

Robust one-sided self-testing of two-qubit states via quantum steeringYukun Wang^{1,2}, Xinjian Liu,¹ Shaoxuan Wang,¹ Haoying Zhang,¹ and Yunguang Han^{3,*}¹*Beijing Key Laboratory of Petroleum Data Mining, China University of Petroleum, Beijing 102249, China*²*State Key Laboratory of Cryptology, P.O. Box 5159, Beijing 100878, China*³*College of Computer Science and Technology, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China*

(Received 26 May 2022; revised 26 September 2022; accepted 27 September 2022; published 18 October 2022)

Entangled two-qubit states are the core building blocks for constructing quantum communication networks. Their accurate verification is crucial to the functioning of the networks, especially for untrusted networks. In this work we study the self-testing of two-qubit entangled states via steering inequalities, with robustness analysis against noise. More precisely, steering inequalities are constructed from the tilted Clauser-Horne-Shimony-Holt inequality and its general form, to verify the general two-qubit entangled states. The study provides a good robustness bound, using both local extraction map and numerical semidefinite-programming methods. In particular, optimal local extraction maps are constructed in the analytical method, which yields the theoretical optimal robustness bound. To further improve the robustness of one-sided self-testing, we propose a family of three measurement settings steering inequalities. The result shows that three-setting steering inequality demonstrates an advantage over two-setting steering inequality on robust self-testing with noise. Moreover, to construct a practical verification protocol, we clarify the sample efficiency of our protocols in the one-sided device-independent scenario.

DOI: [10.1103/PhysRevA.106.042424](https://doi.org/10.1103/PhysRevA.106.042424)**I. INTRODUCTION**

Quantum entangled states are the key resource of quantum information technologies, such as quantum networks [1], cryptography [2], computation [3], and metrology [4]. As we advance towards the second quantum revolution [5], the characterization and certification of quantum devices become extremely important topics in practical applications of quantum technologies [6,7].

To ensure the proper functioning of a quantum network, it is essential to certify the entangled state deployed in the network accurately and efficiently. Besides the traditional quantum state tomography method, various methods have been proposed to improve the efficiency and apply to different scenarios such as direct fidelity estimation [8], compressed sensing tomography [9], and shadow tomography [10]. In the past few years, quantum state verification (QSV) has attracted much attention by achieving remarkably low sample efficiency [11,12]. One drawback of the quantum state verification method is that it requires perfect characterization of the measurements performed by the quantum devices, and thus it is device dependent and not applicable to the untrusted quantum network. Self-testing [13,14] is a prominent candidate of quantum state certification in the device-independent (DI) scenario, in which all quantum devices are treated as black boxes. Taking advantage of Bell nonlocality [15], many important results on self-testing have been achieved such as self-testing various quantum entangled states [16–18], self-testing entangled quantum measurement

[19,20], and parallel self-testing [21,22]. Self-testing has wide applications in device-independent quantum information tasks such as device-independent quantum random number generation [23,24] and quantum key distribution [25,26].

Lying between standard QSV and self-testing, there is a semi-device-independent (SDI) scenario [27] in which some parties are honest while some others may be dishonest. The certification in this scenario can be called SDI self-testing or SDI state verification. This scenario has wide applications in quantum information processing such as one-sided device-independent (1SDI) quantum key distribution [28], quantum random number generation [29], verifiable quantum computation [30], and anonymous communication [31–33]. Meanwhile, the certification in the SDI scenario is closely related to the foundational studies on quantum steering in the untrusted quantum networks [34–37]. However, not much is known about the quantum certification in the SDI scenario despite its significance. In [30,38] the authors studied the one-sided self-testing of a maximally entangled two-qubit state based on two-setting quantum steering inequality. In [39] the authors proposed various verification protocols for a Bell state based on multiple settings. For nonmaximally entangled two-qubit states, the authors in [40] realized the one-sided certification by combining fine-grained inequality [41] and analogous Clauser-Horne-Shimony-Holt (CHSH) inequalities [42], which is more complicated than traditional self-testing. In [27] the authors proposed a tilted steering inequality analogous to the tilted CHSH inequality [43] for one-sided self-testing of two-qubit states. Then they generalized the one-sided certification to general pure bipartite states by adopting the subspace method in the DI scenario [18]. In Ref. [44] a class of steering inequalities concentrating on the

*hanyunguang@nuaa.edu.cn

nonmaximally entangled bipartite-qudit state was constructed, where they achieve the bipartite-qudit state self-testing by performing only two measurements, while in Ref. [45] steering inequalities with $d + 1$ measurement settings were used for self-testing the same states. However, the robustness analysis there follows the norm inequality method in [16,38] (if it is not missed); thus the result is quite weak. For the multipartite case, the studies of SDI certification are mainly focused on Greenberger-Horne-Zeilinger (GHZ) states as the generalization of the Bell state [39,46,47].

In this paper we focus on the robust one-sided self-testing of two-qubit entangled states. We construct two types of two-setting steering inequalities for general two-qubit entangled states based on tilted CHSH inequality and its general form. For the first type, an analytical and optimal robustness bound is obtained using the local extraction channel method introduced in [48]. For the second type, we get a nearly linear robustness bound using a numerical method based on the SWAP trick [17] and semidefinite programming (SDP). To put our work in perspective, we compare the robustness result in the ISDI scenario with both DI and device-dependent scenarios. Our result can be applied to the certification of high-dimensional quantum devices as building blocks.

Furthermore, we construct three measurement settings steering inequalities for general two-qubit states, which is beyond the conventional one-sided self-testing based on two settings. In [39] the authors studied the optimal verification of the Bell state and GHZ states in the ISDI scenario using multiple measurement settings. However, their study is limited to the maximally entangled state in the bipartite case. Based on the three-setting steering inequalities, it is shown that the robustness bound can be further improved. This opens the question of how much the resistance to noise can be improved using multiple measurement settings. Finally, to construct a practical verification protocol, we clarify the sample efficiency for our protocols in the ISDI scenario. It is shown that approximately optimal sample efficiency can be obtained based on the steering inequalities we construct.

II. PRELIMINARIES

A. Steering scenario and steering inequalities

Let us start by recalling the steering theory. Two distant parties, Alice and Bob, are considered, and between them are many copies of the state $\rho_{AB} \in H_A \otimes H_B$. Bob performs two measurements, labeled y , on his particle and obtains the binary outcome b . Meanwhile, Alice receives the corresponding unnormalized conditional states $\rho_{b|y}$ and performs measurements randomly, labeled x , and obtains the binary outcome a . If Alice cannot explain the assemblage of received states by assuming preexisting states at her location and some pre-shared random numbers with Bob, she has to believe that Bob has steerability of her particle from a distance. To determine whether Bob has steerability of her, Alice asks Bob to run the experiment many times with her. Finally, they obtain the measurement statistics. If the statistics admit the description

$$p(a, b|x, y; \rho_{AB}) = \sum_{\lambda} p(\lambda) p(a|x, \rho_{\lambda}) p(b|y, \lambda), \quad (1)$$

then Alice knows Bob does not have steerability of her. This nonsteerable correlation model is the so-called local hidden variable (LHV)–local hidden state (LHS) model [42]. The LHV-LHS decomposition is based on the idea that Bob's outcomes are determined by a local hidden random λ and Alice's outcomes are determined by local measurements on quantum state ρ_{λ} .

The combination of the statistics will give a steering inequality, where the LHV-LHS model can be used to establish local bounds for the steering inequality; violation of such inequalities implies steering. In Ref. [36] the authors introduced a family of steering inequalities for the Bell state

$$S_n \equiv \frac{1}{n} \sum_{k=1}^n \langle \hat{\sigma}_k^A B_k \rangle \leq C_n, \quad (2)$$

where C_n is the LHS bound

$$C_n = \max_{\{A_k\}} \left\{ \lambda_{\max} \left(\frac{1}{n} \sum_{k=1}^n \hat{\sigma}_k^A B_k \right) \right\}, \quad (3)$$

with $\lambda_{\max}(\hat{O})$ denoting the largest eigenvalue of \hat{O} .

An approach to constructing this family of steering inequalities is transforming from Bell inequalities. Bell states are shown to maximally violate the analogous CHSH inequality [30,38,42]. For partially entangled two-qubit states, the authors in Ref. [27] constructed tilted steering inequalities from tilted CHSH inequalities [43]. In this paper we study the more general tilted steering inequalities constructed from tilted CHSH inequalities and study the robustness of one-sided self-testing based on analogous steering inequalities. Furthermore, we consider the construction of three-measurement-setting steering inequalities for general two-qubit states.

B. SDI certification and local extraction channel

In this paper we focus on a one-sided self-testing two-qubit entangled state based on the steering inequalities. To this end, we first review the concept of self-testing.

Self-testing was originally known as a DI state verification, where some observed statistics $p(a, b|x, y)$ from quantum devices can determine uniquely the underlying quantum state and the measurements, up to a local isometry. As an example, the maximal violation of CHSH inequality uniquely identifies the maximally entangled two-qubit state [14,16]. Usually, self-testing relies on the observed extremal correlations. If the quantum systems that achieve the extremal correlations are unique up to local isometries, we say the extremal correlations $p(a, b|x, y)$ self-test the target system $\{|\bar{\psi}\rangle, \bar{M}_{a|x}, \bar{N}_{b|y}\}$. Defining the local isometry as $\Phi = \Phi_{AA'} \otimes \Phi_{BB'}$, self-testing can be formally defined as

$$\begin{aligned} \Phi |\psi\rangle_{AB} |00\rangle_{A'B'} &= |\text{junk}\rangle_{AB} |\bar{\psi}\rangle_{A'B'}, \\ \Phi M_{a|x} \otimes N_{b|y} |\psi\rangle_{AB} |00\rangle_{A'B'} &= |\text{junk}\rangle_{AB} \bar{M}_{a|x} \otimes \bar{N}_{b|y} |\bar{\psi}\rangle_{A'B'}. \end{aligned} \quad (4)$$

For the ISDI scenario, only the existence of an isometry Φ_B on Bob's side is required,

$$\begin{aligned} \Phi |\psi\rangle_{AB} |0\rangle_{B'} &= |\text{junk}\rangle_B \otimes |\bar{\psi}\rangle_{AB'}, \\ \Phi M_{b|y} |\psi\rangle_{AB} |0\rangle_{B'} &= |\text{junk}\rangle_B \otimes \bar{M}_{b|y} |\bar{\psi}\rangle_{AB'}, \end{aligned} \quad (5)$$

where $M_{b|y}$ acts on \mathcal{H}_B and $\bar{M}_{b|y}$ acts on $\mathcal{H}_{B'}$.

In addition to the above ideal definition of self-testing, it is essential to study the robustness of self-testing in the imperfect case when the obtained data deviate from the ideal value. There are two frameworks in the robustness analysis of self-testing. The first approach is based on the SWAP method by introducing an ancilla system. The desired state can be swapped out of the real quantum system and then the distance from the target state can be calculated. One way to calculate this closeness is based on the analytic method involving mathematical inequality techniques first proposed in [16]. The second one is the numerical method based on semidefinite programming combining the hierarchy strategy, which is proposed by Navascués, Pironio and Acín in Ref. [49] and called Navascués-Pironio-Acín (NPA) method. Usually, the numerical method gives much higher robustness.

The second approach is based on operator inequalities first introduced in Ref. [48], which is now widely used in the robustness analysis of self-testing. For a self-testing Bell state using CHSH inequality and a self-testing GHZ state using Mermin inequality, the operator inequalities give a nearly optimal bound. Robustness analysis of self-testing with operator inequalities can recur for local extraction map, which hinges on the idea that local measurements can be used to virtually construct a local extraction channel to extract the desired state from the real quantum system. The local extractability of the target ψ_{AB} from ρ_{AB} is quantified

$$\Xi(\rho_{AB} \rightarrow \psi_{AB}) := \max_{\Lambda_A, \Lambda_B} F((\Lambda_A \otimes \Lambda_B)(\rho_{AB}), \psi_{AB}), \quad (6)$$

where the maximum is taken over all possible local channels constructed with local measurements. For the 1SDI scenario, Alice's side is trusted, and thus the extraction channel on Alice's side is $\Lambda_A = I_A$. The lower bound of the fidelity between ρ and the target state under the observed steering inequality can be defined as one-sided extractability

$$F(\rho_{AB}, \psi_{AB}) := \inf_{\rho_{AB}: S(\rho) \geq S_{\text{obs}}} \max_{\Lambda_B} F(\Lambda_B(\rho_{AB}), \psi_{AB}), \quad (7)$$

where $S(\cdot)$ is the steering expression and S_{obs} is observed violation. To derive a linear bound of the fidelity about the observed steering inequality violation, real parameters s and τ must be fixed such that $F \geq sS_{\text{obs}} + \tau$. This is equivalent to finding Λ_B (constructed by Bob's local measurement operators M_y^b) to make

$$K \geq sS + \tau \mathbb{I}, \quad (8)$$

where $K := (I_A \otimes \Lambda_B^+)(\psi_{AB})$ and Λ^+ refers to the dual channel of quantum channel Λ . By taking the trace with the input state ρ_{AB} on both sides of Eq. (8), one can get $F \geq sS_{\text{obs}} + \tau$, in view of $\langle \Lambda_B^+(\psi_{AB}), \rho_{AB} \rangle = \langle \psi_{AB}, \Lambda_B(\rho_{AB}) \rangle$.

In the 1SDI scenario, Bob's side is untrusted; thus Eq. (8) is required to hold for Alice in two dimensions and Bob in arbitrary dimension. Since the measurements we consider in this paper is dichotomic, consideration in qubit space will be sufficient on Bob's side.

III. ONE-SIDED SELF-TESTING BASED ON TWO-SETTING STEERING INEQUALITIES

In the device-independent scenario, a general pure entangled two-qubit state

$$|\Phi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle \quad (9)$$

has been proven to be self-tested [50,51] by the maximal violation of tilted CHSH inequalities [43], which can be parametrized as

$$\hat{I}_\alpha = \alpha A_0 + A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq \alpha + 2, \quad (10)$$

where $\sin 2\theta = \sqrt{\frac{4-\alpha^2}{4+\alpha^2}}$. The maximum quantum value is $\sqrt{8+2\alpha^2}$. The quantum measurements used to achieve the maximal quantum violation are $\{\sigma_z; \sigma_x\}$ for Alice and $\{\cos \mu \sigma_z + \sin \mu \sigma_x; \cos \mu \sigma_z - \sin \mu \sigma_x\}$ for Bob, where $\tan \mu = \sin 2\theta$ and $\sigma_{x,z}$ are Pauli X and Z measurements.

Having $\alpha = 0$ corresponds to CHSH inequality and the state can be self-tested as a Bell state. The self-testing criterion based on this tilted CHSH inequality is robust against noise. The best robustness bound to date can be found in [48,51], in which the authors introduced the local extraction channel method. However, as claimed in [48], the theoretical optimal upper bound is not achievable. Theoretically, the optimal bound is tied to the maximum classical violation which starts to achieve nontrivial fidelity. The nontrivial fidelity that demonstrates entanglement for the target state is $F > \cos^2 \theta$. Kaniewski guessed that it might be related to the fact that the quantum value of the CHSH inequality does not reach its algebraic limit of 4. Here in the 1SDI scenario, we will show that the theoretical optimal bound can be achieved.

To achieve the 1SDI self-testing criterion, we construct two types of two-setting steering inequalities, which are based on above tilted CHSH inequality by taking the measurements on Alice's side as trusted.

A. One-sided self-testing based on standard tilted CHSH steering inequality

Taking the measurements on Alice's side as trusted, the standard tilted CHSH inequality in Eq. (10) can be transformed to the analog of the tilted CHSH steering inequality

$$\begin{aligned} \hat{S}_\alpha &= \alpha A_0 + A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \\ &= \alpha Z + Z(B_0 + B_1) + X(B_0 - B_1) \\ &\leq \alpha + 2, \end{aligned} \quad (11)$$

which maintains the maximum quantum violation $S_\alpha^Q = \sqrt{8+2\alpha^2}$ as in the DI scenario. We prove that partially entangled two-qubit states can be self-tested using this analogous tilted CHSH steering inequality in a 1SDI manner. The proof is similar to DI self-testing using a tilted CHSH inequality, except that we can trust Alice's measurements now. The trustworthiness of Alice's side can simplify the proof as an advantage. Another advantage is that the theoretical optimal robustness bound can be obtained in the 1SDI scenario with this steering inequality. By contrast, the optimal bound cannot be achieved in DI self-testing with a tilted CHSH inequality. In the following we will show both the analytical proof and the robustness analysis.

1. Self-testing based on analogous tilted CHSH steering inequality

We provide the simple proof here. Though Alice's side is trustworthy, by definition only the existence of isometry on Bob's side will be sufficient to determine uniquely the state and the measurements. However, for simplicity,

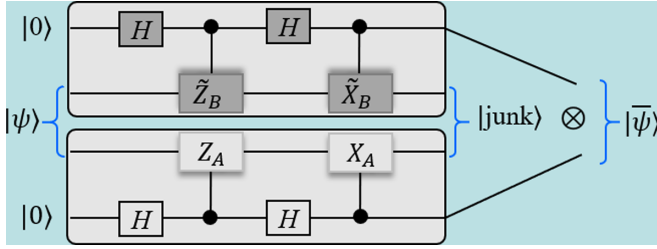


FIG. 1. The SWAP isometry applied on Alice and Bob's side, where the operators Z_A and X_A are exactly the Pauli Z and X operators.

we also introduce one isometry on Alice's side, which has been widely used in the DI scenario, shown in Fig. 1. As shown in below, with the sum of squares decomposition of a positive-semidefinite matrix [52], it is easy to find the algebraic relations that are necessarily satisfied by the target quantum state and measurements to complete the proof.

After the isometry, the systems will be

$$\begin{aligned} \Phi(|\psi\rangle) &= \frac{1}{4}[(I + Z_A)(I + \tilde{Z}_B)|\psi\rangle|00\rangle \\ &+ X_A(I + Z_A)(I - \tilde{Z}_B)|\psi\rangle|01\rangle \\ &+ \tilde{X}_B(I - Z_A)(I + \tilde{Z}_B)|\psi\rangle|10\rangle \\ &+ X_A\tilde{X}_B(I - Z_A)(I - \tilde{Z}_B)|\psi\rangle|11\rangle]. \end{aligned} \quad (12)$$

To derive an underlying state $|\psi\rangle$ that is equivalent to the target one, the algebraic relations between the operators acting on the state should be given. We notice that the analogous tilted CHSH steering inequality \hat{S}_α has the maximum quantum value S_α^Q . This implies that the operator $\hat{S}_\alpha := S_\alpha^Q \mathbb{I} - \hat{S}_\alpha$ should be positive semidefinite (PSD) for all possible quantum states and measurement operators on Bob's side. This can be proven by providing a set of operators $\{P_i\}$ which are polynomial functions of A_x (Z_A and X_A) and B_y such that $\hat{S}_\alpha = \sum_i P_i^\dagger P_i$ holds for any set of measurement operators satisfying the algebraic properties $A_x^2 = \mathbb{I}$ and $B_y^2 = \mathbb{I}$. The decomposition form of $\hat{S}_\alpha = \sum_i P_i^\dagger P_i$ is called a sum of squares (SOS). By a SOS decomposition one can provide a direct certificate that the upper quantum bound of \hat{S}_α is S_α^Q from its PSD, as well as some relations between the projectors on the states, which will be used to give a self-testing statement. This method was first introduced in [50] for the family of CHSH-like Bell inequalities. Given SOS decompositions, if one observes the maximal quantum violation of the steering inequality (CHSH-like one) under state $|\psi\rangle$, then each squared term in SOS decompositions acting on $|\psi\rangle$ should be zero, i.e., $P_i|\psi\rangle = 0$. Then useful relations for the measurements operators acting on underlying state can be obtained from these zero terms.

Similar to the CHSH inequality scenario, two types of SOS decompositions for the analogous tilted CHSH operator in Eq. (11) can be given. The first one is

$$\hat{S}_\alpha = \frac{1}{2S_\alpha^Q} \{ \hat{S}_\alpha^2 + (\alpha X_A - S_0)^2 \} \quad (13)$$

and the second one is

$$\begin{aligned} \hat{S}_\alpha &= \frac{1}{2S_\alpha^Q} \left\{ \left(2Z_A - S_\alpha^Q \frac{B_0 + B_1}{2} + \frac{\alpha}{2} S_1 \right)^2 \right. \\ &\left. + \left(2X_A - S_\alpha^Q \frac{B_0 - B_1}{2} + \frac{\alpha}{2} S_2 \right)^2 \right\}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} S_0 &= Z_A(B_0 - B_1) + X_A(B_0 + B_1), \\ S_1 &= Z_A(B_0 + B_1) - X_A(B_0 - B_1), \\ S_2 &= Z_A(B_0 - B_1) - X_A(B_0 + B_1). \end{aligned} \quad (15)$$

Based on the maximal violation of the analogous tilted CHSH inequality, the existence of the SOS decomposition for \hat{S}_α implies that

$$Z_A|\psi\rangle - \tilde{Z}_B|\psi\rangle = 0, \quad (16)$$

$$\sin(\theta)X_A(I + \tilde{Z}_B)|\psi\rangle - \cos(\theta)\tilde{X}_B(I - Z_A)|\psi\rangle = 0, \quad (17)$$

where $\tilde{Z}_B := \frac{B_0 + B_1}{2 \cos \mu}$ and $\tilde{X}_B := \frac{B_0 - B_1}{2 \sin \mu}$. Then with the algebraic relations (16) and (17) and the fact that $Z_A X_A = -X_A Z_A$, Eq. (12) can be rewritten as

$$\Phi(|\psi\rangle) = |\text{junk}\rangle[\cos \theta|00\rangle + \sin \theta|11\rangle],$$

where $|\text{junk}\rangle = \frac{1}{2 \cos \theta}(I + Z_A)|\psi\rangle$. This means the underlying state is unique to the target one up to local isometries, and thus completes the self-testing statement.

2. Self-testing robustness

Here we mainly focus on the self-testing of quantum states. For the self-testing of quantum measurements, the analysis can be related to quantum states according to Ref. [17]. The procedure is similar, starting with $\Phi M_B(|\psi\rangle)$ instead of $\Phi(|\psi\rangle)$. In this case, the figure of merit should quantify how $M_B|\psi\rangle$ is close to the ideal measurements acting on the target state.

As introduced in Sec. II B, to obtain a better self-testing robustness bound for the state, we should find the smallest value of s while keeping $K - s\hat{S} - \tau\mathbb{I}$ PSD. To this end, we first give the spectral decomposition of \hat{S}_α . Without loss of generality, we write Bob's measurements as

$$B_r = \cos \mu \sigma_z + (-1)^r \sin \mu \sigma_x, \quad (18)$$

with $r \in \{0, 1\}$ and $\mu \in [0, \pi/2]$. Then the spectral decomposition of \hat{S}_α is

$$\hat{S}_\alpha = \sum \lambda_i |\psi_i\rangle\langle\psi_i|, \quad i = 1, 2, 3, 4, \quad (19)$$

with $\lambda_1^2 + \lambda_2^2 = 8 + 2\alpha^2$, $\lambda_3 = -\lambda_2$, and $\lambda_4 = -\lambda_1$.

According to different value ranges of μ , the following two cases are discussed.

Case I: $\cos 2\mu \geq \frac{\alpha^2}{4}$ or equivalently $\mu \in [0, \arcsin \sqrt{\frac{4-\alpha^2}{8}}]$.

The eigenvalues of \hat{S}_α have the form

$$\lambda_{1/2} = \pm \sqrt{\alpha^2 + 4 \sin^2 \mu} + 2 \cos \mu.$$

The eigenvectors and the constraints for γ and μ are

$$\begin{aligned} |\psi_1\rangle &= \cos \gamma |00\rangle + \sin \gamma |11\rangle, \\ |\psi_2\rangle &= \sin \gamma |00\rangle - \cos \gamma |11\rangle, \end{aligned}$$

$$\begin{aligned}
 |\psi_3\rangle &= \cos \gamma |01\rangle + \sin \gamma |10\rangle, \\
 |\psi_4\rangle &= -\sin \gamma |01\rangle + \cos \gamma |10\rangle, \\
 \lambda_1 \cos^2 \gamma + \lambda_2 \sin^2 \gamma &= \alpha + 2 \cos \mu, \\
 \lambda_2 \cos^2 \gamma + \lambda_1 \sin^2 \gamma &= -\alpha + 2 \cos \mu, \\
 (\lambda_1 - \lambda_2) \cos \gamma \sin \gamma &= 2 \sin \mu,
 \end{aligned}$$

$$\text{with } \sin 2\gamma = \frac{2 \sin \mu}{\sqrt{\alpha^2 + 4 \sin^2 \mu}}.$$

To obtain the optimal robustness bound, we consider the following local extraction channel on Bob's side: With the probability of q_1 , he performs the identity operation I on his qubit; with the probability of q_2 , he performs σ_z on his qubit. By this local extraction channel, the ideal state is transformed into $K = q_1 |\psi\rangle\langle\psi| + q_2 \sigma_z |\psi\rangle\langle\psi| \sigma_z$. Denote $K - s\hat{I}_\alpha - \tau\mathbb{I}$ by G . The PSD condition of G requires that all the eigenvalues of it are non-negative, which gives

$$\frac{2 \sin \mu s - C}{2 \cos \theta \sin \theta} + \frac{1}{2} \leq q_1 \leq \frac{2 \sin \mu s + C}{2 \cos \theta \sin \theta} + \frac{1}{2}, \quad (20)$$

where

$$\begin{aligned}
 C &= \sqrt{\cos^2 \theta + [\beta_Q - (\alpha + 2 \cos \mu)]s - 1} \\
 &\times \sqrt{\sin^2 \theta + [\beta_Q - (-\alpha + 2 \cos \mu)]s - 1},
 \end{aligned}$$

with $\beta_Q = \sqrt{8 + 2\alpha^2}$.

We can choose q_1 in the suitable range to saturate its upper bound, which makes G PSD. Meanwhile, we obtain the smallest value of s as

$$s = \frac{1 - \cos^2 \theta}{\beta_Q - (2 + \alpha)} \quad (21)$$

and the corresponding value of τ is

$$\tau = 1 - \sqrt{8 + 2\alpha^2} s, \quad (22)$$

which is exactly equal to the theoretical optimal value. Thus we obtain the optimal robustness bound in the 1SDI scenario using the given extraction channel. Therefore, it gives the optimal robustness bound of self-testing based on the analogous tilted CHSH steering inequality

$$\begin{aligned}
 F &= (\beta - \sqrt{8 + 2\alpha^2})s + 1 \\
 &= (\beta - \sqrt{8 + 2\alpha^2}) \frac{1 - \frac{\sqrt{2}\alpha}{\sqrt{4-\alpha^2}}}{2\sqrt{8 + 2\alpha^2} - (4 + 2\alpha)} + 1 \quad (23)
 \end{aligned}$$

for observed violation β .

Case 2: $0 \leq \cos 2\mu \leq \frac{\alpha^2}{4}$ or equivalently $\mu \in (\arcsin \sqrt{\frac{4-\alpha^2}{8}}, \frac{\pi}{4}]$. The following is the local extraction channel in this case. Bob performs identity operation I with probability q_1 and performs σ_x with probability q_2 . Then the ideal state is transformed into $K = q_1 |\psi\rangle\langle\psi| + q_2 \sigma_x |\psi\rangle\langle\psi| \sigma_x$. The PSD condition of $G := K - s\hat{I}_\alpha - \tau\mathbb{I} \geq 0$ gives

$$q_1 = \max \left\{ 0, \frac{4 \sin^2 \mu s^2 + (C_1 s + \tau)(C_2 s - \tau)}{(\beta_Q + 2 \sin 2\theta \sin \mu + \cos^2 \theta C_2 - \sin^2 \theta C_1) s - 1} \right\},$$

where $\beta_Q = \sqrt{8 + 2\alpha^2}$. It also gives $s = \frac{1 - \cos^2 \theta}{\beta_Q - (2 + \alpha)}$ and $\tau = 1 - \sqrt{8 + 2\alpha^2} s$, which turn out to obtain the same robustness bound as in case 1. (See Appendix A for details.)

In conclusion, the theoretical linear optimal robustness bound can be obtained for self-testing of two-qubit entangled states using the analogous tilted CHSH steering inequality. Different from self-testing in the DI scenario, theoretical optimal robustness bound can be obtained using the local extraction channel method. The reason might be that the extraction channel is needed only on one side in the steering scenario without coordination.

Comparison with the DI and device-dependent scenarios. To put our work into perspective, we compare the certification in the 1SDI scenario with both DI and device-dependent (DD) scenario.

In the DD scenario, the measurements on both sides are trusted and equal to the ideal measurements. In this case, we have

$$\hat{I}_\alpha = \alpha A_0 + A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \quad (24)$$

$$= \alpha Z + 2 \cos \mu ZZ + 2 \sin \mu XX, \quad (25)$$

where $\sin 2\theta = \sqrt{\frac{4-\alpha^2}{4+\alpha^2}}$ and $\tan \mu = \sin 2\theta$. It could be shown that

$$|\Psi\rangle\langle\Psi| \geq \frac{\hat{I}_\alpha}{\sqrt{8 + 2\alpha^2}}. \quad (26)$$

Thus, in the trusted measurement scenario, we have the lower bound of the fidelity

$$F_{\text{DD}} \geq \frac{\beta}{\sqrt{8 + 2\alpha^2}}. \quad (27)$$

In the DI scenario, the authors in [51] conjectured the lower bound of fidelity

$$F_{\text{DI}} \geq s_\alpha \beta + \mu_\alpha, \quad (28)$$

with

$$s_\alpha = \frac{(\sqrt{8 + 2\alpha^2} + 2 + \alpha)(3\sqrt{8 + 2\alpha^2} - \sqrt{4 - \alpha^2} - \alpha\sqrt{2})}{4(2 - \alpha)^2 \sqrt{8 + 2\alpha^2}}, \quad (29)$$

$$\mu_\alpha = 1 - s_\alpha \sqrt{8 + 2\alpha^2}. \quad (30)$$

Their comparison with the SDI scenario is given in Fig. 2. In the case of $\alpha = 0$, it corresponds to the CHSH inequality and the target state is a singlet. The two other cases correspond to the tilted CHSH inequality and partially entangled two-qubit states. Obviously, it has $F_{\text{DD}} > F_{\text{1SDI}} > F_{\text{DI}}$ for all three cases. For $\alpha = 0$, the nontrivial fidelity bound of the singlet state is 0.5. The results show that the nontrivial fidelity bound can be obtained in the DI scenario when the quantum value is larger than 2.105, while for the 1SDI and DD scenarios the bounds are 2 and $\sqrt{2}$, respectively. For $\alpha = 0.5$, the nontrivial fidelity bound of the target state is 0.672. The results show that the nontrivial fidelity bound can be obtained in the DI scenario when the quantum value is larger than 2.655, while for the 1SDI and DD scenarios the bounds are 2.5 and 1.958, respectively. For $\alpha = 1$, the nontrivial fidelity bound of the

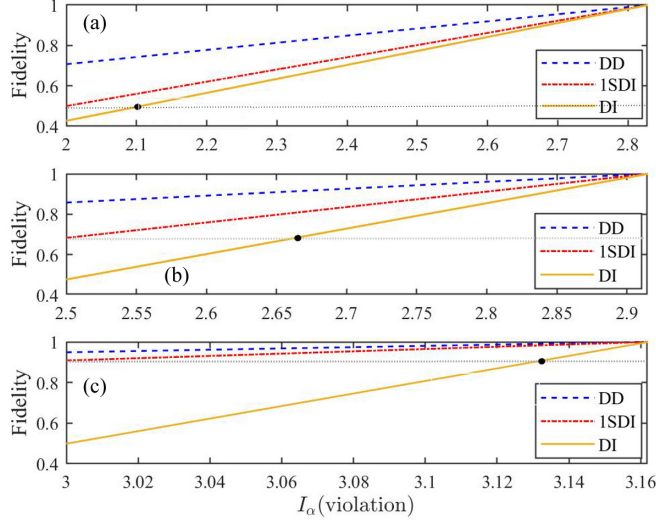


FIG. 2. Comparison of robustness bound between the DI (yellow solid line), 1SDI (red dash-dotted line), and DD (blue dotted line) scenarios for (a) $\alpha = 0$ and $\beta = 2.015$, (b) $\alpha = 0.5$ and $\beta = 2.655$, and (c) $\alpha = 0.1$ and $\beta = 3.103$.

target state is 0.816. The results show that the nontrivial fidelity bound can be obtained in the DI scenario when the quantum value is larger than 3.103, while for the 1SDI and DD scenarios the bounds are 3 and 2.581, respectively. It is shown that with the increase of α , especially for $\alpha = 1$, the 1SDI self-testing bound is much better than in the DI scenario and closer to the DD scenario. Thus our method achieves significant improvement in the 1SDI certification of less entangled two-qubit states, which is comparable to the device-dependent scenario.

B. One-sided self-testing based on general tilted CHSH inequality

In this section we construct two-setting steering inequalities from the general tilted CHSH inequality [43]

$$\hat{S}_{\alpha,\beta} = \alpha A_0 + \beta A_0 B_0 + \beta A_0 B_1 + A_1 B_0 - A_1 B_1. \quad (31)$$

The maximal classical and quantum bounds are $\alpha + 2(1 + \beta)$ and $\sqrt{(4 + \alpha^2)(1 + \beta^2)}$, respectively. The quantum bound can be achieved by pure two-qubit states (9) and corresponding measurements settings $\{\sigma_z; \sigma_x\}$ for Alice and $\{\cos \mu \sigma_z + \sin \mu \sigma_x; \cos \mu \sigma_z - \sin \mu \sigma_x\}$ for Bob, with $\sin 2\theta = \sqrt{\frac{4 - \alpha^2 \beta^2}{4 + \alpha^2}}$ and $\tan \mu = \frac{\sin 2\theta}{\beta}$.

Taking the measurements on Alice's side as trusted, this Bell inequality can be transformed into

$$\hat{S}_{\alpha,\beta} = \alpha Z + \beta Z(B_0 + B_1) + X(B_0 - B_1), \quad (32)$$

which is a steering inequality. However, we can also introduce two other measurements to represent $B_0 + B_1$ and $B_0 - B_1$, thus rewriting the steering inequality as

$$S_{\alpha,\beta}^{(1)} = \alpha \langle Z \rangle + \beta \langle ZB_0 \rangle + \langle XB_1 \rangle \leq \sqrt{1 + (\alpha + \beta)^2}, \quad (33)$$

with $\beta > 0$. The maximal quantum violation is $\beta + \sqrt{1 + \alpha^2} := S_Q$.

This form of steering inequality allows us to compare the construction with the one proposed in Ref. [27], which changes the marginal term to Bob's side,

$$S_{\alpha,\beta}^{(2)} = \alpha \langle B_0 \rangle + \beta \langle ZB_0 \rangle + \langle XB_1 \rangle \leq \alpha + \sqrt{1 + \beta^2}, \quad (34)$$

with $\beta^2 = \alpha^2 + 1$, and keeps the quantum bound as in Eq. (33). It should be remarked that the constraints of β and α given in [27] can be relaxed to $\beta^2 \geq \alpha^2 + 1$, which we prove in Appendix D with SOS decomposition related to the steering operators.

Both of these steering inequalities of $S_{\alpha,\beta}^{(1)}$ and $S_{\alpha,\beta}^{(2)}$ can be used to self-test a pure partially entangled state with $\sin(2\theta) = \frac{1}{\sqrt{1 + \alpha^2}}$. The only difference between our construction and the one in [27] is the exchanging roles of Alice and Bob. The advantage of our construction will be shown later. Before that, we should prove that the maximum violation of both $S_{\alpha,\beta}^{(1)}$ and $S_{\alpha,\beta}^{(2)}$ can be used to self-test a pure partially entangled state, though the proof for self-testing based on $S_{\alpha,\beta}^{(2)}$ was already given in (33). However, a different proof is provided here which is based on the SOS decomposition related to the steering inequality and the isometry given in Fig. 1. The benefit of this proof is that the constraints of $\beta^2 = \alpha^2 + 1$ can be relaxed (details are in Appendix D).

In the following we study the robustness of the self-testing based on these two steering inequalities. In Ref. [27] the robustness of one-sided self-testing was studied only for maximally entangled states based on operator inequalities. For the case $\alpha = 0$, when the violation of the steering inequality is $S = 2 - \epsilon$, the actual state is $24\sqrt{\epsilon} + \epsilon$, close to the target state (see also Ref. [13]). More precisely, the relation between the fidelity and the steering inequality value is

$$F \geq 1 - 24\sqrt{2 - S} - (2 - S), \quad (35)$$

which is quite loose. A nontrivial fidelity bound $f > \frac{1}{2}$ can only be obtained when the violation is larger than 1.99957, which makes the robustness analysis in the one-sided self-testing impractical. Here we have improved this bound to be

$$F \geq \frac{S - 2}{4 - 2\sqrt{2}} + 1, \quad (36)$$

which is the theoretical optimal linear bound. The local extraction channel to achieve this bound is constructed in Appendix B, and this channel coincides with the extraction channel in the DI scenario introduced in Ref. [48]. However, the reason this channel is used was not explained in Ref. [48]. Here we point out that the channel is the optimal local channel that the local party can take.

For the other case of α , we give the robustness analysis of one-sided self-testing based on the numerical method. The details are given in Appendix C. The method works for general, pure two-qubit states and the results show that the robustness bound is nearly linear.

The comparison of the robustness bound of self-testing based of Eqs. (33) and (34) is given in Fig. 3, where we take $\alpha = 1$ and $\beta = \sqrt{2}$ as an example. As shown, the one with trusted partial information gives a better robustness bound. The reason is that construction of the steering inequality (33) shows a smaller LHS bound compared with the inequality

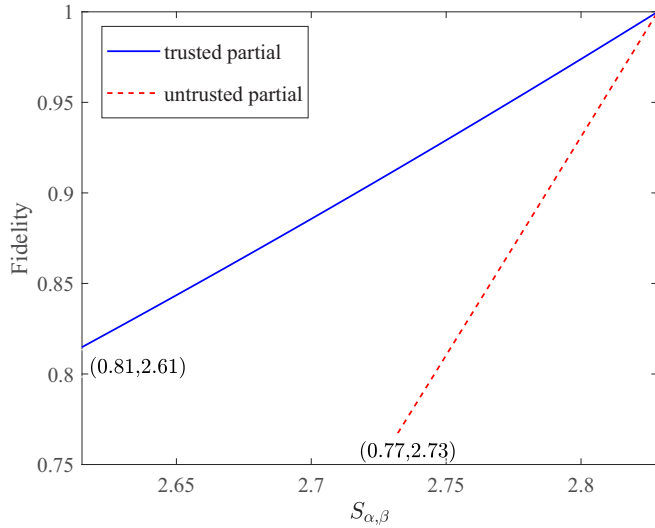


FIG. 3. Comparison of the robustness bound of self-testing based on the two-setting steering inequality $S_{\alpha,\beta}$ of Eqs. (33) and (34), where $\alpha = 1$ and $\beta = \sqrt{2}$.

(34); however, it keeps the quantum maximum bound. Thus the inequality (34) demonstrates an advantage for self-testing; it is more robust than using an untrusted party's partial measurement expectation. Actually, in addition to the advantage in self-testing, the steering inequality constructed with trusted partial expectation can also have fewer constraints on variants α and β , and thus could provide more reasonable steering inequalities (see Appendix D for details).

IV. ONE-SIDED SELF-TESTING BASED ON THREE-SETTING STEERING INEQUALITIES

So far the steering inequalities we have considered are all of two measurement settings. In this section we introduce more measurements settings in constructing steering inequalities. Later we show that adding more measurement settings can help increase the robustness in one-sided self-testing. We construct a family of three-setting steering inequalities

$$I_{\alpha,\beta} \equiv \alpha \langle Z \rangle + \beta \langle ZB_0 \rangle + \langle XB_1 \rangle + \langle YB_2 \rangle \leq \sqrt{2 + (\alpha + \beta)^2}, \quad (37)$$

where $\beta \geq 0$. These inequalities can be viewed as a generalization of analogous tilted CHSH steering inequalities (34). A third measurement involving the Pauli Y measurement is added. Similar to the discussion of the two-setting scenario, the partial expectation in the construction can also be untrusted party Bob's measurement B_0 . Thus $I_{\alpha,\beta} \equiv \alpha \langle B_0 \rangle + \beta \langle ZB_0 \rangle + \langle XB_1 \rangle + \langle YB_2 \rangle$ is constructed. These two slightly different inequalities have a different LHS bound while keeping the same quantum bound (for a detailed discussion and their proof for self-testing a two-qubit partially entangled state see Appendix D).

Here we just consider the first case in the main text for simplicity and give its self-testing robustness bound. The LHS bound is the maximum violation that we can have, assuming Bob has a preexisting state known to Alice, rather than half of an entangled state shared with Alice. Bob's system may be derived from a classical system; thus we can denote his cor-

responding declared result by a random variable $B_k \in \{-1, 1\}$ for $k = 0, 1$. As shown in [36], it is easy to see that

$$I_{\text{LHS}} = \max_{B_k} \lambda_{\max}(I_{\alpha,\beta}), \quad (38)$$

where $\lambda_{\max}(\hat{O})$ denotes the largest eigenvalue of \hat{O} . Then the LHS bound of Eq. (37) is shown to be $\sqrt{2 + (\alpha + \beta)^2}$.

The maximum quantum bound is $\beta + \sqrt{4 + \alpha^2} := S_Q$. This can be verified by the fact that $S_Q \mathbb{I} - I_{\alpha,\beta}$ is PSD. More precisely,

$$\begin{aligned} S_Q \mathbb{I} - \hat{I}_{\alpha,\beta} &= \frac{\beta}{2} (\mathbb{I} - ZB_0)^2 \\ &+ \frac{\sqrt{\alpha^2 + 4}}{4} \left(\mathbb{I} - \frac{\alpha}{\sqrt{4 + \alpha^2}} Z - \frac{2}{\sqrt{4 + \alpha^2}} XB_1 \right)^2 \\ &+ \frac{\sqrt{\alpha^2 + 4}}{4} \left(\mathbb{I} - \frac{\alpha}{\sqrt{4 + \alpha^2}} Z - \frac{2}{\sqrt{4 + \alpha^2}} YB_2 \right)^2. \end{aligned} \quad (39)$$

The quantum systems used to achieve the maximal quantum violation are $B_0 = Z$, $B_1 = X$, $B_2 = -Y$, and $|\Phi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle$, with $\sin 2\theta = \frac{2}{\sqrt{4 + \alpha^2}}$, which in turn can be self-tested when the maximum violation is reached (see Appendix D).

Here, for simplicity, we just consider the case of $\alpha = 0$ and $\beta = 1$. Assuming Bob's measurements are untrusted, without loss of generality, they can be written as $B_{0,1} = \cos \mu \sigma_z \pm \sin \mu \sigma_x$ and $B_2 = \cos \mu_1 \cos \mu_2 \sigma_z + \cos \mu_1 \sin \mu_2 \sigma_x + \sin \mu_1 \sigma_y$. Due to the asymmetry of $I_{\alpha,\beta}$ introduced by the form of B_2 , the spectral decomposition of it is not easy, which leads to the difficulty in constructing a local extraction channel making G PSD. We divide G into two parts. If each part is PSD, then the whole matrix G is PSD,

$$\begin{aligned} G &:= K - [s(ZB_0 + XB_1 + YB_2) + \tau \mathbb{I}] \\ &= K_1 - s(ZB_0 + XB_1) - \tau_1 \mathbb{I} + K_2 - sYB_2 - \tau_2 \mathbb{I}, \end{aligned} \quad (40)$$

where $K_1 + K_2 = K$ defines the two parts.

We consider the local extraction channel which ensures the parts of $G_1 := K_1 - s(ZB_0 + XB_1) - \tau_1 \mathbb{I}$ and $G_2 := K_2 - sYB_2 - \tau_2 \mathbb{I}$ PSD simultaneously (see Appendix F for details of the channel construction). The following robustness bound of self-testing in the three-setting steering scenario is obtained:

$$F \geq sS_{\text{obs}} + \tau \geq \frac{3}{12 - 4\sqrt{2}} (S_{\text{obs}} - 3) + 1. \quad (41)$$

It should be noted that here we did not get the expected robustness bound of $F \geq \frac{(S_{\text{obs}} - 3)}{2(3 - \sqrt{3})} + 1$. This may be because the local extraction channel strategy we consider here is not optimal. It may be possible to find a better extraction strategy to obtain that bound. However, though the bound we give is optimal, it is still better than two-setting analogous CHSH steering scenarios.

For a straightforward comparison between different inequalities, we transform the steering inequalities into the games characterized by the guessing probability which belongs to the same interval $[\frac{1}{2}, 1]$. In the case of $\alpha = 0$, we have $P = \sum_{i=0,1} p(a = b|A_i B_i) = \frac{1}{2} + \frac{s}{2S_Q}$, which is the successful probability of the nonlocal game guessing the other

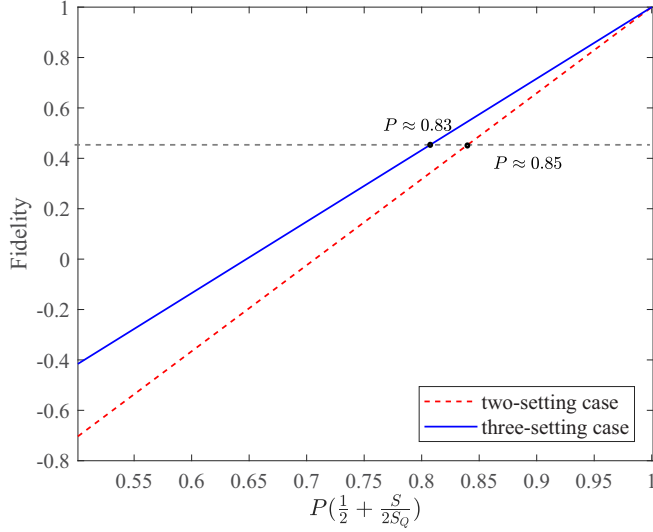


FIG. 4. Comparison of robustness bounds for one-sided self-testing of a singlet based on three-setting and two-setting steering inequalities.

party's outcomes. For the other case, we can also find a nonlocal game, namely, the guessing score is related to the inequalities (33) and (37), respectively. (See Appendix E for details.) We define the guessing probability as the probability for untrusted parties to successfully guess the trusted parties' outcomes, which is also important for the sample efficiency analysis in the next section. Based on the guessing probability, we can compare the robustness bound for one-sided self-testing of a singlet based on three-setting and two-setting steering inequalities. The result is shown in Fig. 4, where the three-setting steering inequality we constructed gives a better robustness bound. It is worth studying whether steering inequalities with more measurement settings can be constructed and further improve the robustness of one-sided self-testing.

V. SAMPLE EFFICIENCY

To construct a practical quantum verification protocol, it is crucial to study the sample efficiency [11,12,39,53]. Sample efficiency is used to study the performance of the self-testing criteria in the finite copy regime in a way that some of the state copies are measured to warrant the rest of the states being close to the target state.

Consider a quantum device producing the states $\rho_1, \rho_2, \dots, \rho_N$ in N runs. Our task is to verify whether these states are sufficiently close to the target state $|\Phi\rangle \in \mathcal{H}$ on average. Here the one-sided extractability is a natural choice for quantifying the closeness in the one-sided self-testing scenario.

For the extraction channel method, we obtain a linear relation between the extractability and the observed value of the steering inequalities

$$F \geq sS_{\text{obs}} + \tau. \quad (42)$$

Since $\tau = 1 - sS_Q$, we have

$$s(S_Q - S_{\text{obs}}) \geq 1 - F. \quad (43)$$

The first step in constructing the verification protocol is to view the steering inequalities as testing games. (Details of the transformation of steering inequalities to testing games are shown in Appendix E.) Based on this, results of unmeasured copies can be guaranteed based on the measured copies. Define p as the guessing probability of the game for a single state. For the steering inequalities in Eqs. (33) and (37), when $\alpha = 0$, which corresponds to the singlet state, the testing game is straightforward based on the outcomes of the same Pauli measurements. When $\alpha > 0$, which corresponds to the nonmaximally entangled state, virtual testing games are constructed from the steering inequalities in Appendix E. For these testing games, we have

$$p = \frac{1}{4} \sum_{i=0,1} p(a = b|A_i B_i) = \frac{1}{2} + \frac{S}{2S_Q}. \quad (44)$$

This relation between the guessing probability and the violation of steering inequalities is essential for the study of sample efficiency. For the analogous CHSH steering inequality in Eq. (11), we have $p = \frac{1}{4} \sum_{a \otimes b = ij} p(a, b|A_i B_j) = \frac{1}{2} + \frac{S}{4}$. This probability corresponds to the successful probability to win the game of $a \otimes b = ij$ for Alice and Bob. For steering inequalities in Eq. (11) for $\alpha \neq 0$ and Eq. (34), we have not found corresponding testing games. One may resort to other theories to study its performance in the finite regime, such as [54].

Defining $\epsilon = 1 - F$ as the infidelity and combining Eqs. (43) and (44), we have

$$p \leq 1 - \frac{\epsilon}{2sS_Q}. \quad (45)$$

Defining $c = \frac{1}{2sS_Q}$, in general we have

$$p \leq 1 - c\epsilon. \quad (46)$$

Now for these inequalities which correspond to a testing game, we are ready to estimate the number of copies sufficient to exceed a certain bound on the average one-sided extractability. Suppose the states in the test are independently distributed. The goal is to guarantee that the average one-sided extractability of the states $\rho_1, \rho_2, \dots, \rho_N$ is larger than $1 - \epsilon$ with significance level δ (confidence level $1 - \delta$). According to Ref. [53], the scaling of sample efficiency depends on whether the quantum bound and algebraic bound coincide for the games between participants. When the quantum bound and algebraic bound coincide, the number of copies satisfies

$$N \geq \frac{\ln \delta^{-1}}{\ln(1 - c\epsilon)^{-1}} \approx \frac{\ln \delta^{-1}}{c\epsilon}. \quad (47)$$

For all the steering inequalities we have considered in this paper, the two-setting inequality (33) and the three-setting inequality (37) satisfy this condition. In that case, the maximal guessing probability 1 can be obtained in the testing games according to the strategy given in Appendix E. Thus we obtain the approximately optimal sample efficiency for one-sided self-testing of general two-qubit states in both the two-setting and three-setting cases, which is comparable to the number needed in quantum state verification.

For the analogous CHSH steering inequality in Eq. (11), the quantum bound and algebraic bound are different. The

number of copies needed satisfies

$$N = O\left(\frac{\ln \delta^{-1}}{c^2 \epsilon^2}\right), \quad (48)$$

according to Ref. [53].

In this section we studied the sample efficiency for one-sided self-testing of two-qubit entangled states. Based on the steering inequalities we constructed, approximately optimal sample efficiency can be obtained in the SDI scenario, which is comparable to the device-dependent scenario. For the general DI scenario, the scaling of testing number is usually in quadratic form. Thus our strategies demonstrate a significant advantage over DI self-testing in sample efficiency.

VI. CONCLUSION

In this paper we studied the one-sided self-testing of general, pure two-qubit states in the untrusted quantum network in which one party is not honest. The self-testing strategies are based on the violation of quantum steering inequalities. To achieve this goal, we first studied two setting scenarios, where the steering inequalities can be constructed from standard tilted CHSH inequalities and its general form. Based on these steering inequalities, we studied the robustness of one-sided self-testing using both the local extraction map method and the numerical semidefinite-programming method. In particular, the local extraction map method has been shown to provide the analytical and theoretical optimal linear bound. Our result also demonstrates an explicit approach to construct the local extraction channel. The comparison with the device-independent scenario and the device-dependent scenario shows clearly that the robustness of SDI certification lies in the middle. The numerical method involving SDP and the SWAP trick gives a nearly linear robustness bound for general, pure two-qubit states. To construct a practical certification protocol, we also clarified the sample efficiency of our 1SDI self-testing protocols. The results show that approximately optimal sample efficiency can be obtained based on the steering inequalities we constructed.

Furthermore, we constructed three-measurement-setting steering inequalities for general two-qubit states, for a partially entangled state. It was shown that the robustness bound

can be further improved by introducing the third measurement setting. It is worth studying whether steering inequalities with more measurement settings can be constructed and further improve the robustness of one-sided self-testing. This question is also of interest in foundational studies on quantum steering. The improvement of the robustness bound in our work can be applied to the certification of high-dimensional quantum devices as building blocks. In the future, our results may be generalized to generic bipartite pure states, multipartite GHZ states, and other quantum states.

ACKNOWLEDGMENTS

This research was supported by National Nature Science Foundation of China (Grants No. 62101600, No. 62201252, and No. 61901218), Science Foundation of China University of Petroleum, Beijing (Grant No. 2462021YJRC008), State Key Laboratory of Cryptology (Grant No. MMK-FKT202109), and Natural Science Foundation of Jiangsu Province, China (Grant No. BK20190407).

APPENDIX A: LOCAL EXTRACTION CHANNEL METHOD FOR SELF-TESTING BASED ON AN ANALOGOUS TILTED CHSH INEQUALITY

This Appendix provides the robust bound of the self-testing based on an analogous tilted CHSH inequality in case 2, i.e., $0 \leq \cos 2\mu \leq \frac{\alpha^2}{4}$ or equivalently $\mu \in (\arcsin \sqrt{\frac{4-\alpha^2}{8}}, \frac{\pi}{4}]$. In this case, the eigenvalues of the decomposition of $\hat{S}_\alpha = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$ are $\lambda_{1,2} = \sqrt{\alpha^2 + 4 \sin^2 \mu} \pm 2 \cos \mu$. The constraints between γ and μ are

$$\begin{aligned} \lambda_1 \cos^2 \gamma - \lambda_2 \sin^2 \gamma &= \alpha + 2 \cos \mu, \\ \lambda_2 \cos^2 \gamma - \lambda_1 \sin^2 \gamma &= \alpha - 2 \cos \mu, \\ (\lambda_1 + \lambda_2) \cos \gamma \sin \gamma &= 2 \sin \mu. \end{aligned}$$

$$\text{Still } \sin 2\gamma = \frac{2 \sin \mu}{\sqrt{\alpha^2 + 4 \sin^2 \mu}}.$$

The following is the local extraction channel in this case. Bob takes rotation operation I with probability q_1 and takes σ_z with probability q_2 . Then the ideal state is transformed into $K = q_1 |\psi\rangle\langle\psi| + q_2 \sigma_x |\psi\rangle\langle\psi| \sigma_x$. The PSD requirement of $G := K - s \hat{I}_\alpha - \tau \mathbb{I} \geq 0$ gives

$$\begin{pmatrix} q_1 \cos^2(\theta) - C_1 s - \tau & 0 & 0 & q_1 \frac{\sin 2\theta}{2} - 2 \sin \mu s \\ 0 & q_2 \cos^2 \theta - C_2 s - \tau & q_2 \frac{\sin 2\theta}{2} - 2 \sin \mu s & 0 \\ 0 & q_2 \frac{\sin 2\theta}{2} - 2 \sin \mu s & q_2 \sin^2 \theta + C_1 s - \tau & 0 \\ q_1 \frac{\sin 2\theta}{2} - 2 \sin \mu s & 0 & 0 & q_1 \sin^2 \theta + C_2 s - \tau \end{pmatrix} \geq 0, \quad (A1)$$

where $C_1 = \alpha + 2 \cos \mu$ and $C_2 = \alpha - 2 \cos \mu$. The eigenvalues of G are

$$\lambda_{1,2} = \frac{G_{11} + G_{44} \pm \sqrt{(G_{11} - G_{44})^2 + 4G_{14}^2}}{2}, \quad (A2)$$

$$\lambda_{3,4} = \frac{G_{22} + G_{33} \pm \sqrt{(G_{22} - G_{33})^2 + 4G_{23}^2}}{2}, \quad (A3)$$

which should be positive to make G PSD,

$$q_1 \geq \frac{4 \sin^2 \mu s^2 + (C_1 s + \tau)(C_2 s - \tau)}{(\beta_Q + 2 \sin 2\theta \sin \mu + \cos^2 \theta C_2 - \sin^2 \theta C_1) s - 1},$$

$$q_2 \geq \frac{4 \sin^2 \mu s^2 + (C_2 s + \tau)(C_1 s - \tau)}{(\beta_Q + 2 \sin 2\theta \sin \mu + \cos^2 \theta C_1 - \sin^2 \theta C_2) s - 1},$$

where $\beta_Q = \sqrt{8 + 2\alpha^2}$.

We can also set $s = \frac{1 - \cos^2 \theta}{\beta_Q - (2 + \alpha)}$ and $\tau = 1 - \sqrt{8 + 2\alpha^2} s$, keeping q_1 in the above range. This gives the same bound as in case 1. To this end, we take q_1 to be the maximum between 0 and the values which saturate the above two inequalities about q_1 .

APPENDIX B: LOCAL EXTRACTION CHANNEL METHOD FOR SELF-TESTING BASED ON REVERSE CHSH INEQUALITY

The analogous CHSH steering operator $\hat{S} = ZB_0 + XB_1$ has the spectral decomposition

$$\hat{S} = \sum \lambda_i |\psi_i\rangle\langle\psi_i|, \quad (\text{B1})$$

with $\lambda_1^2 + \lambda_2^2 = 4$, $\lambda_3 = -\lambda_2$, and $\lambda_4 = -\lambda_1$. Precisely,

$$\lambda_1 = \sqrt{2}(\cos \mu + \sin \mu), \quad \lambda_2 = \sqrt{2}(\cos \mu - \sin \mu), \quad (\text{B2})$$

where Bob's measurements are written as $B_r = \cos \mu \sigma_z + (-1)^r \sin \mu \sigma_x$, with $r = 0, 1$. In the case of $\mu \in (0, \pi/4]$ we have $\lambda_1, \lambda_2 \geq 0$ and

$$\begin{aligned} |\psi_1\rangle &= \frac{|00_B\rangle + |11_B\rangle}{\sqrt{2}}, & |\psi_2\rangle &= \frac{|00'_B\rangle + |11'_B\rangle}{\sqrt{2}}, \\ |\psi_3\rangle &= \frac{|01'_B\rangle - |10'_B\rangle}{\sqrt{2}}, & |\psi_4\rangle &= \frac{|01_B\rangle - |10_B\rangle}{\sqrt{2}}, \end{aligned} \quad (\text{B3})$$

where

$$\begin{aligned} 0_B &= \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} |1\rangle, & 1_B &= \sin \frac{\pi}{8} |0\rangle - \cos \frac{\pi}{8} |1\rangle, \\ 0'_B &= \cos \frac{\pi}{8} |0\rangle - \sin \frac{\pi}{8} |1\rangle, & 1'_B &= \sin \frac{\pi}{8} |0\rangle + \cos \frac{\pi}{8} |1\rangle. \end{aligned}$$

We consider the following local extraction channel. Bob takes the rotation operation $R_1 = I$ on his qubit with the probability of q_1 and takes $R_2 = \sigma_z$ on his qubit with the probability of q_2 . The ideal state is transformed into the mixture of the Bell operator eigenvectors $|\psi\rangle := q_1 |\psi_1\rangle\langle\psi_1| + q_2 |\psi_2\rangle\langle\psi_2|$. In this case, $G := K - s\hat{S} - \tau\mathbb{I}$ is diagonal and the PSD requirement gives

$$\begin{aligned} q_1 - s\lambda_1 - \tau &\geq 0, \\ q_2 - s\lambda_2 - \tau &\geq 0, \\ \text{Tr}(\rho) &= p_1 + p_2 = 1, \\ \text{Tr}(\rho\hat{S}) &= \lambda_1 p_1 + \lambda_2 p_2 = S, \end{aligned}$$

where we set $\tau = 1 - 2s$. By simplifying, we have $s\lambda_1 - 2s + 1 \leq q_2 \leq -s\lambda_2 + 2s$, which gives us $s \geq \frac{1}{4 - (\lambda_1 + \lambda_2)} \geq \frac{1}{4 - 2\sqrt{2}}$. This gives the following robustness bound of self-testing via the steering inequality:

$$F \geq sS + \tau \geq \frac{S - 2}{4 - 2\sqrt{2}} + 1. \quad (\text{B4})$$

In addition, we get the constraints on the rotation probability

$$(1 + \sqrt{2})(\cos \mu + \sin \mu + 1) \leq q_1 \leq 1. \quad (\text{B5})$$

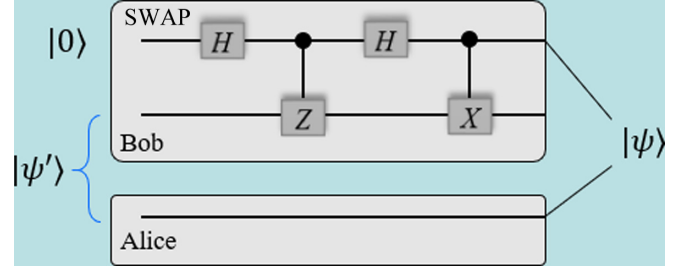


FIG. 5. One-sided SWAP isometry applied on Bob's side.

For the case of $\mu \in (\frac{\pi}{4}, \frac{\pi}{2})$, the local extraction channel is considered as follows. Bob takes the rotation $R_1 = I$ with the probability of q_1 and $R_2 = \sigma_x$ with the probability of q_2 . This gives the same robustness bound.

Above we found that the optimal linear bound and nontrivial fidelity can be obtained as long as the steering inequality is violated. However, as shown in Ref. [48], that nontrivial fidelity bound could not be obtained for an inequality violation at 2, with this local extraction channel. The reason might be that to define the appropriate extraction channel, the two local sites need coordinating. In the DI scenario, both sides are not trusted. The decomposition of the Bell operator is related to both Alice's and Bob's local measurement directions.

Once Alice and Bob could inform each other what measurement directions they choose (do classical communication), it is possible for them to define the appropriate local rotation channel which could rotate the idea states to be the eigenvectors of the Bell operator with positive eigenvalues. It could make $G := K - s\hat{I} - \tau\mathbb{I}$ PSD. In this case, it is easy to find that s and t are the optimal ones. However, allowing communication is not usually device independent. Thus, in the DI scenario, when coordination is needed, the nontrivial fidelity could not be reached.

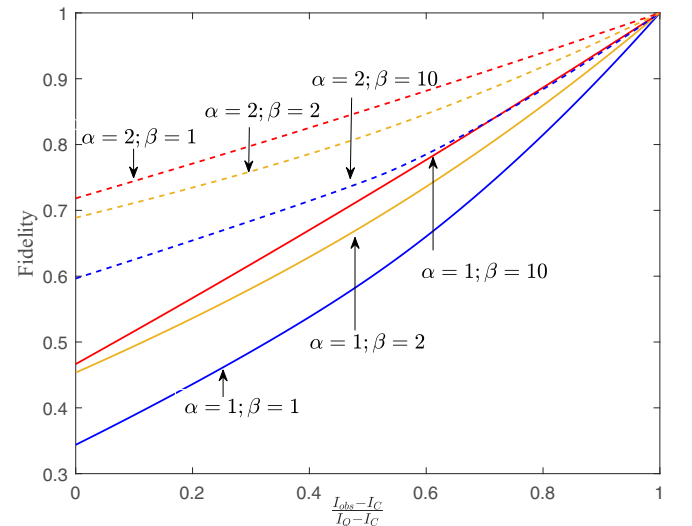


FIG. 6. Robustness bound of self-testing based on the three-setting steering inequality for six scenarios of (α, β) , where $\alpha = 1, 2$ and $\beta = 1, 2, 10$.

APPENDIX C: NUMERICAL RESULTS UTILIZING THE SWAP ISOMETRY

In this Appendix we consider the numerical method based on SDP to show the robustness of the self-testing via steering inequality, which has been widely used in DI frameworks [17,55]. A detailed robustness analysis is given for three-setting steering inequalities. For two-setting scenarios, we only need to remove the third measurement in the code.

The target state is $|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$. Bob's measurements can be written as $B_0 = 2E_{0|0} - I$, $B_1 = 2E_{0|1} - I$,

and $B_2 = 2E_{0|2} - I$, where $B_0^2 = B_1^2 = B_2^2$. After applying the isometry given in Fig. 5 to the physical state $|\psi'\rangle$, we obtain the state

$$|\psi'\rangle = E_{0|0}|\psi\rangle|0\rangle_{A'} + XE_{1|0}|\psi\rangle|1\rangle_{A'}. \quad (C1)$$

We trace the desired system out

$$\rho_{\text{SWAP}} = \text{tr}_A(|\psi'\rangle\langle\psi'|). \quad (C2)$$

Utilizing the SWAP isometry on Bob's side, the fidelity can be bounded as

$$\begin{aligned} f &= \langle\psi|\rho_{\text{SWAP}}|\psi\rangle \\ &= \cos^2\theta\langle 0|\text{tr}_A(E_{0|0}\rho_{AB})|0\rangle + \sin^2\theta\langle 1|\text{tr}_A(E_{1|0}\rho_{AB})|1\rangle + \frac{\sin 2\theta}{2}[\langle 0|\text{tr}_A(E_{1|0}XE_{0|0}\rho_{AB})|1\rangle + \langle 1|\text{tr}_A(E_{0|0}XE_{1|0}\rho_{AB})|0\rangle] \\ &= \cos^2\theta\langle 0|\text{tr}_A(E_{0|0}\rho_{AB})|0\rangle + \sin^2\theta\langle 1|\text{tr}_A(\rho_B - \text{tr}_A E_{0|0})|1\rangle + \sin 2\theta[\langle 0|\text{tr}_A(E_{0|1}E_{0|0} - E_{0|0}E_{0|1}E_{0|0})|1\rangle \\ &\quad + \langle 1|\text{tr}_A(E_{0|0}E_{0|1} - E_{0|0}E_{0|1}E_{0|0})|0\rangle] \\ &= \cos^2\theta\langle 0|\sigma_{0|0}|0\rangle + \sin^2\theta\langle 1|\text{tr}_A(\rho_B - \sigma_{0|0})|1\rangle + \sin 2\theta[\langle 0|\text{tr}_A(\sigma_{0|1,0|0} - \sigma_{0|0,0|1,0|0})|1\rangle + \langle 1|\text{tr}_A(\sigma_{0|0,0|1} - \sigma_{0|0,0|1,0|0})|0\rangle]. \end{aligned} \quad (C3)$$

The goal is now to give a lower bound to f . The numerical method of minimizing the fidelity for a given steering inequality value is given by the SDP

$$\begin{aligned} &\text{minimize } f := \text{Tr}(M\Gamma) \\ &\text{subject to } \Gamma \geq 0, \quad I_{\alpha,\beta} = Q, \end{aligned} \quad (C4)$$

where M is a zero matrix (14,14), with $M_{2,2} = \sin^2\theta$, $M_{2,9} = M_{9,2} = \sin 2\theta$, $M_{3,3} = \cos^2\theta$, $M_{4,4} = -\sin^2\theta$, and $M_{9,10} = M_{10,9} = -\sin 2\theta$;

$$\Gamma = \begin{pmatrix} \rho_C & \sigma_{0|0} & \sigma_{0|1} & \sigma_{0|2} & \sigma_{0|1,0|0} & \sigma_{0|2,0|0} & \sigma_{0|2,0|1} \\ \sigma_{0|0} & \sigma_{0|0} & \sigma_{0|0,0|1} & \sigma_{0|0,0|2} & \sigma_{0|0,0|1,0|0} & \sigma_{0|0,0|2,0|0} & \sigma_{0|0,0|2,0|1} \\ \sigma_{0|1} & \sigma_{0|1,0|0} & \sigma_{0|1} & \sigma_{0|1,0|2} & \sigma_{0|1,0|0} & \sigma_{0|1,0|2,0|0} & \sigma_{0|1,0|2,0|1} \\ \sigma_{0|2} & \sigma_{0|2,0|0} & \sigma_{0|2,0|1} & \sigma_{0|2} & \sigma_{0|2,0|1,0|0} & \sigma_{0|2,0|0} & \sigma_{0|2,0|1} \\ \sigma_{0|0,0|1} & \sigma_{0|0,0|1,0|0} & \sigma_{0|0,0|1} & \sigma_{0|0,0|1,0|2} & \sigma_{0|0,0|1,0|0} & \sigma_{0|0,0|1,0|2,0|0} & \sigma_{0|0,0|1,0|2,0|1} \\ \sigma_{0|0,0|2} & \sigma_{0|0,0|2,0|0} & \sigma_{0|0,0|2,0|1} & \sigma_{0|0,0|2} & \sigma_{0|0,0|2,0|1,0|0} & \sigma_{0|0,0|2,0|0} & \sigma_{0|0,0|2,0|1} \\ \sigma_{0|1,0|2} & \sigma_{0|1,0|2,0|0} & \sigma_{0|1,0|2,0|1} & \sigma_{0|1,0|2} & \sigma_{0|1,0|2,0|1,0|0} & \sigma_{0|1,0|2,0|0} & \sigma_{0|1,0|2,0|1} \end{pmatrix}; \quad (C5)$$

and $I_{\alpha,\beta} = \alpha\langle Z\rangle + \beta\langle ZB_0\rangle + \langle XB_1\rangle + \langle YB_2\rangle = \text{Tr}[(\alpha - \beta)Z\rho_C - (X + Y)\rho_C + 2\beta Z\sigma_{0|0} + 2X\sigma_{0|1} + 2Y\sigma_{0|2}]$ or $I_{\alpha,\beta} = \alpha\langle B_0\rangle + \beta\langle ZB_0\rangle + \langle XB_1\rangle + \langle YB_2\rangle = \text{Tr}[-(\alpha + \beta Z + X + Y)\rho_C + (2\alpha I + 2\beta Z)\sigma_{0|0} + 2X\sigma_{0|1} + 2Y\sigma_{0|2}]$.

We constrain Γ in the optimization to be positive semidefinite and note that each submatrix of Γ corresponding to something like an element of an assemblage is a valid quantum object. It actually turns out that all assemblages that satisfy no-signaling can be realized in quantum theory [56]. Discussion of this point is beyond the scope of this paper, as all we wish to do is give a lower bound on the value of G ; therefore just imposing $\Gamma \geq 0$ gives such a bound. Based on the SDP of Eq. (C4), we show several robustness bounds of self-testing based on the three-setting steering inequality for six scenarios of (α, β) , where $\alpha = 1, 2$ and $\beta = 1, 2, 10$ (see Fig. 6).

APPENDIX D: ANALYSIS OF DIFFERENT TYPES OF TWO-SETTING AND THREE-SETTING STEERING INEQUALITIES

Here we study the maximal quantum violation of the steering inequalities involved in the main text and show that the maximal violation of these inequalities can be used for self-testing. For the two-setting steering inequality

$$S_{\alpha,\beta}^2 = \alpha\langle B_0\rangle + \beta\langle ZB_0\rangle + \langle XB_1\rangle \leq \alpha + \sqrt{1 + \beta^2}. \quad (D1)$$

The maximum quantum bound is $\beta + \sqrt{1 + \alpha^2} := S_Q$. This can be confirmed by showing $S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(2)} \geq 0$ to be true for all the possible underlying states and the measurements. To do so, we provide the following SOS decompositions of $S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(2)}$

to illustrate its PSD: The first SOS decomposition is

$$\begin{aligned} S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(2)} &= \alpha_1^2 (\mathbb{I} - cB_0 - sX_A B_1)^2 + \alpha_2^2 (Z_A - B_0)^2 \\ &+ \alpha_3^2 (-cB_1 + sX_A B_0 + Z_A B_1)^2 \\ &+ \alpha_4^2 (S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(2)})^2, \end{aligned} \quad (\text{D2})$$

where $c = \frac{\alpha}{\sqrt{1+\alpha^2}}$, $s = \frac{1}{\sqrt{1+\alpha^2}}$, $\alpha_4^2 = \frac{1}{4\beta}$, $\alpha_3^2 = \frac{\beta\sqrt{1+\alpha^2}}{1}\alpha_4^2 = \frac{\sqrt{1+\alpha^2}}{4}$, $\alpha_1^2 = (\frac{\beta\sqrt{1+\alpha^2}}{1} - \frac{1+\alpha^2}{1})\alpha_4^2$, and $\alpha_2^2 = \frac{\beta-\sqrt{1+\alpha^2}}{4}$, and the second one is

$$\begin{aligned} S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(2)} &= \alpha_1^2 (\mathbb{I} - cB_0 - sX_A B_1)^2 + \alpha_2^2 (Z_A - B_0)^2 \\ &+ \alpha_3^2 [(\Delta + s^2)B_0 - (\Delta + 1)Z_A + cZ_A B_0 - csX_A B_1]^2 \\ &+ \alpha_4^2 [-(\Delta + s^2)B_1 + s(\Delta + 1)X_A + \Delta cZ_A B_1 - csX_A B_0]^2, \end{aligned} \quad (\text{D3})$$

where α_1 and α_2 are the same as the first SOS decomposition, $\alpha_3^2 = \Delta\alpha_4^2$, $\alpha_4^2 = \frac{S_Q}{4s\beta(\Delta^2+s^2)(\Delta^2+1)}$, and $\Delta = \frac{\beta}{\sqrt{1+\alpha^2}}$.

It is easy to verify that the left-hand sides of Eqs. (D2) and (D3) are equal to the SOS forms on the right. In addition, to make the SOS decompositions positive semidefinite, we should have $\alpha_i \geq 0$, and thus $\beta \geq \sqrt{1+\alpha^2}$. Apparently, S_Q is the upper bound of the steering inequality $S_{\alpha,\beta}^2$ under this constraint, although we do not know whether the quantum can reach the bound. Provided $B_0 = Z$, $B_1 = X$, and $|\Phi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ with $\sin 2\theta = \frac{1}{\sqrt{1+\alpha^2}}$ can make $S_{\alpha,\beta}^2$ achieve S_Q , we conclude that S_Q is the maximum quantum violation.

Next we show that the maximal violation of this steering inequality will self-test the partially entangled state. The local isometry used to determine the equivalence of the states is the same as in the main text, but with $\tilde{Z}_B = B_0$ and $\tilde{X}_B = B_1$. As shown in the main text, the relations required to show that this isometry works are

$$Z_A|\psi\rangle - B_0|\psi\rangle = 0, \quad (\text{D4})$$

$$\sin\theta X_A(I + B_0)|\psi\rangle - \cos\theta B_1(I - Z_A)|\psi\rangle = 0. \quad (\text{D5})$$

To obtain these relations, we let each side of Eqs. (D2) and (D3) take action on $|\psi\rangle$, a state that is supposed to reach the maximum violation of the steering inequality. Then seven terms of $P_i|\psi\rangle = 0$ will be obtained; among them the second squared term in Eq. (D2) gives Eq. (D4), while the linear combination of the third squared term in Eq. (D2) and the fourth squared term in Eq. (D3) leads to Eq. (D4). Then, similar to the proof for the analog of tilted CHSH steering inequality given in the main text, by the isometry given in Fig. 1, we complete the self-testing statement via the two-setting steering inequality $S_{\alpha,\beta}^2$.

For the two-setting steering inequality

$$S_{\alpha,\beta}^{(1)} = \alpha(Z) + \beta(ZB_0) + \langle XB_1 \rangle \leq \sqrt{1 + (\alpha + \beta)^2}, \quad (\text{D6})$$

which keeps the same maximal quantum violation as in Eq. (D1). For this steering inequality, three different types of SOS decompositions related to $S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(1)}$ can be given: The

first one is

$$\frac{\beta}{2} (\mathbb{I} - Z_A B_0)^2 + \frac{\sqrt{\alpha^2 + 1}}{2} (\mathbb{I} - cZ_A - sX_A B_1)^2, \quad (\text{D7})$$

the second one is

$$\frac{1}{2S_Q} (-cX_A + sZ_A B_1 + X_A B_0)^2 + \frac{\beta\sqrt{\alpha^2 + 1}}{2S_Q} (S_Q \mathbb{I} - \hat{S}_{\alpha,\beta}^{(1)})^2, \quad (\text{D8})$$

and the third one is

$$\begin{aligned} \alpha_1^2 [(\Delta + s^2)Z_A - (\Delta + 1)B_0 + cZ_A B_0 - csX_A B_1]^2 \\ + \alpha_2^2 [-(\Delta + s^2)X_A + s(\Delta + 1)B_1 + \Delta cX_A B_0 - csZ_A B_1]^2, \end{aligned} \quad (\text{D9})$$

where $\alpha_1^2 = \Delta\alpha_2^2$, $\alpha_2^2 = \frac{(1+\alpha^2)^2}{2(\beta^2\sqrt{1+\alpha^2} + \beta(1+\alpha^2) + S_Q)}$, and $\Delta = \frac{\beta}{\sqrt{1+\alpha^2}}$. The PSD requirements only require $\beta > 0$. In addition, each squared term in Eqs. (D7)–(D9) acting on $|\psi\rangle$ being zero can lead to the relations our self-testing proofs heavily rely on, namely, Eqs. (D4) and (D5) [the first term in Eq. (D7) leads to Eq. (D4); the first term in Eq. (D7) and the second term in Eq. (D9) lead to Eq. (D5)]. Then we can complete the proof of self-testing based on $S_{\alpha,\beta}^{(1)}$.

For the three-setting scenario, the partial part expectation can be changed into the untrusted part's measurement. Thus there are two three-setting steering inequalities: the one in the main text,

$$I_{\alpha,\beta}^{(1)} \equiv \alpha\langle Z \rangle + \beta\langle ZB_0 \rangle + \langle XB_1 \rangle + \langle YB_2 \rangle \leq \sqrt{2 + (\alpha + \beta)^2}, \quad (\text{D10})$$

and

$$I_{\alpha,\beta}^{(2)} \equiv \alpha\langle B_0 \rangle + \beta\langle ZB_0 \rangle + \langle XB_1 \rangle + \langle YB_2 \rangle \leq \alpha + \sqrt{2 + \beta^2}. \quad (\text{D11})$$

The advantage of this change is that its LHS bound is lower than using Alice's Z measurement in the three-setting inequality, while the quantum bound is maintained. It extends the gap between the LHS bound and steering bound, which is a benefit of the practical experiment. Denoting Bob's corresponding declared result by the random variable $B_k \in \{-1, 1\}$ for $k = 0, 1$, it is easy to obtain the LHS bound $\alpha + \sqrt{2 + \beta^2}$.

The quantum bounds of the both three-setting steering inequalities are the same, $\beta + \sqrt{4 + \alpha^2}$. However, an extra condition should be satisfied for $I_{\alpha,\beta}^{(2)}$, that is, $\beta \geq \sqrt{4 + \alpha^2}$. For $I_{\alpha,\beta}^{(2)}$ it only requires $\beta \geq 0$. This can be obtained from the following SOS: The first one is

$$\begin{aligned} (\beta + \sqrt{4 + \alpha^2}) \mathbb{I} - \hat{I}_{\alpha,\beta}^{(2)} \\ = \alpha_1^2 (\mathbb{I} - cB_0 - sX_A B_1)^2 + \alpha_2^2 (Z_A - B_0)^2 \\ + \alpha_3^2 (\mathbb{I} - cB_0 - sY_A B_2)^2 \\ + \alpha_4^2 (-cB_1 + sX_A B_0 + Z_A B_1)^2 \\ + \alpha_5^2 (-cB_2 + sY_A B_0 + Z_A B_2)^2 \\ + \alpha_6^2 ((\beta + \sqrt{4 + \alpha^2}) \mathbb{I} - I_{\alpha,\beta}^{(2)})^2 \\ + \alpha_7^2 (X_A B_1 - Y_A B_2)^2, \end{aligned} \quad (\text{D12})$$

where $c = \frac{\alpha}{\sqrt{4+\alpha^2}}$, $s = \frac{2}{\sqrt{4+\alpha^2}}$, $\alpha_6^2 = \alpha_7^2 = \frac{1}{4\beta}$, $\alpha_4^2 = \alpha_5^2 = \frac{\beta\sqrt{4+\alpha^2}}{2}\alpha_6^2 = \frac{\sqrt{4+\alpha^2}}{8}$, $\alpha_1^2 = \alpha_3^2 = (\frac{\beta\sqrt{4+\alpha^2}}{2} - \frac{4+\alpha^2}{2})\alpha_6^2$, and

$\alpha_2^2 = \frac{\beta - \sqrt{4 + \alpha^2}}{4}$, and the second one is

$$\begin{aligned}
& (\beta + \sqrt{4 + \alpha^2})\mathbb{I} - \hat{f}_{\alpha, \beta}^{(2)} \\
&= \alpha_1^2 (\mathbb{I} - cB_0 - sX_A B_1)^2 + \alpha_2^2 (Z_A - B_0)^2 \\
&+ \alpha_3^2 (\mathbb{I} - cB_0 - sY_A B_2)^2 \\
&+ \alpha_4^2 [(\Delta + s^2)B_0 - (\Delta + 1)Z_A + cZ_A B_0 - csX_A B_1]^2 \\
&+ \alpha_5^2 [(\Delta + s^2)B_0 - (\Delta + 1)Z_A + cZ_A B_0 - csY_A B_2]^2 \\
&+ \alpha_6^2 [-(\Delta + s^2)B_1 + s(\Delta + 1)X_A \\
&+ \Delta cZ_A B_1 - csX_A B_0]^2 \\
&+ \alpha_7^2 [-(\Delta + s^2)B_2 + s(\Delta + 1)Y_A \\
&+ \Delta cZ_A B_2 - csY_A B_0]^2, \tag{D13}
\end{aligned}$$

where $c = \frac{\alpha}{\sqrt{4 + \alpha^2}}$, $s = \frac{2}{\sqrt{4 + \alpha^2}}$, $\alpha_6^2 = \alpha_7^2 = \frac{1}{4s\Delta(\Delta^2 + s)}$, $\alpha_4^2 = \alpha_5^2 = \Delta\alpha_6^2 = \frac{1}{4s(\Delta^2 + s)}$, $\alpha_1^2 = \alpha_3^2 = \frac{1}{2S} - (\Delta + 1)(\Delta + s^2)\alpha_6^2$, $\alpha_2^2 = \frac{\beta}{2} - \frac{\Delta^2 + 1}{s(\Delta + 1)}$, and $\Delta = 1$.

Making the SOS decomposition positive semidefinite requires each $\alpha_i \geq 0$ and thus $\beta \geq \sqrt{4 + \alpha^2}$. In addition, some squared terms in (D12) and (D13) acting on $|\psi\rangle$ being zero also can lead to the relations (D4) and (D5). Thus, with the isometry given in the main text, we can complete the proof of self-testing based on $S_{\alpha, \beta}^{(2)}$.

For the first three-setting steering inequality, three types of SOS decompositions can be given: The first one is

$$\begin{aligned}
& (\beta + \sqrt{4 + \alpha^2})\mathbb{I} - \hat{f}_{\alpha, \beta}^{(1)} \\
&= \frac{\beta}{2} (\mathbb{I} - Z_A B_0)^2 + \frac{\sqrt{\alpha^2 + 4}}{4} (\mathbb{I} - cZ_A - sX_A B_1)^2 \\
&+ \frac{\sqrt{\alpha^2 - 4}}{4} (\mathbb{I} - cZ_A - sY_A B_2)^2, \tag{D14}
\end{aligned}$$

the second one is

$$\begin{aligned}
& (\beta + \sqrt{4 + \alpha^2})\mathbb{I} - \hat{f}_{\alpha, \beta}^{(1)} \\
&= \alpha_1^2 (-cX_A + sZ_A B_1 + X_A B_0)^2 \\
&+ \alpha_2^2 (-cY_A + sZ_A B_2 + Y_A B_0)^2 + \alpha_3^2 (S_Q \mathbb{I} - \hat{f}_{\alpha, \beta}^{(1)})^2, \tag{D15}
\end{aligned}$$

where $\alpha_1^2 = \alpha_2^2 = \frac{\alpha^2 + \beta^2 + \beta\sqrt{4 + \alpha^2} + 3}{4S_Q}$ and $\alpha_3^2 = \frac{1}{2S_Q}$, and the third one is

$$\begin{aligned}
& (\beta + \sqrt{4 + \alpha^2})\mathbb{I} - \hat{f}_{\alpha, \beta}^{(1)} \\
&= \alpha_1^2 [(\Delta + s^2)Z_A - (\Delta + 1)B_0 + cZ_A B_0 - csX_A B_1]^2 \\
&+ \alpha_2^2 [-(\Delta + s^2)X_A + s(\Delta + 1)B_1 + \Delta cX_A B_0 - csZ_A B_1]^2 \\
&+ \alpha_3^2 [(\Delta + s^2)Z_A - (\Delta + 1)B_0 + cY_A B_0 - csZ_A B_2]^2 \\
&+ \alpha_4^2 [-(\Delta + s^2)Y_A + s(\Delta + 1)B_2 + \Delta cY_A B_0 - csZ_A B_2]^2, \tag{D16}
\end{aligned}$$

where $\alpha_1^2 = \alpha_3^2 = \frac{\beta}{4(\Delta + s^2)(\Delta + 1)}$, $\alpha_2^2 = \alpha_4^2 = \frac{1}{2s(\Delta + s^2)(\Delta + 1)}$, and $\Delta = \frac{\beta}{\sqrt{1 + \alpha^2}}$.

The PSD condition requires $\beta \geq 0$. In addition, the first squared term in (D14) acting on $|\psi\rangle$ being zero ($|\psi\rangle$ is the state which maximally violates the steering inequality) gives the relations (D4), while the linear combination of the second squared term in (D15) and the first squared term in (D16) gives the relation (D5). Thus, with the isometry given in the main text, we can complete the proof of self-testing based on $S_{\alpha, \beta}^{(1)}$.

Self-testing for the measurements. Above we mainly focused on the states self-testing; the self-testing of the corresponding measurements (for analysis refer to [17]) will be similar. We start with $\Phi M_B(|\psi\rangle)$ instead of $\Phi(|\psi\rangle)$ and show it for one of the three measurements in three-setting steering inequality cases, for example. After the isometry, the systems will be

$$\begin{aligned}
\Phi(\tilde{Z}_B|\psi\rangle) &= \frac{1}{4} [(I + Z_A)(I + \tilde{Z}_B)\tilde{Z}_B|\psi\rangle|00\rangle \\
&+ X_A(I + Z_A)(I - \tilde{Z}_B)\tilde{Z}_B|\psi\rangle|01\rangle \\
&+ \tilde{X}_B(I - Z_A)(I + \tilde{Z}_B)\tilde{Z}_B|\psi\rangle|10\rangle \\
&+ X_A\tilde{X}_B(I - Z_A)(I - \tilde{Z}_B)\tilde{Z}_B|\psi\rangle|11\rangle]. \tag{D17}
\end{aligned}$$

With the relations (D4) and (D5) and the fact that $Z_A X_A = -X_A Z_A$, we find $\tilde{Z}_B \tilde{X}_B |\psi\rangle = -\tilde{X}_B \tilde{Z}_B |\psi\rangle$. By using this anti-commutation relation between Bob's two measurements, we move \tilde{Z}_B to the left in the first, second, third, and fourth lines while changing the sign of the fourth line. The analysis is then the same as the state self-testing and the result is

$$\begin{aligned}
\Phi(\tilde{Z}_B|\psi\rangle) &= |\text{junk}\rangle [\cos\theta|00\rangle - \sin\theta|11\rangle] \\
&= |\text{junk}\rangle [(I \otimes \sigma_z) \cos\theta|00\rangle + \sin\theta|11\rangle]. \tag{D18}
\end{aligned}$$

In addition, from the SOS decomposition we can also find the relation $\sin\theta Y_A(I + B_0)|\psi\rangle - \cos\theta B_2(I - Z_A)|\psi\rangle = 0$. Thus we have $\tilde{Z}_B \tilde{Y}_B |\psi\rangle = -\tilde{Y}_B \tilde{Z}_B |\psi\rangle$. Following the above idea, we can finally conclude that the measurements on Bob's side are $B_0 = Z$, $B_1 = X$, and $B_2 = -Y$.

APPENDIX E: TRANSFORMATION OF A STEERING INEQUALITY INTO A GAME

In this Appendix we relate the constructed steering inequality to a game which two parties play to increase the score and build the relation between the quantum violation and success probability of the game defined. This is helpful for a direct comparison between different steering inequalities and it is necessary in the analysis of sample efficiency. For simplicity, here we only consider the three-setting steering inequality.

In principle, to obtain the maximum violation of the three-setting steering inequality (37), the state between Alice and Bob should be $\cos\theta|00\rangle + \sin\theta|11\rangle$, which can be further written as $\frac{1}{\sqrt{2}}(|\psi_0\rangle|+\rangle + |\psi_1\rangle|-\rangle)$, where we define $|\psi_0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ and $|\psi_1\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$. We define two measurements on Alice's side $\{|\psi_0\rangle, |\psi_0^\dagger\rangle; |\psi_1\rangle, |\psi_1^\dagger\rangle\}$, which actually are measurements introduced to replace the measurements chosen in the main text in the real experiments. The measurements can also be written in the Pauli operator form $\{A_0 = \cos(2\theta)\sigma_z + \sin(2\theta)\sigma_x; A_1 = \cos(2\theta)\sigma_z - \sin(2\theta)\sigma_x\}$.

We notice that, if Bob gets $|+\rangle$, Alice takes A_0 and Bob can conclude that Alice's qubit must be projected into $|\psi_0\rangle$; if Bob gets $|-\rangle$, Alice takes A_1 and then Bob can conclude that Alice's qubit must be projected into $|\psi_1\rangle$. Since in the steering scenario Bob can send information to Alice, such measurements result. Thus, this allows us to define the success probability of Bob guessing Alice's measurement result as

$$P_{\text{virtual}}^x = p(A_0^0, B_1^0) + P(A_1^0, B_1^1), \quad (\text{E1})$$

which actually is related to the operators in the three-setting steering inequality (34). More precisely, $\frac{\alpha}{2}Z + XB_1 = (\frac{\alpha}{2}Z + X)B_1^0 + (\frac{\alpha}{2}Z - X)B_1^1 = \frac{\sqrt{4+\alpha^2}}{2}(A_0B_1^0 + A_1B_1^1) = \frac{\sqrt{4+\alpha^2}}{2}(2A_0^0B_1^0 + 2A_1^0B_1^1 - I_B)$ for $\sin(2\theta) = \frac{2}{\sqrt{4+\alpha^2}}$. Thus P_{virtual}^x is related to $\alpha\langle Z \rangle + \langle XB_1 \rangle$. Similarly, we can define P_{virtual}^y for the σ_y measurement scenario, which is related to $\frac{\alpha}{2}\langle Z \rangle + \langle YB_2 \rangle$. Together with the guessing probability for (Z_A, B_0) , we define the total average passing probability as

$$P_{\text{virtual}} = \frac{\sqrt{4 + \alpha^2} \left(\frac{P_{\text{virtual}}^x + P_{\text{virtual}}^y}{2} \right) + \beta p(a = b|Z_A, B_0)}{\sqrt{4 + \alpha^2} + \beta}. \quad (\text{E2})$$

Thus we have

$$P_{\text{virtual}} = \frac{\sqrt{4 + \alpha^2} + \beta + S}{2(\sqrt{4 + \alpha^2} + \beta)} = \frac{1}{2} + \frac{S}{2S_Q}. \quad (\text{E3})$$

This relation between the guessing probability and the violation holds for steering inequalities (30) and (34). Thus the steering inequalities are transformed to testing games.

APPENDIX F: ROBUST SELF-TESTING OF THREE-SETTING INEQUALITY

In this Appendix we provide an analytical robustness bound for self-testing via the three-setting steering inequality. We first consider the part of $G_1 := K_1 - s(ZB_0 + XB_1) - \tau_1 \mathbb{I}$ for $\mu \in (0, \pi/4]$; the spectral decomposition is already given in Eq. (B3). To make $G_1 \geq 0$, we consider the following local extraction channel. Bob takes $R_1 = I$ with probability q_1 and $R_2 = \sigma_z$ with probability q_2 ; meanwhile, with the rest of the probability $1 - q_1 - q_2 := q_3$ Bob takes some other local extraction channel subject to the choice of B_2 . Then we have

$$q_1 - s\lambda_1 - \tau_1 \geq 0,$$

$$q_2 - s\lambda_2 - \tau_1 \geq 0,$$

$$s\lambda_{(1/2)} - \tau_1 \geq 0,$$

$$\text{Tr}(\rho) = q_1 + q_2 + q_3 = 1,$$

$$\text{Tr}(\rho \hat{B}) = \lambda_1 q_1 + \lambda_2 q_2 + q_3 \text{Tr}(\rho Y B_2) = S,$$

where $\tau_1 = 1 - \gamma s$, with $\gamma \in [2, 3]$. In addition, τ_1 should be less than zero. We obtain $s \geq \frac{1+q_3}{2\gamma - (\lambda_1 + \lambda_2)} \geq \frac{1+q_3}{2\gamma - 2\sqrt{2}}$.

Next we determine the value of q_3 to make K_2 PSD. We notice $s\lambda_1 - \tau_1$ and $s\lambda_2 - \tau_1$, which, according to the coefficients of $|\psi_3\rangle$ and $|\psi_4\rangle$, are greater than zero. That is, if only the coefficients of $|\psi_1\rangle$ and $|\psi_2\rangle$ are greater than zero, the K_1 part will be PSD. Therefore, we put $|\psi_3\rangle$ and $|\psi_4\rangle$ into the K_2 part to make it PSD. Now the K_2 part becomes

$$G_2 := q_3 \Lambda_B^+(\psi_1) + (s\lambda_1 - \tau_1)|\psi_3\rangle\langle\psi_3| + (s\lambda_2 - \tau_1)|\psi_4\rangle\langle\psi_4| - sYB_2 - (\gamma - 3)s\mathbb{I}, \quad (\text{F1})$$

which is equivalent to

$$G_2 := q_3 \Lambda_B^+(\psi_1) + (s\lambda_1 - \tau_1)|\psi_3\rangle\langle\psi_3| + (s\lambda_2 - \tau_1)|\psi_4\rangle\langle\psi_4| - s(\gamma - 2)(U|\phi_1\rangle\langle\phi_1|U^T + U|\phi_2\rangle\langle\phi_2|U^T) + s(4 - \gamma)(U|\phi_3\rangle\langle\phi_3|U^T + U|\phi_4\rangle\langle\phi_4|U^T), \quad (\text{F2})$$

where $U = \begin{bmatrix} V & 0 \\ 0 & V^* \end{bmatrix}$ and $U^T = \begin{bmatrix} V^* & 0 \\ 0 & V \end{bmatrix}$, where

$$V = \begin{bmatrix} \frac{-\sin \mu_1 i - \cos \mu_1 \sin \mu_2}{\sqrt{2-2\cos \mu_1 \cos \mu_2}} & \frac{-\sin \mu_1 i - \cos \mu_1 \sin \mu_2}{\sqrt{2+2\cos \mu_1 \cos \mu_2}} \\ \frac{\cos \mu_1 \cos \mu_2 - 1}{\sqrt{2-2\cos \mu_1 \cos \mu_2}} & \frac{\cos \mu_1 \cos \mu_2 + 1}{\sqrt{2+2\cos \mu_1 \cos \mu_2}} \end{bmatrix}, \quad (\text{F3})$$

with $\phi_1 = [\frac{-i}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0]$, $\phi_2 = [0, \frac{i}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]$, $\phi_3 = [\frac{i}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0]$, and $\phi_4 = [0, \frac{-i}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]$. The requirement of $G_2 \geq 0$ gives (\mathcal{O} denotes overlap)

$$\frac{q_3(1+c)}{2} + (s\lambda_2 - \tau_1)\mathcal{O}^2(\psi_3, U^{-1}\phi_1) + (s\lambda_1 - \tau_1)\mathcal{O}^2(\psi_4, U^{-1}\phi_1) - s(\gamma - 2) \geq 0, \quad (\text{F4})$$

$$\frac{q_3(1-c)}{2} + (s\lambda_2 - \tau_1)\mathcal{O}^2(\psi_3, U^{-1}\phi_2) + (s\lambda_1 - \tau_1)\mathcal{O}^2(\psi_4, U^{-1}\phi_2) - s(\gamma - 2) \geq 0, \quad (\text{F5})$$

that is,

$$\begin{aligned} & \frac{q_3(1-c)}{2} \\ & + C_2 \frac{\cos^2(\frac{\pi}{8})(\sin \mu_1 - 1)^2 + \cos^2 \mu_1 \sin^2(\frac{\pi}{8} + \mu_2)}{4} \\ & + C_1 \frac{\cos^2(\frac{\pi}{8})(\sin \mu_1 + 1)^2 + \cos^2 \mu_1 \sin^2(\frac{\pi}{8} - \mu_2)}{4} \\ & - s(\gamma - 2) \geq 0, \\ & \frac{q_3(1+c)}{2} \\ & + C_2 \frac{\sin^2(\frac{\pi}{8})(\sin \mu_1 - 1)^2 + \cos^2 \mu_1 \cos^2(\frac{\pi}{8} + \mu_2)}{4} \\ & + C_1 \frac{\sin^2(\frac{\pi}{8})(\sin \mu_1 + 1)^2 + \cos^2 \mu_1 \cos^2(\frac{\pi}{8} - \mu_2)}{4} \\ & - s(\gamma - 2) \geq 0, \end{aligned}$$

where $C_1 = s\lambda_1 - \tau_1$ and $C_2 = s\lambda_2 - \tau_1$. With this channel, we have

$$\frac{q_3(1+c)}{2} + \frac{2-\sqrt{2}}{8}(s+\gamma s-1) - s(\gamma-2) \geq 0$$

and

$$\frac{q_3(1-c)}{2} + \frac{2+\sqrt{2}}{8}(s+\gamma s-1) - s(\gamma-2) \geq 0,$$

which gives us $q_3 c = \frac{\sqrt{2}}{4}(s+\gamma s-1)$ for $\gamma > 2$ and $q_3 \geq \frac{-5\gamma+2\sqrt{2}+9}{-\gamma+4\sqrt{2}-9}$. We can choose $\gamma = 3$, which gives $q_3 = \frac{1}{2}$; in addition, $s = \frac{3}{12-4\sqrt{2}} = 0.4730$ and $\tau = 1 - 3s$. Thus we give the following robustness bound of one-sided self-testing based

on the three-setting steering inequality:

$$F \geq sS_{\text{obs}} + \tau \geq \frac{3}{12 - 4\sqrt{2}}(S_{\text{obs}} - 3) + 1. \quad (\text{F6})$$

Although this does not reach the theoretical bound $s = \frac{1}{2(3-\sqrt{3})}$, the result is better than that of the two-setting inequality. This shows that adding more measurement settings can help increase the robustness in one-sided self-testing.

-
- [1] H. J. Kimble, The quantum internet, *Nature (London)* **453**, 1023 (2008).
- [2] F. Xu, X. Ma, Q. Zhang, H.-K. Lo, and J.-W. Pan, Secure quantum key distribution with realistic devices, *Rev. Mod. Phys.* **92**, 025002 (2020).
- [3] E. T. Campbell, B. M. Terhal, and C. Vuillot, Roads towards fault-tolerant universal quantum computation, *Nature (London)* **549**, 172 (2017).
- [4] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photon.* **5**, 222 (2011).
- [5] I. H. Deutsch, Harnessing the power of the second quantum revolution, *PRX Quantum* **1**, 020101 (2020).
- [6] J. Eisert, D. Hangleiter, N. Walk, I. Roth, D. Markham, R. Parekh, U. Chabaud, and E. Kashefi, Quantum certification and benchmarking, *Nat. Rev. Phys.* **2**, 382 (2020).
- [7] M. Kliesch and I. Roth, Theory of quantum system certification, *PRX Quantum* **2**, 010201 (2021).
- [8] S. T. Flammia and Y.-K. Liu, Direct Fidelity Estimation from Few Pauli Measurements, *Phys. Rev. Lett.* **106**, 230501 (2011).
- [9] D. Gross, Y.-K. Liu, S. T. Flammia, S. Becker, and J. Eisert, Quantum State Tomography via Compressed Sensing, *Phys. Rev. Lett.* **105**, 150401 (2010).
- [10] H.-Y. Huang, R. Kueng, and J. Preskill, Predicting many properties of a quantum system from very few measurements, *Nat. Phys.* **16**, 1050 (2020).
- [11] S. Pallister, N. Linden, and A. Montanaro, Optimal Verification of Entangled States with Local Measurements, *Phys. Rev. Lett.* **120**, 170502 (2018).
- [12] H. Zhu and M. Hayashi, Efficient Verification of Pure Quantum States in the Adversarial Scenario, *Phys. Rev. Lett.* **123**, 260504 (2019).
- [13] I. Šupić and J. Bowles, Self-testing of quantum systems: A review, *Quantum* **4**, 337 (2020).
- [14] D. Mayers and A. Yao, Self testing quantum apparatus, *Quantum Inf. Comput.* **4**, 273 (2004).
- [15] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* **86**, 419 (2014).
- [16] M. McKague, T. H. Yang, and V. Scarani, Robust self-testing of the singlet, *J. Phys. A: Math. Theor.* **45**, 455304 (2012).
- [17] T. H. Yang, T. Vértesi, J.-D. Bancal, V. Scarani, and M. Navascués, Robust and Versatile Black-Box Certification of Quantum Devices, *Phys. Rev. Lett.* **113**, 040401 (2014).
- [18] A. Coladangelo, K. T. Goh, and V. Scarani, All pure bipartite entangled states can be self-tested, *Nat. Commun.* **8**, 15485 (2017).
- [19] M. O. Renou, J. Kaniewski, and N. Brunner, Self-Testing Entangled Measurements in Quantum Networks, *Phys. Rev. Lett.* **121**, 250507 (2018).
- [20] J.-D. Bancal, N. Sangouard, and P. Sekatski, Noise-Resistant Device-Independent Certification of Bell State Measurements, *Phys. Rev. Lett.* **121**, 250506 (2018).
- [21] B. W. Reichardt, F. Unger, and U. Vazirani, Classical command of quantum systems, *Nature (London)* **496**, 456 (2013).
- [22] X. Wu, J.-D. Bancal, M. McKague, and V. Scarani, Device-independent parallel self-testing of two singlets, *Phys. Rev. A* **93**, 062121 (2016).
- [23] S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, *Nature (London)* **464**, 1021 (2010).
- [24] Y. Liu, Q. Zhao, M.-H. Li, J.-Y. Guan, Y. Zhang, B. Bai, W. Zhang, W.-Z. Liu, C. Wu, X. Yuan, H. Li, W. J. Munro, Z. Wang, L. You, J. Zhang, X. Ma, J. Fan, Q. Zhang, and J.-W. Pan, Device-independent quantum random-number generation, *Nature (London)* **562**, 548 (2018).
- [25] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography against Collective Attacks, *Phys. Rev. Lett.* **98**, 230501 (2007).
- [26] U. Vazirani and T. Vidick, Fully Device-Independent Quantum Key Distribution, *Phys. Rev. Lett.* **113**, 140501 (2014).
- [27] H. Shrotriya, K. Bharti, and L.-C. Kwek, Robust semi-device-independent certification of all pure bipartite maximally entangled states via quantum steering, *Phys. Rev. Res.* **3**, 033093 (2021).
- [28] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, *Phys. Rev. A* **85**, 010301(R) (2012).
- [29] E. Passaro, D. Cavalcanti, P. Skrzypczyk, and A. Acín, Optimal randomness certification in the quantum steering and prepare-and-measure scenarios, *New J. Phys.* **17**, 113010 (2015).
- [30] A. Gheorghiu, P. Wallden, and E. Kashefi, Rigidity of quantum steering and one-sided device-independent verifiable quantum computation, *New J. Phys.* **19**, 023043 (2017).
- [31] A. Unnikrishnan, I. J. MacFarlane, R. Yi, E. Diamanti, D. Markham, and I. Kerencis, Anonymity for Practical Quantum Networks, *Phys. Rev. Lett.* **122**, 240501 (2019).
- [32] F. Hahn, J. de Jong, and A. Pappa, Anonymous quantum conference key agreement, *PRX Quantum* **1**, 020325 (2020).
- [33] Y. Wang, X. Li, Y. Han, and K. Zhang, Practical anonymous entanglement with noisy measurement, *Quantum Inf. Process.* **21**, 49 (2022).
- [34] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, *Rev. Mod. Phys.* **92**, 015001 (2020).
- [35] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, *Phys. Rev. Lett.* **98**, 140402 (2007).
- [36] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Experimental EPR-steering using bell-local states, *Nat. Phys.* **6**, 845 (2010).
- [37] D. Cavalcanti, P. Skrzypczyk, G. H. Aguilar, R. V. Nery, P. H. S. Ribeiro, and S. P. Walborn, Detection of entanglement in asym-

- metric quantum networks and multipartite quantum steering, *Nat. Commun.* **6**, 7941 (2015).
- [38] I. Šupić and M. J. Hoban, Self-testing through EPR-steering, *New J. Phys.* **18**, 075006 (2016)
- [39] Y.-G. Han, Z. Li, Y. Wang, and H. Zhu, Optimal verification of the Bell state and Greenberger-Horne-Zeilinger states in untrusted quantum networks, *npj Quantum Inf.* **7**, 164 (2021).
- [40] S. Goswami, B. Bhattacharya, D. Das, S. Sasmal, C. Jebaratnam, and A. S. Majumdar, One-sided device-independent self-testing of any pure two-qubit entangled state, *Phys. Rev. A* **98**, 022311 (2018).
- [41] T. Pramanik, M. Kaplan, and A. S. Majumdar, Fine-grained Einstein-Podolsky-Rosen-steering inequalities, *Phys. Rev. A* **90**, 050305(R) (2014).
- [42] E. G. Cavalcanti, C. J. Foster, M. Fuwa, and H. M. Wiseman, Analog of the Clauser-Horne-Shimony-Holt inequality for steering, *J. Opt. Soc. Am. B* **32**, A74 (2015).
- [43] A. Acín, S. Massar, and S. Pironio, Randomness versus Nonlocality and Entanglement, *Phys. Rev. Lett.* **108**, 100402 (2012).
- [44] S. Sarkar, J. J. Borkala, C. Jebarathinam, O. Makuta, D. Saha, and R. Augusiak, Self-testing of any pure entangled state with minimal number of measurements and optimal randomness certification in one-sided device-independent scenario, [arXiv:2110.15176](https://arxiv.org/abs/2110.15176).
- [45] P. Skrzypczyk and D. Cavalcanti, Maximal Randomness Generation from Steering Inequality Violations Using Qudits, *Phys. Rev. Lett.* **120**, 260401 (2018).
- [46] A. Pappa, A. Chailloux, S. Wehner, E. Diamanti, and I. Kerenidis, Multipartite Entanglement Verification Resistant against Dishonest Parties, *Phys. Rev. Lett.* **108**, 260502 (2012).
- [47] W. McCutcheon, A. Pappa, B. A. Bell, A. McMillan, A. Chailloux, T. Lawson, M. Mafu, D. Markham, E. Diamanti, I. Kerenidis, J. G. Rarity, and M. S. Tame, Experimental verification of multipartite entanglement in quantum networks, *Nat. Commun.* **7**, 13251 (2016).
- [48] J. Kaniewski, Analytic and Nearly Optimal Self-Testing Bounds for the Clauser-Horne-Shimony-Holt and Mermin Inequalities, *Phys. Rev. Lett.* **117**, 070402 (2016).
- [49] M. Navascués, S. Pironio, and A. Acín, A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations, *New J. Phys.* **10**, 073013 (2008).
- [50] C. Bamps and S. Pironio, Sum-of-squares decompositions for a family of Clauser-Horne-Shimony-Holt-like inequalities and their application to self-testing, *Phys. Rev. A* **91**, 052111 (2015).
- [51] T. Coopmans, J. Kaniewski, and C. Schaffner, Robust self-testing of two-qubit states, *Phys. Rev. A* **99**, 052123 (2019).
- [52] H. Peyrl and P. A. Parrilo, Computing sum of squares decompositions with rational coefficients, *Theor. Comput. Sci.* **409**, 269 (2008).
- [53] A. Gočanin, C. Šupić, and B. Dakić, Sample-Efficient device-independent quantum state verification and certification, *PRX Quantum* **3**, 010317 (2022).
- [54] J.-D. Bancal, K. Redeker, P. Sekatski, W. Rosenfeld, and N. Sangouard, Self-testing with finite statistics enabling the certification of a quantum network link, *Quantum* **5**, 401 (2021).
- [55] Y. Wang, X. Wu, and V. Scarani, All the self-testings of the singlet for two binary measurements, *New J. Phys.* **18**, 025021 (2016).
- [56] L. P. Hughston, R. Jozsa, and W. K. Wootters, A complete classification of quantum ensembles having a given density matrix, *Phys. Lett. A* **183**, 14 (1993).