in S_l^x} \rho_{nn'} a_n^x a_{n'}^{x*} e^{i[h_n^x(\theta) - h_{n'}^x(\theta)]} \tag{A8}$$

and

$$\begin{aligned}\partial_\theta \tilde{P}(x|\theta)|_{\theta_0} = \sum_l \sum_{n,n'\in S_l^x} &\big\{\rho_{nn'} a_n^x a_{n'}^{x*} e^{i[h_n^x(\theta_0) - h_{n'}^x(\theta_0)]}\\ &\times i\big[\partial_\theta h_n^x(\theta)\big|_{\theta_0} - \partial_\theta h_{n'}^x(\theta)\big|_{\theta_0}\big]\big\}.\end{aligned} \tag{A9}$$

Defining $\gamma = 2k\theta$, then

$$\tilde{P}(x|\theta) = \sum_l \sum_{n,n'\in S_l^x} \rho_{nn'} a_n^x a_{n'}^{x*} \exp\left\{i\left[h_n^x\left(\frac{\gamma}{2k}\right) - h_{n'}^x\left(\frac{\gamma}{2k}\right)\right]\right\}, \tag{A10}$$

and thus $\tilde{P}(x|\theta)$ could be rewritten as $P(x|\gamma)$. Defining $g_n^x(\gamma) = h_n^x(\frac{\gamma}{2k})$, then $\partial_\gamma g_n^x(\gamma) = \frac{\partial_\theta h_n^x(\theta)}{2k}$, and thus $|\partial_\gamma g_n^x(\gamma)| \leqslant \frac{1}{2}$. In addition,

$$\begin{aligned}&\partial_\gamma P(x|\gamma)|_{\gamma_0}\\ &= \sum_l \sum_{n,n'\in S_l^x} \left(\rho_{nn'} a_n^x a_{n'}^{x*} \exp\left\{i\left[h_n^x\left(\frac{\gamma_0}{2k}\right) - h_{n'}^x\left(\frac{\gamma_0}{2k}\right)\right]\right\}\right.\\ &\quad\left.\times i\left[\partial_\gamma h_n^x\left(\frac{\gamma}{2k}\right)\bigg|_{\gamma_0} - \partial_\gamma h_{n'}^x\left(\frac{\gamma}{2k}\right)\bigg|_{\gamma_0}\right]\right)\\ &= \frac{1}{2k}\sum_l \sum_{n,n'\in S_l^x} \big\{\rho_{nn'} a_n^x a_{n'}^{x*} e^{i[h_n^x(\theta_0) - h_{n'}^x(\theta_0)]}\\ &\quad\times i\big[\partial_\theta h_n^x(\theta)\big|_{\theta_0} - \partial_\theta h_{n'}^x(\theta)\big|_{\theta_0}\big]\big\}\\ &= \frac{1}{2k}\partial_\theta \tilde{P}(x|\theta)|_{\theta_0},\end{aligned} \tag{A11}$$

where $\gamma_0 = 2k\theta_0$. Thus

$$\begin{aligned}F(P, \gamma_0) &= \sum_x \frac{[\partial_\gamma P(x|\gamma)|_{\gamma_0}]^2}{P(x|\gamma_0)}\\ &= \sum_x \frac{[\partial_\theta \tilde{P}(x|\theta)|_{\theta_0}]^2}{4k^2\tilde{P}(x|\theta_0)} = \frac{F(\tilde{P}, \theta_0)}{4k^2}.\end{aligned} \tag{A12}$$

Combining this with Eq. (A7), we have

$$C^{\gamma_0}(\rho) = \frac{1}{4k^2}\tilde{C}^{\theta_0}(\rho). \tag{A13}$$

■

The above proof shows that if $\partial_\theta h_n^x(\theta)$ is finite, the investigation under the condition $\partial_\theta h_n^x(\theta) \in [0, 1]$ could cover all other general cases. However, if $\partial_\theta h_n^x(\theta)$ is infinite, the FI and

$C^{\theta_0}(\rho)$ will be infinite. In addition, physical models generally lead to a finite $\partial_\theta h_n^x(\theta)$; for example, the parametrization Ramsey interferometer could be written as $U_\theta = \exp(-i\theta J_z)$ [42–44] and the corresponding $\partial_\theta h_n^x(\theta)$ is finite (J_z is the z component of the total angular momentum). Thus we could focus on the finite case.

APPENDIX B: DILATION OF THE OPTIMAL CHANNEL IN G

For the estimation process in Eq. (3), the optimal channel in G_1 could be written as

$$E_x(\theta) = \sum_n b_n^x(\theta)|1\rangle\langle n|. \tag{B1}$$

We denote by $\mathcal{H}_A$ (d dimension) the space for it. Assuming $|x_B\rangle$ are basis vectors in another space $\mathcal{H}_B$ (L dimension), we construct the following states in $\mathcal{H}_B$:

$$|\psi_B^n\rangle = \sum_{x=1}^{L} b_n^x(\theta)|x_B\rangle, \quad n = 1, 2, \ldots, d. \tag{B2}$$

From $\sum_x E_x^\dagger E_x = \mathcal{I}$ we have $\langle\psi_B^m|\psi_B^n\rangle = \delta_{nm}$. With states in Eq. (B2), we can always find the other $|\psi_B^x\rangle$ ($x = d + 1, \ldots, L$) to form a basis set together in $\mathcal{H}_B$. Defining

$$U^B = \sum_x |\psi_B^x\rangle\langle x_B|, \tag{B3}$$

clearly $U^{B\dagger}U^B = U^B U^{B\dagger} = I^B$. Then we can construct a controlled unitary [65–67] in $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$U^{AB} = |1_A\rangle\langle 1_A| \otimes U^B + \mathbb{I}_1^A \otimes \mathbb{I}^B, \tag{B4}$$

where $\mathbb{I}_1^A$ is the identity operator in the residual subspace of H^A. Obviously, U^{AB} is a unitary operator in $H_A \otimes H_B$. Defining

$$V = \sum_n (|1_A\rangle\langle n_A| \otimes |n_B\rangle\langle 1_B| + |n_A\rangle\langle 1_A| \otimes |1_B\rangle\langle n_B|)$$
$$- |1_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 1_B| \tag{B5}$$

and

$$W = V + \mathbb{I}_2, \tag{B6}$$

where $\mathbb{I}_2$ is the identity operator from the residual subspace which eliminates $VV^\dagger$ in $H^A \otimes H^B$ and W is a unitary operator swapping specified states, it is easy to see that

$$E_x(\theta) = \langle x_B|U^{AB}W|1_B\rangle. \tag{B7}$$

Based on the Stinespring dilation theorem [64], $\{E_x(\theta)\}$ could be implemented by a unitary $U^{AB}W$ on $\rho \otimes |1_B\rangle\langle 1_B|$ and projective measurement $\{|x_B\rangle\langle x_B|\}$.

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