Quantum speed limits for observables

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In the Schrödinger picture, the state of a quantum system evolves in time and the quantum speed limit describes how fast the state of a quantum system evolves from an initial state to a final state. However, in the Heisenberg picture the observable evolves in time instead of the state vector. Therefore, it is natural to ask how fast an observable evolves in time. This can impose a fundamental bound on the evolution time of the expectation value of quantum-mechanical observables. We obtain the quantum speed limit time bound for observable for closed systems, open quantum systems, and arbitrary dynamics. Furthermore, we discuss various applications of these bounds. Our results can have several applications ranging from setting the speed limit for operator growth, correlation growth, quantum thermal machines, quantum control, and many-body physics.

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I. INTRODUCTION

Time is one of the fundamental notions in the physical world and it plays a significant role in almost every existing physical theory. However, understanding time has been a challenging task and often it is treated like a parameter. Even though time is not an operator, there is a geometric uncertainty relation between time and energy fluctuation which imposes inherent limitation on how fast a quantum system can evolve in time. This was first discovered in an attempt to operationalize the time-energy uncertainty relation. This concept is now known as the quantum speed limit (QSL) [1,2]. Even though how fast a quantum system evolves in time was addressed in Ref. [1], the notion of speed of transportation of the state vector was formally defined using the Fubini-Study metric in Ref. [2] and using the Riemannian metric in Ref. [3]. Subsequently, an alternate speed limit for quantum state evolution was proved involving the average energy above the ground state of the Hamiltonian [4]. The QSL determines the minimal time of evolution of the quantum system. It entirely depends on intrinsic quantities of evolving quantum systems, such as the shortest path connecting the initial and final states of the quantum system and the uncertainty in the Hamiltonian.

The QSL bounds were first investigated for the unitary dynamics of pure states [1,2,4–27]. Later, the QSL was studied for the case of the unitary dynamics of mixed states [28–37], the unitary dynamics of multipartite systems [38–41], and more general dynamics [42–51]. The study of the QSL is significant for a theoretical understanding of quantum dynamics and has relevance in developing quantum technologies and devices, etc. The QSL has applications in several fields, such as quantum computation [52], quantum thermodynamics [53,54], quantum control theory [55,56], and quantum metrology [57].

In quantum physics, during the arbitrary dynamical evolution of the quantum system, we often encounter the situation where a time-evolved state becomes perfectly distinguishable (orthogonal) from the initial state. At the same time, the expectation value of a given observable does not change or changes at a slower rate. For instance, let us consider a closed quantum system with internal Hamiltonian σ_z and initial state $|+\rangle$. This initial state evolves to its orthogonal state $|-\rangle$ (up to a phase) by an external Hamiltonian σ_y, where $|+\rangle$ and $|-\rangle$ are the eigenstates of σ_x. In this scenario, the initial state and final state of the system are distinguishable. However, both the initial and final states are energetically indistinguishable, so the evolution time for the average energy of the quantum system is zero. Similarly, one can consider that if a given quantum system interacts with a pure dephasing environment (dephasing in σ_z), then its state evolves to a decohered state. However, the expectation value of energy does not change in this process. The above discussion suggests that observables of a system can have different quantum speed limit bounds.

In the Schrödinger picture, the state vector evolves in time, while in the Heisenberg picture, the observable of the quantum system evolves; both of these formalisms are equivalent. In quantum mechanics, there is also an interaction picture where both the state and the observables can change in time. This has important applications in quantum field theory and many-body physics. In this paper we will use the Heisenberg picture for most of our discussion. The natural question that then arises is how fast an observable evolves in time. Specifically, we will answer the question of how to obtain a lower bound on the evolution time of a quantum observable and define the quantum speed limit for the observable. Thus, a seemingly technical difference between the Schrödinger and the Heisenberg pictures becomes rather important in the context of many-body physics. Here, often one cannot describe a state analytically due to its immense computational complexity.

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However, it is possible to compute expectation values of local observables in an efficient manner. Thus, it is desirable to bound how fast the expectation value corresponding to an observable changes in time in the Heisenberg picture.

Another motivation for studying an observable's speed limit is that this can have application in understanding the operator growth in many-body physics. In the context of complex systems, one of the pressing questions is to understand the universal operator growth hypothesis [58]. The observables of a system which may be represented as operators in quantum systems tend to grow over time, i.e., they become more complicated as the system evolves in time. If we start with a simple many-body operator at some initial time, then because of the interaction Hamiltonian, the operator may become complex at a later time. The quantum speed limit for the observable can answer the fundamental question of how fast a many-body operator is to tend to be complex. It is important to know the rate of operator growth and we believe that the quantum speed limit for the observable can throw light on this question.

In complex quantum systems and many-body physics the bound on the commutator of two operators, one operator being the time-evolved version of an operator with support on some region and the other operator with support on some other region, plays a major role in the derivation of the Lieb-Robinson bound [59,60]. The latter proves that the speed of propagation of perturbation in the quantum system is bounded. Physically, this implies that for small times only small amounts of information can propagate from one region of the many-body system to another region. While the Lieb-Robinson bound leads to a speed limit of information in quantum systems, the quantum speed limit for the observable can answer the question of how fast the commutator changes for observables belonging to two distant regions. This is also important in the physical context where the underlying dynamics is highly chaotic [61]. The growth of the commutator between two operators as a function of their separation in time has been used to quantify the rate of growth of chaos and information scrambling [62–64]. Therefore, the quantum speed limit for the commutator can answer the question of how fast a localized perturbation spreads in time in a quantum many-body system. Since the timescale over which scrambling of information occurs is distinct from the relaxation time of the physical system, the observable quantum speed limit for the two-time commutator can play an important role in giving an estimate of the scrambling time in complex quantum systems.

To answer these fundamental questions, we formally introduce the notion of the quantum speed limit for observables. It is characterized as the maximal evolution speed of the expectation value of the given observable of the quantum system during arbitrary dynamical evolution, which can be unitary or nonunitary. It sets the bound on the minimum evolution time of the quantum system required to evolve between different expectation values of a given observable. We do this for both closed and open quantum dynamics. We illustrate our main results for the ergotropy rate of a quantum battery, the rate of probability which also gives the standard QSL for the state of the system. Moreover, we also compute the QSL for the two-time correlation of an observable, which is a central quantity in the theory of quantum transport and complex quantum systems. We also apply our bound to obtain the quantum speed limit for the commutator of two observables belonging to two distant regions in a many-body system. Our result can be equally important like the Lieb-Robinson bound.

II. OBSERVABLE'S QUANTUM SPEED LIMIT

Here we show how to obtain the QSL for an observable. This will answer the question of how fast the expectation value of a given observable changes in time instead of the state of the quantum system. The observable's QSL is defined as the maximum rate of evolution of the expectation value of an observable of a given quantum system during dynamical evolution. It establishes a limit on the minimum evolution time necessary to evolve between different expected values of a given observable of the quantum system. Here we will derive the observable's QSL for closed and open system dynamics.

A. Unitary dynamics

In this section we derive a bound for the observable undergoing unitary dynamics in the Heisenberg picture. Let us consider a closed quantum system whose initial state is $|\psi\rangle \in \mathcal{H}$ and whose dynamical evolution is dictated by unitary $U(t) = e^{-iHt/\hbar}$, where H is the time-independent driving Hamiltonian of the quantum system. Here we want to determine how fast a quantum observable O of the quantum system evolves in time and lower bound its minimal evolution time. We know that the Heisenberg equation of motion governs the time evolution of an observable, which is given by

$$i\hbar\frac{dO(t)}{dt} = [O(t), H], \tag{1}$$

where H is the Hamiltonian of the system and $O(t) = U^{\dagger}(t)O(0)U(t)$, with $U^{\dagger}(t)U(t) = I$.

Now we take the average of the above in the state $|\psi\rangle$ and take the absolute value of Eq. (1). On using the Heisenberg-Robertson uncertainty relation, i.e., $\Delta A\Delta B \geqslant \frac{1}{2}\langle[A, B]\rangle$, where A and B are two incompatible observables [65], we obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| = \frac{1}{\hbar}|\langle[O(t), H]\rangle| \leqslant \frac{2\Delta O(t)\Delta H}{\hbar}, \tag{2}$$

where $\Delta O(t) = \sqrt{\langle O(t)^2\rangle - \langle O(t)\rangle^2}$ and $\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$.

The above inequality (2) is the upper bound on the rate of change of the expectation value of a given observable of the quantum system evolving under unitary dynamics. After integrating the above inequality with respect to time, we obtain the desired bound

$$T \geqslant \frac{\hbar}{2\Delta H}\int_0^T \frac{|d\langle O(t)\rangle|}{\Delta O(t)}. \tag{3}$$

We call the quantity (right-hand side of the inequality) $T_{\rm QSL}^{O} = \frac{\hbar}{2\Delta H}\int_0^T \frac{|d\langle O(t)\rangle|}{\Delta O(t)}$ the quantum speed limit time of an observable (OQSL), i.e., $T \geqslant T_{\rm QSL}^{O}$.

If an observable O satisfies the condition $O^2 = I$, i.e., for the self-inverse observable, the inequality (3) can be expressed as.

$$T \geqslant \frac{\hbar}{2\Delta H}|\arcsin\langle O(T)\rangle - \arcsin\langle O(0)\rangle|. \tag{4}$$

Here we illustrate the tightness of the above speed limit for the observable with a simple example. Consider a qubit in a pure state $|\psi\rangle = \alpha|0\rangle + \sqrt{1-\alpha^2}|1\rangle$ $(0 \leqslant \alpha \leqslant 1)$ which does not evolve in time. The Hamiltonian of the system is given by $H = \hat{m}\cdot\vec{\sigma}$, with $\hat{m}$ a unit vector. In the Heisenberg picture, an observable evolves in time and we would like to evaluate the minimum evolution time of the expectation value of a given observable $O(0) = \hat{n}\cdot\vec{\sigma}$, with $\hat{n}$ the unit vector in the state $|\psi\rangle$. We can calculate the quantities $\Delta H = 1$, $\langle O(0)\rangle = 1$, and $\langle O(T)\rangle = -1$, where we assume $\hat{n} = (1, 0, 0)$, $\hat{m} = (0, 0, 1)$, $\alpha = \frac{1}{\sqrt{2}}$, and $T = \frac{\pi}{2}$. By using Eq. (4), we can obtain $T^O_{\rm QSL} = \frac{\pi}{2} = T$. Hence, the bound given by Eq. (4) is indeed tight. This simple example illustrates the usefulness of the OQSL. The bound given in (3) may be thought of as the analog of the Mandelstam-Tamm (MT) bound for the observable.

1. Quantum speed limit for states

Here we discuss how the QSL for an observable is connected to the standard QSL for the state of a quantum system. We will show that the standard state-speed limit for the state may be viewed as a special case of the observable's speed limit when the observable is chosen to be a projector on the initial state. Let us consider the initial state of a quantum system which is prepared in a state $|\psi\rangle = \sum_i a_i|i\rangle$. If we choose our observable to be a projector, i.e., $O(0) = P$, then the probability of finding the quantum system in state $|i\rangle$ at $t = 0$ is $p(0) = |a_i|^2$ (if we measure a projector $P = |i\rangle\langle i|$). Here we want to obtain the speed limit for the projector for the unitarily evolving quantum system, i.e., how fast the probability of finding the quantum system in state $|i\rangle$ changes in time. From (3) we can then obtain the inequality

$$T \geqslant \frac{\hbar}{2\Delta H}\int_0^T \frac{|d\langle P(t)\rangle|}{\sqrt{\langle P(t)\rangle[1-\langle P(t)\rangle]}},$$

where $P(t) = U^\dagger(t)P(0)U(t)$ and $\langle P(t)\rangle = p(t)$ is the probability of the quantum system being in state $|i\rangle$ at a later time

$$T \geqslant \frac{\hbar}{\Delta H}|\arcsin[\sqrt{p(T)}] - \arcsin[\sqrt{p(0)}]|. \tag{5}$$

If $p(0) = 1$ i.e., $|\psi\rangle = |i\rangle$, then the inequality (5) yields the well-known Mandelstam-Tamm bound of the QSL for state evolution

$$T \geqslant \frac{\hbar}{\Delta H}\arccos[\sqrt{p(T)}]. \tag{6}$$

This is the usual QSL obtained by Mandelstam and Tamm [1] and Anandan and Aharaonov [2]. Thus, the observable QSL also leads to a standard QSL for the state change. In this sense, our approach also unifies the existing QSLs. Note that this is an expression for survival probability and it is related to fidelity decay, which is an important quantity in quantum chaos [66].

Since the bound (3) is harder to compute, one can derive the alternate bound for an arbitrary initial state which may be pure or mixed (see Appendix A), which is easier to compute,

$$T \geqslant \frac{\hbar}{2\sqrt{\mathrm{tr}(\rho^2)}}\frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\|O(0)H\|_{\rm HS}}, \tag{7}$$

where $\mathrm{tr}(\rho^2)$ is the purity of the initial state, $\|A\|_{\rm HS} = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of the operator A, and the right-hand side is defined as $T^O_{\rm QSL}$. The bound (7) suggests that if the initial state of the system is mixed, then the observable evolves slower (the OQSL depends on the initial state). However, this bound (7) may not always be tight compared to the bound (3).

The above result can have an interesting application in physical systems where one tries to estimate approximate conserved quantities. We know that if O is the generator of a symmetry operation that acts on the physical system and if it commutes with the Hamiltonian, then it is conserved. Suppose that O does not commute with the Hamiltonian. Then we know that the observable will not be conserved. However, the bound (7) can suggest, over some time interval T, how much $\langle O(T)\rangle$ differs from $\langle O(0)\rangle$. For a pure state, this difference is upper bounded by $\frac{2T}{\hbar}\|O(0)H\|_{\rm HS}$.

Next we obtain another QSL for the observable. Let us consider a quantum system with a pure state $\rho = |\psi\rangle\langle\psi|$. The time evolution of the expectation value of any system observable O is given as

$$\langle O(t)\rangle = \mathrm{tr}[U^\dagger(t)O(0)U(t)\rho]. \tag{8}$$

To find the rate of change of expectation value of the observable, we need to differentiate Eq. (8) with respect to time. By taking the absolute value of the rate equation and applying the triangular inequality and the Hölder inequality [67–69], we obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{2}{\hbar}\|HO(t)\|_{\rm op}. \tag{9}$$

This inequality (9) provides the upper bound on the rate of change of the expectation value of a given observable of the quantum system evolving under unitary dynamics.

The operator norm, the Hilbert-Schmidt norm, and the trace norm of an operator satisfy the inequality $\|A\|_{\rm op} \leqslant \|A\|_{\rm HS} \leqslant \|A\|_{\rm tr}$ and the operator norm is unitary invariant $\|U^\dagger AU\|_{\rm op} = \|A\|_{\rm op}$. Then we can obtain the bound as (see Appendix B)

$$T \geqslant \frac{\hbar}{2}\left\{\frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\min\{\|O(0)H\|_{\rm op}, \|O(0)H\|_{\rm tr}\}}\right\}, \tag{10}$$

where $T^O_{\rm QSL} = \frac{\hbar}{2}\{\frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\min\{\|O(0)H\|_{\rm op}, \|O(0)H\|_{\rm tr}\}}\}$ is the OQSL time.

One should consider the maximum of (3), (7), and (10) for the tighter bound. For the unitary evolution, the bounds (3), (7), and (10) determine how fast the expectation value of an observable of the quantum system changes in time. If a given observable's initial and final expectation values do not change undergoing unitary evolution, then the OQSL is zero. Since the minimal evolution time for state evolution cannot be zero in the above scenario, the aforementioned scenario is the major difference between OQSL bounds and standard QSL bounds (MT and Margolus-Levitin bounds) for unitary dynamics.

B. Arbitrary dynamics

In general, for an arbitrary observable O one can obtain the following inequality using the triangle inequality for the absolute value and the Hölder inequality (see Appendix C):

$$|\langle O\rangle_\rho - \langle O\rangle_\sigma| \leqslant 2\|O\|_{\mathrm{op}} l(\rho,\sigma). \tag{11}$$

Here $l(\rho,\sigma) = \frac{1}{2}\mathrm{tr}|\rho-\sigma|$ is the trace distance between states ρ and σ. With the help of the inequality relation (11), we can define a distance that captures the change in the expectation of an observable during the arbitrary dynamical evolution (in general, the evolution governed by the master equation $\dot{O}(t) = L_t^\dagger[O(t)]$, where $L_t^\dagger$ is adjoint of the Liouvillian superoperator, which can be unitary or nonunitary), and it is given by

$$\mathcal{D}(O(t), O(0)) = \frac{|\mathrm{tr}[(O(t)-O(0))\rho]|}{2\|O(0)\|_{\mathrm{op}}}, \tag{12}$$

where $\|O(0)\|_{\mathrm{op}}$ is a rescaling factor because the spectral gap in the observable can be arbitrarily large.

Using Eq. (12), we can obtain the desired QSL bound on the evolution time of the expectation value of an observable for arbitrary dynamics as

$$T \geqslant \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}, \tag{13}$$

where $\Lambda_T = \frac{1}{T}\int_0^T dt \|L_t^\dagger O(t)\|_{\mathrm{HS}}$ is the evolution speed of the observable O and $T_{\mathrm{QSL}}^O = \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}$ is the OQSL time.

Details of the derivation are provided in Appendix D. For arbitrary dynamics, the bound (13) determines how fast the expectation value of an observable of the quantum system changes in time. The derived bound (13) implies that the OQSL is dependent on the purity of the initial state of the quantum system as well as the evolution speed of the observable.

C. Lindblad dynamics

Let us consider a quantum system S that is interacting with its environment E. The total Hilbert space of the combined system is $\mathcal{H}_S \otimes \mathcal{H}_E$ and we assume that the initial state of the combined system is represented by the separable density matrix $\rho_{SE}(0) = \rho \otimes \sigma$, where the quantum system's initial state ρ can be pure or mixed and σ is the state of the environment. The Lindbladian $\mathcal{L}$ governs the reduced dynamics of a quantum system S. Here we aim to determine how fast the expectation value of a given observable O of the reduced quantum system S evolves in time and lower bound its minimal evolution time. The quantum system has an internal Hamiltonian H_S. If the system interacts with its surroundings, then the dynamics of the given observable of the quantum system is governed by the Lindblad master equation in the Heisenberg picture. Hence, the expectation value of the observable O belonging to the system follows the dynamics

$$\langle O(t)\rangle = \mathrm{tr}[O(0)\Phi_t(\rho)] = \mathrm{tr}\{\Phi_t^\dagger[O(0)]\rho\}, \tag{14}$$

where Φ_t is the generator of the dynamics and $O(t) = \Phi_t[O(0)] = e^{\mathcal{L}^\dagger t}O(0)$, with $\mathcal{L}^\dagger$ the adjoint of the Lindbladian.

The time evolution of an observable O is given by

$$\frac{dO(t)}{dt} = \mathcal{L}^\dagger[O(t)], \tag{15}$$

where $\mathcal{L}^\dagger[O(t)] = \frac{i}{\hbar}[H_S, O(t)] + D[O(t)]$, with $D[O(t)] = \sum_k \gamma_k(t)[L_k^\dagger O(t)L_k - \frac{1}{2}\{L_k^\dagger L_k, O(t)\}]$ and L_k the jump operators of the system.

Let us take the average of Eq. (15) in the state ρ and its absolute value. By applying the Cauchy-Schwarz inequality, we obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \sqrt{\mathrm{tr}(\rho^2)}\|\mathcal{L}^\dagger[O(t)]\|_{\mathrm{HS}}, \tag{16}$$

where $\|A\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of operator A.

The inequality (16) is the upper bound on the rate of change of the expectation value of the given observable of the quantum system evolving under Lindblad dynamics. After integrating the inequality (16), we obtain the bound

$$T \geqslant \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}, \tag{17}$$

where $\Lambda_T = \frac{1}{T}\int_0^T dt \|\mathcal{L}^\dagger[O(t)]\|_{\mathrm{HS}}$ is evolution speed of the observable O and $T_{\mathrm{QSL}}^O = \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}$ is the OQSL time.

For the Lindblad dynamics, the bound given in (17) determines how fast the expectation value of an observable of the quantum system changes in time. The obtained bound (17) suggests that the OQSL depends on the purity of the initial state of the evolving quantum system. Note that the OQSL time is zero if the expectation value of a given observable does not change during the dynamics.

1. *Comparison between QSL and OQSL for pure dephasing dynamics*

Let us consider the QSL bound for open quantum systems governed by a Lindblad quantum master equation. For the Markovian dynamics of an open quantum system expressed via a Lindbladian $\mathcal{L}$, the lower bound on the evolution time needed for a quantum system to evolve from initial state ρ_0 to final state ρ_T was given in Ref. [43] as

$$T \geqslant \frac{|\cos\theta - 1|\mathrm{tr}(\rho_0^2)}{\sqrt{\mathrm{tr}[\mathcal{L}^\dagger(\rho_0)]^2}}, \tag{18}$$

where $\theta = \cos^{-1}(\frac{\mathrm{tr}(\rho_0\rho_T)}{\mathrm{tr}(\rho_0^2)})$ is expressed in terms of relative purity between the initial and the final state.

Let us consider a two-level quantum system with the ground state $|1\rangle\langle 1|$ and the excited state $|0\rangle\langle 0|$ interacting with a dephasing bath. The corresponding dephasing Lindblad or jump operator of the system is given by $L_0 = \sqrt{\frac{\gamma}{2}}\sigma_z$, where σ_z is the Pauli-Z operator and γ is a real parameter denoting the strength of dephasing. The Lindblad master equation [70] governs the time evolution of the two-level quantum system and is given by

$$\frac{d\rho_t}{dt} = \mathcal{L}(\rho_t) = \frac{\gamma}{2}(\sigma_z\rho_t\sigma_z - \rho_t). \tag{19}$$

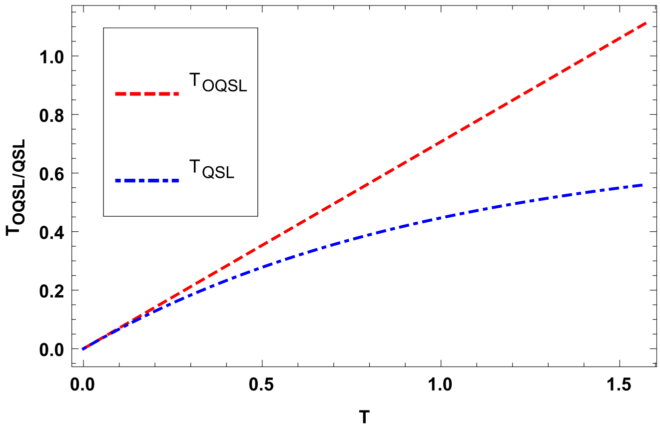

FIG. 1. Plot of $T_{\mathrm{QSL/OQSL}}$ vs T for $\gamma = 1$.

If the quantum system is initially in a state $\rho_0 = |+\rangle\langle+|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then the solution of the Lindblad equation is given by

$$\rho_t = \tfrac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1| + e^{-\gamma t}(|1\rangle\langle 0| + |0\rangle\langle 1|)]. \tag{20}$$

If the given observable is $O(0) = \sigma_x$, then the solution of (15) for the dephasing dynamics is given as

$$O(t) = e^{-\gamma t}\sigma_x. \tag{21}$$

To estimate the bounds (17) and (18), we require the quantities

$$\mathrm{tr}(\rho_0^2) = 1, \tag{22}$$

$$\cos\theta = \frac{1 + e^{-\gamma t}}{2}, \tag{23}$$

$$\mathrm{tr}[\mathcal{L}^\dagger(\rho_0)]^2 = \frac{\gamma^2}{2}, \tag{24}$$

$$\langle O(0)\rangle = \mathrm{tr}(\sigma_z\rho_0) = 1, \tag{25}$$

$$\langle O(t)\rangle = \mathrm{tr}(e^{-\gamma t}\sigma_z\rho_0) = e^{-\gamma t}, \tag{26}$$

$$\|\mathcal{L}^\dagger[O(t)]\|_{\mathrm{HS}} = \sqrt{2}\gamma e^{-\gamma t}. \tag{27}$$

In Fig. 1 we plot $T_{\mathrm{QSL/OQSL}}$ vs $T \in [0, \frac{\pi}{2}]$ for pure dephasing dynamics and for $\gamma = 1$. Figure 1 shows that our OQSL bound (17) is tighter than the QSL bound (18) for the pure dephasing process. Both bounds [our bound (17) and bound (18)] are obtained by employing the Cauchy-Schwarz inequality. Therefore, one expects both these bounds to be equally tight, but it is not true. It turns out that the bound (18) is loose. It happens because while deriving the bound (18) in Ref. [43] the authors used an additional inequality along with the Cauchy-Schwarz inequality, i.e., $\mathrm{tr}(\rho_t^2) \leqslant 1$ [see Eq. (7) of Ref. [43]]. They did this to obtain a time-independent bound on the rate of change of the purity.

D. Dynamical map

We can also express the QSL for the observable using the Kraus operator evolution. Suppose a given quantum system has initial state ρ and its dynamical evolution is governed by a completely positive and trace-preserving (CPTP) map $\mathcal{E}$, which is described by a set of Kraus operators $\{K_i(t)\}$ and $\sum_i K_i^\dagger(t)K_i(t) = \mathcal{I}_S$. The dynamics of the observable O in the Heisenberg picture is described as

$$O(t) = \sum_i K_i^\dagger(t)O(0)K_i(t). \tag{28}$$

Using Eq. (28), we obtain the QSL for the observable as given by

$$T \geqslant \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{2\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}, \tag{29}$$

where $\Lambda_T = \frac{1}{T}\int_0^T dt \|K_i^\dagger(t)O(0)\dot{K}_i(t)\|_{\mathrm{HS}}$ is the evolution speed of the observable O and $T_{\mathrm{QSL}}^O = \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{2\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}$ is the OQSL time.

Details of the derivation are provided in Appendix E. For dynamical map dynamics, the bound (29) determines how fast the expectation value of an observable of the quantum system changes in time. According to the obtained bound (29), the OQSL depends on the purity of the initial state of the evolving quantum system and on the speed of the observable's evolution.

E. State-independent QSL for an observable

The bounds for the QSL for an observable that have been proved in previous sections are state-dependent bounds. One may be curious to know if we can prove some state-independent bounds, i.e., whether we can derive the bounds which are given merely in terms of properties of the observables themselves. Here we make an attempt to formulate a bound without optimizing over states. To derive the state-independent speed limit for the observable, consider the Hilbert-Schmidt inner product for observables. The Hilbert-Schmidt inner product of two observables $O(0)$ and $O(t)$ is defined as

$$\langle O(0), O(t)\rangle = \mathrm{tr}[O(0)O(t)], \tag{30}$$

where $O(t) = e^{L^\dagger t}O(0)$ ($L^\dagger$ is the adjoint of the Liouvillian superoperator).

After differentiating Eq. (30) with respect to time, we obtain

$$\frac{d}{dt}\langle O(0), O(t)\rangle = \mathrm{tr}[O(0)\dot{O}(t)] = \mathrm{tr}[O(0)L^\dagger(O(t))]. \tag{31}$$

Let us take the absolute value of this equation. Then, by applying the Cauchy-Schwarz inequality, we can obtain the inequality

$$\left|\frac{d}{dt}\langle O(0), O(t)\rangle\right| \leqslant \|O(0)\|_{\mathrm{HS}}\|L^\dagger[O(t)]\|_{\mathrm{HS}}. \tag{32}$$

After integrating this inequality, we obtain the bound

$$T \geqslant \frac{|\langle O(0), O(T)\rangle - \langle O(0), O(0)\rangle|}{\|O(0)\|_{\mathrm{HS}}\Lambda_T}, \tag{33}$$

where $\Lambda_T = \frac{1}{T}\int_0^T dt \|L^\dagger[O(t)]\|_{\mathrm{HS}}$ is the evolution speed of the observable O and $T_{\mathrm{QSL}}^O = \frac{|\langle O(0)|O(T)\rangle - \langle O(0)|O(0)\rangle|}{\|O(0)\|_{\mathrm{HS}}\Lambda_T}$ is the OQSL time.

The bound (33) is independent of the state of the quantum system and it is applicable for arbitrary dynamics, which can be unitary or nonunitary. In the future, it will be worth exploring if it is possible to obtain some bounds using a different approach, for example, which may involve separation between extreme eigenvalues of the operators or optimizing over possible initial states. The state-independent bound will have its own merit as it could be understood as representing some best-case scenario where the time is the shortest possible to modify the expectation value of a given observable when optimizing over all possible states. This kind of bound may find applications in the context of quantum metrology, where we optimize over the probe states in order to obtain the fastest change of the state from the point of view of certain parameters.

III. APPLICATIONS

In this section we illustrate the usefulness of the OQSL for a quantum battery, growth of a two-time correlation function, and connection to the Lieb-Robinson bound.

A. Quantum batteries

A quantum battery is a microscopic energy storage device introduced by Alicki and Fannes [71]. Several theoretical works have been done to strengthen this novel idea of quantum battery and enhance its nonclassical features [72–86]. Quantum batteries can easily outperform classical batteries because of several quantum advantages. Here our main aim is to obtain a minimal unitary charging time of the quantum battery using the OQSL.

The quantum battery consists of many quantum systems with several degrees of freedom in which we can deposit work to or extract work from it. Let us consider the battery with Hamiltonian H_B and charged by field H_C. The total Hamiltonian of the quantum battery is described by

$$H_T = H_B + H_C. \tag{34}$$

The amount of extractable energy from the quantum system by unitary operations is termed the ergotropy of quantum battery [75], which is given by

$$\varepsilon(t) = \langle\psi(t)|H_B|\psi(t)\rangle - \langle\psi(0)|H_B|\psi(0)\rangle, \tag{35}$$

where $|\psi(0)\rangle$ and $|\psi(t)\rangle$ are the initial and final states of the quantum battery while charging.

Note that the expression (35) holds true in the Schrödinger picture. However, in the Heisenberg picture the ergotropy can be rewritten as

$$\varepsilon(t) = \langle\psi(0)|[H_B(t) - H_B(0)]|\psi(0)\rangle,$$

where $H_B(t) = e^{iHt/\hbar}H_B(0)e^{-iHt/\hbar}$ and $H_B(0) = H_B$.

The rate of change ergotropy of the quantum battery during the charging process can be obtained by differentiating the above equation with respect to time, which is given by

$$\frac{d\varepsilon(t)}{dt} = \frac{d}{dt}\langle\psi(0)|H_B(t)|\psi(0)\rangle.$$

Using our bound, we can write the QSL for ergotropy as

$$T \geqslant \frac{\hbar}{2\Delta H_T}\int_0^T \frac{|d\langle H_B(t) - H_B(0)\rangle|}{\Delta H_B(t)}, \tag{36}$$

where T is the charging time period of the quantum battery. Also, an alternative unified bound can be obtained by using bounds (7) and (10),

$$T \geqslant \frac{\hbar}{2}\frac{|\langle H_B(T)\rangle - \langle H_B(0)\rangle|}{\min\{\|\bullet\|_{\rm op}, \|\bullet\|_{\rm HS}, \|\bullet\|_{\rm tr}\}}, \tag{37}$$

where $\bullet$ stands for $H_B(0)H_T$ and the operator norm, the Hilbert-Schmidt norm, and the trace norm of an operator satisfy the inequality $\|A\|_{\rm op} \leqslant \|A\|_{\rm HS} \leqslant \|A\|_{\rm tr}$.

Since previously obtained bounds [73,75] on the charging time of the quantum battery are based on distinguishability of the initial and final state vectors of the quantum battery, the bounds we have presented in this section are based on the difference between the initial ergotropy and the final ergotropy of the quantum battery. The bounds obtained in this section can easily outperform previously obtained bounds, especially when the battery has degenerate energy levels.

For example, let us consider the model of a qubit quantum battery which has Hamiltonian $H_B = \sigma_z$ and let us consider the battery to initially be in state $|\phi^+\rangle = a|0\rangle + b|1\rangle$ (which has nonzero ergotropy). Then, by applying some charging field $H_C(t)$, we reach the final state $|\phi^-\rangle = a|0\rangle - b|1\rangle$. In this process we neither extract any work from the quantum battery nor store any work in the quantum battery because both initial and final states have the same ergotropy according to Eq. (35). Note that if we calculate the charging time according to the standard QSL (6) or bounds presented in [73,75], we obtain a nonzero minimal charging time, but according to our bounds (36) and (37) the minimal charging time is zero. This happens because the standard QSL and bounds presented in [73,75] are based on the notion of state distinguishability while our bounds (36) and (37) depend on a change in the ergotropy. Therefore, our bounds (36) and (37) yield the correct minimal charging time.

B. Transport properties

A crucial quantity in the theory of quantum transport in many-body physics is the two-time correlation function of an observable. This section aims to obtain a speed limit for a two-time correlation of an observable and its time-evolved observable. For an arbitrary pure quantum state ρ, we can define the two-time correlation function $C(A(t), A(0))$ between observables $A(t)$ and $A(0)$ as

$$C(t) = \langle A(t)A(0)\rangle - \langle A(t)\rangle\langle A(0)\rangle. \tag{38}$$

For the closed dynamics case $A(t) = U^\dagger(t)A(0)U(t)$ $[U(t) = e^{-iHt/\hbar}]$ and for the open dynamics case $A(t) = e^{\mathcal{L}^\dagger t}A(0)$ ($\mathcal{L}^\dagger$ is the adjoint of the Lindbladian). We can derive the following speed limit bound on the two-time correlation function for closed dynamics:

$$T \geqslant \frac{\hbar}{2}\frac{|C(T) - C(0)|}{\|A(0)\|_{\rm op}\frac{1}{T}\int_0^T dt\|[H, A(t)]\|_{\rm op}}. \tag{39}$$

Similarly, we can derive the following speed limit bound on the two-time correlation function for open dynamics:

$$T \geqslant \frac{\hbar}{2}\frac{|C(T) - C(0)|}{\|A(0)\|_{\rm op}\frac{1}{T}\int_0^T dt\|\mathcal{L}^\dagger[A(t)]\|_{\rm op}}. \tag{40}$$

Details of the derivation of bounds (39) and (40) are provided in Appendixes F and G.

C. Relation to the Lieb-Robinson bound

The Lieb-Robinson bound [59,60] provides the speed limit for information propagation about the perturbation. This gives an upper bound for the operator norm of the commutator of $A(t)$ and B, where A and B are spatially separated operators of a many-body quantum system. This bound implies that even in the nonrelativistic quantum dynamics one has some kind of locality structure analogous to the notion of finiteness of the speed of light in the relativistic theory.

This section aims to derive a distinct speed limit bound for the commutator of $A(t)$ and B, i.e., how fast the commutator changes in the Heisenberg picture. The commutator of two observables in two different regions of a many-body system is defined as

$$O(t) = [B(0), A(t)]. \tag{41}$$

The average of the commutator in the state ρ is given by $\langle O(t)\rangle = \mathrm{tr}[O(t)\rho]$, where ρ is a pure state of the given quantum system.

Here we want to obtain the speed limit bound for the commutator for both the closed system dynamics and open system dynamics. For the closed dynamics $A(t) = U^\dagger(t)A(0)U(t)$ $[U(t) = e^{-iHt/\hbar}]$ and for the open dynamics $A(t) = e^{\mathcal{L}^\dagger t}A(0)$ ($\mathcal{L}^\dagger$ is the adjoint of the Lindbladian). We can derive the following speed limit bound on the commutator for closed dynamics:

$$T \geqslant \frac{2}{\hbar}\frac{|\langle O(T)\rangle|}{\|B(0)\|_{\mathrm{op}}\frac{1}{T}\int_0^T dt\,\|[H, A(t)]\|_{\mathrm{op}}}. \tag{42}$$

Similarly, we can derive the following speed limit bound on the commutator for open dynamics:

$$T \geqslant \frac{2}{\hbar}\frac{|\langle O(T)\rangle|}{\|B(0)\|_{\mathrm{op}}\frac{1}{T}\int_0^T dt\,\|\mathcal{L}^\dagger[A(t)]\|_{\mathrm{op}}}. \tag{43}$$

Details of the derivation of bounds (42) and (43) are provided in Appendixes H and I. Note that our bounds are state dependent while the Lieb-Robinson bound is state independent. Also, to prove the Lieb-Robinson bound, one needs bounded interactions such as those encountered in quantum spin systems, whereas the quantum speed limit for the commutator does not require any assumption about the underlying Hamiltonian.

IV. CONCLUSION

The standard quantum speed limit for the evolution of a state plays an important role in quantum theory, quantum information, quantum control, and quantum thermodynamics. However, if we describe the quantum dynamics in the Heisenberg picture, then we cannot use the QSL for the state evolution. We need to define the evolution speed of the observable for a quantum system in the Heisenberg picture. In this paper we have derived the quantum speed limits for general observables for the unitary, the Lindbladian dynamics, and the completely positive dynamics. Along with this, we have presented several possible applications of these bounds such as in the quantum battery, probability dynamics, growth of the two-point correlation function, and time development of the commutator and its connection to the Lieb-Robinson bound. A salient outcome of our approach is that the standard QSL for the state can be viewed as a special case of the QSL for an observable. In the future, we hope that these bounds can have useful applications in quantum metrology, quantum control, detection of non-Markovianity, quantum thermodynamics, charging and discharging of quantum batteries, and many other areas as well.

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APPENDIX A: DERIVATION OF EQ. (7)

To obtain the alternate OQSL given in Eq. (7), let the state of a quantum system be described by a density operator ρ (not necessarily pure). The time evolution of the expectation value of any system observable O is given as

$$\langle O(t)\rangle = \mathrm{tr}[U^\dagger(t)O(0)U(t)\rho]. \tag{A1}$$

After differentiating this equation with respect to time, we obtain

$$\frac{d\langle O(t)\rangle}{dt} = \mathrm{tr}[\dot{U}^\dagger(t)O(0)U(t)\rho] + \mathrm{tr}[U^\dagger(t)O(0)\dot{U}(t)\rho].$$

Let us take the absolute value of this equation and use the triangular inequality $|A+B| \leqslant |A| + |B|$ to obtain

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant |\mathrm{tr}[\dot{U}^\dagger(t)O(0)U(t)\rho]| + |\mathrm{tr}[U^\dagger(t)O(0)\dot{U}(t)\rho]|.$$

Now, using the Cauchy-Schwarz inequality $|\mathrm{tr}(AB)| \leqslant \sqrt{\mathrm{tr}(A^\dagger A)\mathrm{tr}(B^\dagger B)}$, we can obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant 2\sqrt{\mathrm{tr}[\dot{U}^\dagger(t)O^2(0)\dot{U}(t)]\mathrm{tr}(\rho^2)}.$$

This inequality can be further simplified as

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{2}{\hbar}\sqrt{\mathrm{tr}(\rho)^2}\|O(0)H\|_{\mathrm{HS}},$$

where $\|A\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of operator A.

After integrating with respect to time, we obtain the bound

$$T \geqslant \frac{\hbar}{2\sqrt{\mathrm{tr}(\rho^2)}}\frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\|O(0)H\|_{\mathrm{HS}}}. \tag{A2}$$

If an observable satisfies $O^2 = I$, then for the pure state case the bound (A2) can be expressed as

$$T \geqslant \frac{\hbar}{2\|H\|_{\mathrm{HS}}}|\langle O(T)\rangle - \langle O(0)\rangle|. \tag{A3}$$

This completes the proof of Eq. (7).

APPENDIX B: DERIVATION OF EQ. (10)

To derive the bound given in Eq. (10), let us assume that a quantum system has a state ρ (pure state). The time evolution of the expectation value of any system observable O is given as

$$\langle O(t)\rangle = \mathrm{tr}[U^\dagger(t)O(0)U(t)\rho]. \tag{B1}$$

To find the rate of change of the expectation value of the observable, we need to differentiate Eq. (B1) with respect to time, which is given by

$$\frac{d\langle O(t)\rangle}{dt} = \mathrm{tr}[\dot{U}^\dagger(t)O(0)U(t)\rho] + \mathrm{tr}[U^\dagger(t)O(0)\dot{U}(t)\rho].$$

Let us take the absolute value of this equation. Then, by applying the triangular inequality $|A+B| \leqslant |A| + |B|$, we can obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{1}{\hbar}\{|\mathrm{tr}[HO(t)\rho]| + |\mathrm{tr}[O(t)H\rho]|\}. \tag{B2}$$

Next we use the Hölder inequality $|\mathrm{tr}(AB)| \leqslant \|A\|_{\mathrm{p}}\|B\|_{\mathrm{q}}$, where $p, q \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$ [67–69]. This leads to the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{2}{\hbar}\|HO(t)\|_{\mathrm{op}}.$$

We know that the operator norm, the Hilbert-Schmidt norm, and the trace norm of an operator satisfy the inequality $\|A\|_{\mathrm{op}} \leqslant \|A\|_{\mathrm{HS}} \leqslant \|A\|_{\mathrm{tr}}$ and the operator norm is unitary invariant $\|U^\dagger AU\|_{\mathrm{op}} = \|A\|_{\mathrm{op}}$. Then we can express the above inequality as

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{2}{\hbar}\|HO(0)\|_{\mathrm{tr}}.$$

After integrating this inequality, we obtain the desired bound

$$T \geqslant \frac{\hbar}{2}\frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\|HO(0)\|_{\mathrm{tr}}}.$$

In general, we can write the above bound as

$$T \geqslant \frac{\hbar}{2}\left\{\frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\min\{\|O(0)H\|_{\mathrm{op}}, \|O(0)H\|_{\mathrm{tr}}\}}\right\}. \tag{B3}$$

This completes the proof of Eq. (10).

APPENDIX C: TRACE DISTANCE BOUNDS ON THE OBSERVABLE DIFFERENCE

We know that we can use the trace distance to figure out how close two density operators are. We can ask a similar question for the expectation value of the observable.

Note that

$$\begin{aligned}|\langle O\rangle_\rho - \langle O\rangle_\sigma| &\equiv |\mathrm{tr}(\rho-\sigma)O| \leqslant \mathrm{tr}|(\rho-\sigma)O| \\ &= \|(\rho-\sigma)O\|_{\mathrm{tr}} \overset{\text{Hölder}}{\leqslant} \|(\rho-\sigma)\|_{\mathrm{tr}}\|O\|_{\mathrm{op}} \\ &= 2\|O\|_{\mathrm{op}} l(\rho,\sigma).\end{aligned}$$

These inequalities are obtained by using the triangle inequality for the absolute value and the Hölder inequality

$$\|AB\|_{\mathrm{tr}} = \|A\|_{\mathrm{p}}\|B\|_{\mathrm{q}}, \quad \frac{1}{p}+\frac{1}{q}=1,$$

with $p=1$ and $q=\infty$. Note that $\|X\|_\infty = \|X\|_{\mathrm{op}}$ is the maximal absolute value of all eigenvalues of X when X is Hermitian. Here $p=1$ corresponds to the trace norm.

APPENDIX D: OQSL FOR ARBITRARY DYNAMICS

By using the Cauchy-Schwarz inequality $|\mathrm{tr}(AB)| \leqslant \sqrt{\mathrm{tr}(A^\dagger A)\mathrm{tr}(B^\dagger B)}$ in Eq. (12), we can obtain the inequality

$$\mathcal{D} \leqslant \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O(0)\|_{\mathrm{op}}}\sqrt{\mathrm{tr}[\{O(t)-O(0)\}^\dagger\{O(t)-O(0)\}]}.$$

This inequality can be written in the form

$$\mathcal{D} \leqslant \mathcal{D}' = \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O(0)\|_{\mathrm{op}}}\|\{O(t)-O(0)\}\|_{\mathrm{HS}}. \tag{D1}$$

The rate of change of distance $\mathcal{D}'$ can be obtained by differentiating Eq. (D1) with respect to time. Thus, we obtain

$$\dot{\mathcal{D}}' = \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O(0)\|_{\mathrm{op}}}\frac{\mathrm{tr}[\dot{O}(t)\{O(t)-O(0)\} + \{O(t)-O(0)\}\dot{O}(t)]}{2\|O(t)-O(0)\|_{\mathrm{HS}}}.$$

This inequality can be further simplified as

$$\dot{\mathcal{D}}' = \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O(0)\|_{\mathrm{op}}}\frac{\mathrm{tr}[\dot{O}(t)\{O(t)-O(0)\}]}{\|O(t)-O(0)\|_{\mathrm{HS}}}.$$

If we take the absolute value of $\dot{\mathcal{D}}'$ and again apply the Cauchy-Schwarz inequality $|\mathrm{tr}(AB)| \leqslant \sqrt{\mathrm{tr}(A^\dagger A)\mathrm{tr}(B^\dagger B)}$, then we can obtain the inequality

$$|\dot{\mathcal{D}}'| \leqslant \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O\|_{\mathrm{op}}}\|\dot{O}(t)\|_{\mathrm{HS}} = \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O\|_{\mathrm{op}}}\|L_t^\dagger(O(t))\|_{\mathrm{HS}}.$$

After integrating the above inequality, we obtain

$$\mathcal{D}'(O(T), O(0)) \leqslant \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O\|_{\mathrm{op}}}T\Lambda_T,$$

where $\Lambda_T = \frac{1}{T}\int_0^T dt\|L_t^\dagger(O(t))\|_{\mathrm{HS}}$ is the evolution speed of the observable O. If we use Eq. (D1), then we obtain

$$\mathcal{D}(O(T), O(0)) \leqslant \frac{\sqrt{\mathrm{tr}(\rho^2)}}{2\|O\|_{\mathrm{op}}}T\Lambda_T.$$

Finally, we obtain the desired bound on the evolution time of the expectation value of an observable as

$$T \geqslant \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}, \tag{D2}$$

where $T_{\mathrm{QSL}}^O = \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}$. This completes the proof of Eq. (13).

APPENDIX E: OQSL FOR THE DYNAMICAL MAP

If the given quantum system has the initial state ρ and its evolution is governed by a CPTP map $\mathcal{E}$ which is described by a set of Kraus operators $\{K_i(t)\}$ and $\sum_i K_i^\dagger(t)K_i(t) = \mathcal{I}_S$, the dynamics of the observable in the Heisenberg picture is described as

$$O(t) = \sum_i K_i^\dagger(t)O(0)K_i(t). \tag{E1}$$

The time evolution of the expectation value of the observable O is given by

$$\langle O(t)\rangle = \sum_i \mathrm{tr}[K_i^\dagger(t)O(0)K_i(t)\rho].$$

The rate of change of the expectation value of the observable O can be obtained by differentiating the above equation with respect to time, which is given by

$$\frac{d\langle O(t)\rangle}{dt} = \sum_i \{\mathrm{tr}[\dot{K}_i^\dagger(t)O(0)K_i(t)\rho] + \mathrm{tr}[K_i^\dagger(t)O(0)\dot{K}_i\rho]\}.$$

If we take its absolute value, then we can apply the triangle inequality $|A+B| \leqslant |A| + |B|$ and the Cauchy-Schwarz inequality $|\mathrm{tr}(AB)| \leqslant \sqrt{\mathrm{tr}(A^\dagger A)\mathrm{tr}(B^\dagger B)}$. Finally, we have obtained the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \sum_i \{\sqrt{\mathrm{tr}[\dot{K}_i^\dagger(t)O(0)K_i(t)K_i^\dagger(t)O(0)\dot{K}_i(t)]\mathrm{tr}(\rho^2)} + \sqrt{\mathrm{tr}[K_i^\dagger(t)O(0)\dot{K}_i(t)\dot{K}_i^\dagger(t)O(0)K_i(t)]\mathrm{tr}(\rho^2)}\}.$$

The simplified form of this inequality is given as

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant 2\sqrt{\mathrm{tr}(\rho^2)}\sum_i \|K_i^\dagger(t)O(0)\dot{K}_i(t)\|_{\mathrm{HS}},$$

where $\|A\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of operator A.

After integrating the above inequality, we obtain the bound

$$T \geqslant \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{2\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}, \tag{E2}$$

where $\Lambda_T = \frac{1}{T}\int_0^T dt\|K_i^\dagger(t)O(0)\dot{K}_i(t)\|_{\mathrm{HS}}$ is the evolution speed of the observable O and $T_{\mathrm{QSL}}^O = \frac{|\langle O(T)\rangle - \langle O(0)\rangle|}{2\sqrt{\mathrm{tr}(\rho^2)}\Lambda_T}$, as given in Eq. (29).

For open dynamics, the bounds (17) and (29) determine how fast the expectation value of an observable of the quantum system changes in time. This completes the proof of Eq. (29).

APPENDIX F: QSL OF THE TWO-POINT FUNCTION (UNITARY CASE)

Let us consider the two-point correlation function, which is defined as

$$C(t) = \langle A(t)A(0)\rangle - \langle A(t)\rangle\langle A(0)\rangle, \tag{F1}$$

where $A(t) = U^\dagger(t)A(0)U(t)$ and $U^\dagger(t)U(t) = I$. After differentiating Eq. (F1) with respect to time, we obtain

$$\frac{dC(t)}{dt} = \langle\dot{A}(t)A(0)\rangle - \langle\dot{A}(t)\rangle\langle A(0)\rangle. \tag{F2}$$

If we take the absolute value of Eq. (F2) and use the triangular inequality $|A+B| \leqslant |A| + |B|$, we obtain

$$\left|\frac{dC(t)}{dt}\right| \leqslant |\langle\dot{A}(t)A(0)\rangle| + |\langle\dot{A}(t)\rangle||\langle A(0)\rangle|. \tag{F3}$$

Now, using the fact that $|\mathrm{tr}(A\rho)| \leqslant \|A\|_{\mathrm{op}}$ (where ρ is pure state), we can obtain the inequality

$$\left|\frac{dC(t)}{dt}\right| \leqslant \|\dot{A}(t)A(0)\|_{\mathrm{op}} + \|\dot{A}(t)\|_{\mathrm{op}}\|A(0)\|_{\mathrm{op}}. \tag{F4}$$

This equation can be expressed as

$$\left|\frac{dC(t)}{dt}\right| \leqslant \|\dot{A}(t)\|_{\mathrm{op}}\|A(0)\|_{\mathrm{op}} + \|\dot{A}(t)\|_{\mathrm{op}}\|A(0)\|_{\mathrm{op}}, \tag{F5}$$

which leads to

$$\left|\frac{dC(t)}{dt}\right| \leqslant 2\|A(0)\|_{\mathrm{op}}\|\dot{A}(t)\|_{\mathrm{op}}. \tag{F6}$$

Therefore, we have

$$\left|\frac{dC(t)}{dt}\right| \leqslant \frac{2}{\hbar}\|A(0)\|_{\mathrm{op}}\|[H, A(t)]\|_{\mathrm{op}}. \tag{F7}$$

After integrating, we obtain the bound

$$T \geqslant \frac{\hbar}{2}\frac{|C(T) - C(0)|}{\|A(0)\|_{\mathrm{op}}\frac{1}{T}\int_0^T dt\|[H, A(t)]\|_{\mathrm{op}}}. \tag{F8}$$

This completes the proof of Eq. (39).

APPENDIX G: QSL OF THE TWO-POINT FUNCTION (OPEN-SYSTEM CASE)

Let us consider the two-point function, which is defined as

$$C(t) = \langle A(t)A(0)\rangle - \langle A(t)\rangle\langle A(0)\rangle, \tag{G1}$$

where $A(t) = e^{\mathcal{L}^\dagger t}A(0)$ and $\mathcal{L}^\dagger$ is the adjoint of the Lindbladian. After differentiating Eq. (G1) with respect to time, we obtain

$$\frac{dC(t)}{dt} = \langle\dot{A}(t)A(0)\rangle - \langle\dot{A}(t)\rangle\langle A(0)\rangle. \tag{G2}$$

If we take the absolute value of Eq. (G2) and use the triangular inequality $|A+B| \leqslant |A| + |B|$, we can obtain

$$\left|\frac{dC(t)}{dt}\right| \leqslant |\langle\dot{A}(t)A(0)\rangle| + |\langle\dot{A}(t)\rangle||\langle A(0)\rangle|. \tag{G3}$$

Now, using the fact that $|\mathrm{tr}(A\rho)| \leqslant \|A\|_{\mathrm{op}}$ (where ρ is pure state), we can obtain the inequality

$$\left|\frac{dC(t)}{dt}\right| \leqslant \|\dot{A}(t)A(0)\|_{\mathrm{op}} + \|\dot{A}(t)\|_{\mathrm{op}}\|A(0)\|_{\mathrm{op}}, \tag{G4}$$

which leads to

$$\left|\frac{dC(t)}{dt}\right| \leqslant \|\dot{A}(t)\|_{\mathrm{op}}\|A(0)\|_{\mathrm{op}} + \|\dot{A}(t)\|_{\mathrm{op}}\|A(0)\|_{\mathrm{op}}. \tag{G5}$$

Equation (G5) can be rewritten as

$$\left|\frac{dC(t)}{dt}\right| \leqslant 2\|A(0)\|_{\mathrm{op}}\|\dot{A}(t)\|_{\mathrm{op}}. \tag{G6}$$

Therefore, we have

$$\left|\frac{dC(t)}{dt}\right| \leqslant \frac{2}{\hbar}\|A(0)\|_{\mathrm{op}}\|\mathcal{L}^\dagger[A(t)]\|_{\mathrm{op}}. \tag{G7}$$

After integrating, we obtain the bound

$$T \geqslant \frac{\hbar}{2}\frac{|C(T) - C(0)|}{\|A(0)\|_{\mathrm{op}}\frac{1}{T}\int_0^T dt\|\mathcal{L}^\dagger[A(t)]\|_{\mathrm{op}}}. \tag{G8}$$

This completes the proof of Eq. (40).

APPENDIX H: QSL BOUND FOR COMMUTATORS (UNITARY CASE)

The commutator of operators in two different regions of a many-body system is defined as

$$\langle O(t)\rangle = \langle [B(0), A(t)]\rangle, \tag{H1}$$

where $A(t) = U^\dagger(t)A(0)U(t)$, with $U^\dagger(t)U(t) = I$. After differentiating Eq. (H1) with respect to time, we obtain

$$\frac{d\langle O(t)\rangle}{dt} = \langle [B(0), \dot{A}(t)]\rangle. \tag{H2}$$

Let us take the absolute value of Eq. (H2),

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| = |\langle [B(0), \dot{A}(t)]\rangle|. \tag{H3}$$

Now, by using $|\mathrm{tr}(A\rho)| \leqslant \|A\|_{\mathrm{op}}$ (where ρ is a pure state), we can obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \|[B(0), \dot{A}(t)]\|_{\mathrm{op}}. \tag{H4}$$

Let us use the inequality $\|[O_1, O_2]\|_{\mathrm{op}} \leqslant 2\|O_1\|_{\mathrm{op}}\|O_2\|_{\mathrm{op}}$ to obtain

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant 2\|B(0)\|_{\mathrm{op}}\|\dot{A}(t)\|_{\mathrm{op}}. \tag{H5}$$

Equation (H5) can be rewritten as

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{2}{\hbar}\|B(0)\|_{\mathrm{op}}\|[H, A(t)]\|_{\mathrm{op}}. \tag{H6}$$

After integrating, we obtain the bound

$$T \geqslant \frac{2}{\hbar}\frac{|\langle O(T)\rangle|}{\|B(0)\|_{\mathrm{op}}\frac{1}{T}\int_0^T dt\|[H, A(t)]\|_{\mathrm{op}}}. \tag{H7}$$

This completes the proof of Eq. (42).

APPENDIX I: QSL BOUND FOR COMMUTATORS (OPEN-SYSTEM CASE)

The commutator of operators in two different regions of a many-body system is defined as

$$\langle O(t)\rangle = \langle [B(0), A(t)]\rangle, \tag{I1}$$

where $A(t) = e^{\mathcal{L}^\dagger t}A(0)$ and $\mathcal{L}^\dagger$ is the adjoint of the Lindbladian. After differentiating Eq. (I1)) with respect to time, we obtain

$$\frac{d\langle O(t)\rangle}{dt} = \langle [B(0)\dot{A}(t)]\rangle. \tag{I2}$$

Let us take the absolute value of Eq. (I2),

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| = |\langle [B(0), \dot{A}(t)]\rangle|. \tag{I3}$$

Now, by using $|\mathrm{tr}(A\rho)| \leqslant \|A\|_{\mathrm{op}}$ (where ρ is pure state), we can obtain the inequality

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \|[B(0), \dot{A}(t)]\|_{\mathrm{op}}. \tag{I4}$$

Let us use the inequality $\|[O_1, O_2]\|_{\mathrm{op}} \leqslant 2\|O_1\|_{\mathrm{op}}\|O_2\|_{\mathrm{op}}$ to obtain

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant 2\|B(0)\|_{\mathrm{op}}\|\dot{A}(t)\|_{\mathrm{op}}. \tag{I5}$$

This equation can be written as

$$\left|\frac{d\langle O(t)\rangle}{dt}\right| \leqslant \frac{2}{\hbar}\|B(0)\|_{\mathrm{op}}\|\mathcal{L}^\dagger[A(t)]\|_{\mathrm{op}}. \tag{I6}$$

After integrating, we obtain the bound

$$T \geqslant \frac{2}{\hbar}\frac{|\langle O(T)\rangle|}{\|B(0)\|_{\mathrm{op}}\frac{1}{T}\int_0^T dt\|\mathcal{L}^\dagger[A(t)]\|_{\mathrm{op}}}. \tag{I7}$$

This completes the proof of Eq. (43).

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