Orthogonal product sets with strong quantum nonlocality on a plane structure

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(Received 29 May 2022; revised 14 September 2022; accepted 27 October 2022; published 15 November 2022)

In this paper, we consider the orthogonal product set (OPS) with strong quantum nonlocality in multipartite quantum systems. Based on the decomposition of plane geometry, we present a sufficient condition for the triviality of orthogonality-preserving positive operator-valued measures on fixed subsystem and partially answer an open question given by Yuan *et al.* [Phys. Rev. A **102**, 042228 (2020)]. The connection between the nonlocality and the plane structure of OPSs is established. We successfully construct a strongly nonlocal OPS in $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ $(d_A, d_B, d_C \geqslant 4)$, which contains fewer quantum states, and generalize the structures of known OPSs to any possible three and four-partite systems. In addition, we propose several entanglement-assisted protocols for perfectly local discrimination of the sets. It is shown that the protocols without teleportation use less entanglement resources that on average and these sets can always be discriminated locally with multiple copies of two-qubit maximally entangled states. These results also exhibit nontrivial signification of maximally entangled states in the local discrimination of quantum states.

DOI: 10.1103/PhysRevA.106.052209

I. INTRODUCTION

Quantum nonlocality, as one fundamental property and the most celebrated manifestations of quantum mechanics, arises from entangled states. Quantum entanglement has received extensive attention, and many results have been obtained [1–3]. Since entangled pure states violate Bell-type inequalities, they are nonlocal [4–11]. However, in 1999, Bennett *et al.* [12] proposed complete orthogonal product bases with nonlocality, i.e., each of which cannot be reliably discriminated by local operations and classical communication (LOCC) while it can only be identified by a global measurement. It means that nonlocal properties are no longer restricted only to entangled systems. Later, this phenomenon, quantum nonlocality without entanglement, has aroused wide research attention [13–21]. Zhang *et al.* [14] gave a class of nonlocal orthogonal product bases in the quantum system of $\mathcal{C}^d \otimes \mathcal{C}^d$, where d is odd. Wang *et al.* [15] obtained a small set with only $3(m+n)-9$ orthogonal product states in an arbitrary bipartite quantum system $\mathcal{C}^m \otimes \mathcal{C}^n$ and proved that these states are LOCC indistinguishable. Xu *et al.* [18] presented a locally indistinguishable set of multipartite orthogonal product states of size $2n$, which can be projected to the quantum system $\otimes_{i=1}^n \mathcal{C}^2$ in essence. Jiang *et al.* [21] proposed a simple method to construct a nonlocal set of orthogonal product states in a $\otimes_{i=1}^n \mathcal{C}^{d_i} (n \geqslant 3, d_i \geqslant 2)$ quantum system. It is also shown that local indistinguishability is a crucial primitive for quantum data hiding [22–24] and quantum secret sharing [25–30].

Recently, the concept of quantum nonlocality without entanglement was further developed [31–41]. Halder *et al.* [31] presented a stronger manifestation of this kind of nonlocality in multiparty systems. Specifically, an orthogonal product set (OPS) on $\otimes_{i=1}^n \mathcal{C}^{d_i} (n \geqslant 3, d_i \geqslant 3)$ is defined to be strongly nonlocal if it is locally irreducible in every bipartition. The local irreducibility means that it is not possible to eliminate one or more states from the set by orthogonality-preserving local measurements [31]. Immediately, Zhang *et al.* [32] gave a more general definition of strong quantum nonlocality for multipartite quantum states, where the set is strongly nonlocal if it is locally irreducible in every $(n-1)$ partition. Naturally, the set of orthogonal quantum states which is locally irreducible in every bipartition is the strongest manifestation of nonlocality.

It is well known that entanglement is a very valuable resource which allows remote parties to communicate [42,43], as in teleportation [44–46]. In fact, the set of orthogonal quantum states with quantum nonlocality can always be perfectly discriminated by sharing additional entangled resources among the parties [33,47–52]. Most generally, by using enough entanglement resource, we can teleport the full multipartite states to one of the parties by LOCC, then these states can be determined by performing suitable measurement. In 2008, Cohen [48] proposed protocols using entanglement more efficiently than teleportation to distinguish certain classes of unextendible product bases (UPBs), where less entanglement was consumed in comparison with the teleportation-based method. Rout *et al.* [33] studied local state discrimination protocols with Einstein-Podolsky-Rosen (EPR) states and Greenberger-Horne-Zeilinger (GHZ) states. Zhang *et al.* [50,52] presented several protocols to locally distinguish particular UPBs by using different

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entanglement resources and proved that some sets can also be locally distinguished with multiple copies of EPR states.

In this paper, we investigate OPSs with strong nonlocality. In Sec. II, we introduce some notations and required preliminary concepts and results. In Sec. III, we study the sufficient condition for local irreducibility of OPSs and illustrate the smallest size of OPS under some specific constraints. Next, in Sec. IV, we generalize the structure of given sets to higher dimension systems and construct a smaller OPS with the strongest quantum nonlocality in $\mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}\otimes\mathcal{C}^{d_C}$ $(d_A, d_B, d_C \geqslant 4)$. Furthermore, we also investigate local distinguishability of our OPSs by using different entanglement resources in Sec. V. Finally, we conclude with a brief summary in Sec. VI.

II. PRELIMINARIES

In this section, we introduce some definitions and notations needed in the rest of the paper.

Definition 1 [53]. A measurement is trivial if all the positive operator-valued measure (POVM) elements are proportional to the identity operator. Otherwise, the measurement is nontrivial.

In an n-partite system, a set $\{|\varphi\rangle\}$ of orthogonal states is locally irreducible if the orthogonality-preserving POVM [31] on any party can only be trivial. The inverse does not hold in general. Let $X_1=\{2,3,\ldots,n\}$, $X_2=\{3,\ldots,n,1\}$, $X_3=\{4,\ldots,n,1,2\},\ldots,X_n=\{1,2,\ldots,n-1\}$.

Lemma 1 [36]. If X_i party can only perform a trivial orthogonality-preserving POVM for all $1\leqslant i\leqslant n$, then the set $\{|\varphi\rangle\}$ is of the strongest nonlocality [32].

Let the $d\times d$ matrix $E=(a_{ij})_{i,j\in\mathcal{Z}_d}$ be the matrix representation of the operator $E=M^\dagger M$ in the basis $\mathcal{B}=\{|0\rangle,\ldots,|d-1\rangle\}$. Define

$$ {}_{\mathcal{S}}E_{\mathcal{T}}=\sum_{|i\rangle\in\mathcal{S}}\sum_{|j\rangle\in\mathcal{T}}a_{ij}|i\rangle\langle j|, \quad (1) $$

where $\mathcal{S}$ and $\mathcal{T}$ are two nonempty subsets of $\mathcal{B}$. Especially, ${}_{\mathcal{T}}E_{\mathcal{T}}$ is represented by $E_{\mathcal{T}}$. Let $\{|\psi_i\rangle\}_{i=0}^{s-1}$ and $\{|\phi_j\rangle\}_{j=0}^{t-1}$ be two orthogonal sets spanned by $\mathcal{S}$ and $\mathcal{T}$, respectively, where $s=|\mathcal{S}|$ and $t=|\mathcal{T}|$.

Lemma 2 [36]. If subsets $\mathcal{S}$ and $\mathcal{T}$ are disjoint and $\langle\psi_i|E|\phi_j\rangle=0$ for any $i\in\mathcal{Z}_s$, $j\in\mathcal{Z}_t$, then ${}_{\mathcal{S}}E_{\mathcal{T}}=\mathbf{0}$ and ${}_{\mathcal{T}}E_{\mathcal{S}}=\mathbf{0}$.

Lemma 3 [36]. Suppose that $\langle\psi_i|E|\psi_j\rangle=0$ for any $i\neq j\in\mathcal{Z}_s$. If there exists a state $|i_0\rangle\in\mathcal{S}$ such that ${}_{\{|i_0\rangle\}}E_{\mathcal{S}\setminus\{|i_0\rangle\}}=\mathbf{0}$ and $\langle i_0|\psi_j\rangle\neq 0$ for any $j\in\mathcal{Z}_s$, then $E_{\mathcal{S}}\propto\mathcal{I}_s$, i.e., $E_{\mathcal{S}}$ is proportional to the identity matrix.

Consider an n-partite quantum system $\mathcal{H}=\otimes_{i=1}^n\mathcal{C}^{d_i}$. The computational basis of the whole quantum system is denoted by $\mathcal{B}=\{|i\rangle\}_{i=0}^{d_1d_2\cdots d_n-1}=\{\otimes_{k=1}^n|i_k\rangle\mid i_k=0,1,\ldots,d_k-1\}=\mathcal{B}^{\{1\}}\otimes\mathcal{B}^{\{2\}}\otimes\cdots\otimes\mathcal{B}^{\{n\}}$, where $\mathcal{B}^{\{k\}}=\{|i_k\rangle\}_{i_k=0}^{d_k-1}$ is the computational basis of the kth subsystem. Let

$$ \mathcal{B}_r=\mathcal{B}_r^{\{1\}}\otimes\mathcal{B}_r^{\{2\}}\otimes\cdots\otimes\mathcal{B}_r^{\{n\}} \quad (2) $$

be a subset of basis $\mathcal{B}$ with $\mathcal{B}_r^{\{i\}}\subset\mathcal{B}^{\{i\}}$. Suppose that $\mathcal{B}_r$ $(1\leqslant r\leqslant q)$ are disjoint subsets of $\mathcal{B}$, then, there is a class of OPSs

$$ S=\cup_{r\in Q}S_r,\quad Q=\{1,2,\ldots,q\} \quad (3) $$

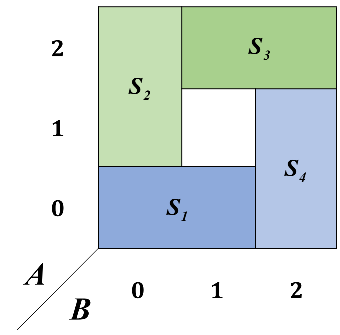

FIG. 1. The plane structure of OPS given by Eq. (5) in bipartition.

in $\mathcal{H}$, where S_r expresses the orthogonal product basis of the subspace spanned by $\mathcal{B}_r$, and each component of the vector in S_r is nonzero under the computational basis $\mathcal{B}_r$, that is, each vector $|\phi\rangle_r$ in S_r has the following form:

$$ |\phi\rangle_r=\left(\sum_{|j_1\rangle\in\mathcal{B}_r^{\{1\}}}a_{j_1}^{(1)}|j_1\rangle\right)\otimes\cdots\otimes\left(\sum_{|j_n\rangle\in\mathcal{B}_r^{\{n\}}}a_{j_n}^{(n)}|j_n\rangle\right), \quad (4) $$

with nonzero complex numbers $a_{j_k}^{(k)}$ for $k=1,2,\ldots,n$. If the set S is invariant under cyclic permutation of all subsystems, then we call it symmetric.

A plane structure of the set S refers to a two-dimensional grid diagram and each subset S_r corresponds to a domain in the diagram.

Example 1. In $\mathcal{C}^3\otimes\mathcal{C}^3$, let

$$ \begin{aligned} &S_1=|0\rangle|0\pm1\rangle,\quad S_2=|1\pm2\rangle|0\rangle,\\ &S_3=|2\rangle|1\pm2\rangle,\quad S_4=|0\pm1\rangle|2\rangle \end{aligned} \quad (5) $$

be the plane structure of the OPS [12]. $S=\cup_{i=1}^4S_i$ is depicted in Fig. 1. The four dominos in this geometry structure represent the four subsets S_1, S_2, S_3, and S_4, respectively.

To facilitate the establishment of the connection between the nonlocality and the plane structure of the given set S, some symbols are introduced. Given a subset X of $\{1,2,\ldots,n\}$ and its complement $Y=\bar{X}$, we use $\mathcal{B}^X=\{|i\rangle_X\}_{i=0}^{d_X-1}$ with $d_X=\prod_{j\in X}d_j$ to represent the computation basis of the Hilbert space $\mathcal{H}_X=\otimes_{j\in X}\mathcal{C}^{d_j}$ corresponding to the X party and analogously $\mathcal{B}^Y=\{|i\rangle_Y\}_{i=0}^{d_Y-1}$ corresponding to the Y party. Under the basis $\mathcal{B}$, the projection set of S_r on the τ $(\tau=X,Y)$ party is expressed as $S_r^{(\tau)}=\{\mathrm{Tr}_{\bar{\tau}}(|i\rangle\langle i|)\mid|i\rangle\in\mathcal{B}\text{ and }\langle i|\phi^r\rangle\neq0\text{ for any }|\phi^r\rangle\in S_r\}$. Naturally, the projection set $S_r^{(\tau)}$ is a subset of basis $\mathcal{B}^\tau$. For a fixed $i\in\mathcal{Z}_{d_X}$, let $\mathcal{B}_i^X:=\{|k\rangle_X\}_{k=i}^{d_X-1}$, $V_i:=\{\bigcup_v S_v^{(Y)}\mid|i\rangle_X\in S_v^{(X)}\}$, and $\widetilde{S}_{V_i}:=\{\bigcup_j S_j^{(X)}\mid S_j^{(Y)}\cap V_i\neq\emptyset\}$.

Example 2. Consider the OPS given by Eq. (5). X and Y represent B and A, respectively. Observe its plane structure shown in Fig. 1, the projection set of a subset on the B (or A) party is actually the coordinate of the corresponding grid on the B (or A) party. We have

$$ \begin{aligned} &S_1^{(B)}=\{|0\rangle_B,|1\rangle_B\},\quad S_2^{(B)}=\{|0\rangle_B\},\\ &S_3^{(B)}=\{|1\rangle_B,|2\rangle_B\},\quad S_4^{(B)}=\{|2\rangle_B\}, \end{aligned} \quad (6) $$

and

$$S_1^{(A)}=\{|0\rangle_A\},\quad S_2^{(A)}=\{|1\rangle_A,|2\rangle_A\},$$
$$S_3^{(A)}=\{|2\rangle_A\},\quad S_4^{(A)}=\{|0\rangle_A,|1\rangle_A\}. \tag{7}$$

For all $i\in\mathcal{Z}_3$, $\mathcal{B}_i^B$ is a subset of basis $\mathcal{B}^B$ and $\mathcal{B}_0^B$ is equal to $\mathcal{B}^B$. It is easy to know that

$$\mathcal{B}_0^B=\{|0\rangle_B,|1\rangle_B,|2\rangle_B\},$$
$$\mathcal{B}_1^B=\{|1\rangle_B,|2\rangle_B\},$$
$$\mathcal{B}_2^B=\{|2\rangle_B\}.$$

Since V_i expresses the union of the projection sets $S_v^{(A)}$ of S_v on the A party, where all corresponding projection sets $S_v^{(B)}$ of S_v on the B party contain quantum state $|i\rangle_B$, then there are

$$V_0=S_1^{(A)}\cup S_2^{(A)}=\{|0\rangle_A,|1\rangle_A,|2\rangle_A\},$$
$$V_1=S_1^{(A)}\cup S_3^{(A)}=\{|0\rangle_A,|2\rangle_A\},$$
$$V_2=S_3^{(A)}\cup S_4^{(A)}=\{|0\rangle_A,|1\rangle_A,|2\rangle_A\}.$$

Note that each projection set $S_j^{(A)}$ contains a quantum state in V_i, $\widetilde{S}_{V_i}$ is the union of all the projection sets $S_j^{(B)}$ of S_j on the B party. That is,

$$\widetilde{S}_{V_0}=\widetilde{S}_{V_1}=\widetilde{S}_{V_2}=\cup_{j=1}^4S_j^{(B)}=\{|0\rangle_B,|1\rangle_B,|2\rangle_B\}.$$

Definition 2. A family of projection sets $\{S_r^{(\tau)}\}_{r\in Q}$ is connected if it cannot be divided into two groups of sets $\{S_k^{(\tau)}\}_{k\in T}$ ($T\subsetneqq Q$) and $\{S_l^{(\tau)}\}_{l\in Q\setminus T}$ such that

$$\left(\bigcup_{k\in T}S_k^{(\tau)}\right)\bigcap\left(\bigcup_{l\in Q\setminus T}S_l^{(\tau)}\right)=\emptyset. \tag{8}$$

Definition 3. $R_r=\bigcup_{k\in T}S_k$ ($r\notin T\subset Q$) is called the projection inclusion (PI) set of S_r on the X party if the projection sets satisfy $\bigcap_{k\in T}S_k^{(Y)}\neq\emptyset$ and $S_r^{(X)}\subset\bigcup_{k\in T}S_k^{(X)}$. Specifically, R_r is called a more useful projection inclusion (UPI) set if there exists a subset $S_k\subset R_r$ such that $|S_r^{(X)}\bigcap S_k^{(X)}|=1$.

From the definition, both the PI set and the UPI set of a subset S_r of an OPS S may not be unique. By observing the plane tile as shown in Fig. 1, it is easy to know that both S_1 and $S_1\cup S_4$ are PI sets of S_2 in (5) on the B party, and $S_2\cup S_3$ is the PI set of S_1 on the B party. Due to $|S_1^{(B)}\cap S_2^{(B)}|=1$, these PI sets are also UPI sets.

For the set S in (3), we construct a set sequence $G_1,G_2,\dots,G_s$. The set G_1 denoted as $\cup_{r_1\in T_1}S_{r_1}$ is the union of all subsets S_{r_1} that have UPI sets. The remaining sets $G_2,\dots,G_s$ are expressed by $\cup_{r_2\in T_2}S_{r_2},\dots,\cup_{r_s\in T_s}S_{r_s}$, respectively. Moreover, this sequence also satisfies the following two conditions:

(1) The sets G_x are pairwise disjoint and the union of all sets is S.

(2) For any $S_{r_{x+1}}\subset G_{x+1}$ ($x=1,\dots,s-1$), there is always a subset $S_{r_x}\subset G_x$ such that $S_{r_x}^{(X)}\cap S_{r_{x+1}}^{(X)}\neq\emptyset$.

Note that such a set sequence $G_1,G_2,\dots,G_s$ satisfying (1) and (2) above does not necessarily exist. In addition, we call S_{r_x} an included (IC) subset about set G_x ($x=1,\dots,s$), if there is a subset $S_{r'_x}\subset G_x$ such that $S_{r_x}^{(X)}\subsetneqq S_{r'_x}^{(X)}$. Otherwise, it is called a nonincluded (NIC) subset.

Example 3. We consider the OPS in (5), where each subset has a corresponding UPI set

$$R_1=S_2\cup S_3,\quad R_2=S_1,\quad R_3=S_1\cup S_4,\quad R_4=S_3. \tag{9}$$

So, there is only one set in its set sequence, which happens to be this OPS. That is, $G_1=\cup_{i=1}^4S_i$.

III. THE SUFFICIENT CONDITION FOR THE TRIVIALITY OF ORTHOGONALITY-PRESERVING POVM AND THE SMALLEST SIZE OF OPS UNDER SOME CONSTRAINTS

It is an important way to illustrate the irreducibility of OPS by proving that the orthogonality-preserving POVM on the subsystems can only be trivial [21,31,32,34,36,40]. Here, we present a sufficient condition for orthogonality-preserving POVM being trivial. On a plane structure, the condition is efficient for constructing an OPS with strong nonlocality and demonstrating the irreducibility of the given OPS.

Theorem 1. For the given set S in (3), any orthogonality-preserving POVM performed on the X party can only be trivial if the following conditions are satisfied:

(i) There is an inclusion relationship $\mathcal{B}_i^X\subset\widetilde{S}_{V_i}$ for any $i\in\mathcal{Z}_{d_X-1}$.

(ii) For any subset S_r, there exists a corresponding PI set R_r on the X party.

(iii) There is a set sequence $G_1,\dots,G_s$ satisfying (1) and (2). Moreover, for each NIC subset $S_{r_{x+1}}\subset G_{x+1}$, there exist a subset $S_{r_x}\subset G_x$ and a subset $S_{r'_{x+1}}\subset R_{r_{x+1}}$ such that $S_{r_x}^{(X)}\cap S_{r_{x+1}}^{(X)}\supset S_{r_{x+1}}^{(X)}\cap S_{r'_{x+1}}^{(X)}$ with $x=1,2,\dots,s-1$.

(iv) The family of sets $\{S_r^{(X)}\}_{r\in Q}$ is connected.

Proof. Let $\{E\}$ be an any orthogonality-preserving POVM performed on X. Without loss of generality, we assume

$$E=\begin{pmatrix}a_{00}&a_{01}&\cdots&a_{0(d_X-1)}\\a_{10}&a_{11}&\cdots&a_{1(d_X-1)}\\\vdots&\vdots&\ddots&\vdots\\a_{(d_X-1)0}&a_{(d_X-1)1}&\cdots&a_{(d_X-1)(d_X-1)}\end{pmatrix}, \tag{10}$$

in the computation basis $\mathcal{B}^X$. Because the postmeasurement states should be mutually orthogonal, for any two states $|\psi_1\rangle_X|\phi_1\rangle_Y$ and $|\psi_2\rangle_X|\phi_2\rangle_Y$ in S, we have ${}_X\langle\psi_1|_Y\langle\phi_1|E\otimes\mathcal{I}|\psi_2\rangle_X|\phi_2\rangle_Y=0$. If $\langle\phi_1|\phi_2\rangle_Y\neq0$, then ${}_X\langle\psi_1|E|\psi_2\rangle_X=0$.

Let $S_r^\tau=\{\mathrm{Tr}_{\bar{\tau}}(|\phi^r\rangle\langle\phi^r|)\,|\,|\phi^r\rangle\in S_r\}$ $(\tau=X,Y)$ express the set of reduced density matrices. For any two different subsets S_{q_1} and S_{q_2}, if $S_{q_1}^{(Y)}\cap S_{q_2}^{(Y)}\neq\emptyset$, then $S_{q_1}^{(X)}\cap S_{q_2}^{(X)}=\emptyset$ and there always exists two states $|\phi_{q_1}\rangle_Y\in S_{q_1}^Y$ and $|\phi_{q_2}\rangle_Y\in S_{q_2}^Y$ such that $\langle\phi_{q_1}|\phi_{q_2}\rangle_Y\neq0$. Due to the orthogonality-preserving property, we obtain ${}_X\langle\psi_{q_1}|E|\psi_{q_2}\rangle_X=0$ for all $|\psi_{q_1}\rangle_X\in S_{q_1}^X$ and $|\psi_{q_2}\rangle_X\in S_{q_2}^X$. According to Lemma 2, we deduce ${}_{S_{q_1}^{(X)}}E_{S_{q_2}^{(X)}}=\mathbf{0}$. Using this result, we can prove that $E\propto\mathcal{I}$ by the following four steps. Here, Figs. 2–5 depict the process of proving.

Step 1. When $i=0$, we know $V_0=\{\cup_vS_v^{(Y)}\,|\,|0\rangle_X\in S_v^{(X)}\}$. For each $|j\rangle_Y\in V_0$, let $\{S_{j_s}\}_{j_s\in Q_j}$ ($Q_j\subset Q$) represent the all subsets whose projection sets on party Y contain the state $|j\rangle_Y$. Suppose S_{j_1} is the subset such that $|0\rangle_X\in S_{j_1}^{(X)}$, then one has ${}_{S_{j_1}^{(X)}}E_{S_{j_s}^{(X)}}=\mathbf{0}$ for any s ($s\neq1$). By the definition and condition (i), it is easy to derive $\cup_{j,s}S_{j_s}^{(X)}=\widetilde{S}_{V_0}=\mathcal{B}_0^X=$

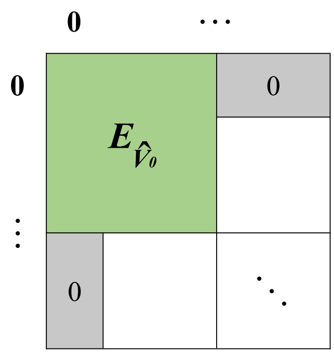

FIG. 2. In step 1, taking $i = 0$ as an example, we show that all the elements of E in the first row and in the first column except $E_{\widehat{V}_0}$ are zero.

$\{|i\rangle_X\}_{i=0}^{d_X-1}$. Thus, we get $a_{0k_0} = a_{k_0 0} = 0$ for $|k_0\rangle_X \notin \widehat{V}_0$, where $\widehat{V}_0 = \{\cup_v S_v^{(X)} \mid |0\rangle_X \in S_v^{(X)}\}$. See Fig. 2.

Similarly, when $i = 1, \ldots, d_X - 2$, we obtain $a_{ik_i} = a_{k_i i} = 0$ for $|k_i\rangle_X \notin \widehat{V}_i$ and $k_i > i$. Here $\widehat{V}_i = \{\cup_v S_v^{(X)} \mid |i\rangle_X \in S_v^{(X)}\}$.

Step 2. According to the condition (ii), for each $r \in Q$, there exists a PI set $R_r = \bigcup_{t\in T_r} S_t$ ($r \notin T_r \subset Q$) of S_r on party X, where $\bigcap_{t\in T_r} S_t^{(Y)} \neq \emptyset$ and $S_r^{(X)} \subset \bigcup_{t\in T_r} S_t^{(X)}$. For any two different indexes t_1 and t_2 in T_r, it is not difficult to deduce that $a_{kl} = a_{lk} = 0$ with $|k\rangle_X \in S_{t_1}^{(X)} \cap S_r^{(X)}$ and $|l\rangle_X \in S_{t_2}^{(X)} \cap S_r^{(X)}$ for $k \neq l$.

Step 3. For any subset S_{r_1} in G_1, the corresponding set R_{r_1} is the UPI set. From Definition 3, there is a subset $S_{r_1'} \subset R_{r_1}$ such that $|S_{r_1}^{(X)} \cap S_{r_1'}^{(X)}| = 1$. It is a special case in step 2. Let $|k\rangle_X$ be the only element of $S_{r_1}^{(X)} \cap S_{r_1'}^{(X)}$, then $a_{kl} = 0$ for all $|l\rangle_X \in S_{r_1}^{(X)} \setminus \{|k\rangle_X\}$. Since each component of the vector in S_{r_1} is nonzero under the computation basis $\mathcal{B}_{r_1}$ from (4), it is easy to know $\langle k|\psi\rangle_X \neq 0$ for any $|\psi\rangle_X \in S_{r_1}^X$. According to Lemma 3, we deduce $E_{r_1} = E_{S_{r_1}^{(X)}} \propto \mathcal{I}$.

By condition (iii), for each NIC subset $S_{r_2} \subset G_2$, there exist a subset $S_{r_1} \subset G_1$ and a subset $S_{r_2'} \subset R_{r_2}$ such that $S_{r_1}^{(X)} \cap S_{r_2}^{(X)} \supset S_{r_2}^{(X)} \cap S_{r_2'}^{(X)}$. Then $a_{kl} = 0$ for $|k\rangle_X, |l\rangle_X \in S_{r_2}^{(X)} \cap S_{r_2'}^{(X)}$ and $k \neq l$. Combining this with the step 2 produces $a_{kl} = 0$ for $|k\rangle_X \in S_{r_2}^{(X)} \cap S_{r_2'}^{(X)}$, $|l\rangle_X \in S_{r_2}^{(X)}$ and $k \neq l$. It follows from Lemma 3 that $E_{r_2} = E_{S_{r_2}^{(X)}} \propto \mathcal{I}$. For each IC subset $S_{r_2''}$, there is always a corresponding NIC subset S_{r_2} that satisfies the inclusion relationship $S_{r_2''}^{(X)} \subsetneqq S_{r_2}^{(X)}$, which implies $E_{r_2''} = E_{S_{r_2''}^{(X)}} \propto \mathcal{I}$. Similarly, $E_r \propto \mathcal{I}$ for each r. That is, there is a positive real number b_r such that $E_r = b_r \mathcal{I}$. See also Fig. 3.

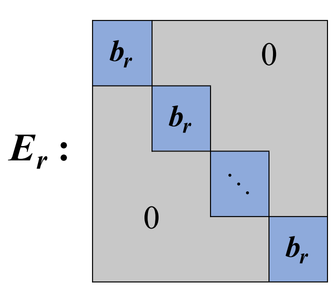

FIG. 3. In steps 2 and 3, it is proved that the operator $E_r = E_{S_r^{(X)}}$ corresponding to subset S_r is proportional to the unit operator for all $r \in Q$.

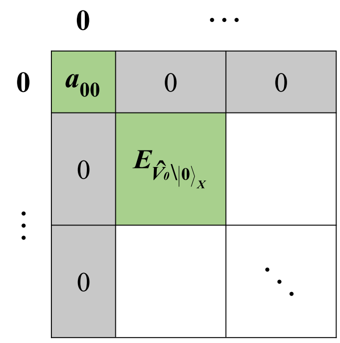

FIG. 4. Consider the operator $E_{\widehat{V}_0}$. Because each $E_v \propto \mathcal{I}$, only element a_{00} in the first row is nonzero. We can get the similar result for other $\widehat{V}_i$ ($i = 1, \ldots, d_X - 2$). Therefore, we deduce that the off-diagonal elements of E are all zero.

Step 4. Consider the set $\widehat{V}_0 = \{\cup_v S_v^{(X)} \mid |0\rangle_X \in S_v^{(X)}\}$ of step 1. Due to each $E_v \propto \mathcal{I}$, we have $a_{0k_0} = 0$ for $|k_0\rangle_X \in \widehat{V}_0$ and $k_0 \neq 0$. Combining this with the step 1 produces $a_{0k_0} = 0$ for all $k_0 > 0$. We can obtain the similar result for other $\widehat{V}_i$ ($i = 1, \ldots, d_X - 2$). So, we deduce that the off-diagonal elements of E are all zero. It is shown in Fig. 4. In addition, for any $x, y \in Q$, if $S_x^{(X)} \cap S_y^{(X)} \neq \emptyset$, then $b_x = b_y$. The condition (iv) indicates that the family of sets $\{S_r^{(X)}\}_{r\in Q}$ is connected. This means that these scalars b_r are all equal. Therefore, the POVM element can only be proportional to the unit operator $\mathcal{I}$. See also Fig. 5. ■

Corollary 1. If the conditions (i)–(iv) in Theorem 1 are satisfied for $X = X_1, X_2, \ldots, X_n$ with $X_j = \{1, 2, \ldots, j-1, j+1, \ldots, n\}$, then the set (3) is an OPS of the strongest quantum nonlocality.

Note that it is obvious that $E_r \propto \mathcal{I}$ for each $r \in Q$, if the set G_1 is equal to the set S. That is, when the set sequence has only one set G_1, we still say that the condition (iii) is valid. Next we provide an example to show the application of this theorem on plane structure.

Example 4. We revisit the quantum nonlocality of the following OPS [34] in $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$:

$$\begin{aligned} S_1 &= \{|0\rangle|1\rangle|0 \pm 1\rangle\}, & S_7 &= \{|0\rangle|2\rangle|0 \pm 2\rangle\},\\ S_2 &= \{|1\rangle|0 \pm 1\rangle|0\rangle\}, & S_8 &= \{|2\rangle|0 \pm 2\rangle|0\rangle\},\\ S_3 &= \{|0 \pm 1\rangle|0\rangle|1\rangle\}, & S_9 &= \{|0 \pm 2\rangle|0\rangle|2\rangle\}, \end{aligned}$$

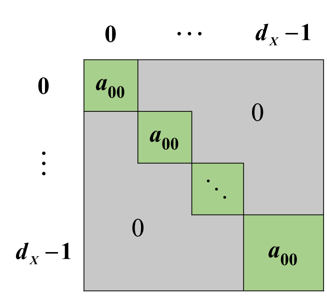

FIG. 5. It follows from condition (iv) that the scalars b_r are all equal. Then the diagonal entries of the POVM element E are all equal; that is, $E = a_{00}\mathcal{I}$ for some positive real number a_{00}, where $\mathcal{I}$ is the identity matrix.

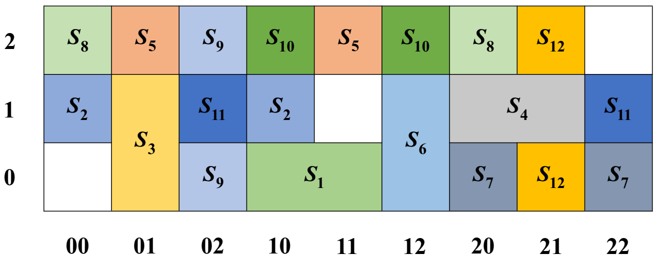

FIG. 6. The corresponding 3×9 grid of $\{S_r\}_{r=1}^{12}$ given by Eq. (11) in $A|BC$ bipartition.

$$\begin{aligned} S_4 &= \{|1\rangle|2\rangle|0\pm 1\rangle\}, & S_{10} &= \{|2\rangle|1\rangle|0\pm 2\rangle\},\\ S_5 &= \{|2\rangle|0\pm 1\rangle|1\rangle\}, & S_{11} &= \{|1\rangle|0\pm 2\rangle|2\rangle\},\\ S_6 &= \{|0\pm 1\rangle|1\rangle|2\rangle\}, & S_{12} &= \{|0\pm 2\rangle|2\rangle|1\rangle\}. \end{aligned} \tag{11}$$

Due to Lemma 1 and the symmetry of the OPS given by Eq. (11), we only need to consider the orthogonality-preserving POVM performed on party BC. Figure 6 is the plane structure of OPS in the $A|BC$ bipartition. By observing this tile graph, we can easily obtain the four conditions in Theorem 1.

First, the projection set $\cup_r S_r^{(ABC)}$ differs from the computation basis $\mathcal{B}$ only by states $|000\rangle$, $|111\rangle$, and $|222\rangle$. It is obvious that $\widetilde{S}_{V_{ij}} = \mathcal{B}^{BC}$ for $i, j \in \mathcal{Z}_3$. Here $\mathcal{B}^{BC}$ is the computation basis on subsystem BC. Naturally, $\mathcal{B}_{ij}^{BC} \subset \widetilde{S}_{V_{ij}}$. The condition (i) holds.

Second, for each subset S_r, we have the corresponding PI sets $R_1 = S_5 \cup S_{10}$, $R_2 = S_8 \cup S_{10}$, $R_3 = S_5$, $R_4 = S_8 \cup S_{12}$, $R_5 = S_1 \cup S_3$, $R_6 = S_{10}$, $R_7 = S_4 \cup S_{11}$, $R_8 = S_2 \cup S_4$, $R_9 = S_{11}$, $R_{10} = S_2 \cup S_6$, $R_{11} = S_7 \cup S_9$, and $R_{12} = S_4$. The condition (ii) is demonstrated.

Furthermore, for any two subsets S_x and S_y, we have $|S_x^{(BC)} \cap S_y^{(BC)}| \leqslant 1$. So, each R_r is an UPI set, i.e., G_1 is the union of all subsets. It is obvious that condition (iii) holds.

Finally, we find a sequence of projection sets $(S_5^{(BC)}, S_{10}^{(BC)}) \rightarrow S_1^{(BC)} \rightarrow S_2^{(BC)} \rightarrow S_8^{(BC)} \rightarrow S_4^{(BC)} \rightarrow S_7^{(BC)} \rightarrow S_{11}^{(BC)}$. In this sequence, the intersection of the sets on both sides of the arrow is not empty and the union of these sets is the computation basis $\mathcal{B}^{BC}$. So, it is impossible to divide all projection sets into disjoint two groups of projection sets. That is, the family of projection sets $\{S_r^{(BC)}\}_{r=1}^{12}$ is connected. The condition (iv) is satisfied.

According to Theorem 1, we deduce the POVM performed on party BC can only be trivial. Therefore, the OPS given by Eq. (11) is of the strongest quantum nonlocality.

For the same set as stated in Theorem 1, we have the following corollary:

Corollary 2. If any orthogonality-preserving POVM element performed on party X can only be proportional to the identity operator, then the set $\cup_{r\in Q} S_r^{(X)}$ is the basis $\mathcal{B}^X$ and the family of projection sets $\{S_r^{(X)}\}_{r\in Q}$ is connected.

By using Corollary 2, in systems $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$ and $\mathcal{C}^4 \otimes \mathcal{C}^4 \otimes \mathcal{C}^4$, we can discuss the minimum size of the OPS given by Eq. (3) under the specific restrictions. Let N express the maximum size of all subsets, i.e., $N = \max_r |S_r|$. We have the following two theorems:

Theorem 2. In $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$, for the set S (3), if the set S is symmetric and any orthogonality-preserving POVM performed on party BC can only be trivial, then the set S is an OPS of the strongest nonlocality. The smallest size of this set is 24.

Theorem 3. In $\mathcal{C}^4 \otimes \mathcal{C}^4 \otimes \mathcal{C}^4$, for the set S in (3), if S is symmetric with $N = 2$ and any orthogonality-preserving POVM element performed on party BC can only be proportional to identity, then the set S is an OPS of the strongest nonlocality. The smallest size of this set S is 48.

The detailed proofs are given in Appendixes A and B, respectively. Theorems 2 and 3 show the minimum sizes of two kinds of OPSs with strong nonlocality, respectively. They are partial answers to the open question in Ref. [34], "Can we find the smallest strongly nonlocal set in $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$, and more generally in any tripartite systems?"

IV. OPS WITH THE STRONGEST QUANTUM NONLOCaLITY IN $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ AND $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C} \otimes \mathcal{C}^{d_D}$

From Theorem 1, we know that the nonlocality of OPS is closely related to its plane structure. In this section, we provide several strongly nonlocal OPSs in three- and four-partite systems.

By extending the dimension of the grid in Fig. 6, we can generalize the structure of the set (11) to any finite dimension. The OPS in $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ is described as

$$\begin{aligned} H_1 &= \{|0\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C\}_{i,j},\\ H_2 &= \{|\xi_i\rangle_A|\eta_j\rangle_B|0\rangle_C\}_{i,j},\\ H_3 &= \{|\eta_j\rangle_A|0\rangle_B|\xi_i\rangle_C\}_{i,j},\\ H_4 &= \{|\xi_i\rangle_A|d_B'\rangle_B|\eta_j\rangle_C\}_{i,j},\\ H_5 &= \{|d_A'\rangle_A|\eta_j\rangle_B|\xi_i\rangle_C\}_{i,j},\\ H_6 &= \{|\eta_j\rangle_A|\xi_i\rangle_B|d_C'\rangle_C\}_{i,j},\\ H_7 &= \{|0\rangle_A|d_B'\rangle_B|0\pm d_C'\rangle_C\},\\ H_8 &= \{|d_A'\rangle_A|0\pm d_B'\rangle_B|0\rangle_C\},\\ H_9 &= \{|0\pm d_A'\rangle_A|0\rangle_B|d_C'\rangle_C\},\\ H_{10} &= \{|d_A'\rangle_A|\xi_i\rangle_B|0\pm d_C'\rangle_C\}_i,\\ H_{11} &= \{|\xi_i\rangle_A|0\pm d_B'\rangle_B|d_C'\rangle_C\}_i,\\ H_{12} &= \{|0\pm d_A'\rangle_A|d_B'\rangle_B|\xi_i\rangle_C\}_i, \end{aligned} \tag{12}$$

where $|\xi_i\rangle_\tau = \sum_{u=0}^{d_\tau-3} \omega_{d_\tau-2}^{iu}|u+1\rangle$, $|\eta_j\rangle_\tau = \sum_{u=0}^{d_\tau-2} \omega_{d_\tau-1}^{ju}|u\rangle$, $d_\tau' = d_\tau - 1$ for $i \in \mathcal{Z}_{d_\tau-2}$, $j \in \mathcal{Z}_{d_\tau-1}$, and $\tau \in \{A, B, C\}$. Here and below we use the notation $\omega_n := e^{\frac{2\pi i}{n}}$ for any positive integer n. Figure 7 is a geometric representation of this OPS in the $A|BC$ bipartition. We explain the strong nonlocality of the OPS (12) in the following theorem:

Theorem 4. In $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$, the set $\cup_{i=1}^{12} H_i$ given by Eq. (12) is an OPS of the strongest nonlocality. The size of this set is $2[(d_Ad_B + d_Bd_C + d_Ad_C) - 2(d_A + d_B + d_C) + 3]$.

Proof. We only need to discuss the orthogonality-preserving POVM performed on party BC. The tile structure is depicted in Fig. 7. Because the set $\cup_{i=1}^{12} H_i$ has the same

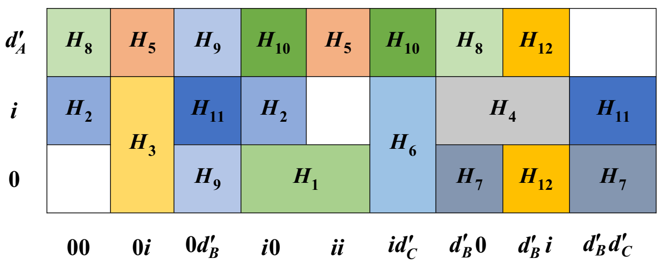

FIG. 7. The corresponding $d_A \times d_B d_C$ grid of $\{H_i\}_{i=1}^{12}$ given by Eq. (12) in the $A|BC$ bipartition.

structure as the set $\cup_{i=1}^{12} S_i$ given by Eq. (11), the conditions (i), (ii), and (iv) of Theorem 1 are obvious. Here $R_1 = H_5 \cup H_{10}, R_2 = H_8 \cup H_{10}, R_3 = H_5, R_4 = H_8 \cup H_{12}, R_5 = H_1 \cup H_3, R_6 = H_{10}, R_7 = H_4 \cup H_{11}, R_8 = H_2 \cup H_4, R_9 = H_{11}, R_{10} = H_2 \cup H_6, R_{11} = H_7 \cup H_9$, and $R_{12} = H_4$. Now consider the condition (iii).

It is not difficult to show that the set sequence

$$G_1 = H_2 \cup H_4 \cup H_7 \cup H_8 \cup H_9 \cup H_{11},$$
$$G_2 = H_1 \cup H_{10} \cup H_{12},$$
$$G_3 = H_5 \cup H_6,$$
$$G_4 = H_3$$

satisfies (1) and (2). Here each subset contained in G_x ($x = 2, 3, 4$) is a NIC subset. For $H_1 \subset G_2$, we find that there are $H_2 \subset G_1$ and $H_{10} \subset R_1$ such that $H_1^{(BC)} \cap H_2^{(BC)} = H_1^{(BC)} \cap H_{10}^{(BC)}$. For the subsets H_{10}, H_{12}, H_5, H_6, and H_3, there are $H_2 = G_1 \cap R_{10}, H_4 = G_1 \cap R_{12}, H_1 = G_2 \cap R_5, H_{10} = G_2 \cap R_6$, and $H_5 = G_3 \cap R_3$, respectively. It follows that the condition (iii) in Theorem 1 holds.

According to Theorem 1, the orthogonality-preserving POVM performed on party BC can only be trivial. Therefore, the set $\cup_{i=1}^{12} H_i$ given by Eq. (12) is of the strongest nonlocality. ■

Applying Theorem 1, we propose a strongly nonlocal OPS in $\mathcal{C}^4 \otimes \mathcal{C}^4 \otimes \mathcal{C}^4$. The newly constructed OPS contains fewer quantum states than in Refs. [34,36]. The specific OPS is given by

$$\begin{aligned}
&S_{11} = \{|0\rangle|1\rangle|2 \pm 3\rangle\}, && S_{51} = \{|1\rangle|3\rangle|2 \pm 3\rangle\},\\
&S_{12} = \{|1\rangle|2 \pm 3\rangle|0\rangle\}, && S_{52} = \{|3\rangle|2 \pm 3\rangle|1\rangle\},\\
&S_{13} = \{|2 \pm 3\rangle|0\rangle|1\rangle\}, && S_{53} = \{|2 \pm 3\rangle|1\rangle|3\rangle\},\\
&S_{21} = \{|0\rangle|2\rangle|1 \pm 2\rangle\}, && S_{61} = \{|2\rangle|3\rangle|1 \pm 2\rangle\},\\
&S_{22} = \{|2\rangle|1 \pm 2\rangle|0\rangle\}, && S_{62} = \{|3\rangle|1 \pm 2\rangle|2\rangle\},\\
&S_{23} = \{|1 \pm 2\rangle|0\rangle|2\rangle\}, && S_{63} = \{|1 \pm 2\rangle|2\rangle|3\rangle\},\\
&S_{31} = \{|0\rangle|3\rangle|0 \pm 2\rangle\}, && S_{71} = \{|3\rangle|0\rangle|2 \pm 3\rangle\},\\
&S_{32} = \{|3\rangle|0 \pm 2\rangle|0\rangle\}, && S_{72} = \{|0\rangle|2 \pm 3\rangle|3\rangle\},\\
&S_{33} = \{|0 \pm 2\rangle|0\rangle|3\rangle\}, && S_{73} = \{|2 \pm 3\rangle|3\rangle|0\rangle\},\\
&S_{41} = \{|1\rangle|0\rangle|0 \pm 1\rangle\}, && S_{81} = \{|3\rangle|1\rangle|0 \pm 1\rangle\},\\
&S_{42} = \{|0\rangle|0 \pm 1\rangle|1\rangle\}, && S_{82} = \{|1\rangle|0 \pm 1\rangle|3\rangle\},\\
&S_{43} = \{|0 \pm 1\rangle|1\rangle|0\rangle\}, && S_{83} = \{|0 \pm 1\rangle|3\rangle|1\rangle\}.
\end{aligned} \quad (13)$$

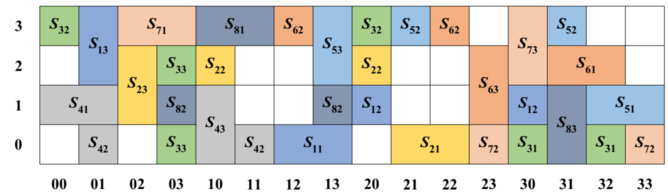

FIG. 8. The corresponding 4×16 grid of $\{S_{ij}\}$ given by Eq. (13) in $A|BC$ bipartition.

A geometric representation of this OPS in $A|BC$ bipartition is depicted in Fig. 8.

Theorem 5. In $\mathcal{C}^4 \otimes \mathcal{C}^4 \otimes \mathcal{C}^4$, the set $\cup_{i=1}^{8}(\cup_{j=1}^{3} S_{ij})$ given by Eq. (13) is of the strongest nonlocality. The size of this set is 48.

The detailed proof is shown in Appendix C. Up to now, we have constructed a strongly nonlocal OPS containing 48 states in $\mathcal{C}^4 \otimes \mathcal{C}^4 \otimes \mathcal{C}^4$, which is six and eight fewer than states presented in Refs. [34] and [36], respectively.

Next, we generalize the structures of OPSs given by Eq. (13) and Ref. [34] to systems $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ and $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C} \otimes \mathcal{C}^{d_D}$, respectively.

In quantum system $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ ($d_A, d_B, d_C \geqslant 4$), consider the following OPS:

$$\begin{aligned}
H_{11} &= \{|0\rangle_A|1\rangle_B|\alpha_3^l\rangle_C\}_l,\\
H_{12} &= \{|1\rangle_A|\alpha_3^l\rangle_B|0\rangle_C\}_l,\\
H_{13} &= \{|\alpha_3^l\rangle_A|0\rangle_B|1\rangle_C\}_l,\\
H_{21} &= \{|0\rangle_A|\alpha^i\rangle_B|\alpha_1^k\rangle_C\}_{i,k},\\
H_{22} &= \{|\alpha^i\rangle_A|\alpha_1^k\rangle_B|0\rangle_C\}_{i,k},\\
H_{23} &= \{|\alpha_1^k\rangle_A|0\rangle_B|\alpha^i\rangle_C\}_{i,k},\\
H_{31} &= \{|0\rangle_A|d'_B\rangle_B|\alpha_0^j\rangle_C\}_j,\\
H_{32} &= \{|d'_A\rangle_A|\alpha_0^j\rangle_B|0\rangle_C\}_j,\\
H_{33} &= \{|\alpha_0^j\rangle_A|0\rangle_B|d'_C\rangle_C\}_j,\\
H_{41} &= \{|1\rangle_A|0\rangle_B|0 \pm 1\rangle_C\},\\
H_{42} &= \{|0\rangle_A|0 \pm 1\rangle_B|1\rangle_C\},\\
H_{43} &= \{|0 \pm 1\rangle_A|1\rangle_B|0\rangle_C\},\\
H_{51} &= \{|1\rangle_A|d'_B\rangle_B|\alpha_3^l\rangle_C\}_l,\\
H_{52} &= \{|d'_A\rangle_A|\alpha_3^l\rangle_B|1\rangle_C\}_l,\\
H_{53} &= \{|\alpha_3^l\rangle_A|1\rangle_B|d'_C\rangle_C\}_l,\\
H_{61} &= \{|\alpha^i\rangle_A|d'_B\rangle_B|\alpha_1^k\rangle_C\}_{i,k},\\
H_{62} &= \{|d'_A\rangle_A|\alpha_1^k\rangle_B|\alpha^i\rangle_C\}_{i,k},\\
H_{63} &= \{|\alpha_1^k\rangle_A|\alpha^i\rangle_B|d'_C\rangle_C\}_{i,k},\\
H_{71} &= \{|d'_A\rangle_A|0\rangle_B|\alpha_3^l\rangle_C\}_l,\\
H_{72} &= \{|0\rangle_A|\alpha_3^l\rangle_B|d'_C\rangle_C\}_l,\\
H_{73} &= \{|\alpha_3^l\rangle_A|d'_B\rangle_B|0\rangle_C\}_l,
\end{aligned}$$

$$H_{81}=\{|d'_A\rangle_A|1\rangle_B|0\pm1\rangle_C\},$$
$$H_{82}=\{|1\rangle_A|0\pm1\rangle_B|d'_C\rangle_C\},$$
$$H_{83}=\{|0\pm1\rangle_A|d'_B\rangle_B|1\rangle_C\}. \tag{14}$$

Here $|\alpha^i\rangle_\tau=\sum_{u=0}^{d_\tau-4}\omega_{d_\tau-3}^{iu}|u+2\rangle$, $|\alpha_0^j\rangle_\tau=|0\rangle+\sum_{u=1}^{d_\tau-3}\omega_{d_\tau-2}^{ju}|u+1\rangle$, $|\alpha_1^k\rangle_\tau=\sum_{u=0}^{d_\tau-3}\omega_{d_\tau-2}^{ku}|u+1\rangle$, $|\alpha_3^l\rangle_\tau=\sum_{u=0}^{d_\tau-3}\omega_{d_\tau-2}^{lu}|u+2\rangle$, $d'_\tau=d_\tau-1$ for $i\in\mathcal{Z}_{d_\tau-3}$, $j,k,l\in\mathcal{Z}_{d_\tau-2}$, and $\tau=A,B,C$. Since the above OPS has the same structure as the set (13), we find that it is strongly nonlocal.

Theorem 6. In $\mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}\otimes\mathcal{C}^{d_C}$, the set $\cup_{i=1}^{8}(\cup_{j=1}^{3}H_{ij})$ given by Eq. (14) is an OPS of the strongest nonlocality. The size of this set is $2[(d_Ad_B+d_Bd_C+d_Ad_C)-3(d_A+d_B+d_C)+12]$.

The detailed proof is in Appendix D. In $\mathcal{C}^d\otimes\mathcal{C}^d\otimes\mathcal{C}^d$, the size $6[(d-1)^2-d+3]$ of the strongly nonlocal OPS of Theorem 4 is strictly fewer, $6(d-3)$ fewer to be precise, than the size $6(d-1)^2$ of the strongly nonlocal OPS in Ref. [34]. Similarly, we propose the following OPS in $\mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}\otimes\mathcal{C}^{d_C}\otimes\mathcal{C}^{d_D}$:

$$U_{11}=\{|0\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C|0\pm d'_D\rangle_D\}_{i,j},$$
$$U_{12}=\{|\xi_i\rangle_A|\eta_j\rangle_B|0\pm d'_C\rangle_C|0\rangle_D\}_{i,j},$$
$$U_{13}=\{|\eta_j\rangle_A|0\pm d'_B\rangle_B|0\rangle_C|\xi_i\rangle_D\}_{i,j},$$
$$U_{14}=\{|0\pm d'_A\rangle_A|0\rangle_B|\xi_i\rangle_C|\eta_j\rangle_D\}_{i,j},$$
$$U_{21}=\{|\xi_i\rangle_A|d'_B\rangle_B|\gamma_k\rangle_C|\eta_j\rangle_D\}_{i,j,k},$$
$$U_{22}=\{|d'_A\rangle_A|\gamma_k\rangle_B|\eta_j\rangle_C|\xi_i\rangle_D\}_{i,j,k},$$
$$U_{23}=\{|\gamma_k\rangle_A|\eta_j\rangle_B|\xi_i\rangle_C|d'_D\rangle_D\}_{i,j,k},$$
$$U_{24}=\{|\eta_j\rangle_A|\xi_i\rangle_B|d'_C\rangle_C|\gamma_k\rangle_D\}_{i,j,k},$$
$$U_{31}=\{|d'_A\rangle_A|0\rangle_B|0\pm d'_C\rangle_C|\gamma_k\rangle_D\}_k,$$
$$U_{32}=\{|0\rangle_A|0\pm d'_B\rangle_B|\gamma_k\rangle_C|d'_D\rangle_D\}_k,$$
$$U_{33}=\{|0\pm d'_A\rangle_A|\gamma_k\rangle_B|d'_C\rangle_C|0\rangle_D\}_k,$$
$$U_{34}=\{|\gamma_k\rangle_A|d'_B\rangle_B|0\rangle_C|0\pm d'_D\rangle_D\}_k,$$
$$U_{41}=\{|\xi_i\rangle_A|\xi_i\rangle_B|0\rangle_C|\gamma_k\rangle_D\}_{i|_A,i|_B,k},$$
$$U_{42}=\{|\xi_i\rangle_A|0\rangle_B|\gamma_k\rangle_C|\xi_i\rangle_D\}_{i|_A,i|_D,k},$$
$$U_{43}=\{|0\rangle_A|\gamma_k\rangle_B|\xi_i\rangle_C|\xi_i\rangle_D\}_{i|_C,i|_D,k},$$
$$U_{44}=\{|\gamma_k\rangle_A|\xi_i\rangle_B|\xi_i\rangle_C|0\rangle_D\}_{i|_B,i|_C,k},$$
$$U_{51}=\{|d'_A\rangle_A|d'_B\rangle_B|\xi_i\rangle_C|0\pm d'_D\rangle_D\}_i,$$
$$U_{52}=\{|d'_A\rangle_A|\xi_i\rangle_B|0\pm d'_C\rangle_C|d'_D\rangle_D\}_i,$$
$$U_{53}=\{|\xi_i\rangle_A|0\pm d'_B\rangle_B|d'_C\rangle_C|d'_D\rangle_D\}_i,$$
$$U_{54}=\{|0\pm d'_A\rangle_A|d'_B\rangle_B|d'_C\rangle_C|\xi_i\rangle_D\}_i,$$
$$U_{61}=\{|0\rangle_A|0\rangle_B|d'_C\rangle_C|\eta_j\rangle_D\}_j,$$
$$U_{62}=\{|0\rangle_A|d'_B\rangle_B|\eta_j\rangle_C|0\rangle_D\}_j,$$
$$U_{63}=\{|d'_A\rangle_A|\eta_j\rangle_B|0\rangle_C|0\rangle_D\}_j,$$
$$U_{64}=\{|\eta_j\rangle_A|0\rangle_B|0\rangle_C|d'_D\rangle_D\}_j,$$
$$U_{71}=\{|0\rangle_A|\xi_i\rangle_B|0\rangle_C|\xi_i\rangle_D\}_{i|_B,i|_D},$$
$$U_{72}=\{|\xi_i\rangle_A|0\rangle_B|\xi_i\rangle_C|0\rangle_D\}_{i|_A,i|_C},$$
$$U_{81}=\{|0\rangle_A|d'_B\rangle_B|0\rangle_C|d'_D\rangle_D\},$$
$$U_{82}=\{|d'_A\rangle_A|0\rangle_B|d'_C\rangle_C|0\rangle_D\},$$
$$U_{91}=\{|\xi_i\rangle_A|d'_B\rangle_B|\xi_i\rangle_C|d'_D\rangle_D\}_{i|_A,i|_C},$$
$$U_{92}=\{|d'_A\rangle_A|\xi_i\rangle_B|d'_C\rangle_C|\xi_i\rangle_D\}_{i|_B,i|_D}, \tag{15}$$

where $|\xi_i\rangle_\tau=\sum_{u=0}^{d_\tau-3}\omega_{d_\tau-2}^{iu}|u+1\rangle$, $|\eta_j\rangle_\tau=\sum_{u=0}^{d_\tau-2}\omega_{d_\tau-1}^{ju}|u\rangle$, $|\gamma_k\rangle_\tau=\sum_{u=0}^{d_\tau-2}\omega_{d_\tau-1}^{ku}|u+1\rangle$, $d'_\tau=d_\tau-1$ for $i\in\mathcal{Z}_{d_\tau-2}$, $j,k\in\mathcal{Z}_{d_\tau-1}$, and $\tau=A,B,C,D$.

Theorem 7. In the system $\mathcal{C}^{d_A}\otimes\mathcal{C}^{d_B}\otimes\mathcal{C}^{d_C}\otimes\mathcal{C}^{d_D}$, the set $\{\cup_{i=1}^{6}(\cup_{j=1}^{4}U_{ij})\}\cup\{\cup_{i=7}^{9}(\cup_{j=1}^{2}U_{ij})\}$ given by Eq. (15) is an OPS of the strongest nonlocality. The size of this set is $d_Ad_Bd_Cd_D-(d_A-2)(d_B-2)(d_C-2)(d_D-2)-2$.

The detailed proof is shown in Appendix E. It is worth noting that the set (15) is still of the strongest nonlocality even though it contains fewer quantum states than the set in Ref. [34]. Moreover, its size is smaller than that of the strongly nonlocal OPS in Ref. [36].

Each of Theorems 2–7 gives a positive answer to one open problem in Ref. [31] of "whether incomplete orthogonal product bases can be strongly nonlocal."

V. ENTANGLEMENT-ASSISTED DISCRIMINATION

The above OPSs cannot be distinguished under LOCC even if any $n-1$ parties are allowed to come together. However, it is possible while one equips enough entanglement resource. Let $|\phi^+(d)\rangle$ denote the maximally entangled state $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$ in $\mathcal{C}^d\otimes\mathcal{C}^d$. Let $(s,|\phi^+(d)\rangle_{AB})$ express a resource configuration, which means that on average an amount s of the two-qudit maximally entangled state is consumed between Alice and Bob. In this section, we present several different entanglement-assisted discrimination protocols. Without loss of generality, from now, we only consider the case $d_A\geqslant d_B\geqslant d_C\geqslant d_D$.

Theorem 8. The entanglement resource configuration $\{(1,|\phi^+(2)\rangle_{AB});(1,|\phi^+(d_C)\rangle_{BC})\}$ is sufficient for local discrimination of the set (12).

The detailed process is provided in Appendix F. In this protocol, we use quantum teleportation one time and consume $(1+\log_2 d_C)$-ebit entanglement resources in total. It is strictly less than the amount consumed in the protocol which teleports all subsystems to one party. Next, we discuss the local discrimination of OPS (12) without teleportation.

Theorem 9. When all the parties are separated, the set $\cup_{i=1}^{12}H_i$ given by Eq. (12) can be locally distinguished by using the entanglement resource $\{(s,|\phi^+(2)\rangle_{AB});\ (1,|\phi^+(2)\rangle_{AC})\}$, where $s=1+\frac{e-3f+6}{2e-4f+6}$ for $e=d_Ad_B+d_Ad_C+d_Bd_C$ and $f=d_A+d_B+d_C$.

The specific process is given in Appendix G. The entanglement consumed in this protocol is $(1+s)$ ebits, due to $s<1.5<\log_2 d_C$, which is less than the resources used in Theorem 8. Since the set (13) is a special case of (14) and they have the same structure, we only need to consider the entanglement-assisted discrimination protocols for the set (14).

Theorem 10. The set $\cup_{i=1}^{8}(\cup_{j=1}^{3}H_{ij})$ given by Eq. (14) can be locally distinguished by using the entanglement resource configuration $\{(1, |\phi^+(2)\rangle_{AB}); (1, |\phi^+(d_C)\rangle_{BC})\}$.

Theorem 11. The set $\cup_{i=1}^{8}(\cup_{j=1}^{3}H_{ij})$ given by Eq. (14) can be locally distinguished by using the entanglement resource configuration $\{(1, |\phi^+(4)\rangle_{AB}); (1, |\phi^+(2)\rangle_{AC})\}$.

The detailed proofs of Theorems 10 and 11 are given in Appendixes H and I, respectively. The protocol in Theorem 10 uses teleportation while the protocol in Theorem 11 does not. Clearly, $1 + \log_2 d_C$ ebits of entanglement are consumed in the previous protocol, which is not less than the amount used of three ebits in the latter protocol because $d_C \geqslant 4$. In other words, the latter resource configuration is more effective when the smallest dimension d_C is greater than four. Next, by the method presented by Zhang *et al.* in Ref. [52], using multiple copies of EPR states instead of high-dimensional entangled states, we can get a new resource configuration.

Theorem 12. The entanglement resource configuration $\{(2, |\phi^+(2)\rangle_{AB}); (1, |\phi^+(2)\rangle_{AC})\}$ is sufficient for local discrimination of the set $\cup_{i=1}^{8}(\cup_{j=1}^{3}H_{ij})$ given by Eq. (14).

In fact, using two EPR states has the same effect as using one maximally entangled state $|\phi^+(4)\rangle_{AB}$. In the ancillary system of one party, $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ can correspond to $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$, respectively. For the detailed procedure please refer to Appendix J. This also shows that, in the similar discrimination protocol, we can replace a maximally entangled state $|\phi^+(d)\rangle$ with n EPR states when $2^n \geqslant d$. Although more resources may be used, the method should be relatively easier to implement in a real experiment because it only requires a device which can produce two-qubit maximally entangled states. Besides, we also get several entanglement resource configurations to discriminate the set (15) by LOCC.

Theorem 13. The entanglement resource configuration $\{(1, |\phi^+(3)\rangle_{AB}); (1, |\phi^+(d_C)\rangle_{BC}); (1, |\phi^+(d_D)\rangle_{BD})\}$ is sufficient for local discrimination of the set $\{\cup_{i=1}^{6}(\cup_{j=1}^{4}U_{ij})\} \cup \{\cup_{i=7}^{9}(\cup_{j=1}^{2}U_{ij})\}$ given by Eq. (15).

The protocol of Theorem 13 is given in Appendix K.

Theorem 14. Any one of the resource configurations $\{(1, |\phi^+(3)\rangle_{AB}); (1, |\phi^+(3)\rangle_{AC}); (1, |\phi^+(3)\rangle_{AD})\}$ and $\{(2, |\phi^+(2)\rangle_{AB}); (2, |\phi^+(2)\rangle_{AC}); (2, |\phi^+(2)\rangle_{AD})\}$ is sufficient for local discrimination of the set (15).

We will not repeat the protocol of Theorem 14, because it is similar to that of Theorems 11 and 12. In Theorem 13, we perform quantum teleportation twice and consume $\log_2 3d_Cd_D$ ebits of entanglement resource. In comparison, the first configuration of Theorem 14 is more effective because $\log_2 27 \leqslant \log_2 3d_Cd_D$, and the second configuration is simpler because it only needs multiple EPR states.

VI. CONCLUSION

We have investigated the OPS with strong quantum nonlocality in multipartite quantum systems through the decomposition of plane geometry. Sufficient conditions for the triviality of orthogonality-preserving POVM on fixed subsystem are presented. We have shown the minimum size of strongly nonlocal OPSs under some restrictions in $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$ and $\mathcal{C}^4 \otimes \mathcal{C}^4 \otimes \mathcal{C}^4$, which partially answer the open question in Ref. [34], "Can we find the smallest strongly nonlocal set in $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$, and more generally in any tripartite systems?" Furthermore, we successfully constructed a smaller OPS which has the strongest nonlocality in $\mathcal{C}^{d_A} \otimes \mathcal{C}^{d_B} \otimes \mathcal{C}^{d_C}$ $(d_A, d_B, d_C \geqslant 4)$ and generalized the previous known structures of strongly nonlocal OPSs to any possible three and four-partite systems. Interestingly, we studied local discrimination protocols for our OPSs with different types of entangled resources. Among them, we have three protocols which only need multiple copies of EPR states. We found that the protocols without teleportation can be more efficient on average. More than that, our results could also be helpful in better understanding of the properties of maximally entangled states.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grants No. 12071110 and No. 62271189, the Hebei Natural Science Foundation of China under Grant No. A2020205014, the Science and Technology Project of Hebei Education Department under Grants No. ZD2020167 and No. ZD2021066, and funded by School of Mathematical Sciences of Hebei Normal University under Grant No. 2021sxbs002.

APPENDIX A: PROOF OF THEOREM 2

According to Corollary 2, we know the union $\cup_r S_r^{(BC)}$ of all projection sets is the basis $\mathcal{B}^{BC}$ and the family of projection sets $\{S_r^{(BC)}\}_r$ is connected.

When $N = 1$, it is obvious that the set S is locally distinguishable. When $N = 2$, due to the symmetry, there is the collection $\{S_{t_1}, S_{t_2}, S_{t_3}\}$ including six quantum states, which satisfies $|S_{t_1}| = |S_{t_2}| = |S_{t_3}| = 2$, $|S_{t_1}^{(BC)}| = |S_{t_2}^{(BC)}| = 2$, and $|S_{t_3}^{(BC)}| = 1$. Moreover, the collection is invariant under the cyclic permutation of the parties. According to the completeness and connectedness of projection sets, the set S contains at least eight subsets whose projection sets on party BC have two elements. That is, we have no less than four disjoint collections with above form. In other words, when $N = 2$, the size of set S cannot be less than 24.

The case $N = 3$ does not exist. If $N = 3$, then there must be a subset satisfying $|S_t^{(A)}| = 3$ and $|S_t^{(BC)}| = 1$. Meanwhile, $S_t^{(A)} = \mathcal{B}^A$. We have $S_t^{(BC)} \cap (\cup_{t' \in Q\setminus\{t\}} S_{t'}^{(BC)}) = \emptyset$. Hence, the family of projection sets $\{S_r^{(BC)}\}_r$ is unconnected, which is a contradiction. Similarly, the cases $N = 6, 9$ do not exist.

In the case $N = 4$, because of symmetry, there is a collection $\{S_{u_1}, S_{u_2}, S_{u_3}\}$ containing 12 quantum states, which is symmetric and satisfies $|S_{u_1}| = |S_{u_2}| = |S_{u_3}| = 4$, $|S_{u_1}^{(BC)}| = 4$ and $|S_{u_2}^{(BC)}| = |S_{u_3}^{(BC)}| = 2$. Similarly, due to the completeness and connectedness of projection sets, there are at least one other subset whose projection set on party BC has four elements or three additional subsets whose projection sets on party BC have two elements. In either case, it means that the size of set S is not less than 24.

It is obvious that $|S_r| \neq 5, 7$ for any $r \in Q$. If there is a subset such that $|S_r| = 8$, then for arbitrary cyclic permutation P_c of subsystems, the two subspaces spanned by S_r and $P_c(S_r)$, respectively, are not orthogonal. It follows that there must be

TABLE I. Corresponding PI set R_{ij} for each subset S_{ij}.

Subset	PI set	Subset	PI set
S_{11}	$R_{11} = S_{53} \cup S_{62}$	S_{51}	$R_{51} = S_{31} \cup S_{72}$
S_{12}	$R_{12} = S_{32} \cup S_{73}$	S_{52}	$R_{52} = S_{21} \cup S_{83}$
S_{13}	$R_{13} = S_{41}$	S_{53}	$R_{53} = S_{11}$
S_{21}	$R_{21} = S_{52} \cup S_{62}$	S_{61}	$R_{61} = S_{51} \cup S_{83}$
S_{22}	$R_{22} = S_{32} \cup S_{81}$	S_{62}	$R_{62} = S_{11} \cup S_{21}$
S_{23}	$R_{23} = S_{71}$	S_{63}	$R_{63} = S_{72}$
S_{31}	$R_{31} = S_{12} \cup S_{51}$	S_{71}	$R_{71} = S_{23} \cup S_{82}$
S_{32}	$R_{32} = S_{12} \cup S_{41}$	S_{72}	$R_{72} = S_{51} \cup S_{63}$
S_{33}	$R_{33} = S_{82}$	S_{73}	$R_{73} = S_{12}$
S_{41}	$R_{41} = S_{13} \cup S_{32}$	S_{81}	$R_{81} = S_{42} \cup S_{43}$
S_{42}	$R_{42} = S_{13} \cup S_{81}$	S_{82}	$R_{82} = S_{11} \cup S_{33}$
S_{43}	$R_{43} = S_{81}$	S_{83}	$R_{83} = S_{61}$

two nonorthogonal quantum states, one of which belongs to S_r and the other of which belongs to $P_c(S_r)$. This contradicts the fact $P_c(S_r) \subset S$. Consequently, the cases $N = 5, 7, 8$ do not hold.

On the other hand, the strongly nonlocal OPS given by Eq. (11) satisfies all conditions and contains 24 quantum states. Thus, in $\mathcal{C}^3 \otimes \mathcal{C}^3 \otimes \mathcal{C}^3$, the minimum size of the set S is 24. The proof is completed. ■

APPENDIX B: PROOF OF THEOREM 3

Because the set is symmetric and the maximum size of all subsets is two, there is a collection $\{S_{t_1}, S_{t_2}, S_{t_3}\}$ containing six quantum states. It satisfies the same requirements as the proof of Theorem 2. Due to the completeness and connectedness of projection sets, there are at least 15 subsets whose projection sets on party BC have size two. So, we have no less than eight disjoint collections, each of which contains six quantum states. That is, the set S contains at least 48 quantum states. On the other side, we find the OPS given by Eq. (13) satisfies all conditions and the size is 48. Therefore, the minimum size of set S is 48. ■

APPENDIX C: PROOF OF THEOREM 5

According to Lemma 1 and the invariance of the set (13) under cyclic permutations, we only need to discuss the orthogonal-preserving measurement on party BC. The tile structure is illustrated in Fig. 8. It is obvious that $\tilde{S}_{V_{kl}} = \mathcal{B}^{BC}$ for all $k, l \in \mathcal{Z}_4$, which implies that $\mathcal{B}^{BC}_{kl} \subset \tilde{S}_{V_{kl}}$. Hence the condition (i) holds.

For each subset S_{ij}, there is the corresponding PI set R_{ij}, which is shown in Table I. It follows that the condition (ii) is satisfied.

Since $|S^{(BC)}_{ij} \cap S^{(BC)}_{kl}| \leqslant 1$ for any two subsets S_{ij} and S_{kl}, each R_{ij} is a UPI set. Therefore G_1 is the union of all subsets. Thus, the condition (iii) is true.

In addition, we have a sequence of projection sets $S^{(BC)}_{41} \to S^{(BC)}_{42} \to S^{(BC)}_{81} \to S^{(BC)}_{22} \to S^{(BC)}_{12} \to S^{(BC)}_{31} \to S^{(BC)}_{51}$ $(\to S^{(BC)}_{72}) \to S^{(BC)}_{61} \to S^{(BC)}_{52} \to S^{(BC)}_{21} \to S^{(BC)}_{62} \to S^{(BC)}_{11} \to$ $S^{(BC)}_{82} \to S^{(BC)}_{71}$, where the intersection of the sets on both sides of the arrow is not empty and the union of these sets is the computation basis $\mathcal{B}^{BC}$. Here the set $S^{(BC)}_{72}$ in the bracket is only related to the previous set $S^{(BC)}_{51}$. This means that it is impossible to divide all projection sets into disjoint two groups. That is, the family of projection sets $\{S^{(BC)}_{ij}\}_{ij}$ is connected. The condition (iv) holds.

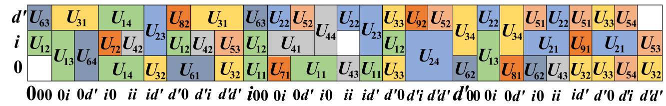

FIG. 9. The corresponding 3×27 grid of $\{U_{ij}\}$ given by Eq. (15) in $A|BCD$ bipartition.

By using Theorem 1, the orthogonality-preserving POVM performed on party BC can only be trivial. Therefore, the OPS (13) is of the strongest quantum nonlocality. ■

APPENDIX D: PROOF OF THEOREM 6

We need only to consider the orthogonality-preserving POVM on party BC. Because the set (14) has the same structure as the set (13), the conditions (i), (ii), and (iv) are obvious. Consider the set sequence

$$\begin{aligned} G_1 &= H_{11} \cup H_{12} \cup H_{13} \cup H_{22} \cup H_{31} \cup H_{32} \cup H_{33} \\ &\quad \cup H_{41} \cup H_{42} \cup H_{43} \cup H_{51} \cup H_{52} \cup H_{53} \cup H_{61} \\ &\quad \cup H_{71} \cup H_{72} \cup H_{73} \cup H_{81} \cup H_{82} \cup H_{83}, \\ G_2 &= H_{21} \cup H_{23} \cup H_{62} \cup H_{63}. \end{aligned} \tag{D1}$$

Here each subset contained in G_2 is a NIC subset. Referring to Table I, we can get the PI set R_{ij} of H_{ij} on party BC. More specifically, H_{ij} is substituted for S_{ij} in Table I, one gets the PI set R_{ij} of H_{ij} on party BC. For the subsets H_{21}, H_{23}, H_{62}, and H_{63}, there are $H_{52} = G_1 \cap R_{21}$, $H_{71} = G_1 \cap R_{23}$, $H_{11} = G_1 \cap R_{62}$, and $H_{72} = G_1 \cap R_{63}$, respectively. It implies that condition (iii) holds.

The set (14) satisfies the four conditions in Theorem 1, therefore it is locally irreducible in every bipartition. That is, the OPS (14) is a set of the strongest nonlocality. ■

APPENDIX E: PROOF OF THEOREM 7

We need to prove that the orthogonality-preserving POVM performed on party BCD can only be trivial. To see this, we prove that the OPS (15) satisfies the four conditions in Theorem 1.

Figure 9 is the tile structure of this OPS. Note that $\tilde{S}_{V_{jkl}} = \mathcal{B}^{BCD}$ for any $j, k, l \in \mathcal{Z}_3$. It is obvious $\mathcal{B}^{BCD}_{jkl} \subset \tilde{S}_{V_{jkl}}$. The condition (i) holds.

For each subset U_{ij}, there is the corresponding PI set R_{ij}, which is shown in Table II. Hence, the condition (ii) holds.

Furthermore, we construct the set sequence

$$\begin{aligned} G_1 &= U_{12} \cup U_{21} \cup U_{31} \cup U_{33} \cup U_{34} \cup U_{61} \\ &\quad \cup U_{62} \cup U_{64} \cup U_{81} \cup U_{82}, \\ G_2 &= U_{11} \cup U_{13} \cup U_{42} \cup U_{51} \cup U_{54} \cup U_{63}, \\ G_3 &= U_{14} \cup U_{22} \cup U_{23} \cup U_{32} \cup U_{41} \cup U_{44} \\ &\quad \cup U_{52} \cup U_{91}, \\ G_4 &= U_{24} \cup U_{43} \cup U_{53} \cup U_{71} \cup U_{72}, \\ G_5 &= U_{92}. \end{aligned} \tag{E1}$$

TABLE II. Corresponding PI set R_{ij} for each subset U_{ij}.

Subset	PI set	Subset	PI set
U_{11}	$R_{11} = U_{12} \cup U_{23} \cup U_{41} \cup U_{44}$	U_{44}	$R_{44} = U_{11}$
U_{12}	$R_{12} = U_{33} \cup U_{63} \cup U_{82}$	U_{51}	$R_{51} = U_{32} \cup U_{62}$
U_{13}	$R_{13} = U_{22} \cup U_{31}$	U_{52}	$R_{52} = U_{11} \cup U_{24}$
U_{14}	$R_{14} = U_{42} \cup U_{72}$	U_{53}	$R_{53} = U_{32}$
U_{21}	$R_{21} = U_{22} \cup U_{33} \cup U_{51} \cup U_{54}$	U_{54}	$R_{54} = U_{21}$
U_{22}	$R_{22} = U_{13} \cup U_{43} \cup U_{71}$	U_{61}	$R_{61} = U_{12} \cup U_{42}$
U_{23}	$R_{23} = U_{11} \cup U_{32}$	U_{62}	$R_{62} = U_{34} \cup U_{51}$
U_{24}	$R_{24} = U_{52} \cup U_{92}$	U_{63}	$R_{63} = U_{12}$
U_{31}	$R_{31} = U_{13} \cup U_{42} \cup U_{53} \cup U_{64}$	U_{64}	$R_{64} = U_{31}$
U_{32}	$R_{32} = U_{23} \cup U_{53} \cup U_{91}$	U_{71}	$R_{71} = U_{41}$
U_{33}	$R_{33} = U_{12} \cup U_{21}$	U_{72}	$R_{72} = U_{14}$
U_{34}	$R_{34} = U_{62} \cup U_{81}$	U_{81}	$R_{81} = U_{34}$
U_{41}	$R_{41} = U_{11} \cup U_{71}$	U_{82}	$R_{82} = U_{12}$
U_{42}	$R_{42} = U_{14} \cup U_{61}$	U_{91}	$R_{91} = U_{51}$
U_{43}	$R_{43} = U_{22}$	U_{92}	$R_{92} = U_{24}$

For the subset $U_{32} \subset G_3$, there are subsets $U_{51} \subset G_2$ and $U_{91} \subset R_{32}$ such that $U_{32}^{(BCD)} \cap U_{51}^{(BCD)} = U_{32}^{(BCD)} \cap U_{91}^{(BCD)}$. For any other subset $U_t \subset G_x$ ($x = 2, \dots, 5$), the intersection of set G_{x-1} and PI set R_t is exhibited in Table III. This shows that the condition (iii) is true.

We find the tree sequence of projection sets $U_{12}^{(BCD)} \to U_{61}^{(BCD)}(\to U_{42}^{(BCD)} \to U_{14}^{(BCD)}) \to U_{31}^{(BCD)} \to U_{32}^{(BCD)} \to U_{51}^{(BCD)}(\to U_{62}^{(BCD)} \to U_{34}^{(BCD)}) \to U_{21}^{(BCD)} \to U_{22}^{(BCD)} \to U_{41}^{(BCD)}(\to U_{11}^{(BCD)}) \to U_{52}^{(BCD)} \to U_{24}^{(BCD)}$, where the subsequence in parentheses is a branch of the previous adjacent set. In this sequence, the intersection of the sets on both sides of the arrow is nonempty and the union of all these sets is the computation basis $\mathcal{B}^{BCD}$. This means that the family of projection sets $\{U_{ij}^{(BCD)}\}_{ij}$ is connected. The condition (iv) is proven.

Therefore, one can only perform a trivial orthogonality-preserving POVM on the BCD party. Combining Lemma 1 with the symmetry of (15) ensures that the OPS (15) is of the strongest quantum nonlocality. ∎

TABLE III. The intersection of set G_{x-1} and PI set R_t about subset $U_t \subset G_x$ ($x = 2, \dots, 5$).

Subset	Intersection	Subset	Intersection
$U_{11} \subset G_2$	$U_{12} = G_1 \cap R_{11}$	$U_{44} \subset G_3$	$U_{11} = G_2 \cap R_{44}$
$U_{13} \subset G_2$	$U_{31} = G_1 \cap R_{13}$	$U_{52} \subset G_3$	$U_{11} = G_2 \cap R_{52}$
$U_{42} \subset G_2$	$U_{61} = G_1 \cap R_{42}$	$U_{91} \subset G_3$	$U_{51} = G_2 \cap R_{91}$
$U_{51} \subset G_2$	$U_{62} = G_1 \cap R_{51}$	$U_{24} \subset G_4$	$U_{52} = G_3 \cap R_{24}$
$U_{54} \subset G_2$	$U_{21} = G_1 \cap R_{54}$	$U_{43} \subset G_4$	$U_{22} = G_3 \cap R_{43}$
$U_{63} \subset G_2$	$U_{12} = G_1 \cap R_{63}$	$U_{53} \subset G_4$	$U_{32} = G_3 \cap R_{53}$
$U_{14} \subset G_3$	$U_{42} = G_2 \cap R_{14}$	$U_{71} \subset G_4$	$U_{41} = G_3 \cap R_{71}$
$U_{22} \subset G_3$	$U_{13} = G_2 \cap R_{22}$	$U_{72} \subset G_4$	$U_{14} = G_3 \cap R_{72}$
$U_{23} \subset G_3$	$U_{11} = G_2 \cap R_{23}$	$U_{92} \subset G_5$	$U_{24} = G_4 \cap R_{92}$
$U_{41} \subset G_3$	$U_{11} = G_2 \cap R_{41}$		

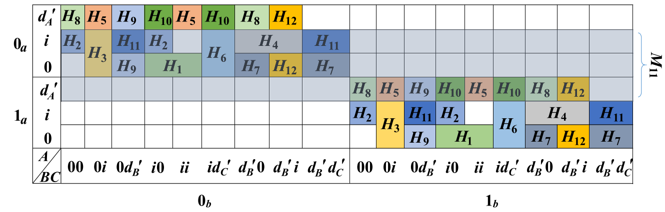

FIG. 10. While Alice and Bob share the EPR state $|\phi^+(2)\rangle_{ab}$, the initial state given by Eq. (F1) can be expressed by the corresponding $2d_A \times 2d_B d_C$ grid. The area covered with light gray represents the measurement effect M_{11} in step 1.

APPENDIX F: PROOF OF THEOREM 8

Suppose that the whole quantum system is shared among Alice, Bob, and Charlie. By taking advantage of entangled resource $|\phi^+(d_C)\rangle$, Charlie first teleports the state in his subsystem C to Bob. Let the subindex $\widetilde{B}$ represent the joint part of B and C. Whereafter, to locally discriminate the states in (12), the EPR state $|\phi^+(2)\rangle_{ab}$ is shared by Alice and Bob. The initial state is

$$|\psi\rangle_{A\widetilde{B}} \otimes |\phi^+(2)\rangle_{ab}, \tag{F1}$$

where $|\psi\rangle_{A\widetilde{B}}$ is one of the states from the set (12), a and b are ancillary systems of Alice and Bob, respectively. Because each subset H_r ($r \in Q$) is LOCC distinguishable, one only needs to locally distinguish these subsets. Now the discrimination protocol proceeds as follows:

Step 1. Alice performs the measurement:

$$\mathcal{M}_1 \equiv \{M_{11} := P[(|0\rangle, \dots, |d_A' - 1\rangle)_A; |0\rangle_a] + P[|d_A'\rangle_A; |1\rangle_a],$$
$$M_{12} := I - M_{11}\},$$

where $P[(|0\rangle, \dots, |d_A' - 1\rangle)_A; |0\rangle_a] := (|0\rangle\langle 0| + \dots + |d_A' - 1\rangle\langle d_A' - 1|)_A \otimes (|0\rangle\langle 0|)_a$, this definition is applicable for all the protocols. Suppose the outcome corresponding to M_{11} clicks (see Fig. 10), then the resulting postmeasurement states are

$$H_1 \to \{|0\rangle_A|\xi_i \circ \eta_j\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_2 \to \{|\xi_i\rangle_A|\eta_j \circ 0\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_3 \to \{|\eta_j\rangle_A|0 \circ \xi_i\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_4 \to \{|\xi_i\rangle_A|d_B' \circ \eta_j\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_5 \to \{|d_A'\rangle_A|\eta_j \circ \xi_i\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_6 \to \{|\eta_j\rangle_A|\xi_i \circ d_C'\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_7 \to \{|0\rangle_A|d_B' \circ (0 \pm d_C')\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_8 \to \{|d_A'\rangle_A|(0 \pm d_B') \circ 0\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_9 \to \{(|0\rangle_A|00\rangle_{ab} \pm |d_A'\rangle_A|11\rangle_{ab})|0 \circ d_C'\rangle_{\widetilde{B}}\},$$
$$H_{10} \to \{|d_A'\rangle_A|\xi_i \circ (0 \pm d_C')\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{11} \to \{|\xi_i\rangle_A|(0 \pm d_B') \circ d_C'\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{12} \to \{(|0\rangle_A|00\rangle_{ab} \pm |d_A'\rangle_A|11\rangle_{ab})|d_B' \circ \xi_i\rangle_{\widetilde{B}}\}.$$

Henceforth, symbol "$\circ$" represents the union of the parties. For example, $|\psi_1 \circ \psi_2\rangle_{\widetilde{B}} = |\psi_1\rangle_B|\psi_2\rangle_C$ for any two quantum states $|\psi_1\rangle_B$ and $|\psi_2\rangle_C$. Specifically, let $|(0, \dots, d_B - 1) \circ (0, \dots, d_C - 1)\rangle_{\widetilde{B}}$ express the set

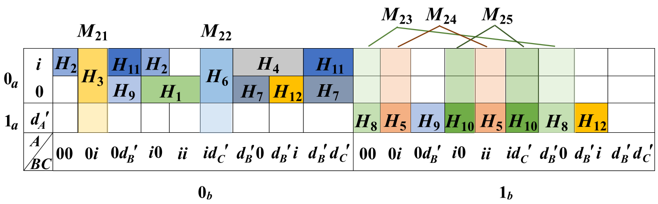

FIG. 11. The $d_A \times 2d_Bd_C$ grid is the states after clicking M_{11}. Areas covered by different light colors denote the different measurement effect.

$\{|ij\rangle_{\widetilde{B}} \,|\, i = 0, 1, \ldots, d_B - 1; j = 0, 1, \ldots, d_C - 1\}$ denoted by $(|00\rangle, \ldots, |0(d_C - 1)\rangle, |10\rangle, \ldots, |(d_B - 1)(d_C - 1)\rangle)_{\widetilde{B}}$.

Step 2. Bob performs the measurement:

$$\begin{aligned}\mathcal{M}_2 \equiv \{M_{21} &:= P[|0 \circ (1, \ldots, d_C' - 1)\rangle_{\widetilde{B}}; |0\rangle_b],\\ M_{22} &:= P[|(1, \ldots, d_B' - 1) \circ d_C'\rangle_{\widetilde{B}}; |0\rangle_b],\\ M_{23} &:= P[|(0, d_B') \circ 0\rangle_{\widetilde{B}}; |1\rangle_b],\\ M_{24} &:= P[|(0, \ldots, d_B' - 1) \circ (1, \ldots, d_C' - 1)\rangle_{\widetilde{B}}; |1\rangle_b],\\ M_{25} &:= P[|(1, \ldots, d_B' - 1) \circ (0, d_C')\rangle_{\widetilde{B}}; |1\rangle_b],\\ M_{26} &:= I - M_1 - M_2 - M_3 - M_4 - M_5\}.\end{aligned}$$

This step is shown in Fig. 11. If the corresponding operations M_{21}, M_{22}, M_{23}, M_{24}, and M_{25} click, we can distinguish the subsets H_3, H_6, H_8, H_5, and H_{10}, respectively. If M_{26} clicks, the given state is belonging to one of the remaining seven subsets $\{H_1, H_2, H_4, H_7, H_9, H_{11}, H_{12}\}$. At this point, we move on to the next step.

Step 3. Alice performs the measurement:

$$\begin{aligned}\mathcal{M}_3 \equiv \{M_{31} &:= P[|0\rangle_A; |0\rangle_a] + P[|d_A'\rangle_A; |1\rangle_a],\\ M_{32} &:= I - M_{31}\}.\end{aligned}$$

Figure 12 shows the intuitive situation. If M_{31} clicks, we can determine the four subsets $\{H_1, H_7, H_9, H_{12}\}$. Otherwise, the subset is one of the remaining three $\{H_2, H_4, H_{11}\}$. Moreover, they are all perfectly LOCC distinguishable.

In addition, if M_{12} clicks in step 1, we can find a similar protocol where these states can be perfectly LOCC distinguished. ∎

APPENDIX G: PROOF OF THEOREM 9

Naturally, we only need to locally distinguish these subsets. To this end, let Alice and Bob share an EPR state $|\phi^+(2)\rangle_{a_1b_1}$, meanwhile Alice and Charlie share the EPR state $|\phi^+(2)\rangle_{a_2c_1}$.

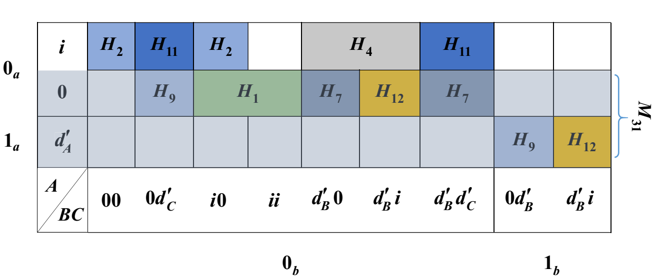

FIG. 12. The remaining states after Bob performs the measurement. Area covered by light gray is the measurement effect M_{31}.

Therefore, the initial state is

$$|\psi\rangle_{ABC} \otimes |\phi^+(2)\rangle_{a_1b_1} \otimes |\phi^+(2)\rangle_{a_2c_1}, \tag{G1}$$

where the state $|\psi\rangle_{ABC}$ is one of the states from the set $\cup_{r=1}^{12} H_r$ (12), a_1 and a_2 are ancillary systems of Alice, b_1 and c_1 are ancillary systems of Bob and Charlie, respectively. The specific process is as follows:

Step 1. Bob performs the measurement:

$$\begin{aligned}\mathcal{M}_1 \equiv \{M_{11} &:= P[(|0\rangle, \ldots, |d_B' - 1\rangle)_B; |0\rangle_{b_1}]\\ &\quad + P[|d_B'\rangle_B; |1\rangle_{b_1}],\\ M_{12} &:= I - M_{11}\},\end{aligned}$$

and Charlie performs the measurement:

$$\begin{aligned}\mathcal{M}_2 \equiv \{M_{21} &:= P[(|0\rangle, \ldots, |d_C' - 1\rangle)_C; |1\rangle_{c_1}]\\ &\quad + P[|d_C'\rangle_C; |0\rangle_{c_1}],\\ M_{22} &:= I - M_{21}\}.\end{aligned}$$

Suppose M_{11} and M_{21} click (refer to Fig. 13), the resulting postmeasurement states are

$$\begin{aligned}H_1 &\to \{|0\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\},\\ H_2 &\to \{|\xi_i\rangle_A|\eta_j\rangle_B|0\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\},\\ H_3 &\to \{|\eta_j\rangle_A|0\rangle_B|\xi_i\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\},\\ H_4 &\to \{|\xi_i\rangle_A|d_B'\rangle_B|\eta_j\rangle_C|11\rangle_{a_1b_1}|11\rangle_{a_2c_1}\},\\ H_5 &\to \{|d_A'\rangle_A|\eta_j\rangle_B|\xi_i\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\},\\ H_6 &\to \{|\eta_j\rangle_A|\xi_i\rangle_B|d_C'\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}\},\\ H_7 &\to \{|0\rangle_A|d_B'\rangle_B|11\rangle_{a_1b_1}(|0\rangle_C|11\rangle_{a_2c_1}\pm\\ &\qquad |d_C'\rangle_C|00\rangle_{a_2c_1})\},\\ H_8 &\to \{|d_A'\rangle_A(|0\rangle_B|00\rangle_{a_1b_1} \pm |d_B'\rangle_B|11\rangle_{a_1b_1})\\ &\qquad |0\rangle_C|11\rangle_{a_2c_1}\},\\ H_9 &\to \{|0 \pm d_A'\rangle_A|0\rangle_B|d_C'\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}\},\\ H_{10} &\to \{|d_A'\rangle_A|\xi_i\rangle_B|00\rangle_{a_1b_1}(|0\rangle_C|11\rangle_{a_2c_1}\pm\\ &\qquad |d_C'\rangle_C|00\rangle_{a_2c_1})\},\\ H_{11} &\to \{|\xi_i\rangle_A(|0\rangle_B|00\rangle_{a_1b_1} \pm |d_B'\rangle_B|11\rangle_{a_1b_1})\\ &\qquad |d_C'\rangle_C|00\rangle_{a_2c_1}\},\\ H_{12} &\to \{|0 \pm d_A'\rangle_A|d_B'\rangle_B|\xi_i\rangle_C|11\rangle_{a_1b_1}|11\rangle_{a_2c_1}\}.\end{aligned}$$

Step 2. Alice performs the measurement:

$$\begin{aligned}\mathcal{M}_3 \equiv \big\{M_{31} &:= P\big[(|0\rangle, \ldots, |d_A' - 1\rangle)_A; |0\rangle_{a_1}; |1\rangle_{a_2}\big],\\ M_{32} &:= P\big[(|1\rangle, \ldots, |d_A' - 1\rangle)_A; |1\rangle_{a_1}; |1\rangle_{a_2}\big],\\ M_{33} &:= I - M_{31} - M_{32}\big\}.\end{aligned}$$

This process is described in Fig. 14. If M_{31} clicks, the given subset is one of $\{H_1, H_2, H_3\}$, which contains $e - 3f + 6$ quantum states in total. Here $e = d_Ad_B + d_Ad_C + d_Bd_C$ and $f = d_A + d_B + d_C$. It is obvious that these three subsets cannot be perfectly distinguished by LOCC. Let Alice and Bob share the maximally entangled state $|\phi^+(2)\rangle_{a_3b_2}$. Moreover, Bob performs the measurement $\mathcal{M}_3' \equiv \{M_{31}' := P[|0\rangle_B; I_{b_1}; |0\rangle_{b_2}] +$

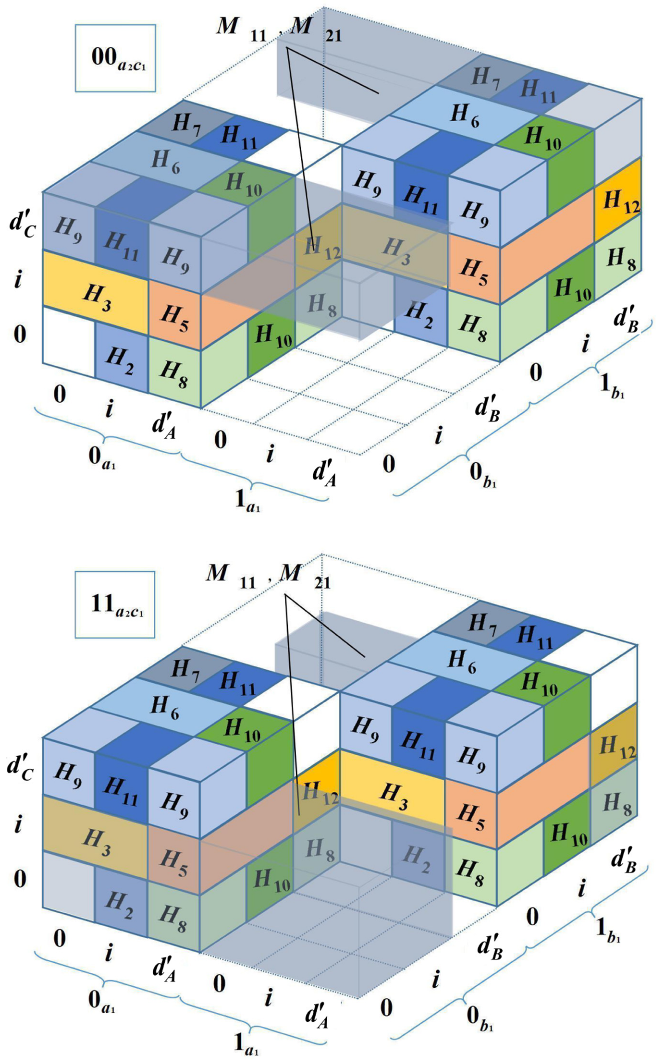

FIG. 13. The two $2d_A \times 2d_B \times d_C$ grids represent the initial states (G1) of auxiliary system as $|00\rangle_{a_2c_1}$ and $|11\rangle_{a_2c_1}$, respectively. Areas covered with light gray represent the measurement effect M_{11} and M_{21} in step 1.

$P[(|1\rangle, \ldots, |d_B'\rangle)_B; I_{b_1}; |1\rangle_{b_2}]$, $M_{32}' := I - M_{31}'\}$. When M_{31}' clicks, Alice performs the measurement $\mathcal{M}_3'' \equiv \{M_{31}'' := P[|0\rangle_A; I_{a_1}; I_{a_2}; |1\rangle_{a_3}]$, $M_{32}'' := I - M_{31}''\}$. The results corresponding to operators M_{31}'' and M_{32}'' are H_1 and $\{H_2, H_3\}$, respectively. The collection $\{H_2, H_3\}$ is LOCC distinguishable. Similarly, when M_{32}' clicks, the task of local discrimination can also be accomplished. The average entanglement consumed in this process is $(e-3f+6)/(2e-4f+6)$ maximally entangled state $|\phi^+(2)\rangle_{a_3b_2}$ [33], because the size of the set (12) is $2e-4f+6$.

If M_{32} clicks, the subset is H_4. Otherwise, the subset is one of the remaining eight.

Step 3. Charlie performs the measurement:

$$\mathcal{M}_4 \equiv \{M_{41} := P[(|1\rangle, \ldots, |d_C' - 1\rangle)_C; |1\rangle_{c_1}],$$
$$M_{42} := I - M_{41}\}.$$

Refer to Fig. 15, if M_{41} clicks, the given subset is one of $\{H_5, H_{12}\}$. Obviously, it is locally distinguishable.

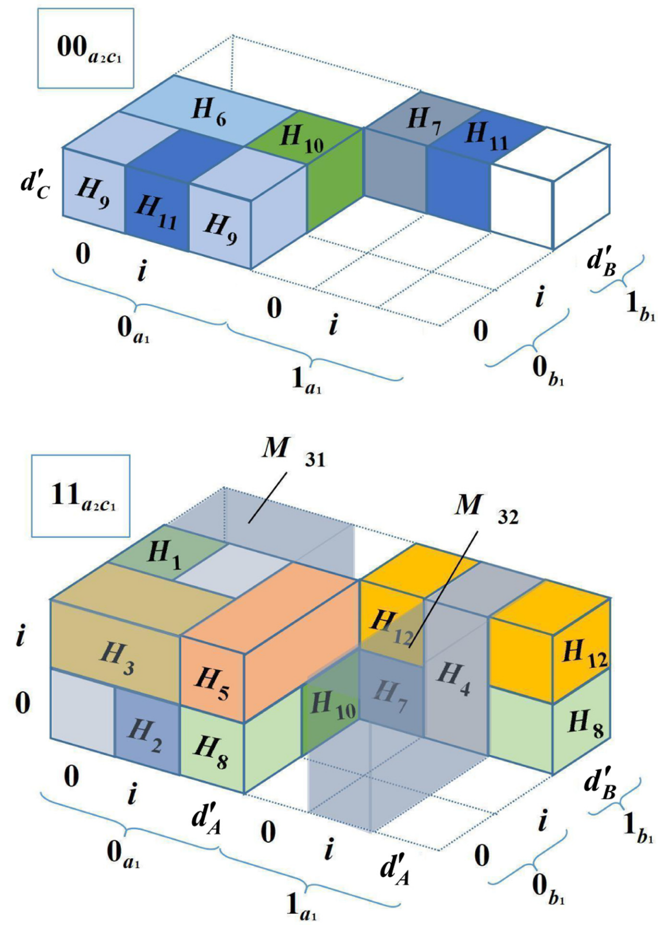

FIG. 14. The states after clicking M_{11} and M_{21}. The two areas covered with light gray express the measurement effect M_{31} and M_{32}, respectively.

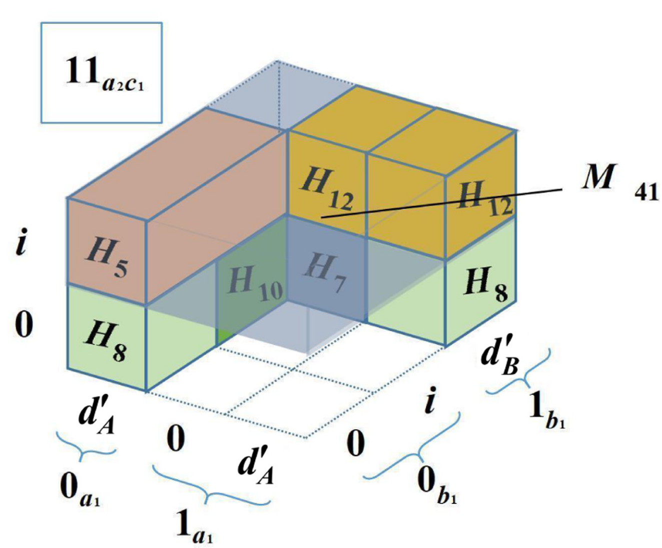

FIG. 15. The states with auxiliary system $|11\rangle_{a_2c_1}$ after clicking M_{33}. The area covered with light gray represents the measurement effect M_{41}.

Step 4. Alice performs the measurement:

$$\mathcal{M}_5 \equiv \{M_{51} := P[|0\rangle_A; |1\rangle_{a_1}; I_{a_2}], M_{52} := I - M_{51}\}.$$

If M_{51} clicks, the subset is H_7. Otherwise, the subset is one of $\{H_6, H_8, H_9, H_{10}, H_{11}\}$.

Step 5. Bob performs the measurement:

$$\mathcal{M}_6 \equiv \{M_{61} := P[|0\rangle_B; |0\rangle_{b_1}] + P[|d'_B\rangle_B; |1\rangle_{b_1}], M_{62} := I - M_{61}\}.$$

The results corresponding to operators M_{61} and M_{62} are $\{H_8, H_9, H_{11}\}$ and $\{H_6, H_{10}\}$, respectively. They are all locally distinguishable.

In summary, we consume a total of $1 + (e - 3f + 6)/(2e - 4f + 6)$ EPR states between Alice and Bob and one EPR state between Alice and Charlie for this distinguishing task. If in the step 1 other operators click, we can find similar protocols to distinguish these subsets perfectly by LOCC. ■

APPENDIX H: PROOF OF THEOREM 10

Suppose that the whole quantum system is shared among Alice, Bob, and Charlie. Since $d_C \leqslant d_B$, the subsystem C is teleported to Bob by using the entanglement resource $|\phi^+(d_C)\rangle_{BC}$, and the new union subsystem is represented by $\widetilde{B}$. To locally discriminate the states, Alice and Bob should share a maximally entangled state $|\phi^+(2)\rangle_{ab}$. The discrimination protocol proceeds as follows:

Step 1. Alice performs the measurement:

$$\mathcal{M}_1 \equiv \{M_{11} := P[(|0\rangle, |1\rangle)_A; |0\rangle_a] + P[(|2\rangle, \ldots, |d'_A\rangle)_A; |1\rangle_a], M_{12} := I - M_{11}\}.$$

Suppose M_{11} clicks, then the resulting postmeasurement states are

$$H_{11} \to \{|0\rangle_A|1 \circ \alpha_3^l\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{12} \to \{|1\rangle_A|\alpha_3^l \circ 0\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{13} \to \{|\alpha_3^l\rangle_A|0 \circ 1\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{21} \to \{|0\rangle_A|\alpha^i \circ \alpha_1^k\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{22} \to \{|\alpha^i\rangle_A|\alpha_1^k \circ 0\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{23} \to \{(|1\rangle_A|00\rangle_{ab} + |\alpha_1^{k,2}\rangle_A|11\rangle_{ab})|0 \circ \alpha^i\rangle_{\widetilde{B}}\},$$
$$H_{31} \to \{|0\rangle_A|d'_B \circ \alpha_0^j\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{32} \to \{|d'_A\rangle_A|\alpha_0^j \circ 0\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{33} \to \{(|0\rangle_A|00\rangle_{ab} + |\alpha_0^{j,2}\rangle_A|11\rangle_{ab})|0 \circ d'_C\rangle_{\widetilde{B}}\},$$
$$H_{41} \to \{|1\rangle_A|0 \circ (0 \pm 1)\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{42} \to \{|0\rangle_A|(0 \pm 1) \circ 1\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{43} \to \{|0 \pm 1\rangle_A|1 \circ 0\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{51} \to \{|1\rangle_A|d'_B \circ \alpha_3^l\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{52} \to \{|d'_A\rangle_A|\alpha_3^l \circ 1\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{53} \to \{|\alpha_3^l\rangle_A|1 \circ d'_C\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{61} \to \{|\alpha^i\rangle_A|d'_B \circ \alpha_1^k\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{62} \to \{|d'_A\rangle_A|\alpha_1^k \circ \alpha^i\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{63} \to \{(|1\rangle_A|00\rangle_{ab} + |\alpha_1^{k,2}\rangle_A|11\rangle_{ab})|\alpha^i \circ d'_C\rangle_{\widetilde{B}}\},$$
$$H_{71} \to \{|d'_A\rangle_A|0 \circ \alpha_3^l\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{72} \to \{|0\rangle_A|\alpha_3^l \circ d'_C\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{73} \to \{|\alpha_3^l\rangle_A|d'_B \circ 0\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{81} \to \{|d'_A\rangle_A|1 \circ (0 \pm 1)\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$H_{82} \to \{|1\rangle_A|(0 \pm 1) \circ d'_C\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$H_{83} \to \{|0 \pm 1\rangle_A|d'_B \circ 1\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$

where $|\alpha_0^{j,2}\rangle_A = \sum_{u=1}^{d_A-3} \omega_{d_A-2}^{ju}|u+1\rangle$ and $|\alpha_1^{k,2}\rangle_A = \sum_{u=1}^{d_A-3} \omega_{d_A-2}^{ku}|u+1\rangle$.

Step 2. Bob performs the measurement:

$$\mathcal{M}_2 \equiv \Big\{M_{21} := P[|(1, \ldots, d'_B - 1) \circ (2, \ldots, d'_C - 1)\rangle_{\widetilde{B}}; |1\rangle_b],$$
$$M_{22} := P[(|(2, \ldots, d'_B) \circ (0, d'_C)\rangle, |d'_B \circ (2, \ldots, d'_C - 1)\rangle)_{\widetilde{B}}; |0\rangle_b] + P[|(2, \ldots, d'_B - 1) \circ d'_C\rangle_{\widetilde{B}}; |1\rangle_b],$$
$$M_{23} := P[|01\rangle_{\widetilde{B}}; |1\rangle_b],$$
$$M_{24} := P[(|(0, \ldots, d'_B - 1) \circ 0\rangle, |11\rangle)_{\widetilde{B}}; |1\rangle_b],$$
$$M_{25} := P[|1d'_C\rangle_{\widetilde{B}}; |1\rangle_b],$$
$$M_{26} := P[(|(2, \ldots, d'_B) \circ 1\rangle, |d'_B2\rangle)_{\widetilde{B}}; |1\rangle_b],$$
$$M_{27} := P[|d'_B0\rangle_{\widetilde{B}}; |1\rangle_b],$$
$$M_{28} := P[(|00\rangle, |01\rangle, |11\rangle)_{\widetilde{B}}; |0\rangle_b],$$
$$M_{29} := P[|10\rangle_{\widetilde{B}}; |0\rangle_b],$$
$$M_{210} := P[|d'_B1\rangle_{\widetilde{B}}; |0\rangle_b],$$
$$M_{211} := P[|(2, \ldots, d'_B - 1) \circ (1, \ldots, d'_C - 1)\rangle_{\widetilde{B}}; |0\rangle_b],$$
$$M_{212} := I - \sum_{i=1}^{11} M_{2i}\Big\}.$$

For the operator M_{2i} $(i = 1, \ldots, 12)$, the result of postmeasurement is

$$M_{21} \Rightarrow H_{62}, \quad M_{22} \Rightarrow H_{12}, H_{31}, H_{51}, H_{72}, H_{63},$$
$$M_{23} \Rightarrow H_{13}, \quad M_{24} \Rightarrow H_{22}, H_{32}, H_{81},$$
$$M_{25} \Rightarrow H_{53}, \quad M_{26} \Rightarrow H_{52}, H_{61},$$
$$M_{27} \Rightarrow H_{73}, \quad M_{28} \Rightarrow H_{41}, H_{42},$$
$$M_{29} \Rightarrow H_{43}, \quad M_{210} \Rightarrow H_{83},$$
$$M_{211} \Rightarrow H_{21}, \quad M_{212} \Rightarrow H_{11}, H_{23}, H_{33}, H_{71}, H_{82}.$$

Clearly, $\{H_{52}, H_{61}\}$ and $\{H_{41}, H_{42}\}$ are locally distinguishable. If M_{22} clicks, Alice performs the measurement $\mathcal{M}'_2 \equiv \{M'_{21} := P[|0\rangle_A; |0\rangle_a]$, $M'_{22} := I - M'_{21}\}$. The outcomes corresponding to the operators M'_{21} and M'_{22} are $\{H_{31}, H_{72}\}$ and $\{H_{12}, H_{51}, H_{63}\}$, respectively. They are also locally distinguishable. If M_{24} clicks, Alice performs the measurement $\mathcal{M}''_2 \equiv \{M''_{21} := P[(|2\rangle, \ldots, |d'_A - 1\rangle)_A; |1\rangle_a]$, $M''_{22} :=$

$I - M''_{21}\}$. The outcomes corresponding to the operators M''_{21} and M''_{22} are H_{22} and $\{H_{32}, H_{81}\}$, respectively. Moreover, $\{H_{32}, H_{81}\}$ is a LOCC distinguishable collection. If M_{212} clicks, we proceed to the next step.

Step 3. Alice performs the measurement:

$$\mathcal{M}_3 \equiv \{M_{31} := P[|d'_A\rangle_A; |1\rangle_a], M_{32} := I - M_{31}\}.$$

If M_{31} clicks, the subset is H_{71}. If M_{32} clicks, the subset is one of the remaining four.

Step 4. Bob performs the measurement:

$$\mathcal{M}_4 \equiv \{M_{41} := P[|0 \circ (2, \ldots, d'_C - 1)\rangle_{\widetilde{B}}; I_b], \quad M_{42} := I - M_{41}\}.$$

If M_{41} clicks, the subset is H_{23}. If M_{42} clicks, the result is one of the three remaining subsets.

Step 5. Alice performs the measurement:

$$\mathcal{M}_5 \equiv \{M_{51} := P[|1\rangle_A; |0\rangle_a], M_{52} := I - M_{51}\}.$$

If M_{51} clicks, the subset is H_{82}. If M_{52} clicks, the subset is one of $\{H_{33}, H_{11}\}$, which is locally distinguishable.

On the other hand, when M_{12} clicks in the step 1, we can find the distinction protocol similarly. ■

APPENDIX I: PROOF OF THEOREM 11

To locally distinguish the set (14), let Alice and Bob share a maximally entangled state $|\phi^+(4)\rangle_{a_1b_1}$, while Alice and Charlie share an EPR state $|\phi^+(2)\rangle_{a_2c_1}$.

Step 1. Bob performs the measurement:

$$\begin{aligned}\mathcal{M}_1 \equiv \big\{M_{11} := &P\big[|0\rangle_B; |0\rangle_{b_1}\big] + P\big[|1\rangle_B; |1\rangle_{b_1}\big]\\ &+ P\big[(|2\rangle, \ldots, |d'_B - 1\rangle)_B; |2\rangle_{b_1}\big]\\ &+ P\big[|d'_B\rangle_B; |3\rangle_{b_1}\big],\\ M_{12} := &P\big[|0\rangle_B; |1\rangle_{b_1}\big] + P\big[|1\rangle_B; |2\rangle_{b_1}\big]\\ &+ P\big[(|2\rangle, \ldots, |d'_B - 1\rangle)_B; |3\rangle_{b_1}\big]\\ &+ P\big[|d'_B\rangle_B; |0\rangle_{b_1}\big],\\ M_{13} := &P\big[|0\rangle_B; |2\rangle_{b_1}\big] + P\big[|1\rangle_B; |3\rangle_{b_1}\big]\\ &+ P\big[(|2\rangle, \ldots, |d'_B - 1\rangle)_B; |0\rangle_{b_1}\big]\\ &+ P\big[|d'_B\rangle_B; |1\rangle_{b_1}\big],\\ M_{14} := &I - M_{11} - M_{12} - M_{13}\big\}.\end{aligned}$$

Charlie performs the measurement:

$$\begin{aligned}\mathcal{M}_2 \equiv \big\{M_{21} := &P\big[(|0\rangle, |1\rangle)_C; |0\rangle_{c_1}\big]\\ &+ P\big[(|2\rangle, \ldots, |d'_C\rangle)_C; |1\rangle_{c_1}\big],\\ M_{22} := &I - M_{21}\big\}.\end{aligned}$$

Suppose the outcomes corresponding to M_{11} and M_{21} click, the resulting postmeasurement states are

$$\begin{aligned}
H_{11} &\to \big\{|0\rangle_A|1\rangle_B\big|\alpha_3^l\big\rangle_C|11\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{12} &\to \big\{|1\rangle_A\big(\big|\alpha_3^{l,1}\big\rangle_B|22\rangle_{a_1b_1} + \big|\alpha_3^{l,2}\big\rangle_B|33\rangle_{a_1b_1}\big)|0\rangle_C|00\rangle_{a_2c_1}\big\},\\
H_{13} &\to \big\{\big|\alpha_3^l\big\rangle_A|0\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}\big\},\\
H_{21} &\to \big\{|0\rangle_A|\alpha^i\rangle_B|22\rangle_{a_1b_1}\big(|1\rangle_C|00\rangle_{a_2c_1} + \big|\alpha_1^{k,2}\big\rangle_C|11\rangle_{a_2c_1}\big)\big\},\\
H_{22} &\to \big\{|\alpha^i\rangle_A\big(|1\rangle_B|11\rangle_{a_1b_1} + \big|\alpha_1^{k,2}\big\rangle_B|22\rangle_{a_1b_1}\big)|0\rangle_C|00\rangle_{a_2c_1}\big\},\\
H_{23} &\to \big\{\big|\alpha_1^k\big\rangle_A|0\rangle_B|\alpha^i\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{31} &\to \big\{|0\rangle_A|d'_B\rangle_B|33\rangle_{a_1b_1}\big(|0\rangle_C|00\rangle_{a_2c_1} + \big|\alpha_0^{j,2}\big\rangle_C|11\rangle_{a_2c_1}\big)\big\},\\
H_{32} &\to \big\{|d'_A\rangle_A\big(|0\rangle_B|00\rangle_{a_1b_1} + \big|\alpha_0^{j,2}\big\rangle_B|22\rangle_{a_1b_1}\big)|0\rangle_C|00\rangle_{a_2c_1}\big\},\\
H_{33} &\to \big\{\big|\alpha_0^j\big\rangle_A|0\rangle_B|d'_C\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{41} &\to \{|1\rangle_A|0\rangle_B|0 \pm 1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}\},\\
H_{42} &\to \big\{|0\rangle_A\big(|0\rangle_B|00\rangle_{a_1b_1} \pm |1\rangle_B|11\rangle_{a_1b_1}\big)|1\rangle_C|00\rangle_{a_2c_1}\big\},\\
H_{43} &\to \big\{|0 \pm 1\rangle_A|1\rangle_B|0\rangle_C|11\rangle_{a_1b_1}|00\rangle_{a_2c_1}\big\},\\
H_{51} &\to \big\{|1\rangle_A|d'_B\rangle_B\big|\alpha_3^l\big\rangle_C|33\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{52} &\to \big\{|d'_A\rangle_A\big(\big|\alpha_3^{l,1}\big\rangle_B|22\rangle_{a_1b_1} + \big|\alpha_3^{l,2}\big\rangle_B|33\rangle_{a_1b_1}\big)|1\rangle_C|00\rangle_{a_2c_1}\big\},\\
H_{53} &\to \big\{\big|\alpha_3^l\big\rangle_A|1\rangle_B|d'_C\rangle_C|11\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{61} &\to \big\{|\alpha^i\rangle_A|d'_B\rangle_B|33\rangle_{a_1b_1}(|1\rangle_C|00\rangle_{a_2c_1} + \big|\alpha_1^{k,2}\big\rangle_C|11\rangle_{a_2c_1})\big\},\\
H_{62} &\to \big\{|d'_A\rangle_A\big(|1\rangle_B|11\rangle_{a_1b_1} + \big|\alpha_1^{k,2}\big\rangle_B|22\rangle_{a_1b_1}\big)|\alpha^i\rangle_C|11\rangle_{a_2c_1}\big\},\\
H_{63} &\to \big\{\big|\alpha_1^k\big\rangle_A|\alpha^i\rangle_B|d'_C\rangle_C|22\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{71} &\to \big\{|d'_A\rangle_A|0\rangle_B\big|\alpha_3^l\big\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}\big\},\\
H_{72} &\to \big\{|0\rangle_A\big(\big|\alpha_3^{l,1}\big\rangle_B|22\rangle_{a_1b_1} + \big|\alpha_3^{l,2}\big\rangle_B|33\rangle_{a_1b_1}\big)|d'_C\rangle_C|11\rangle_{a_2c_1}\big\},\\
H_{73} &\to \big\{\big|\alpha_3^l\big\rangle_A|d'_B\rangle_B|0\rangle_C|33\rangle_{a_1b_1}|00\rangle_{a_2c_1}\big\},\\
H_{81} &\to \big\{|d'_A\rangle_A|1\rangle_B|0 \pm 1\rangle_C|11\rangle_{a_1b_1}|00\rangle_{a_2c_1}\big\},\\
H_{82} &\to \big\{|1\rangle_A(|0\rangle_B|00\rangle_{a_1b_1} \pm |1\rangle_B|11\rangle_{a_1b_1})|d'_C\rangle_C|11\rangle_{a_2c_1}\big\},\\
H_{83} &\to \big\{|0 \pm 1\rangle_A|d'_B\rangle_B|1\rangle_C|33\rangle_{a_1b_1}|00\rangle_{a_2c_1}\big\},
\end{aligned} \tag{I1}$$

where $|\alpha_0^{j,2}\rangle_\tau = \sum_{u=1}^{d_\tau-3} \omega_{d_\tau-2}^{ju}|u+1\rangle$, $|\alpha_1^{k,2}\rangle_\tau = \sum_{u=1}^{d_\tau-3} \omega_{d_\tau-2}^{ku}|u+1\rangle$, $|\alpha_3^{l,1}\rangle_\tau = \sum_{u=0}^{d_\tau-4} \omega_{d_\tau-2}^{lu}|u+2\rangle$, and $|\alpha_3^{l,2}\rangle_\tau = \omega_{d_\tau-2}^{l(d_\tau-3)}|d_\tau - 1\rangle$ for $j, k, l \in \mathcal{Z}_{d_\tau-2}$ and $\tau = B, C$.

Step 2. Alice performs the measurement:

$$\mathcal{M}_3 \equiv \Big\{M_{31} := P\big[|d'_A\rangle_A; |0\rangle_{a_1}; |1\rangle_{a_2}\big], \quad M_{32} := P\big[|1\rangle_A; |0\rangle_{a_1}; |0\rangle_{a_2}\big],$$

$$M_{33} := P\big[(|2\rangle, \dots, |d'_A - 1\rangle)_A; (|1\rangle, |2\rangle)_{a_1}; |0\rangle_{a_2}\big],$$
$$M_{34} := P\big[|0\rangle_A; |1\rangle_{a_1}; |1\rangle_{a_2}\big],$$
$$M_{35} := P\big[|d'_A\rangle_A; |1\rangle_{a_1}; |0\rangle_{a_2}\big],$$
$$M_{36} := P\big[(|1\rangle, \dots, |d'_A - 1\rangle)_A; |2\rangle_{a_1}; |1\rangle_{a_2}\big],$$
$$M_{37} := P\big[|1\rangle_A; |3\rangle_{a_1}; |1\rangle_{a_2}\big],$$
$$M_{38} := I - \sum_{i=1}^{7} M_{3i}\Big\}.$$

The result of postmeasurement, corresponding to the operator M_{3i} $(i = 1, \dots, 7)$ is

$$M_{31} \Rightarrow H_{71}, \quad M_{32} \Rightarrow H_{41}, \quad M_{33} \Rightarrow H_{22}, \quad M_{34} \Rightarrow H_{11},$$
$$M_{35} \Rightarrow H_{81}, \quad M_{36} \Rightarrow H_{63}, \quad M_{37} \Rightarrow H_{51}.$$

If M_{38} clicks, we proceed to the next step.

Step 3. Charlie performs the measurement:

$$\mathcal{M}_4 \equiv \big\{M_{41} := P\big[|d'_C\rangle_C; |1\rangle_{c_1}\big], M_{42} := I - M_{41}\big\}.$$

If M_{41} clicks, the given subset is one of $\{H_{33}, H_{53}, H_{72}, H_{82}\}$. It is locally distinguishable. Otherwise, we continue to the next step.

Step 4. Alice performs the measurement:

$$\mathcal{M}_5 \equiv \Big\{M_{51} := P\big[(|0\rangle, |1\rangle)_A; (|0\rangle, |1\rangle)_{a_1}; |0\rangle_{a_2}\big],$$
$$M_{52} := P\big[(|2\rangle, \dots, |d'_A\rangle)_A; (|1\rangle, |2\rangle)_{a_1}; |1\rangle_{a_2}\big],$$
$$M_{53} := P\big[(|1\rangle, \dots, |d'_A - 1\rangle)_A; |0\rangle_{a_1}; |1\rangle_{a_2}\big],$$
$$M_{54} := P\big[|0\rangle_A; |2\rangle_{a_1}; I_{a_2}\big],$$
$$M_{55} := P\big[(|0\rangle, |1\rangle)_A; |3\rangle_{a_1}; |0\rangle_{a_2}\big] + P\big[|0\rangle_A; |3\rangle_{a_1}; |1\rangle_{a_2}\big] + P\big[|1\rangle_A; |2\rangle_{a_1}; |0\rangle_{a_2}\big],$$
$$M_{56} := I - \sum_{i=1}^{5} M_{5i}\Big\}.$$

Corresponding to the operator M_{5i} $(i = 1, \dots, 6)$, there is the following result

$$M_{51} \Rightarrow H_{43}, H_{42}, \quad M_{54} \Rightarrow H_{21},$$
$$M_{52} \Rightarrow H_{62}, \quad M_{55} \Rightarrow H_{12}, H_{31}, H_{83},$$
$$M_{53} \Rightarrow H_{23}, \quad M_{56} \Rightarrow H_{13}, H_{32}, H_{52}, H_{61}, H_{73}.$$

If M_{55} clicks, then Charlie performs the measurement $\mathcal{M}'_5 \equiv \{M'_{51} := P[|1\rangle_C; |0\rangle_{c_1}],\ M'_{52} := I - M'_{51}\}$. The outcomes corresponding to the operators M'_{51} and M'_{52} are H_{83} and $\{H_{12}, H_{31}\}$, respectively. Obviously, $\{H_{42}, H_{43}\}$ and $\{H_{12}, H_{31}\}$ are locally distinguishable. If M_{56} clicks, we move on to the next step.

Step 5. Charlie performs the measurement:

$$\mathcal{M}_6 \equiv \big\{M_{61} := P\big[|0\rangle_C; |0\rangle_{c_1}\big], M_{62} := I - M_{61}\big\}.$$

Corresponding to the operators M_{61} and M_{62}, the subsets of postmeasurement are $\{H_{32}, H_{73}\}$ and $\{H_{13}, H_{52}, H_{61}\}$, respectively. They are all LOCC distinguishable.

If another operator clicks in the step 1, then also a similar entanglement-assisted discrimination protocol follows. ■

APPENDIX J: PROOF OF THEOREM 12

Let Alice and Bob share two EPR states $|\phi^+(2)\rangle_{a_1b_1}|\phi^+(2)\rangle_{a_2b_2}$, while Alice and Charlie share an EPR state $|\phi^+(2)\rangle_{a_3c_1}$.

Bob performs the measurement:

$$\mathcal{M}_1 \equiv \big\{M_{11} := P\big[|0\rangle_B; |0\rangle_{b_1}; |0\rangle_{b_2}\big] + P\big[|1\rangle_B; |0\rangle_{b_1}; |1\rangle_{b_2}\big] + P\big[(|2\rangle, \dots, |d'_B - 1\rangle)_B; |1\rangle_{b_1}; |0\rangle_{b_2}\big] + P\big[|d'_B\rangle_B; |1\rangle_{b_1}; |1\rangle_{b_2}\big],$$
$$M_{12} := P\big[|0\rangle_B; |0\rangle_{b_1}; |1\rangle_{b_2}\big] + P\big[|1\rangle_B; |1\rangle_{b_1}; |0\rangle_{b_2}\big] + P\big[(|2\rangle, \dots, |d'_B - 1\rangle)_B; |1\rangle_{b_1}; |1\rangle_{b_2}\big] + P\big[|d'_B\rangle_B; |0\rangle_{b_1}; |0\rangle_{b_2}\big],$$
$$M_{13} := P\big[|0\rangle_B; |1\rangle_{b_1}; |0\rangle_{b_2}\big] + P\big[|1\rangle_B; |1\rangle_{b_1}; |1\rangle_{b_2}\big] + P\big[(|2\rangle, \dots, |d'_B - 1\rangle)_B; |0\rangle_{b_1}; |0\rangle_{b_2}\big] + P\big[|d'_B\rangle_B; |0\rangle_{b_1}; |1\rangle_{b_2}\big],$$
$$M_{14} := I - M_{11} - M_{12} - M_{13}\big\}.$$

Charlie performs the measurement:

$$\mathcal{M}_2 \equiv \big\{M_{21} := P\big[(|0\rangle, |1\rangle)_C; |0\rangle_{c_1}\big] + P\big[(|2\rangle, \dots, |d'_C\rangle)_C; |1\rangle_{c_1}\big],$$
$$M_{22} := I - M_{21}\big\}.$$

Similar to the proof of Theorem 11, when $a_1a_2a_3$ and b_1b_2 are substituted for ancillary systems a_1a_2 and b_1 in (I1), respectively, the outcomes are obtained. It is easy to prove that these postmeasurement states are also locally distinguishable. ■

APPENDIX K: PROOF OF THEOREM 13

Notice that $d_C, d_D \leqslant d_B$. The states of subsystems C and D are teleported to Bob using the maximally entangled states $|\phi^+(d_C)\rangle_{BC}$ and $|\phi^+(d_D)\rangle_{BD}$, respectively. Their union is represented by $\widetilde{B}$. In addition, to locally discriminate the set (15), Alice and Bob share a maximally entangled state $|\phi^+(3)\rangle_{ab}$. The specific protocol is as follows.

Alice performs the measurement:

$$\mathcal{M}_1 \equiv \{M_{11} := P[|0\rangle_A; |0\rangle_a] + P[(|1\rangle, \dots, |d'_A - 1\rangle)_A; |1\rangle_a] + P[|d'_A\rangle_A; |2\rangle_a],$$
$$M_{12} := P[|0\rangle_A; |1\rangle_a] + P[(|1\rangle, \dots, |d'_A - 1\rangle)_A; |2\rangle_a] + P[|d'_A\rangle_A; |0\rangle_a],$$
$$M_{13} := I - M_{11} - M_{12}\}.$$

Suppose the outcome corresponding to M_{11} clicks, the resulting postmeasurement states are

$$U_{11} \to \{|0\rangle_A|\xi_i \circ \eta_j \circ (0 \pm d'_D)\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{12} \to \{|\xi_i\rangle_A|\eta_j \circ (0 \pm d'_C) \circ 0\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$

$$U_{13} \to \{(|0\rangle_A|00\rangle_{ab} + |\eta_j^1\rangle_A|11\rangle_{ab})|(0 \pm d_B') \circ 0 \circ \xi_i\rangle_{\widetilde{B}}\},$$
$$U_{14} \to \{(|0\rangle_A|00\rangle_{ab} \pm |d_A'\rangle_A|22\rangle_{ab})|0 \circ \xi_i \circ \eta_j\rangle_{\widetilde{B}}\},$$
$$U_{21} \to \{|\xi_i\rangle_A|d_B' \circ \gamma_k \circ \eta_j\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$U_{22} \to \{|d_A'\rangle_A|\gamma_k \circ \eta_j \circ \xi_i\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$
$$U_{23} \to \{(|\gamma_k^1\rangle_A|11\rangle_{ab} + |\gamma_k^2\rangle_A|22\rangle_{ab})|\eta_j \circ \xi_i \circ d_D'\rangle_{\widetilde{B}}\},$$
$$U_{24} \to \{(|0\rangle_A|00\rangle_{ab} + |\eta_j^1\rangle_A|11\rangle_{ab})|\xi_i \circ d_C' \circ \gamma_k\rangle_{\widetilde{B}}\},$$
$$U_{31} \to \{|d_A'\rangle_A|0 \circ (0 \pm d_C') \circ \gamma_k\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$
$$U_{32} \to \{|0\rangle_A|(0 \pm d_B') \circ \gamma_k \circ d_D'\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{33} \to \{(|0\rangle_A|00\rangle_{ab} \pm |d_A'\rangle_A|22\rangle_{ab})|\gamma_k \circ d_C' \circ 0\rangle_{\widetilde{B}}\},$$
$$U_{34} \to \{(|\gamma_k^1\rangle_A|11\rangle_{ab} + |\gamma_k^2\rangle_A|22\rangle_{ab})|d_B' \circ 0 \circ (0 \pm d_D')\rangle_{\widetilde{B}}\},$$
$$U_{41} \to \{|\xi_i\rangle_A|\xi_i \circ 0 \circ \gamma_k\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$U_{42} \to \{|\xi_i\rangle_A|0 \circ \gamma_k \circ \xi_i\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$U_{43} \to \{|0\rangle_A|\gamma_k \circ \xi_i \circ \xi_i\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{44} \to \{(|\gamma_k^1\rangle_A|11\rangle_{ab} + |\gamma_k^2\rangle_A|22\rangle_{ab})|\xi_i \circ \xi_i \circ 0\rangle_{\widetilde{B}}\},$$
$$U_{51} \to \{|d_A'\rangle_A|d_B' \circ \xi_i \circ (0 \pm d_D')\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$
$$U_{52} \to \{|d_A'\rangle_A|\xi_i \circ (0 \pm d_C') \circ d_D'\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$
$$U_{53} \to \{|\xi_i\rangle_A|(0 \pm d_B') \circ d_C' \circ d_D'\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$U_{54} \to \{(|0\rangle_A|00\rangle_{ab} \pm |d_A'\rangle_A|22\rangle_{ab})|d_B' \circ d_C' \circ \xi_i\rangle_{\widetilde{B}}\},$$
$$U_{61} \to \{|0\rangle_A|0 \circ d_C' \circ \eta_j\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{62} \to \{|0\rangle_A|d_B' \circ \eta_j \circ 0\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{63} \to \{|d_A'\rangle_A|\eta_j \circ 0 \circ 0\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$
$$U_{64} \to \{(|0\rangle_A|00\rangle_{ab} + |\eta_j^1\rangle_A|11\rangle_{ab})|0 \circ 0 \circ d_D'\rangle_{\widetilde{B}}\},$$
$$U_{71} \to \{|0\rangle_A|\xi_i \circ 0 \circ \xi_i\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{72} \to \{|\xi_i\rangle_A|0 \circ \xi_i \circ 0\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$U_{81} \to \{|0\rangle_A|d_B' \circ 0 \circ d_D'\rangle_{\widetilde{B}}|00\rangle_{ab}\},$$
$$U_{82} \to \{|d_A'\rangle_A|0 \circ d_C' \circ 0\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$
$$U_{91} \to \{|\xi_i\rangle_A|d_B' \circ \xi_i \circ d_D'\rangle_{\widetilde{B}}|11\rangle_{ab}\},$$
$$U_{92} \to \{|d_A'\rangle_A|\xi_i \circ d_C' \circ \xi_i\rangle_{\widetilde{B}}|22\rangle_{ab}\},$$

where $|\eta_j^1\rangle_A = \sum_{u=1}^{d_A-2} \omega_{d_A-1}^{ju}|u\rangle$, $|\gamma_k^1\rangle_A = \sum_{u=0}^{d_A-3} \omega_{d_A-1}^{ku}|u+1\rangle$ and $|\gamma_k^2\rangle_A = \omega_{d_A-1}^{k(d_A-2)}|d_A-1\rangle$ for $j,k \in \mathcal{Z}_{d_A-1}$. Evidently, they can be perfectly distinguished by LOCC. For all other cases a similar protocol follows. ■

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