Universal method to estimate quantum coherence

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Coherence is a defining property of quantum theory that accounts for quantum advantage in many quantum information tasks. Although many coherence quantifiers were introduced in various contexts, the lack of efficient methods to estimate them restricts their applications. In this paper, we tackle this problem by proposing one universal method to provide measurable bounds for most current coherence quantifiers. Our method is motivated by the observation that the distance between the state of interest and its diagonal parts in the reference basis, which lies at the heart of the coherence quantifications, can be readily estimated by the disturbance effect and uncertainty of the reference measurement. Thus, our method of bounding coherence provides a feasible and broadly applicable avenue for detecting coherence, facilitating its further practical applications.

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I. INTRODUCTION

Coherence, captured by the superposition principle, is a defining property of quantum theory. It underscores almost all the quantum features, such as symmetry [1], entanglement [2–4], and quantum correlation [5,6]. Coherence also accounts for quantum advantages in various quantum information processing tasks, such as quantum metrology [7–9] and quantum cryptography [10,11]. Within a strictly mathematical framework of resource theory, the significance of coherence as a resource has been fully appreciated in recent years. Many aspects of it, ranging from characterization [12], distillation, and catalytic [13–16], were investigated, along with an intense analysis of how coherence plays a role in fundamental physics (see [17] for a review).

Quantifying coherence lies in the heart of coherence resource theory [12,18–23]. Recently, many methods were proposed. The most compelling method was based on state distance, for example, quantifying coherence with the minimal distance between the state of interest and the closest coherence free state. Typical examples are the relative entropy of coherence and the l_1 norm of coherence [20]. Coherence may also be quantified with the distance between the concerned state and its diagonal parts in the reference basis [24]. One example is the coherence of the trace norm [25,26]. Another method is via the convex-roof measure. That is, provided a quantifier for the pure state, a general mixed state's quantifier is constructed via a roof construction; this method leads to the formation of coherence [27] and the infidelity coherence measure [23]. There are also other quantifiers such as the robustness of coherence [28] and the Wigner-Yanase skew information of coherence [22].

While many theoretical works were devoted to a systematical research of coherence [12,17,29,30], it remains a difficult problem to efficiently estimate coherence in experiments, which limits the applications of the quantifications as common tools for quantum information processing. Clearly, one can perform state tomography and then calculate quantifiers with the derived quantum density matrix or estimate coherence by employing normal witness technique [31–33] or with numerical optimizations [34]. These methods suffer from the complexities of mathematics and the experiment setup, thus lacking efficiency. Another method is based on spectrum estimation [35], which commonly needs a few test measurements to obtain a nontrivial estimation. These methods, unfortunately, are commonly restricted for estimating the convex-roof quantifiers, such as the coherence of formation, the convex roof of infidelity, and the convex roof of the Wigner-Yanase skew information.

To improve the evaluation of coherence in experiments, we report one simple and feasible detection method, which provides both the upper and the lower bounds in terms of the reference measurement's uncertainty and disturbance effect, respectively. We find that coherence quantifiers are closely related to the distance between the state of interest and its diagonal parts (written in the reference basis). The distance can be upper-bounded in terms of uncertainty according to recent uncertainty-disturbance relations (UDR) [36,37] and lower-bounded according to data processing inequality. These bounds are formulated with statistics from a universal experimental scheme. By this method, almost all the current coherence quantifiers of great interest are immediately bounded as long as one universal experiment setup outputs statistics. The quantifiers include the relative entropy of the coherence measure, the coherence of formation, the $l_{1,2}$ norm of coherence, the trace norm of coherence, the convex roof of the infidelity coherence, the Wigner-Yanase skew information of coherence and the convex-roof construction, and the robustness of coherence. Thus the method exhibits merits of simplicity and broad applicability.

The rest of the paper is structured as follows. In Sec. I, we briefly review state-distance-based coherence quantifiers, then provide a framework for upper bounding and lower bounding

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them. In Sec. II, we provide measurable bounds for most current coherence quantifiers, including the distance-based quantifiers and some other quantifiers of general interest. In Sec. III, we compare our method with the one based on spectrum estimation, showing that our method is experimentally less demanding and more efficient.

II. FRAMEWORK OF DETECTING COHERENCE

A. Distance-based quantifications of coherence

Coherence is a quantity characterized with respect to one prefixed reference basis denoted by $\{|j\rangle\}$ with the relevant measurement being referred to as the reference measurement. The free states are the ones of the form $\sigma = \sum_j p_j |j\rangle\langle j|$. Otherwise, a nondiagonal state contains coherence. Coherence is commonly quantified with state distance, for example, with the minimal distance between the state of interest and the closest incoherent one [15]

$$\mathcal{C}(\rho) := \min_{\sigma\in\mathcal{I}} \mathcal{D}(\rho, \sigma), \tag{1}$$

where $\mathcal{I}$ denotes the set of coherence free states and $\mathcal{D}(\rho, \sigma)$ specifies the state distance. Appling Eq. (1) to the state distances of relative entropy, the l_1 norm, and the Tsallis relative α entropies, one can define quantifiers meeting all the criteria of the coherence resource theory [12]. They are referred to as coherence measures. If the chosen distance measures are the trace norm and fidelity [38,39], then Eq. (1) defines the coherence monotones that meet a specific subset of the criteria of the coherence resource theory.

The computability of quantifier $\mathcal{C}(\rho)$ is generally hard except for the state distance, for which $\mathcal{C}(\rho) = \mathcal{D}(\rho, \rho_d)$, where ρ_d specifies the diagonal parts of ρ in the reference basis. One may simply define an easily computable quantifier as the distance between ρ and ρ_d [24]:

$$\tilde{\mathcal{C}}(\rho) := \mathcal{D}(\rho, \rho_d), \tag{2}$$

which can be understood as the disturbance caused by the reference measurement in ρ as ρ_d is just the postmeasurement state. For some distance measures, Eq. (2) defines a better monotone than the one defined by Eq. (1). One example is the trace norm [25], for which Eq. (2) defines a quantifier that can satisfy more criteria [than that defined by Eq. (1)] and also allows a physically well-motivated interpretation as the capability to exhibit interference visibility [26]. $\tilde{\mathcal{C}}(\rho)$ does not involve a minimization process. Therefore,

$$\tilde{\mathcal{C}}(\rho) \geqslant \mathcal{C}(\rho).$$

One may also define the coherence quantifier via a convex-roof technique [27]. That is, provided a quantifier for the pure state, one can define a mixed state's quantifier via a convex-roof construction. For example, one may define the pure state coherence via Eq. (1), then the convex-roof construction is

$$\mathcal{C}'(\rho) := \min_{\{f_i, |\phi_i\rangle\}} \sum_i f_i \cdot \mathcal{C}(\phi_i), \tag{3}$$

where the minimization is taken over all possible pure-state decompositions of $\rho = \sum f_i |\phi_i\rangle\langle\phi_i|$. This definition has its advantage, e.g., when applied to infidelity, Eq. (3) can define a measure [23] for coherence while Eq. (1) defines only a monotone.

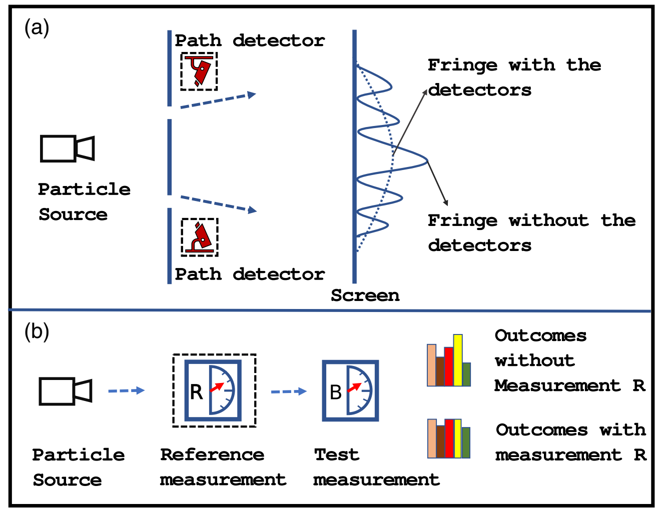

FIG. 1. Two-slit experiment and coherence detection protocol. Particles with and without coherence lead to different statistics when subjected to a test measurement.

The above definitions led to many quantifiers and also induced the bounds for the quantifiers defined in other ways. In the following, we introduce a framework for bounding them.

B. Framework of detecting coherence

Detecting the nonclassical properties, such as entanglement, coherence, and randomness, is to lower-bound them using the statistics coming from experiments. For coherence, it is instructive to recall its early illustration based on the two-slit experiment, which is shown in Fig. 1(a), where the reference measurement is the path detectors that erases the coherence between paths and the screen is an incompatible measurement that verifies the coherence in terms of the change of interference fringes due to the destruction of coherence. Here, we would like to exploit how this idea applies to many other coherence quantifiers. We consider a similar measurement setting as shown in Fig. 1(b). It consists of one reference measurement $\mathcal{R}$ and one following test measurement $\mathcal{B}$. The reference measurement updates an input state ρ into an incoherent state, i.e., its diagonal parts ρ_d. It then is subject to the following measurement $\mathcal{B} = \{|b_j\rangle\langle b_j|\}$, giving rise to a distribution $\mathbf{q}' = \{q'_j = \mathrm{tr}(\rho_d \cdot |b_j\rangle\langle b_j|)\}$. This is a typical sequential measurement scheme that can be readily realized with off-the-shelf instruments [40–43]. If without the measurement $\mathcal{R}$, directly performing $\mathcal{B}$ on ρ yields a distribution $\mathbf{q} = \{q_j = \mathrm{tr}(\rho \cdot |b_j\rangle\langle b_j|)\}$. The distance between $\mathbf{q}$ and $\mathbf{q}'$ can be understood as the disturbance introduced by the reference measurement in $\mathcal{B}$. One may choose $\mathcal{B}$ as the one maximally incompatible with the reference measurement, i.e., $\forall i, j$, $|\langle i|b_j\rangle|^2 = \frac{1}{d}$ with d specifying the dimension of the relevant Hilbert's space. This setting commonly can ensure a significant distance between $\mathbf{q}$ and $\mathbf{q}'$, which is in favor of coherence estimation. We also note that $\mathbf{q}'$ actually does not require performing a real test measurement after the reference measurement. As ρ_d is determined by the reference measurement's distribution as $\rho_d = \sum_i p_i |i\rangle\langle i|$ and $\mathbf{p} = \{p_i = \mathrm{tr}(\rho \cdot |i\rangle\langle i|)\}$, one can directly calculate $\mathbf{q}'$ via Born's rule. For example, the probability when $\mathcal{B} := \{|b_j\rangle\langle b_j|\}$ is $q'_i = \mathrm{tr}(\rho_d \cdot$

$|b_i\rangle\langle b_j|) = \sum_i c_{ij} p_i$ is given by $\mathbf{p}$, where $c_{ij} = |\langle i|b_j\rangle|^2$. In this way, only two independent measurements, namely, $\mathcal{R}$ and $\mathcal{B}$, are sufficient for giving the statistics $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$. In the following, the estimation of coherence quantifiers only involves these distributions.

1. Lower-bounding coherence

First, the coherence quantifiers of the form of Eq. (2) can be estimated according to the data processing inequality, which states that the distance between states, say ρ and ρ_d, are no less than the corresponding classical distance between the statistics coming from another measurement, say, $\mathcal{B}$, performed on them

$$\mathcal{D}(\rho, \rho_d) \geqslant D[\mathbf{q}, \mathbf{q}'].$$

Immediately, we obtain a lower bound for the coherence measure

$$\tilde{\mathcal{C}}(\rho) \geqslant D[\mathbf{q}, \mathbf{q}']. \tag{4}$$

We highlight a useful property of the classical distance, namely, the convexity of classical distance D,

$$\sum_i f_i \cdot D[\mathbf{q}_i, \mathbf{q}_i'] \geqslant D\left[\sum_i f_i \cdot \mathbf{q}_i, \sum_i f_i \cdot \mathbf{q}_i'\right],$$

which will be used for bounding $\mathcal{C}'(\rho)$.

Second, we consider the convex-roof-based coherence quantifier. We note that for the pure state a coherence quantifier $\mathcal{C}'(\phi)$ is always a function of distribution $\mathbf{p}_\phi$. This is because the diagonal elements of a pure state are sufficient to determine the coherence quantifier as they determine a pure state up to some relative phases. These phases are inessential in quantifying coherence as they can be modified freely with a reversible incoherent operation of phase shifting. Note that the maximum coherent state $|\phi\rangle = \frac{1}{d}\sum_i |i\rangle$ and the zero-coherence pure state $|i\rangle\langle i|$ exhibit the maximum and the minimum, respectively, of the uncertainty of the reference measurement in a given basis. It is therefore reasonable to assume that the coherence quantifier $\mathcal{C}'(\phi)$ for the pure state is positively related to the uncertainty $\delta_{\mathcal{D}}(\mathbf{p})$ (the subscript means that uncertainty can be related to the state distance $\mathcal{D}$) and may be lower-bounded with uncertainty or a function of it (whose definition is left to the next section). We find that, if a pure-state coherence $\mathcal{C}'(\phi)$ allows a lower bound in terms of a convex and monotonically increasing function of $\delta_{\mathcal{D}}(\mathbf{p}_\phi)$ specified by $g(\delta_{\mathcal{D}})$, the constructed convex-roof-based coherence quantifier $\mathcal{C}'(\rho)$ can be lower-bounded as

$$\mathcal{C}'(\rho) \geqslant g(D(\mathbf{q}, \mathbf{q}')). \tag{5}$$

It needs to be stressed that $\mathcal{C}'(\rho) \geqslant g(\delta_{\mathcal{D}}(\mathbf{p}))$ generally does not hold for a mixed state ρ due to the concavity of $\delta_{\mathcal{D}}$, namely, $\mathcal{C}'(\rho) \geqslant \sum_i f_i \cdot g(\delta_{\mathcal{D}}(\mathbf{p}_i)) \geqslant g(\sum_i f_i \cdot \delta_{\mathcal{D}}(\mathbf{p}_i))$ while $g(\sum_i f_i \cdot \delta_{\mathcal{D}}(\mathbf{p}_i)) \leqslant g(\delta_{\mathcal{D}}(\mathbf{p}))$, where $\sum_i \mathbf{p}_i = \mathbf{p}$ and $\sum_i f_i \cdot \mathbf{p}_i = \mathbf{p}$. The key idea behind Eq. (5) is to relax a concave uncertainty measure $\delta_{\mathcal{D}}$ into a convex disturbance measure $D(\mathbf{q}, \mathbf{q}')$ using UDRs [37], stating that one measurement's uncertainty in terms of, say $\delta_{\mathcal{D}}(\mathbf{p})$, is no less than its disturbance effect in the measured state ρ and in the subsequent test measurement $\mathcal{B}$:

$$\delta_{\mathcal{D}}(\mathbf{p}) \geqslant \mathcal{D}(\rho, \rho_d) \geqslant D(\mathbf{q}, \mathbf{q}').$$

Then, we have $g(\sum_i f_i \cdot D[\mathbf{q}_i, \mathbf{q}_i']) \geqslant g(D(\mathbf{q}, \mathbf{q}'))$ with $\sum_i f_i \cdot \mathbf{q}_i = \mathbf{q}$ and $\sum_i f_i \cdot \mathbf{q}_i' = \mathbf{q}'$, leading to a lower bound for coherence quantifiers. We left the proof of Eq. (5) to the Appendix. It can be seen that the function of $g(\cdot)$ provides a way of finding the lower bound of the coherence measure in terms of disturbance $D(\mathbf{q}, \mathbf{q}')$. In the next section, we shall show that $g(\cdot)$ can always be found for the existing convex-roof-based coherence quantifiers.

2. Estimation of upper bounds

In general, upper bounds for quantum properties are not as useful as the lower bounds since they may be much larger than the actual value and thus are commonly ignored in the theory and experiment. Here, we can obtain the estimation of the upper bound with the outcome distribution of the reference measurement for free, i.e., without introducing extra experimental settings, and most importantly the resulting upper bound may assist the estimation of coherence in our framework.

It follows from the UDRs that the upper bounds of the quantifiers of the form of Eqs. (1) or (2) are given as

$$\delta(\mathbf{p}) \geqslant \tilde{\mathcal{C}}(\rho) \geqslant \mathcal{C}(\rho). \tag{6}$$

The upper bound of $\mathcal{C}'(\rho)$ is given as

$$\delta(\mathbf{p}) \geqslant \sum_i f_i \cdot \delta(\mathbf{p}_i) \geqslant \sum_i f_i \cdot \mathcal{C}(\phi_i) = \mathcal{C}'(\rho). \tag{7}$$

Thus, one reference measurement is sufficient for upper-bounding the three kinds of coherence quantifiers.

Recently, with the distributions of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$, both the upper and the lower bounds were obtained. A possible large gap between them roughly indicates that (i) the state of interest contains little coherence and (ii) the setting of $\mathcal{B}$ is not well chosen. Then the lower bound may be optimized by choosing other settings of $\mathcal{B}$ or one may almost confirm that the state of interest contains little coherence. In this way the upper bound assists the estimation of the lower bound.

III. DETECTING COHERENCE IN VARIOUS CONTEXTS

In the following, we use the above framework to estimate coherence quantifiers having general interests.

A. Relative entropy of coherence measure and the coherence of formation

First, we consider the relative entropy of coherence [29] and the coherence of formation [18,27,44], which are defined by applying Eqs. (1) and (3) to the relative entropy

$$S(\rho\|\sigma) := \mathrm{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma).$$

The relative entropy of coherence is a legitimate measure. It has operational meaning as the asymptotic coherence distillation rate [20] and also quantifies the quantum randomness under the quantum adversaries (with independent measurements) [27,45,46]. The quantifier is defined as

$$\mathcal{C}_{\mathrm{re}}(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho\|\sigma) = S(\rho\|\rho_d).$$

The coherence of formation has an interpretation of the asymptotic coherence dilution rate [20]. It also quantifies the quantum randomness under the classical adversaries (with independent measurements) [27,45,46]. The quantifier reads

$$\mathcal{C}'_{\rm re}(\rho) := \min_{\{f_i,\phi_i\}} \sum_i f_i \cdot S(\phi_i \| \phi_{i,d}).$$

The UDR corresponding to the relative entropy is given as

$$H(\mathbf{p}) \geqslant S(\rho\|\rho_d) \geqslant H(\mathbf{q}\|\mathbf{q}'),$$

where the Shannon entropy $H(\mathbf{p}) := -\sum_i p_i \log_2 p_i$ defines the measurement uncertainty $\delta_{\rm re}(\mathbf{p})$ and the relative entropy $S(\rho\|\rho_d)$ defines disturbance in the quantum state and the classical relative entropy $H(\mathbf{q}\|\mathbf{q}') := \sum_i q_i \log_2 q_i - \sum_i q_i \log_2 q'_i$ defines the disturbance in measurement $\mathcal{B}$ denoted as $D_{\rm re}(\mathbf{q},\mathbf{q}')$. Based on the general arguments just provided, we immediately have [37]

$$H(\mathbf{p}) \geqslant \mathcal{C}_{\rm re}(\rho), \quad \mathcal{C}'_{\rm re}(\rho) \geqslant H(\mathbf{q}\|\mathbf{q}'). \tag{8}$$

For $\mathcal{C}_{\rm re}(\rho)$, the bounds are obvious. For $\mathcal{C}'_{\rm re}(\rho)$, we have $\mathcal{C}'_{\rm re}(|\phi\rangle) = H(\mathbf{p}_\phi)$, which leads to the definition of the convex and monotonically increasing g function as $g(x) = x$. Then a lower bound for $\mathcal{C}'_{\rm re}(\rho)$, namely, $H(\mathbf{q}\|\mathbf{q}')$, follows from Eq. (5) and the uncertainty disturbance relation [37].

B. l_1 norm, l_2 norm, and the trace-norm of coherence

1. l_1 norm of coherence

The l_1 norm of coherence quantifies the maximum entanglement that can be created from coherence under incoherent operations acting on the system and an incoherent ancilla [4]. It was used to investigate the speed-up of quantum computation [47,48], wave-particle duality [17,49,50], and the uncertainty principle [51]. The quantifier is defined via Eq. (1) as [29]

$$\mathcal{C}_{l_1}(\rho) = \min_{\sigma\in\mathcal{I}} \mathcal{D}_{l_1}(\rho,\sigma) = \mathcal{D}_{l_1}(\rho,\rho_d),$$

where the l_1 norm

$$\mathcal{D}_{l_1}(\rho,\sigma') = \textstyle\sum_{i,j} |\rho_{ij} - \sigma'_{ij}|,$$

with ρ_{ij} and σ'_{ij} specifying the matrices' elements.

The UDR corresponding to the l_1-norm distance is

$$\|\mathbf{p}\|_{\frac{1}{2}} - 1 \geqslant \mathcal{D}_{l_1}(\rho,\rho_d),$$

where $\|\mathbf{p}\|_x = (\sum_i p_i^x)^{\frac{1}{x}}$. The inequality is because $\sum_{i\neq j}|\rho_{ij}| \leqslant \sum_{i\neq j}\sqrt{p_i p_j} = \|\mathbf{p}\|_{\frac{1}{2}} - 1$. Based on Eq. (6) and the UDR, the $\mathcal{C}_{l_1}(\rho)$ is estimated via

$$\|\mathbf{p}\|_{\frac{1}{2}} - 1 \geqslant \mathcal{C}_{l_1}(\rho) \geqslant 2|\mathbf{q}-\mathbf{q}'|, \tag{9}$$

where $|\mathbf{q}-\mathbf{q}'| := \frac{1}{2}\sum_i |q_i - q'_i|$ is the Kolmogorov distance. The lower-bound side is due to $\sum_{i\neq j}|\rho_{ij}| = \sum_{i>j} {\rm tr}\,|\varrho_{ij}| \geqslant {\rm tr}\,|\rho-\rho_d| \geqslant 2|\mathbf{q}-\mathbf{q}'|$, where ${\rm tr}\,|A| = {\rm tr}\sqrt{AA^\dagger}$ and $\varrho_{ij} := |j\rangle\langle j|\rho|i\rangle\langle i| + |i\rangle\langle i|\rho|j\rangle\langle j|$ and $\sum_{i\neq j}{\rm tr}\,|\varrho_{ij}| \geqslant {\rm tr}\,|\rho-\rho_d|$ is due to the convexity.

2. l_2 norm of coherence

The l_2 norm of coherence [12,14] has an operational interpretation as the state uncertainty [52] and is also employed to study nonclassical correlations [53]. The quantifier reads

$$\mathcal{C}_{l_2}(\rho) := \min_{\sigma\in\mathcal{I}} \mathcal{D}_{l_2}(\rho,\sigma) = \mathcal{D}_{l_2}(\rho,\rho_d),$$

where the l_2 norm or the (squared) Hilbert-Schmidt distance $\mathcal{D}_{l_2}(\rho,\sigma) := {\rm tr}(\rho-\sigma)^2$.

The UDR corresponding to the l_2 norm is

$$1 - \|\mathbf{p}\|_2^2 \geqslant \mathcal{D}_{l_2}(\rho,\rho_d),$$

which is due to ${\rm tr}(\rho-\rho_d)^2 = {\rm tr}(\rho^2 - \rho\cdot\rho_d) \leqslant 1 - \|\mathbf{p}\|_2^2$. Based on Eqs. (6) and (4), $\mathcal{C}_{l_2}(\rho)$ is bounded as

$$1 - \|\mathbf{p}\|_2^2 \geqslant \mathcal{C}_{l_2}(\rho) \geqslant \|\mathbf{q}-\mathbf{q}'\|_2^2, \tag{10}$$

where the lower bound is due to the data processing inequality with the measurement $\mathcal{B}$ being required to be projective [54].

3. Trace norm of coherence

The trace norm of coherence has an interpretation of interference visibility and reads [25]

$$\tilde{\mathcal{C}}_{\rm tr}(\rho) = \mathcal{D}_{\rm tr}(\rho,\rho_{\rm d}),$$

where the distance of the trace norm $\mathcal{D}_{\rm tr}(\rho,\sigma) = \frac{1}{2}{\rm tr}|\rho-\sigma|$. According to the UDR corresponding to this distance [37]

$$\sqrt{1-\|\mathbf{p}\|_2^2} \geqslant \mathcal{D}_{\rm tr}(\rho,\rho_{\rm d}),$$

we have

$$\sqrt{1-\|\mathbf{p}\|_2^2} \geqslant \tilde{\mathcal{C}}_{\rm tr}(\rho) \geqslant |\mathbf{q}-\mathbf{q}'|. \tag{11}$$

Again, the lower bound is due to the data processing inequality.

C. Convex-roof coherence of infidelity

Now we consider the convex-roof coherence of the infidelity [23]

$$\mathcal{C}'_{\rm if}(\rho) := \min_{\{f_i,|\phi_i\rangle\}} \textstyle\sum_i f_i \cdot \mathcal{C}_{\rm if}(\phi_i),$$

where the pure-state coherence is quantified as

$$\mathcal{C}_{\rm if}(\phi) = \min_{\sigma\in\mathcal{I}} \mathcal{D}_{\rm if}(\phi,\sigma),$$

with the infidelity $\mathcal{D}_{\rm if}(\rho,\sigma) := \sqrt{1-F(\rho,\sigma)}$ and $F(\rho,\sigma) = [{\rm Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})]^2$. The UDR corresponding to the infidelity is [37]

$$\sqrt{1-\|\mathbf{p}\|_2^2} \geqslant \mathcal{D}_{\rm if}(\rho,\rho_d).$$

By Eq. (7), the coherence measure acquires an upper bound as $\sqrt{1-\|\mathbf{p}\|_2^2}$.

To lower-bound the measure using Eq. (5), we note that $\mathcal{C}_{\rm if}(\phi) \geqslant \frac{\sqrt{2}}{2}\sqrt{1-\|\mathbf{p}\|_2^2} := \frac{\sqrt{2}}{2}\delta_{\rm if}(\mathbf{p})$ (see the Appendix), which leads to a definition of the g function as $g(x) = \frac{\sqrt{2}}{2}x$. Using Eq (5), we have $\mathcal{C}'_{\rm if}(\rho) \geqslant \frac{\sqrt{2}}{2}D_{\rm if}(\mathbf{q},\mathbf{q}')$, where $D_{\rm if}(\mathbf{q},\mathbf{q}') := \sqrt{1-(\sum_i\sqrt{q_i q'_i})^2}$. Thus, we finally have

$$\sqrt{1-\|\mathbf{p}\|_2^2} \geqslant \mathcal{C}'_{\rm if}(\rho) \geqslant \frac{\sqrt{2}}{2}D_{\rm if}(\mathbf{q},\mathbf{q}'). \tag{12}$$

Till now, we estimated many distance-based coherence quantifiers. In the following, we shall use the method to estimate the quantifiers going beyond the above definitions. The following quantifiers shall be specified by C instead of $\mathcal{C}$ for the sake of specification.

D. Coherence of Wigner-Yanase skew information and the convex-roof construction

Coherence can also be quantified based on quantum Fisher information, which is a basic concept in the field of quantum metrology that places the fundamental limit on the information accessible by a performing measurement on the quantum state. Two remarkable quantifiers are the Wigner-Yanase skew information of coherence and the convex-roof construction based on it.

1. Wigner-Yanase skew information of coherence

The Skew information coherence is a legitimate coherence measure and defined as [22]

$$C_s(\rho) = \sum_{1=j}^{d} I(\rho, |j\rangle\langle j|),$$

where $I(\rho, |j\rangle\langle j|) \equiv -\frac{1}{2}\,\mathrm{Tr}([\rho, |j\rangle\langle j|])^2$ represents the Wigner-Yanase skew information subject to the projector $|j\rangle\langle j|$. This measure was never estimated with a spectrum estimation method employing the standard overlap measurement technique [22], where one needs to perform $2d-2$ measurements to estimate the coherence of a d-dimensional system. In the following, our method can reduce the number to 3.

First, we can reexpress the measure as $C_s(\rho) = 1 - \sum_j \langle j|\sqrt{\rho}|j\rangle^2$ [22], then an upper bound readily follows as

$$\begin{aligned} C_s(\rho) &= 1 - \sum_j \langle j|\sqrt{\rho}|j\rangle^2 \leqslant 1 - \textstyle\sum_j \langle j|\rho|j\rangle^2 \\ &= 1 - \|\mathbf{p}\|_2^2. \end{aligned}$$

To derive a lower bound, we use an inequality $C_s(\rho) \geqslant \frac{1}{2}\mathcal{C}_{l_2}(\rho)$ [22] whose bound was already bounded in Eq. (10). Then, we have

$$1 - \|\mathbf{p}\|_2^2 \geqslant C_s(\rho) \geqslant \frac{1}{2}\|\mathbf{q} - \mathbf{q}'\|_2^2. \tag{13}$$

2. Convex-roof construction

With $C_s(\phi)$ quantifying the coherence of a pure state $|\phi\rangle$, the Wigner-Yanase skew information can lead to a convex-roof construction of coherence as [55]

$$C'_s(\rho) = \min_{\{f_i,\phi_i\}} \textstyle\sum_i f_i \cdot C_s(\phi_i).$$

This quantifier can be equivalently defined via the quantum Fisher information (up to an inessential factor) with respect to the reference measurement [55,56]

$$C'_s(\rho) = \frac{1}{4}\sum_j F(\rho, |j\rangle\langle j|),$$

where $F(\rho, |j\rangle\langle j|) := \sum_{k,l} 2\frac{(\lambda_k-\lambda_l)^2}{\lambda_k+\lambda_l}|c_{kl}|^2$ specifies the quantum information of ρ subject to the measurement projector $|j\rangle\langle j|$ and $c_{kl} = \langle\phi_k|j\rangle\langle j|\phi_l\rangle$ with $|\phi_n\rangle$ being the nth eigenvector of ρ and λ_n being the weight.

Given that $C_s(\phi) = 1 - \|\mathbf{p}(\phi)\|_2^2 = \delta_{\mathrm{if}}^2(\mathbf{p})$, we can define $g(\delta_{\mathrm{if}}) = \delta_{\mathrm{if}}^2$. By Eq. (5), the lower bound of $C'_s(\rho)$ follows as

$$C_s(\rho) \geqslant D_{\mathrm{if}}^2(\mathbf{q}, \mathbf{q}').$$

Due to the convexity of $1 - \|\mathbf{a}\|_2^2$ with respect to distribution $\mathbf{a}$, an upper bound is immediately obtained as

$$1 - \|\mathbf{p}\|_2^2 \geqslant C'_s(\rho).$$

Thus, the quantifier is bounded as

$$1 - \|\mathbf{p}\|_2^2 \geqslant C'_s(\rho) \geqslant D_{\mathrm{if}}^2(\mathbf{q}, \mathbf{q}'). \tag{14}$$

E. Robustness of coherence

One important coherence monotone is the robustness of coherence [28], which quantifies the advantage enabled by a quantum state in a phase discrimination task. For a given state ρ, it is defined as the minimal mixing required to make the state incoherent

$$C_{\mathrm{Ro}}(\rho) = \min_{\tau}\left\{ s \geqslant 0 \,\middle|\, \frac{\rho + s\tau}{1+s} := \sigma \in \mathcal{I} \right\},$$

where τ is a general quantum state. With the inequality [28]

$$C_{\mathrm{Ro}}(\rho) \geqslant \frac{\mathcal{D}_{l_2}(\rho, \rho_d)}{\|\rho_d\|_\infty}.$$

where we note that $\|\rho_d\|_\infty = \|\mathbf{p}\|_\infty$ is just the maximum element in $\mathbf{p}$. It follows from Eq. (10) that

$$C_{\mathrm{Ro}}(\rho) \geqslant \frac{\|\mathbf{q} - \mathbf{q}'\|_2^2}{\|\mathbf{p}\|_\infty}.$$

With the inequality $C_{\mathrm{Ro}}(\rho) \leqslant \mathcal{C}_{l_1}(\rho)$ and Eq. (9), we have

$$\|\mathbf{p}\|_{\frac{1}{2}} - 1 \geqslant C_{\mathrm{Ro}}(\rho) \geqslant \frac{\|\mathbf{q} - \mathbf{q}'\|_2^2}{\|\mathbf{p}\|_\infty}. \tag{15}$$

The robustness of coherence was previously detected based on the witness method [31–33], where the mathematics of constructing the witness and experimental setup were generally complex.

Finally, we summarize the obtained measurable lower bounds in Table I. These bounds are all formulated in terms of the statistics $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$. As the essential quantity, the disturbance D can readily be measured by performing a sequential measurements scheme, which was well developed in the study of error-disturbance relations [40–43]. In our approach, as the lower bounds are smooth functions of experimental statistics, the statistics errors due to imperfection of the implementation of measurement or state preparation are one order smaller compared to the lower bounds. These aspects make our protocol quite feasible.

It is also of practical interest to consider the tightness of these bounds. We find that the bounds of coherence measures $\mathcal{C}_{l_2}$ and $\tilde{\mathcal{C}}_{\mathrm{tr}}$ can be saturated for any input state when the test measurement $\mathcal{B}$ is taken as the eigenvectors of $\rho - \rho_d$, and the bounds for the convex-roof-based measure C'_s can be saturated if the concerned state is pure and the bound of $\mathcal{C}_{l_1}$ is tight in the qubit case (when $\mathcal{B}$ is taken as the eigenvectors of $\rho - \rho_d$). The bounds for other quantifiers either cannot be nontrivially saturated or only be saturated for some specific states.

TABLE I. Measurable lower bounds for coherence quantifiers.

Quantifier	$\mathcal{C}_{\rm re}$	$\mathcal{C}_{l_1}$	$\mathcal{C}_{l_2}$	$\tilde{\mathcal{C}}_{\rm tr}$	$\mathcal{C}'_{\rm if}$	$\mathcal{C}_s$	$\mathcal{C}'_s$	$\mathcal{C}_{\rm Ro}$
Upper bound	$H(\mathbf{p})$	$\\|\mathbf{p}\\|_{\frac{1}{2}}-1$	$1-\\|\mathbf{p}\\|_2^2$	$\sqrt{1-\\|\mathbf{p}\\|_2^2}$	$\sqrt{1-\\|\mathbf{p}\\|_2^2}$	$1-\\|\mathbf{p}\\|_2^2$	$1-\\|\mathbf{p}\\|_2^2$	$\\|\mathbf{p}\\|_{\frac{1}{2}}-1$
Lower bound	$H(\mathbf{q}\\|\mathbf{q}')$	$2\|\mathbf{q}-\mathbf{q}'\|$	$\\|\mathbf{q}-\mathbf{q}'\\|_2^2$	$\|\mathbf{q}-\mathbf{q}'\|$	$\frac{\sqrt{2}}{2}D_{\rm if}(\mathbf{q},\mathbf{q}')$	$\frac{1}{2}\\|\mathbf{q}-\mathbf{q}'\\|_2^2$	$D_{\rm if}^2(\mathbf{q},\mathbf{q}')$	$\frac{\\|\mathbf{q}-\mathbf{q}'\\|_2^2}{\\|\mathbf{p}\\|_\infty}$

IV. EFFICIENCY ARGUMENT

The previous coherence estimation protocols commonly apply to only a few coherence quantifiers. The collective measurement protocol [57] applies to the relative entropy of coherence and the l_2 norm of coherence. The witness method [31–33,58] applies to the robustness of coherence, the l_1 norm, and the l_2 norm of coherence. One quite simple and efficient method is based on spectrum estimation via majorization theory [35], which can be used to estimate the l_1 norm, the l_2 norm of coherence, the robustness of coherence, and the relative entropy of coherence. This method employs a similar measurement scheme to ours. In the following, we compare our approach with it.

A. Comparison with the method based on spectrum estimation

The spectrum estimation method is based on the theory of majorization [35]. A probability $\mathbf{a}$ is said to majorize a probability distribution $\mathbf{b}$, specified as $\mathbf{a}\succeq\mathbf{b}$, if their elements satisfy $\sum_{i=1}^k a_i^\downarrow \geqslant \sum_{i=1}^k b_i^\downarrow\ \forall k$, where the superscript means that the elements are arranged in a descending order, namely, $\mathbf{a}^\downarrow=(a_1^\downarrow,a_2^\downarrow,\ldots,\ a_d^\downarrow)$, $\mathbf{b}^\downarrow=(b_1^\downarrow,b_2^\downarrow,\ldots,\ b_d^\downarrow)$ with $a_i\geqslant a_{i+1}$ and $b_i\geqslant b_{i+1}$. Clearly, the spectrum of ρ, specified as $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\ \lambda_d)$, majorizes distribution from any projection measurement, say $\mathcal{B}$, performed on ρ, i.e., $\boldsymbol{\lambda}\succeq\mathbf{q}$. By the Shur convexity theorem, $H(\mathbf{a})\geqslant H(\mathbf{b})$ if $\mathbf{b}\succeq\mathbf{a}$. Thus, $\mathcal{C}_r(\rho)=H(\mathbf{p})-H(\boldsymbol{\lambda})\geqslant H(\mathbf{p})-H(\mathbf{q})$ with $\mathbf{p}$ and $\mathbf{q}$ being distributions from the reference measurement and $\mathcal{B}$, which provides a nontrivial lower bound if $\mathbf{q}\succ\mathbf{p}$. Generally, as the state of interest is unknown one needs to try a few settings of $\mathcal{B}$ to ensure $\mathbf{q}\succ\mathbf{p}$. In this paper, we deal with the disturbance effect, namely, $D(\mathbf{q},\mathbf{q}')$, which is zero iff $\rho-\rho_d$ is perpendicular with all the elements of $\mathcal{B}$ simultaneously. The settings resulting in such a failure lie in a space of measure zero. Therefore, our method almost always works even if the $\mathcal{B}$ is chosen arbitrarily. As a simple illustration, assume that ρ is given as $|\phi\rangle=\sin\frac{\pi}{8}|0\rangle+\cos\frac{\pi}{8}|1\rangle$ and the reference basis is $\{|0\rangle,|1\rangle\}$, immediately, $\mathbf{p}=\{\sin^2\frac{\pi}{8},\cos^2\frac{\pi}{8}\}$. By the spectrum estimation method, a nontrivial estimation, namely, $H(\mathbf{q})-H(\mathbf{p})>0$ requires that $|\langle\phi|\mathcal{B}|\phi\rangle|>\cos^2\frac{\pi}{8}-\sin^2\frac{\pi}{8}=\frac{\sqrt{2}}{2}$. With our method, $D(\mathbf{q},\mathbf{q}')\neq 0$ requires that $\mathcal{B}\neq\sigma_z,\sigma_y$, which is much weaker than the above requirement.

TABLE II. Efficiencies in estimating coherence of qubit states.

Measure	$\mathcal{C}_{\rm re}$	$\mathcal{C}_{l_1}$	$\mathcal{C}_{l_2}$	$\tilde{\mathcal{C}}_{\rm tr}$	$\mathcal{C}'_{\rm if}$	$\mathcal{C}_s$	$\mathcal{C}'_s$	$\mathcal{C}_{\rm Ro}$
Q_D^C	0.36	$\frac{2}{\pi}$	$\frac{1}{2}$	$\frac{2}{\pi}$	0.22	0.25	0.31	0.28
Q_S^C	0.17	0.29	0.20	–	–	–	–	0.20

B. Numerical results for the qubit case

The lower bounds provided in Table I work very well. As the second illustration of efficiency, we consider the qubit case. For the sake of computability, we let ρ be a pure state. How well a quantifier is estimated can be naturally quantified with the ratio of the estimate to the exact value. For each quantifier, we calculate the average of the ratio over the randomly chosen measurement $\mathcal{B}$ and randomly chosen pure state (see the Appendix for details) with two different methods, i.e., our method and the one based on spectrum estimation. The averages are listed in the Table II, where the one based on our method is specified by Q_D^C and by Q_S^C the other method. It can be seen that our method enjoys wider applicability and higher efficiency.

V. CONCLUSION

In conclusion, we provided a universal and straightforward method to estimate coherence. It enables us to give measurable bounds for many quantifiers of general interest, where all the bounds are expressed as functions of the experimentally accessible data $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$ without involving cumbersome mathematics. This is advantageous over the previous methods, which only applies to one or a few measures and cannot apply to the quantifiers based on convex-roof construction. Our approach exhibits many desired features: experimentally friendly, broad applicability, and mathematical simplicity, and therefore serves as an efficient coherence-detecting method.

For possible further research in the quantum foundation, we note that the disturbance effect is one basic concept in the quantum foundation that closely relates to nonlocality, the uncertainty principle, and the security of quantum cryptography. Thus, the framework may inspire alternative connection among these concepts, for example, nonlocality and coherence.

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APPENDIX

1. Proof of Eq. (5)

For any pure-state ensemble $\rho=\sum_i f_i'|\phi_i'\rangle\langle\phi_i'|$, we have $\sum_i f_i'\cdot\mathbf{q_i}=\mathbf{q}$ and $\sum_i f_i'\cdot\mathbf{q_i}'=\mathbf{q}'$. For the ensemble $\sum_i f_i\cdot|\phi_i\rangle\langle\phi_i|$ that achieves the minimal of the convex-roof-based

TABLE III. Comparing the efficiencies of our method and the one based on the spectrum estimation.

Measure	$\mathcal{C}_{\text{re}}$	$\mathcal{C}_{l_1}$	$\mathcal{C}_{l_2}$	$\mathcal{C}'_{\text{if}}$	$\tilde{\mathcal{C}}_{\text{tr}}$	$\mathcal{C}'_s$	$\mathcal{C}_s$	$\mathcal{C}_{\text{Ro}}$
Exact value $C(\phi)$	$\mathrm{H}(\sin^2\frac{\theta}{2})$	$\sin\theta$	$\frac{1}{2}\sin^2\theta$	$\min\{\sin\frac{\theta}{2},\cos\frac{\theta}{2}\}$	$\frac{1}{2}\sin^2\theta$	$\frac{1}{2}\sin^2\theta$	$\sin\theta$	$\sin\theta$
Q_D^C	0.36	$\frac{2}{\pi}$	$\frac{1}{2}$	$\frac{2}{\pi}$	0.22	0.25	0.31	0.28
Q_M^C	0.17	0.29	0.20	–	–	–	–	0.20
$Q_M'^C$	0.20	0.32	0.22	–	–	–	–	0.22

measure $\mathcal{C}'(\rho)$, we have

$$\begin{aligned}\mathcal{C}'(\rho) &= \sum_i f_i\cdot\mathcal{C}(\phi_i) \geqslant \sum_i f_i\cdot g(\delta_{\mathcal{D}}(\mathbf{p}_i))\\ &\geqslant \sum_i f_i\cdot g(D(\mathbf{q}_i,\mathbf{q}'_i)) \geqslant g\left(\sum_i f_i\cdot D(\mathbf{q}_i,\mathbf{q}'_i)\right)\\ &\geqslant g(D(\mathbf{q},\mathbf{q}')).\end{aligned} \tag{A1}$$

The first inequality is due to the definition of the g function. The second inequality is due to UDRs, which relax a concave quantity $\delta_{\mathcal{D}}(\mathbf{p})$ into a convex one $D(\mathbf{q},\mathbf{q}')$. The third is due to the convexity of $g(\cdot)$. The fourth is due to the convexity of the classical distance D and the monotonicity assumption of $g(\cdot)$.

2. Proof of Eq. (12)

To lower-bound the coherence measure, let us first consider a pure-state case, for example, $|\phi\rangle=\sum_i\sqrt{p_i(\phi)}e^{i\psi_i}|i\rangle$, where we have

$$\mathcal{C}_{\text{if}}(\phi)=\min_{\sigma\in\mathcal{I}}\mathcal{D}_{\text{if}}(\rho,\sigma)=\sqrt{1-p},$$

where $p=\max_i\{|p_0(\phi)|,\dots,|p_{d-1}(\phi)|\}$. Note that

$$\sqrt{2(1-p)}\geqslant\sqrt{(1+p)(1-p)}\geqslant\sqrt{1-\|\mathbf{p}(\phi)\|_2^2}.$$

The UDR corresponding to infidelity is

$$\sqrt{1-\|\mathbf{p}\|_2^2}\geqslant D_{\text{if}}(\mathbf{q}(\phi),\mathbf{q}'(\phi_d)),$$

where $D_{\text{if}}[\mathbf{q}(\phi),\mathbf{q}'(\phi_d)]:=\sum_j\sqrt{q_jq'_j}$ is classical infidelity. Then we have $\mathcal{C}_{\text{if}}(\phi)\geqslant\frac{\sqrt{2}}{2}\sqrt{1-\|\mathbf{p}\|_2^2}:=\frac{\sqrt{2}}{2}\delta_{\text{if}}(\mathbf{p})$, which leads to a definition of the g function as $g(x)=\frac{\sqrt{2}}{2}x$. The lower bound of $\mathcal{C}'_{\text{if}}(\rho)$ follows from Eq. (A1)

$$\mathcal{C}'_{\text{if}}(\rho)\geqslant\frac{\sqrt{2}}{2}D_{\text{if}}(\mathbf{q},\mathbf{q}').$$

To obtain an upper bound, we use Eq. (7) and the UDR $D_{\text{if}}(\rho,\rho_d)\leqslant\sqrt{1-\|\mathbf{p}\|_2^2}$ then we have

$$\mathcal{C}'_{\text{if}}(\rho)\leqslant\sum_i f_i\cdot\sqrt{1-\|\mathbf{p}_i\|^2}\leqslant\sqrt{1-\|\mathbf{p}\|_2^2},$$

where we used the concavity of $\sqrt{1-\|\mathbf{p}\|_2^2}$.

3. Tightness argument

For the sake of computability, we consider an arbitrary pure state $\rho=|\phi\rangle\langle\phi|$ with $|\phi\rangle=\sin\frac{\theta}{2}|0\rangle+\cos\frac{\theta}{2}e^{i\psi}|1\rangle$ and $0\leqslant\theta\leqslant\pi$ and $0\leqslant\psi\leqslant2\pi$. The exact value of the coherence is specified by $C(\phi)$. We choose the test measurement $\mathcal{B}$ as the one maximally incompatible with the reference measurement. Its setting thus is determined up to a relative phase as $\{\frac{1}{\sqrt{2}}|0\rangle+\frac{e^{i\psi'}}{\sqrt{2}}|1\rangle;\frac{1}{\sqrt{2}}|0\rangle-\frac{e^{i\psi'}}{\sqrt{2}}|1\rangle\}$, and none of them is of priority. We average the ratio $\frac{L^C_{D,\mathcal{B}}}{C(\phi)}$ over all the possible $\mathcal{B}$ as

$$\frac{\bar{L}^C_{D,\mathcal{B}}}{C(\phi)}:=\frac{1}{2\pi}\int_0^{2\pi}\frac{L^C_{D,\mathcal{B}}}{C(\phi)}d\psi',$$

where one lower bound $L^C_{D,\mathcal{B}}$ is given under the choice of $\mathcal{B}$. As the state of interest is unknown, we access the average performance of the protocol over all the pure states as $\frac{\bar{L}^C_{D,\mathcal{B}}}{C(\phi)}$.

$$Q^C_D:=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\bar{L}^C_{D,\mathcal{B}}}{C(\phi)}\sin\theta d\theta d\psi.$$

The Q^C_D for the various quantifiers are calculated and listed in Table III. Using the above maximally incompatible measurement implies a greater disturbance for $\mathcal{C}_{l_1}$, $\mathcal{C}_{l_2}$, $\tilde{\mathcal{C}}_{\text{tr}}$, $\mathcal{C}_s$, and $\mathcal{C}_{\text{Ro}}$, than using a random choice of the test measurement $\mathcal{B}$ along direction $\vec{B}=\{\sin\alpha\sin\psi'',\sin\alpha\cos\psi'',\cos\alpha\}$ with $0\leqslant\alpha\leqslant\pi$ and $0\leqslant\psi''\leqslant2\pi$. This is because typical quantities such as $|\mathbf{q}-\mathbf{q}'|$ or $\|\mathbf{q}-\mathbf{q}'\|_2^2$ involved in lower bounds, which are $|\sin\alpha\sin\theta\cos(\psi'-\psi'')|$ or $\frac{1}{2}|\sin\alpha\sin\theta\cos(\psi'-\psi'')|^2$, respectively, attain their maximums at $\alpha=\frac{\pi}{2}$ and $\psi'=\psi''$. We also calculate Q^C_D for the other three quantifiers, namely, $\mathcal{C}_{\text{re}}$, $\mathcal{C}'_{\text{if}}$, and $\mathcal{C}'_s$ using the general choice of $\mathcal{B}$, and obtain the average lower bounds as 0.266, 0.365, and 0.234.

We also calculate the average performance Q^C_M using the method based on the spectrum estimation. One key difference is that the test measurement $\mathcal{B}$ does not need to be the one maximum incompatible with the reference measurement. This is because, in the protocol based on spectrum estimation, the closer the setting of $\mathcal{B}$ to the eigenvectors of the state of interest, the better the estimate. However, the state of interest is unknown and so are its eigenvectors. Therefore, there is no reason to choose the test measurement $\mathcal{B}$ as the one maximumally incompatible with the reference measurement. According to this protocol, $\mathcal{B}$ is taken randomly as $\mathcal{B}=\vec{\sigma}\cdot\vec{B}$ and $\vec{B}=\{\sin\alpha\sin\psi'',\sin\alpha\cos\psi'',\cos\alpha\}$ with $0\leqslant\alpha\leqslant\pi$ and $0\leqslant\psi''\leqslant2\pi$ with $\vec{\sigma}$ being the Pauli matrix. The average performance is defined as

$$Q^C_M:=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\bar{L}^C_{M,\mathcal{B}}}{C(\phi)}\sin\theta d\theta d\psi,$$

with $\frac{\bar{L}^C_{M,\mathcal{B}}}{C(\phi)}$ being defined as

$$\frac{\bar{L}^C_{M,\mathcal{B}}}{C(\phi)}:=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{L^C_{M,\mathcal{B}}}{C(\phi)}\sin\alpha d\alpha d\psi''.$$

We calculated the coherence quantifiers to which this method is applicable. As a comparison, we also calculated the performance when $\mathcal{B}$ takes the measurement maximally incompatible with the reference measurement, which is specified by $Q_M'^C$. By Table III, it is shown that our method enjoys higher efficiency and wider applicability.

[1] I. Marvian and R. W. Spekkens, Nat. Commun. **5**, 3821 (2014).
[2] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[3] E. Chitambar and M.-H. Hsieh, Phys. Rev. Lett. **117**, 020402 (2016).
[4] H. Zhu, M. Hayashi, and L. Chen, Phys. Rev. A **97**, 022342 (2018).
[5] M.-L. Hu, X.-M. Wang, and H. Fan, Phys. Rev. A **98**, 032317 (2018).
[6] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, Phys. Rev. Lett. **116**, 160407 (2016).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photon. **5**, 222 (2011).
[8] P. Giorda and M. Allegra, J. Phys. A: Math. Theor. **51**, 025302 (2018).
[9] H. Kwon, K. C. Tan, T. Volkoff, and H. Jeong, Phys. Rev. Lett. **122**, 040503 (2019).
[10] B. Huttner, N. Imoto, N. Gisin, and T. Mor, Phys. Rev. A **51**, 1863 (1995).
[11] J. Ma, Y. Zhou, X. Yuan, and X. Ma, Phys. Rev. A **99**, 062325 (2019).
[12] A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. **89**, 041003 (2017).
[13] E. Chitambar and G. Gour, Phys. Rev. A **94**, 052336 (2016).
[14] E. Chitambar and G. Gour, Phys. Rev. Lett. **117**, 030401 (2016).
[15] E. Chitambar and G. Gour, Rev. Mod. Phys. **91**, 025001 (2019).
[16] J. Åberg, Phys. Rev. Lett. **113**, 150402 (2014).
[17] M.-L. Hu, X. Hu, J. Wang, Y. Peng, Y.-R. Zhang, and H. Fan, Phys. Rep. **762–764**, 1 (2018).
[18] D. K. L. Oi and J. Åberg, Phys. Rev. Lett. **97**, 220404 (2006).
[19] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[20] A. Winter and D. Yang, Phys. Rev. Lett. **116**, 120404 (2016).
[21] A. E. Rastegin, Phys. Rev. A **93**, 032136 (2016).
[22] C.-S. Yu, Phys. Rev. A **95**, 042337 (2017).
[23] C. L. Liu, D.-J. Zhang, X.-D. Yu, Q.-M. Ding, and L. Liu, Quantum Inf. Proc. **16**, 198 (2017).
[24] Z.-W. Liu, X. Hu, and S. Lloyd, Phys. Rev. Lett. **118**, 060502 (2017).
[25] X.-D. Cui, C. L. Liu, and D. M. Tong, Phys. Rev. A **102**, 022420 (2020).
[26] T. Biswas, M. García-Díaz, and A. Winter, Proc. R. Soc. London, Ser. A **473**, 20170170 (2017).
[27] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A **92**, 022124 (2015).
[28] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Phys. Rev. Lett. **116**, 150502 (2016).
[29] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[30] X.-D. Yu, D.-J. Zhang, G. F. Xu, and D. M. Tong, Phys. Rev. A **94**, 060302(R) (2016).
[31] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, Phys. Rev. A **93**, 042107 (2016).
[32] Y.-T. Wang, J.-S. Tang, Z.-Y. Wei, S. Yu, Z.-J. Ke, X.-Y. Xu, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. **118**, 020403 (2017).
[33] W. Zheng, Z. Ma, H. Wang, S.-M. Fei, and X. Peng, Phys. Rev. Lett. **120**, 230504 (2018).
[34] D.-J. Zhang, C. L. Liu, X.-D. Yu, and D. M. Tong, Phys. Rev. Lett. **120**, 170501 (2018).
[35] X.-D. Yu and O. Gühne, Phys. Rev. A **99**, 062310 (2019).
[36] L.-L. Sun, Y.-S. Song, S. Yu, and Z.-B. Chen, Phys. Rev. A **106**, 032213 (2022).
[37] L.-L. Sun, K. Bharti, Y.-L. Mao, X. Zhou, L.-C. Kwek, J. Fan, and S. Yu, arXiv:quant-ph/2202.07251.
[38] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Phys. Rev. A **91**, 042120 (2015).
[39] S. Rana, P. Parashar, and M. Lewenstein, Phys. Rev. A **93**, 012110 (2016).
[40] G. J. Pryde, J. L. O'Brien, A. G. White, T. C. Ralph, and H. M. Wiseman, Phys. Rev. Lett. **94**, 220405 (2005).
[41] F. Kaneda, S.-Y. Baek, M. Ozawa, and K. Edamatsu, Phys. Rev. Lett. **112**, 020402 (2014).
[42] M. M. Weston, M. J. W. Hall, M. S. Palsson, H. M. Wiseman, and G. J. Pryde, Phys. Rev. Lett. **110**, 220402 (2013).
[43] B. Marques, J. Ahrens, M. Nawareg, A. Cabello, and M. Bourennane, Phys. Rev. Lett. **113**, 250403 (2014).
[44] J. Åberg, arXiv:quant-ph/0612146.
[45] Y. Liu, Q. Zhao, and X. Yuan, J. Phys. A: Math. Theor. **51**, 414018 (2018).
[46] X. Yuan, Q. Zhao, D. Girolami, and X. Ma, Adv. Quantum Technol. **2**, 1900053 (2019).
[47] M. Hillery, Phys. Rev. A **93**, 012111 (2016).
[48] H.-L. Shi, S.-Y. Liu, X.-H. Wang, W.-L. Yang, Z.-Y. Yang, and H. Fan, Phys. Rev. A **95**, 032307 (2017).
[49] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Phys. Rev. A **92**, 012118 (2015).
[50] S. Cheng and M. J. W. Hall, Phys. Rev. A **92**, 042101 (2015).
[51] X. Yuan, G. Bai, T. Peng, and X. Ma, Phys. Rev. A **96**, 032313 (2017).
[52] Y. Sun and S. Luo, Phys. Rev. A **103**, 042423 (2021).
[53] Y. Sun, Y. Mao, and S. Luo, Europhys. Lett. **118**, 60007 (2017).
[54] M. Ozawa, Phys. Lett. A **268**, 158 (2000).
[55] L. Li, Q.-W. Wang, S.-Q. Shen, and M. Li, Phys. Rev. A **103**, 012401 (2021).
[56] S. Yu, arXiv:quant-ph/1302.5311.
[57] Y. Yuan, Z. Hou, J.-F. Tang, A. Streltsov, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, npj Quantum Inf. **6**, 46 (2020).
[58] Z. Ma, Z. Zhang, Y. Dai, Y. Dong, and C. Zhang, Phys. Rev. A **103**, 012409 (2021).