Applications of the Hillery-Zubairy entanglement criteria to N-qubit systems: The Tavis-Cummings model

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We consider the application of the entanglement criteria derived by Hillery and Zubairy [Phys. Rev. Lett. **96**, 050503 (2006)] to the detection of entanglement in N-qubit systems. For $N = 2$ qubits we show that, with the natural choice of operators, one of the criteria never detects entanglement; we also derive conditions for the other criterion to work and for it to have a simple relation to the negativity when it does. For general angular momenta we show the Hillery-Zubairy relations can always detect the entanglement of the (pure) states of well-defined total $(\mathbf{J}, J_z)$ if the "test" operators are chosen appropriately. We then show how this may be used, in particular, to develop useful criteria to detect entanglement in a system of N two-level atoms interacting with a field initially in a number state (the Tavis-Cummings model).

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I. INTRODUCTION AND MOTIVATION

Entanglement is a valuable quantum resource, and hence it is useful to have ways to determine experimentally whether two quantum systems are entangled, particularly if this can be done only through measurements carried out on the separate systems. Several criteria were developed to this end over the years; see, e.g., [1–13] for a partial list. Most of these only detected certain types of entanglement and many are only sufficient, but not necessary, conditions for entanglement. This means, in general, one must resort to several different criteria to get as complete a picture as possible of what is going on in the system under study.

In this paper we focus on the two entanglement conditions derived by Hillery and Zubairy in [10], initially for two modes of the radiation field, but since then extended to other systems [12,14]. For a bipartite system, if A and B are non-Hermitian operators, each acting on one of the two subsystems, the joint state is entangled if either one of the following conditions holds:

$$|\langle AB^\dagger\rangle|^2 > \langle A^\dagger AB^\dagger B\rangle, \tag{1a}$$

$$|\langle AB\rangle|^2 > \langle A^\dagger A\rangle\langle B^\dagger B\rangle. \tag{1b}$$

Although sufficient, these conditions are, in general, not necessary for entanglement. In applications to finite-dimensional systems, it was generally found that Eq. (1a) is the more useful of the two criteria, and we will show this to be the case here as well.

Generally speaking, the goal of this paper is to explore the usefulness of the criteria (1) to detect entanglement in systems involving a finite number of qubits. We start by looking at the simplest case, a two-qubit system, to develop an intuition for the way the criterion (1) detects entanglement and for when it fails to do so; we also discuss the similarities and differences with a necessary and sufficient criterion that was presented in [15]. We then use these insights to propose specific choices for the operators A and B to detect different kinds of entanglement in spin systems, extending the results of [14]. Finally, we illustrate their usefulness with one specific example, namely, by looking at entanglement between the atoms in the Tavis-Cummings model with the field initially in a number state. Here again we compare the results obtained with our criteria to others that were derived previously, in particular in [3,7,13].

II. TWO QUBITS

A. Density matrix parametrization

We start with the simplest case of two qubits, or spin-$\frac{1}{2}$ particles, where we will use the standard quantum information convention of identifying the state $|0\rangle$ with the positive-eigenvalue eigenstate of σ_z. For the operators A and B we will consider initially either $\sigma_+ = (\sigma_x + i\sigma_y)/2$ or $\sigma_- = (\sigma_x - i\sigma_y)/2$ and later their transformations by local unitaries.

We will consider in this section only density operators of the following form:

$$\rho = \frac{1}{4}\begin{pmatrix} 1+t_3 & 0 & 0 & t_1-t_2 \\ 0 & 1-t_3 & t_1+t_2 & 0 \\ 0 & t_1+t_2 & 1-t_3 & 0 \\ t_1-t_2 & 0 & 0 & 1+t_3 \end{pmatrix}$$
$$= \lambda_1|\Psi^+\rangle\langle\Psi^+| + \lambda_2|\Phi^+\rangle\langle\Phi^+| + \lambda_3|\Phi^-\rangle\langle\Phi^-| + \lambda_4|\Psi^-\rangle\langle\Psi^-| \tag{2}$$

with $\lambda_1 = \frac{1}{4}(1 + t_1 + t_2 - t_3)$, $\lambda_2 = \frac{1}{4}(1 + t_1 - t_2 + t_3)$, and $\lambda_3 = \frac{1}{4}(1 - t_1 + t_2 + t_3)$, $\lambda_4 = \frac{1}{4}(1 - t_1 - t_2 - t_3)$. The quantities t_i may be assumed to be ordered so that

$$1 \geqslant t_1 \geqslant t_2 \geqslant |t_3|, \tag{3}$$

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and the $|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle$ are the Bell states. The condition (3) ensures all the eigenvalues λ_i are nonnegative, except possibly the last one, which requires the additional condition

$$t_1 + t_2 \leqslant 1 - t_3. \tag{4}$$

The matrix (2) is written in the standard basis. While not the most general form possible for a two-qubit system, it was shown in [16] that most density matrices can be transformed into a form essentially equivalent by local Lorentz transformations acting on each qubit separately. Special instances of density operators of the form (2) have also been shown to have interesting properties, such as maximizing entanglement for a given degree of mixedness [17]; they are, in turn, a subset of states that were originally studied in that context by Ishizaka and Hiroshima [18] and which came to be called later "X states" [19].

It is easy to see that the eigenvalues λ'_i of the partial transpose matrix ρ^{T_B} are $\lambda'_1 = \frac{1}{4}(1 + t_1 + t_2 + t_3)$, $\lambda'_2 = \frac{1}{4}(1 + t_1 - t_2 - t_3)$, $\lambda'_3 = \frac{1}{4}(1 - t_1 + t_2 - t_3)$, and $\lambda'_4 = \frac{1}{4}(1 - t_1 - t_2 + t_3)$. Because of the assumption (3), only the last of these eigenvalues can be negative, which means that the negativity [20] of the state (2) is given by

$$\begin{aligned} N(\rho) &= \|\rho^{T_B}\|_1 - 1 = \tfrac{1}{2}(|\lambda'_4| - \lambda'_4) \\ &= \max\{0, \tfrac{1}{4}(t_1 + t_2 - 1 - t_3)\}. \end{aligned} \tag{5}$$

Put differently, the state (2) is entangled if and only if

$$t_1 + t_2 > 1 + t_3. \tag{6}$$

Note that this condition, together with Eq. (4), forces t_3 to be negative.

B. Hillery-Zubairy criteria with A, B lowering operators

We can now check how well the Hillery-Zubairy conditions do at detecting this entanglement. Starting with the choice $A = \sigma_{-,a} = |1\rangle_a\langle 0|$ and $B = \sigma_{-,b} = |1\rangle_b\langle 0|$, it is easy to see that

$$\langle AB^\dagger\rangle = \mathrm{Tr}(\rho AB^\dagger) = \mathrm{Tr}(\rho|10\rangle\langle 01|) = \tfrac{1}{4}(t_1 + t_2), \tag{7a}$$

$$\begin{aligned} \langle A^\dagger A\rangle &= \mathrm{Tr}(\rho|0\rangle_a\langle 0|) = \langle 00|\rho|00\rangle + \langle 01|\rho|01\rangle \\ &= \tfrac{1}{2} = \langle B^\dagger B\rangle, \end{aligned} \tag{7b}$$

$$\langle A^\dagger AB^\dagger B\rangle = \mathrm{Tr}(\rho|00\rangle\langle 00|) = \tfrac{1}{4}(1 + t_3). \tag{7c}$$

From Eqs. (3), (7a), and (7b), it follows that the second condition, Eq. (1b), can, in fact, never happen in the two-qubit system. On the other hand, the first condition, Eq. (1a), is equivalent to $\frac{1}{16}(t_1 + t_2)^2 > \frac{1}{4}(1 + t_3)$, or

$$t_1 + t_2 > 2\sqrt{1 + t_3}. \tag{8}$$

Since t_3 is always a number between -1 and 1, one always has $2\sqrt{1 + t_3} \geqslant 1 + t_3$, and so Eq. (8) always implies, but is not equivalent to, Eq. (6); that is to say, there are entangled states [satisfying Eq. (6)] that do not satisfy Eq. (8), and therefore are not detected by the entanglement criterion (1a).

Moreover, the quadratic dependence of $|\langle AB^\dagger\rangle|^2$ on $t_1 + t_2$, compared to the negativity (5), means that $|\langle AB^\dagger\rangle|^2 - \langle A^\dagger AB^\dagger B\rangle$ is not an entanglement monotone. For example, suppose you change $t_1 \to t_1 + \epsilon$ and $t_3 \to t_3 + 0.9\epsilon$. Then, if $\epsilon > 0$ the negativity (and hence the entanglement) increases by an amount 0.1ϵ, but $|\langle AB^\dagger\rangle|^2 - \langle A^\dagger AB^\dagger B\rangle$ changes as

$$\begin{aligned} &\frac{1}{16}(t_1 + t_2 + \epsilon)^2 - \frac{1}{4}(1 + t_3 + 0.9\epsilon) \\ &\quad = \frac{1}{16}(t_1 + t_2)^2 - \frac{1}{4}(1 + t_3) + \frac{\epsilon}{4}\left[\frac{1}{2}(t_1 + t_2) + \frac{\epsilon}{4} - 0.9\right], \end{aligned} \tag{9}$$

and the quantity in the square brackets on the right-hand side can be positive or negative depending on the actual value of ϵ (note an overall negative value is perfectly possible since $t_1 + t_2$ can be as small as 1.6 and still register entanglement).

Despite this negative general result, we find there are special situations (typically involving families of states depending on fewer parameters, or with some symmetries) for which the difference $|\langle AB^\dagger\rangle|^2 - \langle A^\dagger AB^\dagger B\rangle$ is an entanglement monotone. A couple of examples are presented in the next subsection.

C. Optimizing the choice of A and B via local unitaries

The results from the previous subsection depend on the density matrix adopting a certain form (which, in general, can only be achieved after performing suitable unitary, local transformations on both subsystems) and also on special choices for the operators A and B. It is natural to ask if some other choices of A and B, related to σ_- by local unitary transformations, could be better at detecting the entanglement of states like Eq. (2), for given values of the parameters t_i. Specifically, we want to look for local unitary operators U_a and U_b to apply to σ_-, so that the inequalities (1) will be satisfied for entangled states by the transformed operators

$$\begin{aligned} A &= U_a^\dagger|1\rangle_a\langle 0|U_a = \begin{pmatrix} -\cos\psi\sin\psi & \cos^2\psi \\ -\sin^2\psi & \cos\psi\sin\psi \end{pmatrix}, \\ B &= U_b^\dagger|1\rangle_b\langle 0|U_b = \begin{pmatrix} -\cos\theta\sin\theta & \cos^2\theta \\ -\sin^2\theta & \cos\theta\sin\theta \end{pmatrix}. \end{aligned} \tag{10}$$

We consider only real transformations since the density matrix form (2) is itself real, and therefore parametrize them by just two angles, ψ and θ. After some algebra, we obtain

$$\begin{aligned} \langle AB^\dagger\rangle &= \tfrac{1}{4}(t_2 + t_1\cos 2\psi\cos 2\theta + t_3\sin 2\psi\sin 2\theta) \\ &= \langle A^\dagger B\rangle, \end{aligned} \tag{11a}$$

$$\langle A^\dagger A\rangle = \tfrac{1}{2} = \langle B^\dagger B\rangle, \tag{11b}$$

$$\begin{aligned} \langle A^\dagger AB^\dagger B\rangle &= \tfrac{1}{4}(1 + t_3\cos 2\psi\cos 2\theta + t_1\sin 2\psi\sin 2\theta) \\ &= \langle AA^\dagger BB^\dagger\rangle. \end{aligned} \tag{11c}$$

Once again, it is easy to see from Eqs. (11) that the second criterion, inequality (1b), never holds, since Eq. (11a) is maximized by $(t_1 + t_2)/4$, which is always $\leqslant \frac{1}{2}$. On the other hand, it is also easy to see that, even with this rather general form of the operators A and B, the criterion (1a) still fails to detect some of the entangled states of the form (2). For a specific example, take $t_1 = t_2 = -t_3 = \frac{1}{2}$. This gives an entangled state, with negativity $\frac{1}{8}$ [according to Eq. (5)], yet $\langle AB^\dagger\rangle^2 = (1 + \cos[2(\psi + \theta)])^2/64 < \langle A^\dagger AB^\dagger B\rangle = (1 - \frac{1}{2}\cos[2(\psi + \theta)])/4$.

On the other hand, one may indeed use the transformed operators (10) to detect entanglement in states that do not necessarily satisfy the inequality (8). A particularly interesting choice is to make $\sin 2\psi \sin 2\theta = -1$, which yields the condition for entanglement

$$t_2 - t_3 > 2\sqrt{1 - t_1}. \tag{12}$$

Like Eq. (8), this implies Eq. (6), but is not equivalent to it, and since it is also different from Eq. (8) it can be used to identify a different set of entangled states. This set includes states that are an incoherent superposition of only two Bell states, for which, with the parametrization (2) and the condition (3), one must have $t_1 = 1$ and $t_2 = -t_3$, in which case one has

$$\langle AB^\dagger\rangle^2 - \langle A^\dagger AB^\dagger B\rangle = \frac{t_2^2}{4} = N(\rho)^2. \tag{13}$$

That is, in this case the quantity $\langle AB^\dagger\rangle^2 - \langle A^\dagger AB^\dagger B\rangle$ is indeed an entanglement monotone, equivalent to the square of the negativity.

Another example of a family of states for which Eq. (1a) yields an entanglement monotone is provided by the reduced-rank density operators considered in [16] and given by

$$\rho = \frac{1}{2a}\begin{pmatrix} a+c & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b-c & 0 \\ d & 0 & 0 & a-b \end{pmatrix}, \tag{14}$$

which do not belong to the same equivalence class as Eq. (2) under local Lorentz transformations and are entangled if and only if $d \neq 0$, with negativity

$$N(\rho) = \frac{a}{4}(\sqrt{4d^2 + (b-c)^2} - (b-c)). \tag{15}$$

Choosing $A = |1\rangle_a\langle 0|$ and $B = |0\rangle_b\langle 1|$, it is easy to see that $\langle A^\dagger AB^\dagger B\rangle = 0$ and $\langle AB^\dagger\rangle| = |d/2a|$.

D. Building a necessary and sufficient criterion: A physical interpretation

The reason the conditions (8) and (12) do not detect all the entangled states of the form (2) is because they are too strong: to have entanglement it is enough to have $t_1 + t_2$ greater than $1 + t_3$, it is not necessary for it to be also larger than $2\sqrt{1 + t_3}$. Equations (7) then suggest, therefore, that a better criterion than Eq. (1) for a system of two qubits might be simply

$$|\langle AB^\dagger\rangle| > \langle A^\dagger AB^\dagger B\rangle, \tag{16}$$

where, for systems described by the density matrix (2), the optimal choice of A and B is $A = B = \sigma_+$. Unfortunately, while Eq. (16) is both a necessary and sufficient entanglement condition for systems whose density matrix is of the form (2), it is not necessary in general, since it is easy to find product states that also satisfy it. For example, letting $|\psi\rangle = (\cos\theta|0\rangle + \sin\theta|1\rangle) \otimes (\cos\phi|0\rangle + \sin\phi|1\rangle)$ and $A = |1\rangle_a\langle 0|$, $B = |1\rangle_b\langle 0|$, we have $|\langle AB^\dagger\rangle| = |\cos\theta \sin\phi \sin\theta \cos\phi|$ and $\langle A^\dagger AB^\dagger B\rangle = \cos^2\theta \cos^2\phi$, so the inequality (16) will be satisfied whenever $|\sin\phi \sin\theta| > |\cos\phi \cos\theta|$, which is not at all difficult to arrange.

At this point, it may be useful to consider what the original criterion (1a) is telling us about an entangled state that it can actually detect, such as $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$. With the choice of A and B above, we find $\langle AB^\dagger\rangle = \langle\Psi^+|(|10\rangle\langle 01|)|\Psi^+\rangle = \frac{1}{2}$ and $\langle A^\dagger AB^\dagger B\rangle = |\langle 00|\Psi^+\rangle|^2 = 0$. So what we are comparing is the following: how much the system's state resembles the original state after we change the value of a variable of system A from 0 to 1 *and simultaneously* a variable of system B from 1 to 0 (left-hand side of the inequality), versus the plain probability to find the value 0 for both variables simultaneously in the separate systems (right-hand side of the inequality).

Now this makes sense for the following reason: in a bipartite entangled state, there must be some variable in each system that does not have a definite value by itself because it is in a correlated superposition with the other variable in the other system. Thus, changing the state in a way that preserves the correlation (as above, $|01\rangle \to |10\rangle$) should still yield a substantial overlap with the initial state, which we then compare to the probability of the two variables having simultaneously well-defined values that violate this correlation (in this case, $|00\rangle$).

In the original criterion (1a), to make sure we do not get "false positives" as with Eq. (16), it is necessary to square the left-hand side, that is to say, to make sure it is large enough. Yet, as we showed in the previous sections, this is actually asking for too much and the criterion (1a) ends up missing a substantial fraction of the entangled states. The above discussion, however, immediately suggests another possibility, based on symmetry: in the example above, we are testing the strength of the coherent superposition or correlation $|10\rangle + |01\rangle$ versus that of the alternative, incoherent ("classical") correlation $|00\rangle$. Yet since the states $|1\rangle_a$ and $|1\rangle_b$ are also involved in the quantum correlation, it would make sense to include the state $|11\rangle$ as well on the classical side of the inequality. In other words, to try a criterion like

$$|\langle\psi|(|10\rangle\langle 01|)|\psi\rangle|^2 > |\langle 00|\psi\rangle|^2|\langle 11|\psi\rangle|^2. \tag{17}$$

With the A and B operators introduced above this "new" criterion can be written

$$|\langle AB^\dagger\rangle|^2 > \langle A^\dagger AB^\dagger B\rangle\langle AA^\dagger BB^\dagger\rangle. \tag{18}$$

It turns out that the inclusion of the factor $|\langle 11|\psi\rangle|^2$ on the right-hand side of Eq. (17) is just enough to eliminate the "false positives," so Eq. (17) can only hold for entangled states; moreover, as it turns out, it is now possible for *every* entangled state of two qubits [whether pure or mixed, and not restricted to the form (2)], to find operators A and B so that Eq. (18) is satisfied, that is, the entanglement is detected. This can again be easily checked for the states of the form (2) that we considered above, but its full generality actually follows from the results in [15] because, in fact, our "new, improved" criterion (18) is not new at all, just a rephrasing of a result in [15].

Indeed, in [15], Wölk, Huber, and Gühne established that the inequality

$$|\langle A_1A_2B_1B_2\rangle|^2 > \langle A_1A_1^\dagger B_2^\dagger B_2\rangle\langle A_2^\dagger A_2B_1B_1^\dagger\rangle \tag{19}$$

(where A_i and B_i are arbitrary operators acting on subsystems A and B of a bipartite system) (i) can only be satisfied by entangled states, (ii) is only satisfied by states with a negative

partial transpose, and (iii) will in fact detect all the entangled states of a two-qubit system, with the choice of operators $A_1 = |a_1\rangle\langle\phi|$, $A_2 = |\phi\rangle\langle a_2|$, $B_1 = |b_1\rangle\langle\xi|$, $B_2 = |\xi\rangle\langle b_2|$, where $|\phi\rangle$ and $|\xi\rangle$ are arbitrary pure states in A and B, respectively, and $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ are appropriate orthogonal (i.e., basis) states in A and B chosen based on the Schmidt decomposition of the eigenvector corresponding to the negative eigenvalue of ρ^{T_A} (see Theorem 4 of [15]). Note that, in terms of these operators, we just have to define $A = A_1A_2 = |a_1\rangle\langle a_2|$ and $B^\dagger = B_1B_2 = |b_1\rangle\langle b_2|$ to recover our criterion (18), from Eq. (19), with A and B raising and/or lowering operators in an appropriate basis.

Additionally, according to Theorem 3 of [15], choosing the A_i and B_i operators to have the form $A_1 = |a_1\rangle\langle\phi|$, $A_2 = |\phi\rangle\langle a_2|$, $B_1 = |b_1\rangle\langle\xi|$, $B_2 = |\xi\rangle\langle b_2|$ is always optimal for use with the criterion Eq. (19) for bipartite systems of any dimension. In that case, Eq. (19) will again take the form (18), with $A = |a_1\rangle\langle a_2|$ and $B^\dagger = |b_1\rangle\langle b_2|$, although the optimal choice of the $|a_i\rangle$ and $|b_i\rangle$ states is not trivial in general when the systems are not qubits. If we choose $|a_1\rangle$ and $|a_2\rangle$ to be mutually orthogonal, and likewise $|b_1\rangle$ and $|b_2\rangle$, we can think of the a_i as representing different values of some observable in system A and likewise for the b_i. Then Eq. (18) becomes an inequality relating the density matrix elements in a basis that includes the states $|a_i\rangle|b_j\rangle$:

$$|\rho_{a_2b_2,a_1b_1}|^2 > \rho_{a_2b_1,a_2b_1}\rho_{a_1b_2,a_1b_2} \tag{20}$$

(cf. Eq. (19) of [15], and inequality I in [21]). In this form our previous interpretation becomes apparent: the criterion compares the strength of a nonclassical correlation (a coherent superposition, expressed by the off-diagonal element on the left-hand side), to the strength of two alternative, incompatible [22], classical correlations, expressed by the diagonal elements on the right-hand side.

E. Discussion

Both the Hillery-Zubairy and Wölk-Huber-Gühne criteria require one to make a good choice of the operators A and B, or equivalently, for the criterion (20), of the basis in which to write the density operator. Note that if the density operator ρ is known already in some basis, there is really no point in pursuing such an "optimization" approach since, in general, it would be faster to just compute the eigenvalues of the partial transpose ρ^{T_A} and we know that the criterion (19) only detects states with a negative partial transpose in the first place. [The original Hillery-Zubairy criteria can be trivially derived from Eq. (19) by setting some of the operators appearing in it equal to the identity.]

These criteria, therefore, are only useful in practice if the full density matrix is not immediately available (as in an experimental situation), and especially, if one already has some idea of what correlations to test for, or alternatively, what the entangled variables are likely to be. Under those conditions, the key difference between Eqs. (1) and (20) is that the former requires one to measure some operators, whereas the latter requires one to measure specific density matrix elements. Although the latter can, in principle, be done, by appropriate quantum tomographic techniques, the former will, in general, be simpler [even though the operators in Eq. (1) are non-Hermitian, one can always obtain the corresponding expectation values by measuring the Hermitian combinations $A + A^\dagger$ and $i(A - A^\dagger)$].

For qubits, this distinction between between measuring elements of ρ and measuring operators is pretty much meaningless since all the elements of the density matrix can be written in terms of the expectation values of appropriate combinations of the Pauli matrices corresponding to the spin components. For larger-dimensional systems, on the other hand, the Hillery-Zubairy criterion, although in general less powerful (as we have seen), may have an advantage in terms of ease of use in experiments, especially if, as mentioned above, one has some notion of what kind of correlations to look for in the first place. In the following section we present an important example in support of this idea.

III. LARGER-DIMENSIONAL BIPARTITE SYSTEMS

A. Qubits and angular momentum

It seems natural, when seeking to extend the criteria (1) to larger-dimensional systems, to look at angular momentum systems and use for the operators A and B the angular momentum raising and lowering operators, as was done for qubits in the previous section. Moreover, groups of qubits can also be treated this way: defining a collective raising operator $J_{a+} = \sum_{i=1}^{N_a} \sigma_{+,i}$, one has an angular momentum algebra corresponding to $j_a = N_a/2$.

In [14], Zheng, Trung Dung, and Hillery followed this approach to investigate entanglement between groups of qubits. They were able to identify situations and special states where the second criterion, Eq. (1b), could be used to detect entanglement; derived a more powerful criterion starting from Eq. (1a) and requiring invariance under local rotations of one of the subsystems; and used their results to study entanglement in spin waves. The remainder of this paper will follow a similar pattern. First, in this section, we will look for criteria that can be used to detect certain types of angular momentum correlations. Then, in the next section, we will apply these criteria to the study of entanglement in a quantum-optical system of some importance, namely, the Tavis-Cumming model.

B. Entangled states of total angular momentum

Consider two angular momentum systems, described by $\mathbf{J}_a$ and $\mathbf{J}_b$, and let the eigenstates of $\mathbf{J}_a^2$ and $\mathbf{J}_b^2$ have angular momentum numbers j_a and j_b, respectively. The eigenstates $|j, m\rangle$ of the total angular momentum $\mathbf{J}^2$ (with $\mathbf{J} = \mathbf{J}_a + \mathbf{J}_b$) and total J_z are always entangled, except for the $m = \pm(j_a + j_b)$ cases. The property that is in an entangled superposition is, of course, the z component of the individual angular momenta. That is to say, if the total $J_z = m\hbar$, with $j_a + j_b > m > 0$ for definiteness, we can always write $m = j_a + j_b - k$ for some $k > 0$, and we will have

$$|j, m\rangle_{\text{total}} = \sum_{n=0}^{k} C_n |j_a, j_a - n\rangle_a |j_b, j_b - k + n\rangle_b. \tag{21}$$

Suppose we have a state that we suspect is close to, but not quite, an eigenstate of the total J_z, and we want to check to see whether it is entangled. If we could measure directly

the arbitrary matrix elements of the system, we could test for entanglement using Eq. (20) in the form

$$|\langle m_a, m_b|\rho|m_a+n, m_b-n\rangle|^2 > \langle m_a, m_b-n|\rho|m_a, m_b-n\rangle \times\langle m_a+n, m_b|\rho|m_a+n, m_b\rangle \tag{22}$$

for appropriate m_a, m_b, and n.

Alternatively, we could try to use one of the Hillery-Zubairy criteria (1), choosing operators A and B that have nonvanishing matrix elements of the form appearing on the left-hand side of Eq. (22). An obvious choice would be to use powers of the raising operators, $A = J_{a+}^n$ and $B = J_{b+}^n$. Additionally, note that Eq. (21) implies that we can always make the right-hand side of Eq. (1a) equal to zero by choosing the power of J_{a+} and $B = J_{b+}$ equal to the constant k that appears there since

$$\begin{aligned} J_{a+}^k|j_a, j_a-n\rangle_a &= 0 \quad \text{except for } n=k, \\ J_{b+}^k|j_b, j_b-k+n\rangle_b &= 0 \quad \text{except for } n=0, \end{aligned} \tag{23}$$

and consequently $J_{a+}^k J_{b+}^k$ acting on the state (21) gives zero. On the other hand,

$$\begin{aligned} J_{a+}^k J_{b-}^k \sum_{n=0}^{k} C_n|j_a, j_a-n\rangle_a|j_b, j_b-k+n\rangle_b \\ \propto C_k|j_a, j_a\rangle_a|j_b-k\rangle_b, \end{aligned} \tag{24}$$

and therefore

$$\left|\left\langle J_{a+}^k J_{b-}^k\right\rangle\right|^2 \propto |C_0 C_k|^2 > 0, \tag{25}$$

which means the criterion (1a) is satisfied with $A = J_{a+}^k$, $B = J_{b+}^k$.

For the case the total m is negative, we can write $m = -j_a - j_b + k$, in which case an analogous reasoning shows that now it is $J_{a-}^k J_{b-}^k$ that gives zero when acting on the state $|j, m\rangle$, whereas $J_{a-}^k J_{b+}^k$ yields a nonzero result.

Equation (25) makes it clear that this choice of the exponent n in $A = J_{a+}^n$ and $B = J_{b+}^n$ amounts to testing the superposition, in the state (21), of the two most distant values of m_a, and similarly for m_b, and one might ask whether this is always necessary. The tests that we ran on small j cases suggest that this is indeed the case. If we know the values of j_a, j_b, j, and m, then k is uniquely determined; on the other hand, if we do not know, say, j or m, we can try testing for entanglement with the criterion (1a) using a hierarchy of $A = J_{a+}^n$, $B = J_{b+}^n$, with $n = 1, 2, \ldots, 2 \times \min(j_a, j_b)$:

$$\left|\left\langle J_{a+}^n J_{b-}^n\right\rangle\right|^2 > \left\langle J_{a-}^n J_{a+}^n J_{b-}^n J_{b+}^n\right\rangle \tag{26}$$

if we believe m is likely to be positive, or

$$\left|\left\langle J_{a-}^n J_{b+}^n\right\rangle\right|^2 > \left\langle J_{a+}^n J_{a-}^n J_{b+}^n J_{b-}^n\right\rangle, \tag{27}$$

if we believe m is more likely to be negative.

In addition to the pure states of well-defined total angular momentum considered above, many other entangled states, both pure and mixed, can be detected by the criteria (26) and (27). For the left-hand side to be nonzero, it suffices that the state have a nonvanishing coherence of the form shown on the left-hand side of Eq. (22), whereas the right-hand side will be zero for all the states that have no populations (i.e., diagonal matrix elements) involving both an $m_a \leqslant j_a - n$ and an $m_b \leqslant j_b - n$. Of course, some entangled states that have such populations will still be detectable, as long as they are sufficiently small.

It is straightforward to extend the approach developed here to the detection of other types of entangled states. Generally speaking, if we want to check for the presence of an entangled superposition of the form

$$|m_a, m_b\rangle + |m_a+p, m_b-q\rangle, \tag{28}$$

with q, p positive or negative, we can use the criterion (1a) with $A = J_{a\pm}^{|p|}$, $B = J_{b\pm}^{|q|}$, and the raising operator is used on A (or B) if $p > 0$ (or $q > 0$), and the lowering operator in the opposite case.

Finally, we note that all of the above has made use only of the criterion (1a), as the obvious generalization of Eq. (1b) appears to be weaker than Eq. (26) for detecting the kind of entanglement considered here: in fact, for the states (21) specifically, the closest we can get to $|\langle J_{a+}^n J_{b-}^n\rangle|^2 > \langle J_{a-}^n J_{a+}^n\rangle\langle J_{b-}^n J_{b+}^n\rangle$ is the equality of both sides. This does not mean Eq. (1b) is useless, however, as pointed out above, in [14] the authors found $|\langle J_{a+}J_{b-}\rangle|^2 > \langle J_{a-}J_{a+}\rangle\langle J_{b-}J_{b+}\rangle$ could be used to detect a different kind of entanglement among angular momentum systems. As this was well covered in [14], we will not dwell any more on it here.

IV. APPLICATION: ENTANGLEMENT IN THE TAVIS-CUMMINGS MODEL

In this section we show how the criteria developed in Sec. III B can be used to study entanglement between the atoms in the Tavis-Cummings model, which consists of N two-level atoms interacting with a single mode of the quantized radiation field in the rotating-wave approximation [23]. This is a system that recently sparked some interest in quantum information, as it could be realized experimentally either in ion traps (with a phonon instead of a photon field; see [24]) and in circuit QED with superconducting qubits [25]. For $N = 2$ entanglement in this system was studied using the tangle formalism in [26], and it was also explored in [12] (for $N = 2$ in Sec. III, and for large N in the Holstein-Primakoff approximation in Sec. VI). This last work also used a variation of the Hillery-Zubairy criteria in a form that allowed the authors to study entanglement between different kinds of systems, notably, in this case, the atoms and the field. See also [27,28] for other recent studies.

As noted in [23], the Hamiltonian for this system (on resonance) can be written in the form

$$H = \hbar\omega + \omega J_z + g(J_+ a + a^\dagger J_-), \tag{29}$$

where the angular momentum operators are the sum of N "spins":

$$J_z = \frac{\hbar}{2}\sum_{i=1}^{N}\sigma_{iz}, \quad J_\pm = \frac{\hbar}{2}\sum_{i=1}^{N}\sigma_{i\pm}. \tag{30}$$

The total angular momentum J^2 has $j = N/2$. The joint eigenstates of J_z (eigenvalue $\hbar m$) and the photon number operator $a^\dagger a$ (eigenvalue n), which we will write as $|m\rangle|n\rangle$,

form a natural basis for the study of this system's dynamics since the Hamiltonian (29) conserves the number of excitations $n+m$. An exact (though complicated) solution was given by Tavis and Cummings in their original paper [23].

Here we will restrict ourselves to the case in which the initial field state is a number state with $|n_0\rangle$ photons and the initial state of the atoms is an eigenstate of J_z with eigenvalue $|m_0\rangle$. Under those circumstances, each value of n in the total state will be associated with one and only one value of m, and therefore, the reduced density operator of the atoms will be of the form

$$\rho_{at} = \sum_{m=-N/2}^{N/2} |c_m(t)|^2 |m\rangle\langle m| \tag{31}$$

At this point we may note that the atomic states $|m\rangle$ introduced above are nothing but the Dicke states [29] for this collection of N pseudospin-$\frac{1}{2}$ systems. These include states of interest for quantum information (such as the W states) as well as metrological applications (see, e.g., [30,31] for relevant work and many references). Our choice of initial state, therefore, ensures that the atomic state at any later time is an evolving mixture of Dicke states and our purpose is to apply the criteria developed here to investigate the entanglement of such mixtures.

In particular, if we divide the N atoms into two groups, A and B, we can use the criteria (26) and (27) to study the evolution of the entanglement between the two groups for different initial conditions. Note that both the operators $J_{a+}^n J_{b-}^n$ and $J_{a-}^n J_{a+}^n J_{b-}^n J_{b+}^n$ [as well as the corresponding ones in Eq. (27)], when acting on a state $|m\rangle$, either destroy it or preserve the total number of spins up and down ($N/2+m$ and $N/2-m$, respectively), so their restrictions to the $j=N/2$ space are diagonal in the $\{|m\rangle\}$ basis. In principle, closed-form expressions for these matrix elements could be obtained for arbitrary partitions into sets of N_a and $N_b = N - N_a$ atoms, using the explicit expressions

$$J_{a\pm} = \frac{\hbar}{2}\sum_{i=1}^{N_a} \sigma_{i\pm}, \quad J_{b\pm} = \frac{\hbar}{2}\sum_{i=N_a+1}^{N} \sigma_{i\pm}. \tag{32}$$

Here, we will only consider two types of partition: one atom versus the rest and (for even N) half the atoms versus the other half. These are all the possibilities through $N=4$, which is the highest N we will treat explicitly. Setting $\hbar = 1$ for convenience (as we did in Sec. III), we find, for the first type ($N_a = 1$):

$$\begin{aligned}
\langle m|J_{a+}J_{b-}|m\rangle &= \frac{1}{N}\left(\frac{N^2}{4} - m^2\right),\\
\langle m|J_{a-}J_{a+}J_{b-}J_{b+}|m\rangle &= \frac{1}{N}\left(\frac{N}{2}+m+1\right)\\
&\quad\times\left(\frac{N}{2}-m\right)\left(\frac{N}{2}-m-1\right),
\end{aligned} \tag{33}$$

and for the second one ($N_a = N/2$)

$$\begin{aligned}
\langle m|J_{a+}J_{b-}|m\rangle &= \frac{N}{4(N-1)}\left(\frac{N^2}{4} - m^2\right),\\
\langle m|J_{a-}J_{a+}J_{b-}J_{b+}|m\rangle &= \frac{1}{N!}(N/2-m)!(N/2+m)!\\
&\quad\times\sum_{k=1}^{N/2-m-1}(m+k+1)^2(N/2-k+1)^2\\
&\quad\times\binom{N/2}{m+k+1}\binom{N/2}{N/2-k+1}.
\end{aligned} \tag{34}$$

Although we have not been able to find a simpler form for the last of these expressions, its evaluation for specific cases is straightforward and our numerical results suggest that, for a given N, it is of the form $(N/2-m)(N/2-m-1)$ times a polynomial of order m^3. The corresponding expressions for $\langle m|J_{a-}J_{b+}|m\rangle$ and $\langle m|J_{a+}J_{a-}J_{b+}J_{b-}|m\rangle$ [to be used with criterion (27)] are obtained from Eqs. (33) and (34) by changing m to $-m$.

Note that Eqs. (33) and (34), and the absence of off diagonal elements mentioned above, indicate that, when restricted to the subspace spanned by the kets $\{|m\rangle\}$ for fixed j (that is, fixed N), both the operator $J_{a+}J_{b-}$ and $J_{a-}J_{a+}J_{b-}J_{b+}$ are, in effect, functions of J_z; for instance, the first of Eqs. (33) essentially implies $J_{a+}J_{b-} = N/4 - J_z^2/N$. This means that the criterion (26) becomes, for this system, an entanglement criterion involving only the operator for the total population inversion. For instance, explicitly, for a system of N atoms the criterion for one of them to be entangled with the rest becomes

$$\frac{1}{N}\left(\frac{N^2}{4} - \langle J_z^2\rangle\right)^2 > \left\langle\left(\frac{N}{2}+1+J_z\right)\left(\frac{N}{2}-J_z\right)\left(\frac{N}{2}-1-J_z\right)\right\rangle, \tag{35}$$

or, if Eq. (27) is used instead, the same expression with $J_z \to -J_z$. A similar explicit form for the criterion for entanglement between two groups of $N/2$ atoms requires evaluating the second of Eqs. (34) for specific values of N, which is not hard; an explicit example is given below for the $N=4$ case. In the following we show that entanglement can, in fact, be detected by these criteria.

A. Two atoms

For $N=2$ the two cases (33) and (34) coincide. The possible values of m are $-1, 0$, and 1, and the only nonzero matrix elements are $\langle 0|J_{a+}J_{b-}|0\rangle = \frac{1}{2}$ and $\langle -1|J_{a-}J_{a+}J_{b-}J_{b+}|-1\rangle = 1$.

As mentioned above, we will assume an initial state of the form $|n_0\rangle|m_0\rangle$. Moreover, we will restrict ourselves to the two most natural initial conditions: all atoms excited with no photons present ($|\psi_0\rangle = |N/2\rangle|0\rangle$) or all atoms in the ground state ($m_0 = -N/2$) with n_0 photons present. Either way, the state at any later time can be written as

$$|\psi(t)\rangle = c_1|1\rangle|p-2\rangle + c_0|0\rangle|p-1\rangle + c_{-1}|-1\rangle|p\rangle, \tag{36}$$

where $p = m_0 + n_0$ is the initial total number of excitations and the entanglement criterion (26) (with $n=1$) can be

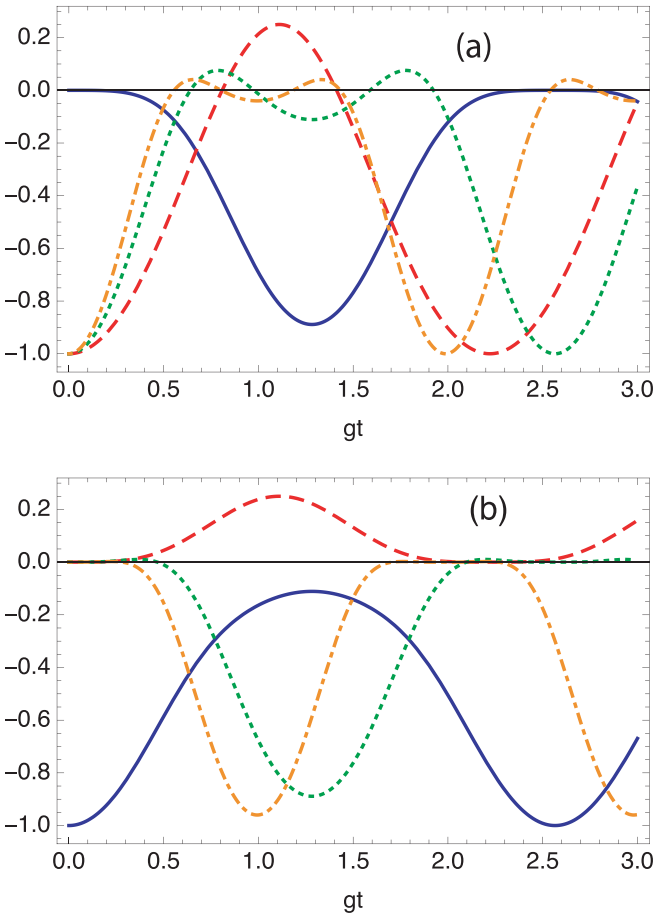

FIG. 1. (a) The difference $\frac{1}{4}|c_0(t)|^4 - |c_{-1}(t)|^2$ (positive values indicate an entangled state of the two atoms) for four different cases: pure spontaneous decay ($m_0 = 1, n_0 = 0$, solid line), and evolution from the ground state with $n_0 = 1$ (dashed line), 2 (dotted line), and 3 (dot-dashed line) photons respectively. (b) The same for $\frac{1}{4}|c_0(t)|^4 - |c_1(t)|^2$.

written

$$\tfrac{1}{4}|c_0(t)|^4 - |c_{-1}(t)|^2 > 0, \tag{37}$$

whereas Eq. (27) yields the alternative

$$\tfrac{1}{4}|c_0(t)|^4 - |c_1(t)|^2 > 0. \tag{38}$$

The equations of motion for the coefficients c_i are

$$\begin{aligned} \dot{c}_1 &= -ig\sqrt{2(p-1)}\,c_0, \\ \dot{c}_0 &= -ig\sqrt{2p}\,c_{-1} - ig\sqrt{2(p-1)}\,c_1, \\ \dot{c}_{-1} &= -ig\sqrt{2p}\,c_0. \end{aligned} \tag{39}$$

They can easily be solved analytically and the expressions in Eqs. (37) and (38) evaluated. Figure 1(a) shows the left-hand side of Eq. (37) for four different cases: pure spontaneous decay ($m_0 = 1, n_0 = 0$) and evolution from the ground state with $n_0 = 1$, 2, and 3 photons, respectively, whereas Fig. 1(b) shows the same for the left-hand side of Eq. (38). As can be seen, the usefulness of Eq. (38) decreases much faster than that of Eq. (37) as the number of photons increases and the probability to find the system in the doubly excited state

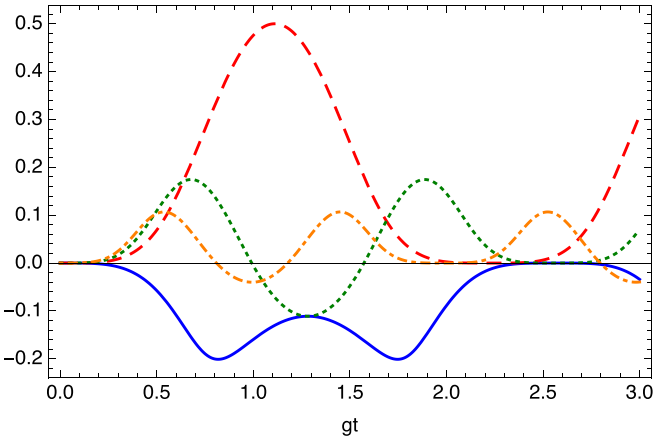

FIG. 2. For the same initial conditions as in Fig. 1, the figure shows $-\lambda_4'$ [where λ_4' is the only eigenvalue of the partial transpose of the matrix (40) that can become negative]. The negativity $N(\rho) = \max\{0, -\lambda_4'\}$; a positive value of the negativity is a necessary and sufficient condition for entanglement in this system.

(and correspondingly $\langle J_z\rangle$) increases: this is to be expected since Eq. (38) was derived from Eq. (27), which detects entanglement for states of negative total m. On the other hand, Eq. (38) is critical in establishing the important fact that the system with only one excitation ($m_0 = -1, n_0 = 1$) is always entangled, except at the times when $c_0 = 0$ (see Sec. IV.C, below, for a general proof of this result for arbitrary N).

In view of the fact that Eqs. (37) and (38) are only sufficient, not necessary conditions for entanglement, the fact that no entanglement is seen for the spontaneous decay case may appear suspect. Fortunately, as this simple case involves only two qubits, we can verify that this is correct by calculating the negativity. In the four-state basis $\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\}$ (equivalent to the one used in Sec. II, but here using e and g instead of 0 and 1 to avoid confusion with the total J_z eigenstates), the reduced density matrix of the two atoms, after tracing over the field state, is

$$\rho = \begin{pmatrix} |c_1|^2 & 0 & 0 & 0 \\ 0 & \frac{1}{2}|c_0|^2 & \frac{1}{2}|c_0|^2 & 0 \\ 0 & \frac{1}{2}|c_0|^2 & \frac{1}{2}|c_0|^2 & 0 \\ 0 & 0 & 0 & |c_{-1}|^2 \end{pmatrix} \tag{40}$$

and the negativity is $N(\rho) = \max\{0, -\lambda_4'\}$, with

$$\begin{aligned} \lambda_4' = \tfrac{1}{2}(1 - |c_0|^2 \\ - \sqrt{(1 - 2|c_{-1}|^2 + 2|c_0|^2(|c_0|^2 + 2|c_{-1}|^2 - 1)}). \end{aligned} \tag{41}$$

Figure 2 shows $-\lambda_4'$ for the same initial conditions illustrated in Fig. 1. As expected from the general considerations in Sect. II, the criteria (37) and (38) do miss some of the entanglement, but in general, and taken together, they can be used to approximately identify the times where entanglement is largest in the two-atom system.

Note that, in this special case, the criterion (37) reduces to the ones we studied in Sec. II for two qubits, if we just write $A = |e\rangle_a\langle g|$ and $B = |e\rangle_b\langle g|$ [and similarly Eq. (38) if e and g are exchanged]. This indicates that, as suggested in Sec. III,

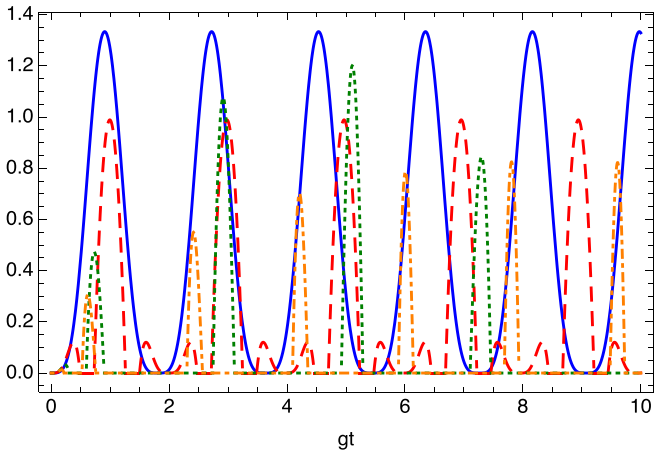

FIG. 3. Entanglement, as indicated by positive values of the left-hand sides of Eqs. (42) and (43), for a system of three atoms starting from the ground state with $n_0 = 1$ (solid line), 2 (dashed line), 3 (dotted line), and 4 (dot-dashed line) photons, respectively. For $n_0 = 1$, only the left-hand side of Eq. (43) contributes (i.e., is greater than zero). For larger n_0, however, its contribution is limited to small secondary minima, here only visible for $n_0 = 2$.

this family of criteria is a natural extension of the two-qubit case to higher-dimensional systems.

B. Three atoms

For $N = 3$ we have $j = \frac{3}{2}$ and $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. The matrix for $N/2 - J_z^2$ has two nonzero diagonal entries, both equal to 2, corresponding to $m = \pm\frac{1}{2}$, whereas the matrix for the operator on the right-hand side of Eq. (35) has only nonvanishing entries equal to 4, for $m = -\frac{1}{2}$ and 6 for $m = -\frac{3}{2}$. The criterion (35), therefore, becomes

$$\tfrac{4}{3}(|c_{-1/2}|^2 + |c_{1/2}|^2)^2 - 4|c_{-1/2}|^2 - 6|c_{-3/2}|^2 > 0 \qquad (42)$$

and the corresponding one with $J_z \to -J_z$,

$$\tfrac{4}{3}(|c_{-1/2}|^2 + |c_{1/2}|^2)^2 - 4|c_{1/2}|^2 - 6|c_{3/2}|^2 > 0, \qquad (43)$$

if the total state of the system is written, as in the previous subsection, in the general form (with p initial quanta)

$$|\psi(t)\rangle = \sum_{m=-j}^{m=j} c_m(t)|m\rangle|p - (m + j)\rangle. \qquad (44)$$

The equations of motion are now

$$\begin{aligned}
\dot{c}_{3/2} &= -ig\sqrt{3(p-2)}\,c_{1/2},\\
\dot{c}_{1/2} &= -ig2\sqrt{p-1}\,c_{-1/2} - ig\sqrt{3(p-2)}\,c_{3/2},\\
\dot{c}_{-1/2} &= -ig\sqrt{3p}\,c_{-3/2} - ig2\sqrt{p-1}\,c_{1/2},\\
\dot{c}_{-3/2} &= -ig\sqrt{3p}\,c_{-1/2}.
\end{aligned} \qquad (45)$$

As was the case for $N = 2$, no entanglement is found either by Eq. (42) or Eq. (43) for the spontaneous decay case $m_0 = \frac{3}{2}$, $n_0 = 0$. The cases in which the system starts from the ground state with $n_0 = 1, 2, 3$, and 4 photons are illustrated in Fig. 3, which shows only the maximum of either 0, the left-hand side of Eq. (42), or the left-hand side of Eq. (43). As before, the usefulness of the second criterion, Eq. (43), is limited to the cases with few excitations: all the visible features in the figure for $n_0 = 3$ and 4 come from Eq. (42), and for $n_0 = 2$ Eq (43) only contributes the small secondary maxima. On the other hand, Eq. (43) is essential to show the entanglement for $n_0 = 1$ since, in that case, Eq. (42) is found never to hold, whereas Eq. (43) shows the state is, in fact, virtually always entangled.

Note that, for $m_0 = -\frac{3}{2}$, if $n_0 = 1$ (or $n_0 = 2$) only the state(s) $m = -\frac{1}{2}$ (or $m = -\frac{1}{2}, \frac{1}{2}$) can be excited. The solution to Eq. (45) in these two cases is sufficiently simple to be included here:

$$\begin{aligned}
n_0 = 1:\quad & |\psi(t)\rangle\\
&= \cos(\sqrt{3}gt)\left|-\frac{3}{2}\right\rangle|1\rangle - i\sin(\sqrt{3}gt)\left|-\frac{1}{2}\right\rangle|0\rangle,\\
n_0 = 2:\quad & |\psi(t)\rangle\\
&= \frac{1}{5}(2 + 3\cos(\sqrt{10}gt))\left|-\frac{3}{2}\right\rangle|2\rangle - i\sqrt{\frac{3}{5}}\sin(\sqrt{10}gt)\\
&\quad\times\left|-\frac{1}{2}\right\rangle|1\rangle + \frac{2\sqrt{6}}{5}\sin^2\left(\frac{\sqrt{10}}{2}gt\right)\left|\frac{1}{2}\right\rangle|0\rangle.
\end{aligned} \qquad (46)$$

In the first case the reduced density operator for the atoms is formally equivalent to a two-qubit system since only two states of the two-atom subsystem are involved ($|gg\rangle$ and $(|eg\rangle + |ge\rangle)/\sqrt{2}$). The second case (and all the others with $n_0 > 2$) is formally equivalent to a qubit-qutrit system since the state $|ee\rangle$ is also involved. This means that, in principle, the negativity would still provide a necessary and sufficient condition for entanglement. Rather than engage in such a laborious calculation, however, we can use Eq. (46) to understand the main features of Fig. 3 with some simple considerations. In particular, the taller peaks in the $n_0 = 2$ case happen when $\sqrt{10}gt$ is close to an odd multiple of π since, in that case, the weight of the state $|-\frac{3}{2}\rangle$ is very small ($1/25 = 0.04$), that of the state $|-\frac{1}{2}\rangle$ is zero, and the state is almost entirely $|\frac{1}{2}\rangle$, which can be written

$$\left|\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|gee\rangle + |ege\rangle + |eeg\rangle). \qquad (47)$$

This is an entangled state, of the W form, like the one observed experimentally in [25]. The preponderance of $m > 0$ in the mix also explains why the entanglement is not so readily detected by the criterion (43), which is instead responsible for the small peaks; these happen at times when $c_{1/2}$ is small and the state $|-\frac{1}{2}\rangle$ dominates, although with a substantial contribution from $|-\frac{3}{2}\rangle$.

C. Four atoms

1. One atom versus the rest

With four atoms, to test for entanglement of any one atom with the other three, Eqs. (33) (and the corresponding ones with $m \to -m$) yield the two entanglement inequalities

$$\begin{aligned}
&\tfrac{1}{4}(3|c_1|^2 + 4|c_0|^2 + 3|c_{-1}|^2)^2 - 6|c_0|^2 - 12|c_{-1}|^2\\
&\quad -12|c_{-2}|^2 > 0,
\end{aligned} \qquad (48a)$$

$$\tfrac{1}{4}(3|c_1|^2+4|c_0|^2+3|c_{-1}|^2)^2-6|c_0|^2-12|c_1|^2 -12|c_2|^2>0. \tag{48b}$$

The equations of motion are now

$$\begin{aligned}\dot{c}_2&=-2ig\sqrt{p-3}\,c_1,\\ \dot{c}_1&=-2ig\sqrt{p-3}\,c_2-ig\sqrt{6(p-2)}\,c_0,\\ \dot{c}_0&=-ig\sqrt{6(p-2)}\,c_1-ig\sqrt{6(p-1)}\,c_{-1},\\ \dot{c}_{-1}&=-ig\sqrt{6(p-1)}\,c_0-2ig\sqrt{p}\,c_{-2},\\ \dot{c}_{-2}&=-2ig\sqrt{p}\,c_{-1}.\end{aligned} \tag{49}$$

As before, neither of the two inequalities detects entanglement at any time for the spontaneous emission case, $m_0=2$, $n_0=0$. When starting from the ground state instead, the inequality (48a) does not detect entanglement for $n_0=1$, but Eq. (48b) does and shows essentially the same result as for $N=2$ and 3 atoms, namely, a regular oscillation that implies entanglement at all times except when $c_{-1}=0$, which here happens for $2gt=n\pi$.

It is, in fact, possible to use the criterion (35), with $J_z\to -J_z$, to show that this is a general result, valid for all N. First, in the case there is only one excitation in the system, ($m_0=-N/2$, $n_0=1$) the equations of motion immediately show that $|c_{-N/2}|^2=\cos^2\theta$ and $|c_{-N/2+1}|^2=\sin^2\theta$, with $\theta=gt\sqrt{N}$. The density operator for the atomic system is therefore the incoherent superposition of an entangled state $|m\rangle=|-N/2+1\rangle$ and the product ground state $|-N/2\rangle$:

$$\rho_{at}=\cos^2\theta|-N/2+1\rangle\langle -N/2+1|+\sin^2\theta|-N/2\rangle\langle -N/2| \tag{50}$$

When the criterion (35), with $J_z\to -J_z$, is applied to this state, a little algebra shows that the inequality reduces to

$$-\frac{(N-1)^2}{N}\sin^4(gt\sqrt{N})<0, \tag{51}$$

which is always satisfied, except when $gt\sqrt{N}=n\pi$. Note that, for $N=2$, the state $|-N/2+1\rangle$ is a Bell state, and for $N>2$ a W state.

Leaving the $n_0=1$ case aside, then, the graphs in Fig. 4(a) focus on the less obvious cases with $n_0=2$, 3, and 4. For $n_0=2$, the large peaks in Fig. 4(a) also come from Eq. (48b), whereas Eq. (48a) contributes only a few small peaks. As expected, as the number of excitations increases and the upper atomic levels become more populated, the situation is reversed, with Eq. (48b) contributing only one small peak to the $n_0=3$ graph and essentially nothing visible to the $n_0=4$ case.

2. Two atoms versus two

If instead of one atom versus three we want to test the entanglement of any two atoms with the other two, Eqs. (34) (and the corresponding ones with $n\to -m$) yield the inequalities

$$\left(|c_1|^2+\tfrac{4}{3}|c_0|^2+|c_{-1}|^2\right)^2-\tfrac{8}{3}|c_0|^2-4|c_{-1}|^2-4|c_{-2}|^2>0, \tag{52a}$$

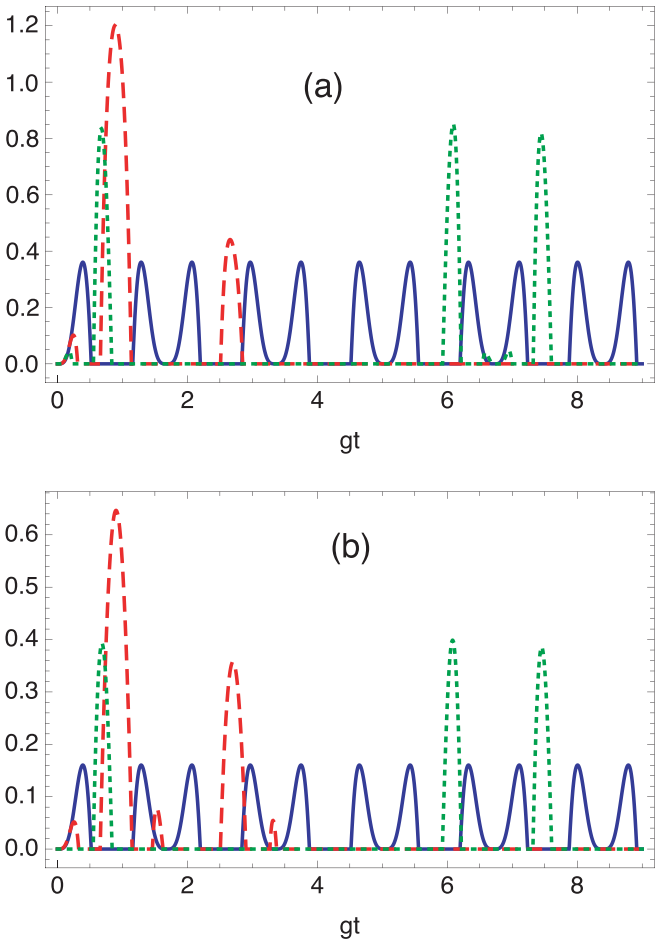

FIG. 4. Entanglement between subsystems for $N=4$ atoms starting in the ground state and interacting with $n_0=2$ (solid line), 3 (dashed line), and 4 (dotted line) photons. (a) One atom in subsystem a, three in subsystem b [inequalities (48)]. (b) Two atoms in each subsystem [inequalities (52)].

$$\left(|c_1|^2+\tfrac{4}{3}|c_0|^2+|c_{-1}|^2\right)^2-\tfrac{8}{3}|c_0|^2-4|c_1|^2-4|c_2|^2>0. \tag{52b}$$

Again no entanglement is detected for the spontaneous emission case and for the system starting from the ground state the results are also remarkably similar to the 1-to-3 case, as Fig. 4(b) shows, the only visible difference being two small peaks for $n_0=3$ (near $gt=1.5$ and 3.4).

On the other hand, this is also the first opportunity to test the criteria (26) and (27) with $n>1$, namely, $n=2$ (clearly, these criteria cannot be applied to the study of the 1-to-$N-1$ split since acting on a single spin $J_\pm^2$ will necessarily give zero). We do not have general formulas for arbitrary N and m, but it is easy to see that for $N=4$ Eqs. (26) and (27) yield the inequalities

$$|c_0|^4-36|c_{-2}|^2>0, \tag{53a}$$

$$|c_0|^4-36|c_2|^2>0. \tag{53b}$$

As usual, no entanglement is detected by either inequality for the spontaneous emission case, whereas starting from the ground-state entanglement is found for $n_0=2$, 3, and 4, with

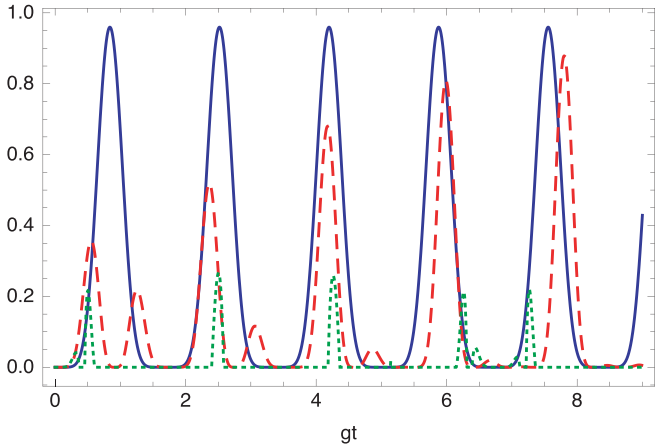

FIG. 5. Entanglement between subsystems of two atoms each, for a total $N = 4$ atoms starting in the ground state and interacting with $n_0 = 2$, 3, and 4 photons, as detected by inequalities (53).

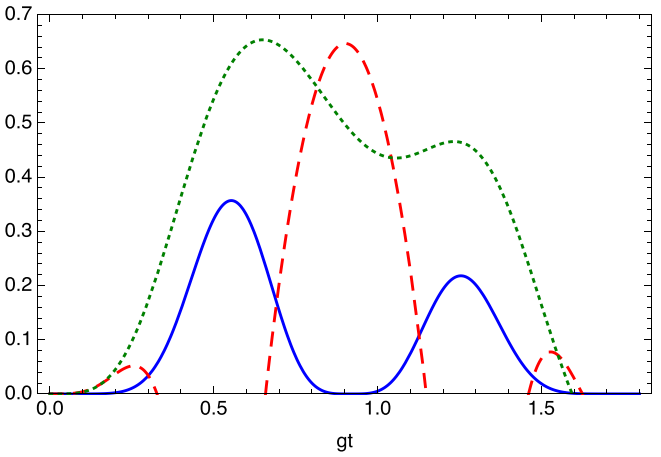

FIG. 6. Entanglement for the case $N = 4$ and $n_0 = 3$ as detected by inequalities (53) (solid line), (57) (bipartite entanglement, dashed line), and Eq. (52) (dotted line). Note that the solid line is actually always above zero except at $gt\sqrt{14} = 2n\pi$, whereas the other two cross the horizontal axis with a nonzero slope. Note also the existence of a small region to the right where the dashed line detects entanglement but the dotted one does not.

inequality (53b) being the strongest for $n_0 = 2$ and 3, and both being comparable for $n = 4$. The result, shown in Fig. 5, is visibly different from Fig. 4, which makes it clear that the criteria (26) and (27) with $n = 1$ and 2 explore different types of entanglement, as discussed in Sec. III. Specifically, the inequalities (53) test for the presence of the entangled superposition $|eegg\rangle + |ggee\rangle$, where the states of the two subgroups of atoms are "as far apart" as possible.

As the figure suggests, the $n_0 = 2$ and $n_0 = 3$ curves, while sometimes getting very close to zero, never actually cross the x axis. This is because, for $n_0 = 2$ and $n_0 = 3$, starting from the ground state, the state $m = 2$ cannot get populated, and thus the inequality (53b) is always satisfied as long as $c_0 \neq 0$. This means that, for these two cases, the atomic state is *always* entangled, with at least some entanglement of the form $|eegg\rangle + |ggee\rangle$, except for the isolated times when $c_0 = 0$. For $n_0 = 2$, it is easy to show analytically that this happens when $gt\sqrt{14} = 2n\pi$.

3. Comparison with other entanglement criteria

As mentioned in the Introduction, a number of criteria to detect entanglement in systems of spins have been developed through the years. In this section we compare, for this particular problem, the performance of our criteria to some of these alternative ones, focusing on the simplest ones, namely, those that only require the measurement of various components of the total angular momentum.

Perhaps the earliest such criterion is that Sørensen *et al.* [3], according to which a state of N qubits is entangled if there exist three orthogonal directions (labeled by k, l, and n below) such that

$$\frac{\langle \Delta^2 J_n \rangle}{\langle J_k \rangle^2 + \langle J_l \rangle^2} < \frac{1}{N}. \tag{54}$$

Because of the obvious symmetry of our problem (the density operator is diagonal in the J_z basis) we expect to find any maxima or minima when one of the directions above is chosen to be z and the other two are indifferently set to x or y. Clearly setting $n = z$ in Eq. (54) will not work in our case since $\langle J_x \rangle = \langle J_y \rangle = 0$, but we can try, e.g., $k = z$ and then $n = x$, $l = y$. For $\langle \Delta^2 J_x \rangle$ with a ρ of the form (31) we find

$$\langle \Delta^2 J_x \rangle = \langle \Delta^2 J_y \rangle = \langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{N}{4}\left(\frac{N}{2} + 1\right) - \frac{1}{2}\langle J_z^2 \rangle, \tag{55}$$

and hence the condition (54) becomes

$$\left(\frac{1}{N} - \frac{1}{2}\right)\langle J_z^2 \rangle - \frac{N}{4}\left(\frac{N}{2} + 1\right) > 0, \tag{56}$$

which is an impossibility for any $N \geqslant 2$.

Korbicz, Cirac, and Lewenstein [7] introduced a necessary and sufficient criterion for bipartite entanglement in symmetric systems: the reduction of the density operator to just two spins will be entangled if and only if there is a direction n along which the total angular momentum satisfies

$$1 - \frac{4\langle J_n \rangle^2}{N^2} - \frac{4\langle \Delta^2 J_n \rangle}{N} > 0. \tag{57}$$

With $n = x$ or y this gives $\langle J_z^2 \rangle > \frac{N^2}{4}$, which is impossible since $N^2/4$ is the largest value J_z^2 can have. With $n = z$, however, the criterion (57) actually shows entanglement most (but not all) of the time our other criteria do, and also sometimes when our criteria fail to detect it, as shown in Fig. 6.

The fact that our criteria sometimes show entanglement when Eq. (57) does not is indicative that we have genuine multipartite entanglement (involving three or more qubits) in those cases. In [7] the authors actually gave several other criteria to test for tripartite entanglement in any reduction of ρ to three qubits. One, Eq. (23) of [7], is specifically for Greenberger-Horne-Zeilinger (GHZ)-type entanglement, and fails to detect any for our system. Two others, Eqs. (24) and (25) of [7], detect either GHZ or W-type entanglement. The

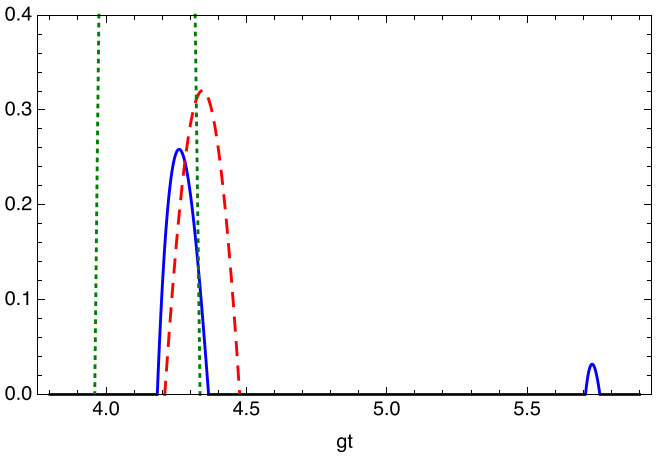

FIG. 7. Entanglement for the case $N = 4$ and $n_0 = 4$ as detected by inequalities (53) (solid line), (57) (bipartite entanglement, dashed line), and (58) (genuine tripartite entanglement, dotted line). Note that the three criteria overlap in places, but there are also times when entanglement is only detected by one of them.

first one, for the optimal choice of z ($z = n$), becomes

$$-3\langle J_z^3\rangle - \frac{3}{2}(N-2)\langle J_z^2\rangle + \frac{3N^2}{4}\langle J_z\rangle - \frac{N}{24} \times (7N^2 - 38N + 32) > 0. \tag{58}$$

For $N = 4$, this fails to detect entanglement for the $n_0 = 1$ case. For $n_0 = 2$, however, it detects it in a narrow region around the tall peaks of Fig. 5, meaning it always overlaps with Eq. (57), but it also shows that at those times the system has both bipartite and tripartite entanglement. For $n_0 = 3$, it sometimes overlaps with Eq. (57) and sometimes it does not, but in general both are worse than Eq. (53b). Lastly, for $n_0 = 4$, each of these three criteria spots some entanglement that the others miss, as can be seen in Fig. 7.

As for the last criterion presented in [7], their Eq. (25), for our system it yields the condition

$$\tfrac{5}{6}\langle J_z^3\rangle - \tfrac{1}{4}(N-2)\langle J_z^2\rangle - \tfrac{1}{24}(3N^2+6N-4)\langle J_z\rangle - \tfrac{1}{16}(N^3 - 4N^2 + 4N) > 0. \tag{59}$$

For $N = 4$, we find this only detects entanglement when $n_0 = 1$. This happens around the times where the left-hand side of the inequality (51) is largest in absolute value, that is, around $gt = (2n+1)\pi/4$. Recall that both our criteria (48b) and (52b) show that the $n_0 = 1$ case is always entangled [except when $c_0(t) = 0$], both across 1-to-3 qubit and 2-2 qubit partitions. From Eq. (59), we get the additional information that the system also has genuinely tripartite entanglement for the reductions of ρ around the times $gt = (2n+1)\pi/4$.

Finally, Tóth *et al.* presented in [13] several other criteria, summarized in their Eqs. (7b) to (7d). When we apply them to our system, we obtain only one inequality that can actually be satisfied: this is derived from their Eq. (7c) with $m = z$, and ends up being equivalent to Eq. (57).

V. CONCLUSION

We presented a number of results concerning the usefulness of the Hillery-Zubairy entanglement criteria for systems of N qubits. For $N = 2$, we showed that the criterion (1b) is in general not useful and that Eq. (1a), while useful, is not in general an entanglement monotone, although we also identified some sets of states for which it does have such a property. By focusing on the kinds of correlation it is best suited to detect, we generalized it to angular momentum systems, or systems with $N > 2$ qubits, and we used these extensions, Eqs. (26) and (27), to explore entanglement among atoms in the Tavis-Cummings model when the field is initially in a number state, so the atomic state is an evolving mixture of Dicke states. By comparing our results with those obtained from other previously derived criteria to detect entanglement in systems of spins, we showed that ours can, in fact, detect entanglement in situations where the others miss it and vice versa. We conclude that, as all of these criteria explore different entanglement possibilities, all are valuable if one wants to obtain as complete a picture as possible of the quantum correlations possible in mixed states of N-qubit systems.

With respect to the Tavis-Cummings model specifically, we were able to prove a number of results, such as that, for any N, when the atoms start in the ground state with only one photon present, the resulting state (a mixture of the ground state and a W-type state) is always entangled, except for a discrete set of times. We also showed that the same is true in the $N = 4$ atom case when only two or three photons are initially present. Generalizations of this last result to larger N systems are almost certainly possible, but we did not pursue them here.

Perhaps our most intriguing result, however, is that all the criteria fail to detect any entanglement, at any time, for the "spontaneous emission" case, where the atoms start all in the excited state with no photons present. We conjecture that this is probably a general result, valid for all N, but we have no way of proving it or explaining it. It implies, among other things, that the probability to find all the atoms in the ground state cannot be equal to 1 at any time since otherwise the subsequent evolution would produce entangled states; hence, the picture it suggests is that as long as the emitted photons "stick around," they can be reabsorbed and the atomic system cannot get rid of all its energy at any time. This, at least, would change in the presence of losses (e.g., in free space or in a leaky cavity), and it might be interesting to see if entanglement appears in that case.

ACKNOWLEDGMENT

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