Geometric property of energetic cost in transitionless quantum driving

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Transitionless quantum driving is a useful approach to mimic the adiabatic evolution of a quantum system so that the system can rapidly evolve from its initial state to the target state without transitions between instantaneous eigenstates of the reference Hamiltonian. To quantify the energetic cost in the transitionless quantum driving process, the instantaneous cost is defined as the Frobenius norm of the driving Hamiltonian and the integral of instantaneous cost is used to describe the total cost in the entire evolution. In this paper, we find that the minimal integral of instantaneous cost has a geometric property being equal to the length of the evolution path on the Riemannian manifold spanned by the control parameters with a positive definite metric, but independent of the evolution details such as the changing rate of the parameters. Based on this property, we further show that the optimal transitionless quantum driving with the minimum total cost corresponds to the geodesic path on the Riemannian manifold, which provides a method for optimizing transitionless quantum driving.

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I. INTRODUCTION

As a fundamental element of quantum technologies, accurate and noise-resistant quantum control is a topic that has attracted much attention. However, controlling the evolution of a quantum system with high fidelity is a difficult task due to the existence of control errors and environment-induced decoherence. The adiabatic theorem [1–3] inspires adiabatic control techniques [4–8], which are inherently resistant to control errors but require the quantum system to evolve adiabatically. Adiabatic evolutions must satisfy the adiabatic condition which requires a long run time [9], and this makes adiabatic control techniques vulnerable to environment-induced decoherence. Therefore, accelerating the adiabatic evolution process is an interesting issue in the development of quantum technologies.

Shortcuts to adiabaticity [10,11], which include transitionless quantum driving [12–18], invariant-based inverse engineering [19,20], and fast-forward approaches [21,22], were used to speed up adiabatic processes in a number of applications, such as quantum computation [23–30], quantum heat engines [31–33], quantum state preparation [23,34–38], and so on [39–44]. This makes it possible to achieve fast and robust quantum control. Among varieties of shortcuts to adiabaticity, transitionless quantum driving is a useful approach to mimic the adiabatic evolution of a quantum system, so that the system can rapidly evolve from its initial state to the target state without any transitions between instantaneous eigenstates of the reference Hamiltonian. Transitionless quantum driving was applied to realize nonadiabatic geometric quantum computation [24], nonadiabatic holonomic quantum computation [25], and the merits of both its rapidity and robustness were verified by experiments in nitrogen-vacancy centers [26] and superconducting circuits [27,28].

The energetic cost associated with a quantum evolution process is often neglected when we are only interested in a particular quantum phenomenon, but it is vital to take such cost into account when developing quantum technologies [45–48]. To quantify the energetic cost in the transitionless quantum driving process, cost functions based on the Frobenius norm of the driving Hamiltonian were proposed [17,45,49–56]. Among these, the instantaneous cost function was defined as the Frobenius norm of the driving Hamiltonian [17,45], and its integral was used to describe the total cost in the entire evolution process [49]. They are closely related to the energy coupling constants and the level gaps of a quantum system, which are typical characteristics of a quantum evolution process. Minimizing the energetic cost is relevant to the quantum technologies based on transitionless quantum driving. The question is how to find the driving Hamiltonian that can minimize the cost of transitionless quantum driving. Several efforts were undertaken to address this issue [45,50,56]. Specifically, a minimal energy-demanding scheme for transitionless quantum driving was proposed in Ref. [50], where the Hamiltonian that minimizes the energetic cost was obtained by properly choosing the quantum phases.

In this paper, we find that the minimal integral of instantaneous cost has a geometric property being equal to the length of the evolution path on the Riemannian manifold spanned by the control parameters with a positive-definite metric, but independent of the evolution details such as the changing rate of the parameters. Based on this property, we further show that the optimal transitionless quantum driving with the minimum total cost corresponds to the geodesic path on the Riemannian manifold. That is, finding the optimal control Hamiltonian for transitionless quantum driving is equivalent to finding the geodesic path on the Riemannian manifold, which provides

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a method for optimizing transitionless quantum driving. Furthermore, we give two examples to illustrate the usefulness of our finding.

II. GEOMETRY OF ENERGETIC COST

First, we introduce the approach of transitionless quantum driving that is used to speed up the adiabatic process of the quantum system.

Consider an N-dimensional quantum system defined by the time-dependent Hamiltonian $H_{\rm ref}(\vec{\lambda}(t))$, known as the reference Hamiltonian, with its instantaneous eigenvalues $\{E_n(\vec{\lambda}(t))\}$ and eigenstates $\{|E_n(\vec{\lambda}(t))\rangle\}$, where $\vec{\lambda}(t) = (\lambda^1(t), \lambda^2(t), \ldots, \lambda^M(t))$ are time-dependent parameters. The Hamiltonian $H_{\rm ref}(\vec{\lambda}(t))$ varies along the curve in parameter space that connects the initial point $\vec{\lambda}(0)$ and the final point $\vec{\lambda}(\tau)$, where τ is the final time. Hereafter, we use $H_{\rm ref}(t)$, $E_n(t)$, and $|E_n(t)\rangle$ to denote $H_{\rm ref}(\vec{\lambda}(t))$, $E_n(\vec{\lambda}(t))$, and $|E_n(\vec{\lambda}(t))\rangle$, respectively, for simplicity. According to the adiabatic theorem, if $H_{\rm ref}(t)$ varies slowly enough, the system initially in the nth eigenstate $|E_n(0)\rangle$ will evolve transitionlessly along the instantaneous eigenstate $|E_n(t)\rangle$ up to a phase factor.

However, a low evolution speed means a long run time from the initial state to the final state, which may make the system vulnerable to environment-induced decoherence. To speed up the adiabatic process of the quantum system, transitionless quantum driving [14] can be used. The key idea of transitionless quantum driving is to find a driving Hamiltonian $H(t)$ instead of $H_{\rm ref}(t)$, which can drive the system exactly along the eigenstates of $H_{\rm ref}(t)$ but need not vary slowly. To find such a Hamiltonian, we can start from the evolution operator

$$U(t) = \sum_n^N e^{i\gamma_n(t)}|E_n(t)\rangle\langle E_n(0)|, \tag{1}$$

which can guarantee that the system initially in $|E_n(0)\rangle$ evolves along the nth instantaneous eigenstate up to the phase factor $e^{i\gamma_n(t)}$. Here, $\gamma_n(t)$ can be an arbitrary real function of t.

By using $H(t) = -iU(t)\dot{U}^\dagger(t)$, we immediately obtain the Hamiltonian,

$$H(t) = i\sum_n^N [|\partial_t E_n(t)\rangle\langle E_n(t)| + i\dot{\gamma}_n(t)|E_n(t)\rangle\langle E_n(t)|]. \tag{2}$$

The system derived by the driving Hamiltonian $H(t)$ will evolve from the initial point $\vec{\lambda}(0)$ to the final point $\vec{\lambda}(\tau)$ in the parameter space along the same path as that defined by the reference Hamiltonian $H_{\rm ref}(t)$ without any transitions between the eigenstates and $|\phi_n(t)\rangle = e^{i\gamma_n(t)}|E_n(t)\rangle$ satisfies the Schrödinger equation defined by $H(t)$.

Second, we show that the minimal integral of instantaneous cost has a geometric property being equal to the length of the evolution path on the Riemannian manifold spanned by the control parameters with a positive-definite metric but independent of the evolution details such as the changing rate of the parameters.

To quantify the energetic cost associated with transitionless quantum driving, cost functions based on the Frobenius norm of the driving Hamiltonian were proposed [17,45,49–56]. Although the concrete expressions of the functions are slightly different depending on the physical implementation, the norm of the driving Hamiltonian plays the most crucial role in the definition of cost. Here, we take the Frobenius norm of the driving Hamiltonian, $\|H(t)\| = \sqrt{{\rm tr}[H^\dagger(t)H(t)]}$, as the instantaneous cost [17,45] and use its integral to describe the total cost in the entire evolution [49], which is expressed as

$$C = \int_0^\tau \|H(t)\|\, dt. \tag{3}$$

The integral C can also be interpreted as the action arising from the driving Hamiltonian [17].

Substituting Eq. (2) into Eq. (3), we can obtain the integral C corresponding to the driving Hamiltonian $H(t)$. Obviously, the integral C is dependent on the choice of $\gamma_n(t)$ in Eq. (1). For a given evolution path $\vec{\lambda}(t)$, $t \in [0, \tau]$, the integral C takes the minimal value if and only if $\gamma_n(t) = -i\int_0^t \langle\partial_{t'}E_n(t')|E_n(t')\rangle dt'$ [50]. Therefore, in the condition of the integral C taking minimal value, the driving Hamiltonian $H(t)$ must have the form

$$H(t) = i\sum_n^N [|\partial_t E_n(t)\rangle\langle E_n(t)| + \langle\partial_t E_n(t)|E_n(t)\rangle|E_n(t)\rangle\langle E_n(t)|]. \tag{4}$$

Since $\sum_n^N |\partial_t E_n(t)\rangle\langle E_n(t)| = \sum_{m,n}^N \langle E_m(t)|\partial_t E_n(t)\rangle|E_m(t)\rangle\langle E_n(t)|$ and $\langle\partial_t E_n(t)|E_n(t)\rangle = -\langle E_n(t)|\partial_t E_n(t)\rangle$, we obtain

$$\begin{aligned} H(t) &= i\sum_{m,n}^N \langle E_m(t)|\partial_t E_n(t)\rangle|E_m(t)\rangle\langle E_n(t)| \\ &\quad + i\sum_n^N \langle\partial_t E_n(t)|E_n(t)\rangle|E_n(t)\rangle\langle E_n(t)| \\ &= i\sum_{m,n\,(m\neq n)}^N \langle E_m(t)|\partial_t E_n(t)\rangle|E_m(t)\rangle\langle E_n(t)|. \end{aligned} \tag{5}$$

It is interesting to note that, for the system driven by $H(t)$, the parallel transport condition, $\langle\phi_n(t)|H(t)|\phi_n(t)\rangle = 0$, i.e., $\langle\phi_n(t)|\partial_t\phi_n(t)\rangle = 0$, for $n = 1, 2, \ldots, N$, is automatically satisfied and hence the phase $\gamma_n(t) = -i\int_0^t \langle\partial_{t'}E_n(t')|E_n(t')\rangle dt'$ is a purely geometric phase, which is useful for nonadiabatic geometric computation [57–63].

Note that the phrase "minimal integral" is used to represent the least value of the integral of instantaneous cost for a given evolution path. Obviously, different evolution paths in parameter space may have different minimal values of the integral, and we will use the phrase "minimum integral" to represent the least value of the integral in all the paths considered.

We now proceed to reveal the geometric property of the minimal integral of instantaneous cost in the transitionless quantum driving process. The Frobenius norm of the driving

Hamiltonian in Eq. (5) can be then expressed as

$$\|H(t)\| = \left(\sum_{m,n\,(m\neq n)}^{N} \langle \partial_t E_n(t)|E_m(t)\rangle\langle E_m(t)|\partial_t E_n(t)\rangle \right)^{\frac{1}{2}} = \left(\sum_{n}^{N} \langle \partial_t E_n(t)|P_{\perp n}|\partial_t E_n(t)\rangle \right)^{\frac{1}{2}}, \tag{6}$$

where $P_{\perp n} = 1 - |E_n(t)\rangle\langle E_n(t)|$ is the projector operator onto the state space orthogonal to $|E_n(t)\rangle$. Since $|E_n(t)\rangle$, i.e., $\{|E_n(\vec{\lambda}(t))\rangle\}$, is time dependent through parameters $\vec{\lambda}(t) = (\lambda^1(t), \lambda^2(t), \dots, \lambda^M(t))$, expression (6) can be further written as

$$\|H(t)\| = \left(\sum_{n}^{N} \langle \partial_\mu E_n(t)|P_{\perp n}|\partial_\nu E_n(t)\rangle \dot{\lambda^\mu}\dot{\lambda^\nu} \right)^{\frac{1}{2}} = \left(\sum_{n}^{N} g_{\mu\nu}^{(n)} \dot{\lambda^\mu}\dot{\lambda^\nu} \right)^{\frac{1}{2}}, \tag{7}$$

where the repeated indices μ and ν are summed and the metric $g_{\mu\nu}^{(n)}$ is defined as

$$g_{\mu\nu}^{(n)} = \mathrm{Re} Q_{\mu\nu}^{(n)}, \tag{8}$$

with $Q_{\mu\nu}^{(n)} = \langle \partial_\mu E_n(t)|P_{\perp n}|\partial_\nu E_n(t)\rangle$ being the quantum geometric tensor of the $|E_n(t)\rangle$-state manifold [64].

We further define the positive-definite metric

$$g_{\mu\nu} = \sum_{n}^{N} g_{\mu\nu}^{(n)}, \tag{9}$$

which induces a Riemannian manifold on the parameter space. As a result, Eq. (7) can be then recast as

$$\|H(t)\| = \sqrt{g_{\mu\nu}\dot{\lambda^\mu}\dot{\lambda^\nu}}. \tag{10}$$

Substituting Eq. (10) into Eq. (3), we immediately obtain

$$C = \int_0^\tau \sqrt{g_{\mu\nu}\dot{\lambda^\mu}\dot{\lambda^\nu}}\, dt = \int_{\vec{\lambda}(0)}^{\vec{\lambda}(\tau)} \sqrt{g_{\mu\nu} d\lambda^\mu d\lambda^\nu}. \tag{11}$$

Equation (11) clearly indicates that the minimal integral of the Frobenius norm of the Hamiltonian in the evolution process has a geometric property. It is equal to the length of the curve $\vec{\lambda}(t)$ ($t \in [0, \tau]$) on the Riemannian manifold induced by the metric $g_{\mu\nu}$ being only dependent on the evolution path but independent of the evolution speed along the path. Moreover, the rate $\partial_t C$, i.e., the instantaneous cost $\|H(t)\|$, is just equal to the evolution speed $\sqrt{g_{\mu\nu}\dot{\lambda^\mu}\dot{\lambda^\nu}}$ along the curve $\vec{\lambda}(t)$.

Third, we demonstrate that the optimal transitionless evolution with the minimum integral of instantaneous cost corresponds to the geodesic path connecting the initial and final points $\vec{\lambda}(0)$ and $\vec{\lambda}(\tau)$ on the Riemannian manifold.

We show that the minimal integral of instantaneous cost is equal to the length of the evolution path $\vec{\lambda}(t)$ of the quantum system in parameter space. However, different evolution paths in parameter space correspond to different values of the integral. There are infinitely many paths that connect the initial point $\vec{\lambda}(0)$ and the final point $\vec{\lambda}(\tau)$ in parameter space. After having obtained the minimal integral for a given path, it is ready for us to find the path that corresponds to the minimum value of the integral.

The above discussion indicates that for the given initial and final points $\vec{\lambda}(0)$ and $\vec{\lambda}(\tau)$, the shorter the evolution path connecting them, the smaller is the integral C. From this, we can conclude that the evolution path corresponding to the minimum value of the integral C is the geodesic path on the Riemannian manifold spanned by the control parameters with the metric $g_{\mu\nu}$. The geodesic path $\vec{\lambda}(t)$ ($t \in [0, \tau]$) determined by the geodesic equation

$$\frac{d^2\lambda^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma}\frac{d\lambda^\nu}{dt}\frac{d\lambda^\sigma}{dt} = 0, \quad \mu = 1, 2, \dots, M, \tag{12}$$

where the Christoffel symbol is defined as

$$\Gamma^\mu_{\nu\sigma} = \tfrac{1}{2} g^{\mu\rho}(\partial_\sigma g_{\rho\nu} + \partial_\nu g_{\rho\sigma} - \partial_\rho g_{\nu\sigma}), \tag{13}$$

with $g^{\mu\rho} = (g^{-1})_{\mu\rho}$.

By resolving the geodesic equation (12) with the boundary condition given by the initial point $\vec{\lambda}(0)$ and the final point $\vec{\lambda}(\tau)$ in parameter space, we can obtain the evolution path that corresponds to the minimum value of the integral C. Substituting the time-dependent parameters satisfying the geodesic equation into Eq. (5), we can obtain the optimal driving Hamiltonian $H(t)$. In passing, we would like to point out that the result about the minimum integral of instantaneous cost can be regarded as a kind of minimum action principle for transitionless quantum driving.

So far, we fulfilled the general discussions related to the geometric property of the cost in transitionless quantum driving process. Our discussions also provide a method for optimizing transitionless quantum driving. Indeed, starting from the parameterized eigenstates $\{|E_n(\lambda^1(t), \lambda^2(t), \dots, \lambda^M(t))\rangle\}$ of a reference Hamiltonian, we can calculate the metric $g_{\mu\nu}$ and the Christoffel symbol $\Gamma^\mu_{\nu\sigma}$ by using Eqs. (8), (9), and (13), and write out the geodesic equation (12). Then, resolving the geodesic equation, we can obtain the parameters as functions of t. Substituting these parameters into Eq. (5), we can finally work out the driving Hamiltonian $H(t)$. Note that the Hamiltonian that can drive the system along the geodesic path as well as lead to the minimum integral C is not unique. $H[\vec{\lambda}(\alpha(t))]$ is also such a Hamiltonian if and only if $H[\vec{\lambda}(t)]$ is such a Hamiltonian, where $\alpha(t)$ is a monotone-increasing real function.

It is interesting to see that the work of optimizing the transitionless quantum driving is equivalent to resolving the geodesic equation. In fact, geometry-based methods were used in other optimal controls, such as designing the optimal and robust control fields of quantum systems [65,66] and finding the optimal scheme for the shortcut to the isothermal process [67].

III. EXAMPLES

A. Example 1: Optimizing the transitionless quantum driving of a two-level system

We take a two-level system to illustrate our method. Suppose the reference Hamiltonian is

$$H_{\text{ref}}(t) = \Omega_0(\sin\theta\cos\varphi\sigma_x + \sin\theta\sin\varphi\sigma_y + \cos\theta\sigma_z), \quad (14)$$

where $\theta = \theta(t)$ and $\varphi = \varphi(t)$ are time-dependent parameters and σ_x, σ_y, σ_z are Pauli matrices. The parameterized eigenstates of $H_{\text{ref}}(t)$ can be expressed as

$$|E_1(t)\rangle = \sin\frac{\theta}{2}e^{-i\varphi}|0\rangle - \cos\frac{\theta}{2}|1\rangle,$$
$$|E_2(t)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle, \quad (15)$$

corresponding to the eigenvalues $E_1 = -\Omega_0$ and $E_2 = \Omega_0$, respectively. The system being initially in the state $|E_n(0)\rangle$, $n = 1, 2$ will evolve to the target state $|E_n(\tau)\rangle$ along the eigenstate $|E_n(t)\rangle$ if $H(t)$ changes slowly enough.

We aim to find the optimal evolution path by which the driving Hamiltonian can transitionlessly drive the system fast from the initial state to the target state as well as with the minimum integral C. To this end, we follow the method proposed in the last section.

Starting from the parameterized eigenstates given in Eq. (15), we can obtain the metric $g_{\mu\nu}$ by using Eqs. (8) and (9)

$$g = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{\sin^2\theta}{2} \end{pmatrix}. \quad (16)$$

Substituting it into Eq. (13), we then have the Christoffel symbols $\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$ and $\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta$. Substituting the Christoffel symbols into Eq. (12), we finally obtain the geodesic equation

$$\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\varphi}{dt}\right)^2 = 0,$$
$$\frac{d^2\varphi}{dt^2} + 2\cot\theta\,\frac{d\theta}{dt}\frac{d\varphi}{dt} = 0. \quad (17)$$

To resolve the geodesic equation with the given boundary conditions, i.e., the starting and terminal points (θ_0, φ_0) and $(\theta_\tau, \varphi_\tau)$, which correspond to the initial and final states, respectively, we can find out the optimal evolution path $[\theta(t), \varphi(t)]$, $t \in [0, \tau]$. For the two-level system, the optimal evolution path is the minor arc of the great circle on the spherical coordinate system spanned by the control parameters. Substituting Eq. (15) with $\theta = \theta(t)$ and $\varphi = \varphi(t)$ into Eq. (5), we can obtain the driving Hamiltonian in the basis $\{|0\rangle, |1\rangle\}$,

$$H(t) = \frac{1}{2}\begin{pmatrix} \Delta(t) & \Omega(t)e^{-i\Phi(t)} \\ \Omega(t)e^{i\Phi(t)} & -\Delta(t) \end{pmatrix}, \quad (18)$$

where $\Delta(t) = \dot{\varphi}(t)\sin^2\theta(t)$ and $\Omega(t)e^{-i\Phi(t)} = e^{-i\varphi(t)}[-i\dot{\theta}(t) - \dot{\varphi}(t)\sin\theta(t)\cos\theta(t)]$. It can be realized experimentally by applying a near-resonate driving field with Rabi frequency $\Omega(t)$, phase $\Phi(t)$, and detuning $\Delta(t)$ to the two-level system, as shown in Fig. 1.

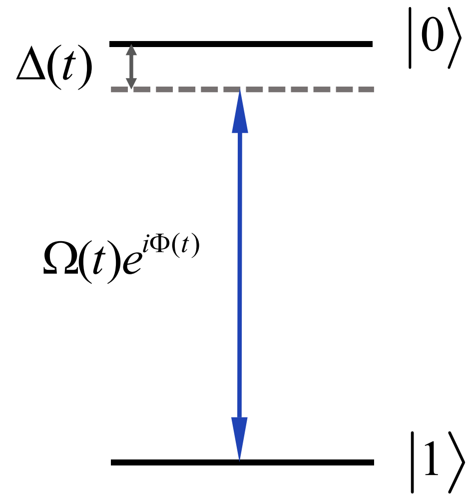

FIG. 1. Energy level diagram of the two-level system driven by external field with Rabi frequency $\Omega(t)$, phase $\Phi(t)$, and detuning $\Delta(t)$.

To further specify the example, we may assume the system is required to evolve along the eigenstate $|E_1(t)\rangle$ from the initial state $|E_1(0)\rangle = -|1\rangle$ to the target state $|E_1(\tau)\rangle = \frac{1}{\sqrt{2}}(e^{-i\varphi(\tau)}|0\rangle - |1\rangle)$, which corresponds to the starting and terminal points $(\theta_0, \varphi_0) = (0, \varphi_\tau)$ and $(\theta_\tau, \varphi_\tau) = (\pi/2, \varphi_\tau)$ in the parameter space. Then the optimal evolution path can be expressed as $\theta(t) = \pi t/2$, $\varphi(t) = \varphi(\tau)$, $t \in [0, 1]$ and the driving Hamiltonian reads $H(t) = \pi/4[-\sin\varphi(\tau)\sigma_x + \cos\varphi(\tau)\sigma_y]$.

B. Example 2: Optimizing the evolution path of nonadiabatic geometric gates

We showed that the minimal integral of instantaneous cost in the transitionless quantum driving process has the geometric property expressed by Eq. (11), i.e., $C = \int_0^\tau \sqrt{g_{\mu\nu}\dot{\lambda}^\mu\dot{\lambda}^\nu}\,dt$. The driving Hamiltonian corresponding to the minimal integral of instantaneous cost is given by Eq. (5), i.e., $H(t) = i\sum_{m\neq n}^N \langle E_m(t)|\partial_t E_n(t)\rangle|E_m(t)\rangle\langle E_n(t)|$. Noting that such a Hamiltonian is exactly the same as the general Hamiltonian proposed for nonadiabatic geometric quantum computation in Ref. [57] [see Eq. (4) in that reference], our finding can be used to optimize the evolution path of nonadiabatic geometric gates.

For the system driving by the Hamiltonian $H(t)$ defined by Eq. (5), if it is initially in the state $|E_n(0)\rangle$, it will evolve along the state $|\phi_n(t)\rangle = \exp[i\gamma_n(t)]|E_n(t)\rangle$ with $\gamma_n(t) = -i\int_0^t \langle\partial_{t'}E_n(t')|E_n(t')\rangle dt'$. When the system undergoes a cyclic evolution in the parameter space, i.e., $\vec{\lambda}(0) = \vec{\lambda}(\tau)$, its final state will be $|\phi_n(\tau)\rangle = \exp[i\gamma_n(\tau)]|E_n(0)\rangle$. $\gamma_n(\tau)$ is purely a geometric phase since the parallel transport condition, $\langle\phi_n(t)|H(t)|\phi_n(t)\rangle = 0$ is satisfied automatically. Consequently, the evolution operator at the final time reads

$$U(\tau) = \sum_n^N e^{i\gamma_n(\tau)}|E_n(0)\rangle\langle E_n(0)|. \quad (19)$$

Therefore, the Hamiltonian $H(t)$ defined in Eq. (5) can be used to realize nonadiabatic geometric quantum gates by encoding logical qubits into span$\{|E_n(0)\rangle\}$.

We again take the two-level system as an example. However, the optimal path in this case cannot be obtained by resolving the geodesic equation in Eq. (12) since the constraint condition is the geometric phase $\gamma(\tau)$ here instead of the starting and target points (θ_0, φ_0) and $(\theta_\tau, \varphi_\tau)$ in Example 1. For simplicity, we continue to use the same symbols as those in Example 1 and directly quote the above equations as well.

By substituting $|E_1(0)\rangle$ and $|E_2(0)\rangle$ into Eq. (19), the unitary operator for the two-level system can be expressed as $U(\tau) = e^{i\gamma(\tau)}|E_1(0)\rangle\langle E_1(0)| + e^{-i\gamma(\tau)}|E_2(0)\rangle\langle E_2(0)|$, where $\gamma(\tau) = \frac{1}{2}\int_0^\tau [1 - \cos\theta(t)]\dot{\varphi}(t)dt$. For a cyclic evolution, the evolution path in the parameter space is a closed curve and hence $\gamma(\tau)$ can be rewritten as the integration along the evolution path $\mathcal{P}$,

$$\gamma = \frac{1}{2}\oint_{\mathcal{P}} (1 - \cos\theta)\, d\varphi. \tag{20}$$

If we let $\mathbf{n} = (\sin\theta_0 \cos\varphi_0, \sin\theta_0 \sin\varphi_0, \cos\theta_0)$ and use $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, the unitary operator can be simply rewritten as

$$U(\tau) = e^{-i\gamma(\tau)\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{21}$$

It plays the role of an arbitrary one-qubit geometric gate with the rotation axis $\mathbf{n}$ and rotation angle $2\gamma(\tau)$.

To find the optimal path for realizing the gate $U(\tau)$ with the minimum integral C, we substitute the metric g in Eq. (16) into the formula (11), obtaining

$$\begin{aligned} C &= \frac{1}{\sqrt{2}}\int_0^\tau \sqrt{\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta}\, dt \\ &= \frac{1}{\sqrt{2}}\oint_{\mathcal{P}} \sqrt{\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta}\, d\varphi. \end{aligned} \tag{22}$$

Here, C describes the total cost corresponding to the path $\mathcal{P}$. Noting that there are infinitely many paths that can realize the same gate $U(\tau)$ and each path corresponds to a different C in general, we aim to pick out the one that corresponds to the minimum value of integral C.

To this end, we need to calculate the extremum of the functional C under the constraint Eq. (20) by using the Lagrange multiplier method. The optimal path $\theta = \theta(\varphi)$ satisfies the differential equation

$$\operatorname{grad} C + \lambda \operatorname{grad} \gamma = 0, \tag{23}$$

where λ is a Lagrangian multiplier to be determined and the gradient is defined by Euler-Lagrange equations

$$\begin{aligned} \operatorname{grad} C &= \frac{\partial F(\theta, \varphi)}{\partial \theta} - \frac{d}{d\varphi}\left(\frac{\partial F(\theta, \varphi)}{\partial \dot{\theta}}\right), \\ \operatorname{grad} \gamma &= \frac{\partial G(\theta, \varphi)}{\partial \theta} - \frac{d}{d\varphi}\left(\frac{\partial G(\theta, \varphi)}{\partial \dot{\theta}}\right), \end{aligned} \tag{24}$$

with $F(\theta, \varphi) = \frac{1}{\sqrt{2}}\sqrt{(\frac{d\theta}{d\varphi})^2 + \sin^2\theta}$, $G(\theta, \varphi) = \frac{1}{2}(1 - \cos\theta)$, and $\dot{\theta} = d\theta/d\varphi$. By substituting $F(\theta, \varphi)$ and $G(\theta, \varphi)$ into Eq. (23), the differential equation of $\theta = \theta(\varphi)$ can be explicitly written as

$$\begin{aligned} &-\sin\theta \frac{d^2\theta}{d\varphi^2} + \frac{\sqrt{2}}{2}\lambda\left[\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta\right]^{\frac{3}{2}} \\ &+ 2\cos\theta\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta\cos\theta = 0. \end{aligned} \tag{25}$$

By resolving the above differential equation with the boundary conditions $\theta(\varphi)|_{\varphi=\varphi_0} = \theta_0$, $\theta(\varphi)|_{\varphi=\varphi_\tau} = \theta_\tau$ and the constrain condition (20), the optimal path $\theta = \theta(\varphi)$ can be obtained. However, it is a difficult task to resolve Eq. (25) analytically and we need to resort to numerical calculation in general.

Fortunately, we can provide an alternative method, which is only based the geometric properties of γ and C but without the need of Eq. (25) to work out the optimal path for realizing the geometric gate $U = e^{-i\gamma(\tau)\mathbf{n}\cdot\boldsymbol{\sigma}}$. In fact, if we take $\theta(t)$ and $\varphi(t)$ as the polar angle and the azimuthal angle of a spherical coordinate system, respectively, γ in Eq. (20) is equal to the half of the solid angle enclosed by the evolution path $\mathcal{P}$ on the sphere while C in Eq. (22) is just equal to the length of the same path except for a constant coefficient. Therefore, the optimal path which encloses a given solid angle but has the shortest length must be the small circle on the sphere. The optimal path for realizing the geometric gate $U = e^{-i\gamma(\tau)\mathbf{n}\cdot\boldsymbol{\sigma}}$ with the minimum integral C is not unique. It can be taken as any one of the small circles that pass through the point (θ_0, φ_0) as well as enclose the solid angle 2γ. A sketch of the optimal paths for the geometric gate $U = \exp[-i\pi\sigma_z/4]$ is shown as Fig. 2.

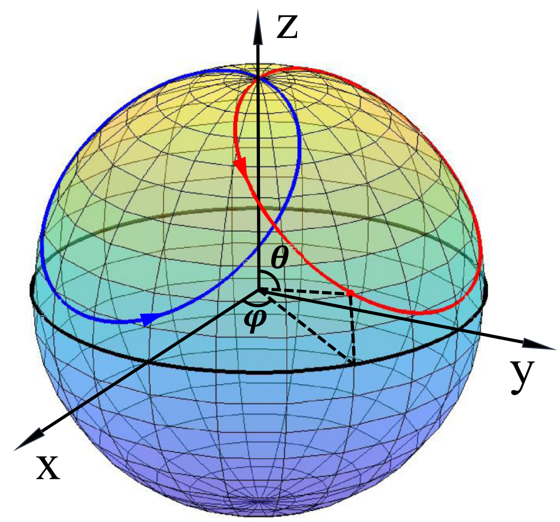

FIG. 2. The optimal evolution paths for nonadiabatic geometric gate $U = \exp[-i\pi\sigma_z/4]$ on the sphere.

IV. CONCLUSION

In conclusion, we proved that the minimal integral of instantaneous cost in the transitionless quantum driving process has a geometric property being only dependent on the evolution path in the space of control parameters, but independent of the evolution details such as the changing rate of the parameters. It is equal to the length of the evolution path on the Riemannian manifold induced by the metric defined in Eq. (9). Based on this property, we further demonstrated that the optimal transitionless evolution with the minimum integral of instantaneous cost corresponds to the geodesic path on the Riemannian manifold, as described by Eq. (12), which can be regarded as a kind of minimum action principle for transitionless quantum driving. Our finding provides a method for optimizing transitionless quantum driving. We illustrated the application of the method by a two-level system. In addition, we also illustrated the usefulness of our finding in the geometric quantum computation and showed that the optimal path, which realizes the geometric gate $U = e^{-i\gamma(\tau)\mathbf{n}\cdot\boldsymbol{\sigma}}$ with the minimum integral of instantaneous cost, is the small circles that pass through the starting point as well as enclose the solid angle 2γ.

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