Capacity of entanglement for a nonlocal Hamiltonian

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The notion of capacity of entanglement is the quantum information theoretic counterpart of the heat capacity which is defined as the second cumulant of the entanglement spectrum. Given any bipartite pure state, we can define the capacity of entanglement as the variance of the modular Hamiltonian in the reduced state of any of the subsystems. Here, we study the dynamics of this quantity under a nonlocal Hamiltonian. Specifically, we address the following question: Given an arbitrary nonlocal Hamiltonian, what is the capacity of entanglement that the system can possess? As a useful application, we show that the quantum speed limit for creating the entanglement is not only governed by the fluctuation in the nonlocal Hamiltonian, but also depends inversely on the time average of the square root of the capacity of entanglement. Furthermore, we discuss this quantity for a general self-inverse Hamiltonian and provide a bound on the rate of capacity of entanglement. Towards the end, we generalize the capacity of entanglement for bipartite mixed states based on the relative entropy of entanglement and show that the above definition reduces to the capacity of entanglement for pure bipartite states. Our results can have several applications in diverse areas of physics.

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I. INTRODUCTION

Entanglement has potential applications in quantum information science, ranging from quantum computing and quantum communication to a host of other areas such as condensed matter physics, high-energy physics, and even string theory [1,2]. It is considered a very useful resource in information processing tasks. For several years, how to create and quantify entanglement has been a subject of major exploration [3,4]. Thanks to technological progress, now we can create entanglement between two or more particles in quantum optical systems [5], ion traps [6], superconducting systems [7,8], and nuclear magnetic resonance (NMR) setups [9]. How to create entanglement between more and more particles and distribute over long distances still continues to be quite challenging [10]. Quantum entanglement between two particles can of course be created depending on the choice of the initial state and suitable nonlocal interaction between them. However, the design of a suitable interacting Hamiltonian is not always easy. This makes the production of entanglement a nontrivial task. Therefore, it is natural to ask the question, for a given nonlocal Hamiltonian, what is the best way to exploit this Hamiltonian to create entanglement? This was addressed in Ref. [11].

Entanglement entropy is quite a useful diagnostic tool which measures the degree of quantum entanglement between subsystems in many-body quantum systems [12]. A different quantity, called the capacity of entanglement, has been proposed to characterize topologically ordered states in the context of the Kitaev model [13]. Given a pure bipartite entangled state ρ_{AB}, the capacity of entanglement is defined as the second cumulant of the entanglement spectrum. Thus, associated to a reduced density matrix, we can define the capacity of entanglement as the variance of the modular Hamiltonian in the mixed state. If $\{\lambda_i\}'$s are the eigenvalues of the reduced density matrix of one of the subsystems, then the entanglement entropy is defined as $S_{EE} = S(\rho_A) = -\,\mathrm{tr}(\rho_A \log_2 \rho_A) = -\sum_i \lambda_i \log_2 \lambda_i$. Now, the capacity of entanglement C_E is defined as the second cumulant of this entanglement spectrum [14], i.e., the variance in the entanglement entropy operator. It is similar to the heat capacity of thermal systems and is given by [14–16]

$$C_E = \sum_i \lambda_i \log^2 \lambda_i - S_{EE}^2.$$

The above quantity can be thought of as the variance of the distribution of $-\log_2 \lambda_i$ with probability λ_i, and thus it contains information about the width of the eigenvalue distribution of the reduced density matrix. We can gain insight on the whole spectrum by studying up to the first two cumulants, i.e., the entanglement entropy and the capacity of entanglement. Defining a modular Hamiltonian as $K_A = -\log_2 \rho_A$, they are the expectation value and the variance of K_A. The capacity of entanglement has found useful applications in conformal and nonconformal quantum field theories [17,18], as well as in models related to gravitational phase transitions [18–21].

The motivation for using the above definition of the capacity of entanglement stemmed from the fact that it is defined as the variance of an operator whose average is the entanglement entropy. In principle, one can define the entanglement

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capacity for other measures of entanglement, provided we can define an operator whose average will give that particular measure of entanglement. As we can see, not all measures of entanglement can be defined as averages of some Hermitian operators. This is the main reason why, in the literature, the capacity of entanglement has been defined for the entanglement entropy. Also, as mentioned above, it has a similarity with the heat capacity in the context of quantum thermodynamics.

In this paper, we address the entanglement capacities for nonlocal Hamiltonians. To be specific, we answer the following question: Given a nonlocal Hamiltonian, what is the capacity of entanglement for bipartite systems? We show that the entanglement rate is bounded by the fluctuation in the nonlocal Hamiltonian and the capacity of entanglement. In addition, the quantum speed limit for creating the entanglement depends inversely on the fluctuation in the nonlocal Hamiltonian, as well as on the time average of the square root of the capacity of entanglement. Thus, the more the capacity of entanglement, the shorter the time duration system may take to produce the desired amount of entanglement. We illustrate the quantum speed limit for a general two-qubit nonlocal Hamiltonian and find that our bound is indeed tight. Furthermore, we discuss the capacity of entanglement for self-inverse Hamiltonians and provide a bound on the rate of capacity of entanglement. Finally, we generalize the capacity of entanglement for bipartite mixed states based on the relative entropy of entanglement measure. This definition reduces to the capacity of entanglement for the pure bipartite states. This will open up its explorations for mixed states in the future. We believe that our results can find applications in diverse areas of physics, ranging from condensed matter systems to conformal field theories and the like.

The present paper is organized as follows. In Sec. II, we provide basic definitions and useful relations for the capacity of entanglement for pure bipartite states. In Sec. III, we discuss the capacity of entanglement for nonlocal Hamiltonians. In Sec. IV, we prove that the entanglement rate is bounded by the capacity of entanglement and the speed of quantum evolution under the nonlocal Hamiltonian. We also provide a quantum speed limit for entanglement production or degradation and discuss how the capacity of entanglement helps in deciding the speed limit. In Sec. V, we discuss the capacity of entanglement for self-inverse Hamiltonians and provide a bound on the rate of the capacity of entanglement. In Sec. VI, we generalize the definition of the capacity of entanglement for bipartite mixed states based on the notion of relative entropy of entanglement. Finally, in Sec. VII, we summarize our findings.

II. DEFINITIONS AND RELATIONS

Let $\mathcal{H}$ represent a separable Hilbert space and $\dim(\mathcal{H})$ be the dimension of Hilbert space. Let us consider a bipartite quantum system described by state vector $|\Psi\rangle_{AB} \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with unit norm. It is possible to express the state vector $|\Psi\rangle_{AB}$ as

$$|\Psi\rangle_{AB} = \sum_n \sqrt{\lambda_n}|\psi_n\rangle_A \otimes |\phi_n\rangle_B, \qquad (1)$$

where $\{|\psi_n\rangle\}_A$ and $\{|\phi_n\rangle\}_B$ are the Schmidt basis in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $\{\lambda_n\}$ are the non-negative real numbers with $\sum_n \lambda_n = 1$. Equation (1) is called the Schmidt decomposition of $|\Psi\rangle_{AB}$ and λ_n are known as the Schmidt coefficients. If the Schmidt decomposition of $|\Psi\rangle_{AB}$ has more than one nonzero Schmidt coefficient, then we say that systems A and B are "entangled." If there is only one nonzero Schmidt coefficient, then the state is not entangled.

Let $\mathcal{B}(\mathcal{H}_{AB})$ denote the algebra of linear operators acting on a finite-dimensional Hilbert space $\mathcal{H}_{AB}$ of dimension $\dim(\mathcal{H}_{AB})$ and let $\mathcal{D}(\mathcal{H}_{AB})$ denote the set of density operators for the bipartite system. The density operators are positive operators of unit trace acting on $\mathcal{H}_{AB}$. For any state $\rho_{AB} \in \mathcal{D}(\mathcal{H})$, if we can express ρ_{AB} as $\rho_{AB} = \sum_i p_i {\rho_i}^A \otimes {\rho_i}^B$, then it is a separable state, otherwise the mixed state is an entangled one. Given a density operator ρ_{AB} associated with a bipartite quantum system AB, the reduced density matrix for subsystem A (or B) is obtained by taking the partial trace over subsystem B (or A), i.e., $\rho_A = \mathrm{tr}_B(\rho_{AB})$. A physical quantity of system A represented by a self-adjoint operator $\mathcal{O}_A$ on $\mathcal{H}_A$ is identified with a self-adjoint operator $\mathcal{O}_A \otimes \mathcal{I}_B$ on $\mathcal{H}_{AB}$, where $\mathcal{I}_B$ is the identity operator on $\mathcal{H}_B$. The expectation value of $\mathcal{O}_A \otimes \mathcal{I}_B$ on state ρ_{AB} is given by $\mathrm{tr}(\rho_A \mathcal{O}_A)$, where ρ_A is the reduced density operator of system A.

The quantum relative entropy between two density operators ρ and σ acting on the same Hilbert space $\mathcal{H}$ is defined as [22]

$$S(\rho\|\sigma) := \begin{cases} \mathrm{tr}[\rho(\ln\rho - \ln\sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{otherwise,} \end{cases} \qquad (2)$$

where $\mathrm{supp}(\rho)$ and $\mathrm{supp}(\sigma)$ are the supports of ρ and σ, respectively. The quantum relative entropy satisfies important properties: (i) $S(\rho\|\sigma) \geqslant 0$ and $S(\rho\|\sigma) = 0$ iff $\rho = \sigma$, (ii) $\sum_i p_i S(\rho_i\|\sigma_i) \geqslant S(\sum_i p_i\rho_i\| \sum_i p_i\sigma_i)$, and (iii) $S(\rho\|\sigma) \geqslant S[\mathcal{E}(\rho)\|\mathcal{E}(\sigma)]$ for any completely positive trace-preserving map $\mathcal{E}$.

Let us consider a composite system AB with pure state $|\Psi\rangle_{AB}$. The amount of entanglement between subsystems A and B can be quantified via the entanglement entropy, which is defined as the von Neumann entropy of the reduced density operator $\rho_A = \sum_n \lambda_n |\psi_n\rangle_A\langle\psi_n|$ (or ρ_B), i.e.,

$$S_{EE} = S(\rho_A) = -\,\mathrm{tr}(\rho_A \log_2 \rho_A) = -\sum_n \lambda_n \log_2 \lambda_n, \qquad (3)$$

which is invariant under local unitary transformations on ρ_A. The von Neumann entropy vanishes when density operator ρ_A is a pure state. For a completely mixed density operator, the von Neumann entropy attains its maximum value of $\log_2 d_A$, where $d_A = \dim(\mathcal{H}_A)$.

For any density operator ρ_A associated with quantum system A, we can define a formal Hamiltonian K_A, called the modular Hamiltonian, with respect to which the density operator ρ_A is a Gibbs-like state (with $\beta = 1$),

$$\rho_A = \frac{e^{-K_A}}{Z},$$

where $Z = \mathrm{tr}(e^{-K_A})$. Note that any density matrix can be written in this form for some choice of Hermitian operator K_A. With slight adjustments in the above equation, the modular

Hamiltonian K_A can be written as $K_A = -\log_2 \rho_A$. In this case, the entanglement entropy of the system is equivalent to the thermodynamic entropy of a system described by Hamiltonian K_A (with $\beta = 1$). Writing in terms of modular Hamiltonian $K_A = -\log_2 \rho_A$, the entanglement entropy becomes the expectation value of the modular Hamiltonian,

$$S_{EE} = -\operatorname{tr}(\rho_A \log_2 \rho_A) = \operatorname{tr}(\rho_A K_A) = \langle K_A \rangle. \tag{4}$$

The capacity of entanglement is another information-theoretic quantity that has gained some interest recently [13,23]. It is defined as the variance of the modular Hamiltonian K_A [13] in the state $|\Psi\rangle_{AB}$ and can be expressed as

$$\begin{aligned} C_E(\rho_A) &= \langle\Psi|(K_A \otimes \mathcal{I}_B)^2|\Psi\rangle - \langle\Psi|(K_A \otimes \mathcal{I}_B)|\Psi\rangle^2 \\ &= \operatorname{tr}[\rho_A(-\log_2 \rho_A)^2] - [\operatorname{tr}(-\rho_A \log_2 \rho_A)]^2 \quad (5) \\ &= \operatorname{tr}\left[\rho_A K_A^2\right] - [\operatorname{tr}(\rho_A K_A)]^2 \\ &= \left\langle K_A^2 \right\rangle - \langle K_A\rangle^2 = \Delta K_A^2. \quad (6) \end{aligned}$$

The capacity of entanglement can also be defined in terms of the variance of the relative surprisal between two density matrices $V(\rho||\sigma)$ [24],

$$V(\rho||\sigma) = \operatorname{tr}[\rho(\log_2 \rho - \log_2 \sigma)^2] - [D(\rho||\sigma)]^2. \tag{7}$$

If one of the density matrices becomes maximally mixed (i.e., either ρ or σ becomes I/d), then the variance of the relative surprisal becomes the capacity of entanglement.

As shown in Ref. [25], uncertainty for any observable is a convex function. Given two or more Hermitian operators such as O_1 and O_2, the standard deviation or the uncertainty for observables satisfies $\Delta(p_1O_1 + p_2O_2) \leqslant p_1\Delta O_1 + p_2\Delta O_2$ for $0 \leqslant p_i \leqslant 1$ $(i = 1, 2)$ with $\Delta O_i = \sqrt{\langle O_i^2\rangle - \langle O_i\rangle^2}$. This shows that adding two or more observables always reduces the uncertainty. If we define the standard deviation in the modular Hamiltonian as uncertainty in the entanglement operator, then for any two modular Hamiltonian K_1 and K_2, we will have

$$\Delta\left(\sum_i p_i K_i\right) \leqslant \sum_i p_i \Delta K_i, \tag{8}$$

where $K_i = -\ln \rho_i$. This property has an interesting implication when we have a modular Hamiltonian undergoing some variation. Suppose we allow a variation in the modular Hamiltonian as $K \to K' = K + xV$, where V is the additional term in the modular Hamiltonian and x is a real parameter. Then, the following relation holds true: $\Delta K' \leqslant \Delta K + x\Delta V$.

For the sake of completeness, we mention the following properties which are applicable for C_E due to having a similar form as the relative surprisal between two density matrices:

(i) Additivity under tensor product:

$$C_E(\rho_A \otimes \rho_B) = C_E(\rho_A) + C_E(\rho_B).$$

(ii) Positivity : $C_E(\rho) \geqslant 0$.

(iii) Uniform continuity:

$$|C_E(\rho) - C_E(\rho')|^2 \leqslant \xi \log^2 d \cdot D(\rho, \rho'),$$

for ξ some constant and l_1 the trace norm between states $D(\rho, \rho')$.

(iv) $C_E(\rho) = 0$ if and only if all nonzero eigenvalues of ρ are the same. Such states are termed as flat states. Examples include any pure state or maximally mixed state.

(v) Corrections to subadditivity:

$$C_E(\rho) \leqslant C_E(\rho_1) + C_E(\rho_2) + \chi \log^2 d \cdot f(I_\rho),$$

for any bipartite state ρ with marginal states ρ_1, ρ_2 and mutual information I_ρ, with constant χ and $f(x) = \max(x^{1/4}, x^2)$.

(vi) For fixed dimensions $d \geqslant 2$, the state ρ_d with maximal variance has the spectrum

$$\operatorname{spec}(\rho_d) = \left(1 - r, \frac{r}{d-1}, \ldots, \frac{r}{d-1}\right),$$

with r being the unique solution to

$$(1 - 2r)\ln\left[\frac{1-r}{r}(d-1)\right] = 2.$$

We get $\frac{1}{4}\log^2(d-1) < C_E(\rho_d) < \frac{1}{4}\log^2(d-1) + \frac{1}{\ln^2(2)}$ and, for the limit of large d, $r \approx \frac{1}{2}$.

For further details and proofs regarding the above properties, see Ref. [24].

III. CAPACITY OF ENTANGLEMENT FOR NONLOCAL HAMILTONIANS

The dynamics of entanglement under a two-qubit nonlocal Hamiltonian has been addressed in Ref. [11]. In this section, we address the following question: What is the capacity of entanglement for an arbitrary two-qubit nonlocal Hamiltonian? Further, we also discuss the rate of the capacity of entanglement for the nonlocal Hamiltonian. For any two-qubit system, the nonlocal Hamiltonian can be expressed as (except for trivial constants)

$$H = \vec{\alpha} \cdot \vec{\sigma}^A \otimes \mathcal{I}_B + \mathcal{I}_A \otimes \vec{\beta} \cdot \vec{\sigma}^B + \sum_{i,j=1}^{3} \gamma_{ij}\sigma_i^A \otimes \sigma_j^B, \tag{9}$$

where $\vec{\alpha}, \vec{\beta}$ are real vectors, γ is a real matrix, and $\mathcal{I}_A$ and $\mathcal{I}_B$ are the identity operator acting on $\mathcal{H}_A$ and $\mathcal{H}_B$. The above Hamiltonian can be rewritten in one of the two standard forms under the action of local unitaries acting on each qubit [11,26]. This is given by

$$H^\pm = \mu_1\sigma_1^A \otimes \sigma_1^B \pm \mu_2\sigma_2^A \otimes \sigma_2^B + \mu_3\sigma_3^A \otimes \sigma_3^B, \tag{10}$$

where $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant 0$ are the singular values of matrix γ [11]. Using the Schmidt decomposition, any two-qubit pure state can be written as

$$|\Psi\rangle_{AB} = \sqrt{p}|\phi\rangle|\chi\rangle + \sqrt{1-p}|\phi^\perp\rangle|\chi^\perp\rangle. \tag{11}$$

We can utilize the form of Hamiltonian in Eq. (10) and choose H^+ [i.e., assuming $\det(\gamma) \geqslant 0$] to evolve the state in Eq. (11) without losing any generality [11]. To further showcase a specific example, let us choose $|\phi\rangle = |0\rangle$ and $|\chi\rangle = |0\rangle$. Thus, the state at time $t = 0$ takes the form

$$|\Psi(0)\rangle_{AB} = \sqrt{p}|0\rangle|0\rangle + \sqrt{1-p}|1\rangle|1\rangle. \tag{12}$$

Under the action of the nonlocal Hamiltonian, the joint state at time t can be written as ($\hbar = 1$)

$$|\Psi(t)\rangle_{AB} = e^{-iHt}|\Psi\rangle_{AB} = \alpha(t)|0\rangle|0\rangle + \beta(t)|1\rangle|1\rangle, \tag{13}$$

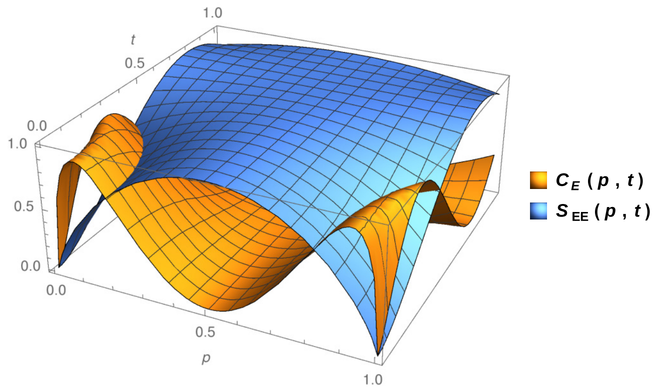

FIG. 1. Plot for capacity of entanglement (C_E) and entanglement entropy (S_{EE}) vs p and t taking $\theta = 1$.

where $\alpha(t) = e^{-i\mu_3 t}[\sqrt{p}\cos(\theta t) - i\sqrt{1-p}\sin(\theta t)]$, $\beta(t) = e^{-i\mu_3 t}[\sqrt{1-p}\cos(\theta t) - i\sqrt{p}\sin(\theta t)]$, and $\theta = (\mu_1 - \mu_2)$. To evaluate the capacity of entanglement, we would require the reduced density matrix of the two-qubit evolved state, $\rho_A(t) = \text{tr}_B[\rho_{AB}(t)]$, which is given by

$$\rho_A(t) = \lambda_1(t)|0\rangle\langle 0| + \lambda_2(t)|1\rangle\langle 1|, \tag{14}$$

where $\lambda_1(t) = |\alpha(t)^2|$ and $\lambda_2(t) = |\beta(t)^2|$, with

$$\lambda_1(t) = \tfrac{1}{2}[1 - (1-2p)\cos(2\theta t)],$$
$$\lambda_2(t) = \tfrac{1}{2}[1 + (1-2p)\cos(2\theta t)].$$

The capacity of entanglement at a later time t can be calculated from the variance of modular Hamiltonian K_A. This is given by

$$C_E(t) = \text{tr}\{\rho_A(t)[-\log_2 \rho_A(t)]^2\} - \{\text{tr}[-\rho_A(t)\log_2 \rho_A(t)]\}^2,$$
$$= \sum_{i=1}^{2} \lambda_i(t)\log^2 \lambda_i(t) - \left[-\sum_{i=1}^{2} \lambda_i(t)\log_2 \lambda_i(t)\right]^2. \tag{15}$$

In Fig. 1 we plot the capacity of entanglement $C_E(t)$ and entanglement entropy $S_{EE}(t)$ for an example case taking $\theta = 1$. In order to quantify the entanglement production, we can define the entanglement rate Γ as defined in Ref. [11], i.e.,

$$\Gamma(t) = \frac{dS_{EE}(t)}{dt} = \frac{dS_{EE}(t)}{dp}\frac{dp}{dt}. \tag{16}$$

The assertion is that this quantity depends upon the entanglement S_{EE}, which depends upon some parameter p and the rate of the Schmidt coefficient. The condition(s) to obtain a maximal entanglement rate are of interest for which two things are of significance. First, for a given value of S_{EE} of a two-qubit system, we find $|\Psi_E\rangle$ for which the interaction produces maximum rate Γ_E and, the maximal achievable entanglement rate $\Gamma_{\max} = \max_E \Gamma_E$ with corresponding state $|\Psi_{\max}\rangle$.

Let us evaluate objects defined above for an arbitrary Hamiltonian H. Using the Schmidt decomposition of the state $|\Psi(t)\rangle$,

$$|\Psi\rangle_{AB} = \sqrt{p}|\phi\rangle|\chi\rangle + \sqrt{1-p}|\phi^\perp\rangle|\chi^\perp\rangle, \tag{17}$$

where $\langle\phi|\phi^\perp\rangle = 0 = \langle\chi|\chi^\perp\rangle$ and $p \leqslant \frac{1}{2}$. The entanglement measure S_{EE} must depend only on the Schmidt coefficient p, given the fact that it must be invariant under local unitary operations. If we choose the entropy of entanglement as S_{EE}, the entropy of the reduced density operator of one of the qubits is given by

$$S_{EE}(p) = -p\log_2(p) - (1-p)\log_2(1-p). \tag{18}$$

Operationally, S_{EE} quantifies the amount of Einstein–Podolsky–Rosen entanglement contained asymptotically in a pure state $|\Psi\rangle_{AB}$, and thus S_{EE} gives a ratio of maximally entangled EPR state $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)$ which can be distilled from $|\Psi\rangle_{AB}$.

Considering the infinitesimal time evolution of the Schmidt coefficient of a two-qubit state, we get

$$|\Psi(t+\delta t)\rangle = e^{iH\delta t}|\Psi(t)\rangle \approx (1 - iH\delta t)|\Psi(t)\rangle.$$

The time evolution of the reduced state for the subsystem A is given by

$$\rho_A(t+\delta t) = \rho_A(t) - i\delta t\ \text{tr}_B\{[H, |\Psi(t)\rangle\langle\Psi(t)|]\}. \tag{19}$$

Starting from $\rho_A|\phi\rangle = p|\phi\rangle$, then using the Schrödinger equation, we have

$$\frac{dp}{dt} = 2\sqrt{p(1-p)}\,\text{Im}(\langle\phi,\chi|H|\phi^\perp,\chi^\perp\rangle). \tag{20}$$

As Γ is to be maximized, we can choose

$$\Gamma = f(p)|h(H,\phi,\chi)|,$$

where

$$f(p) = 2\sqrt{p(1-p)}S'_{EE}(p) \quad \text{and} \quad h(H,\phi,\chi) = \langle\phi,\chi|H|\phi^\perp,\chi^\perp\rangle.$$

Note that fixing S_{EE} means fixing p and so the maximum entropy corresponds to a state with some fixed $|\phi\rangle$ and $|\chi\rangle$. For any value S_{EE} of entanglement, the state $|\phi\rangle$ and $|\chi\rangle$ for which maximum entanglement rate Γ_E is obtained does not depend on S_{EE}, but only on the form of Hamiltonian H.

Let $h_{\max}$ be the maximum value of $|h|$, i.e.,

$$h_{\max} = \max_{||\phi||,||\chi||=1} |\langle\phi,\chi|H|\phi^\perp,\chi^\perp\rangle|. \tag{21}$$

Now we need to drive the two-qubit state with local operators so that for all time, the corresponding state is the one with the maximum rate and we would then know how the capacity of entanglement evolves with time.

Evaluating the capacity for entanglement for general pure bipartite states in the Schmidt-decomposed form as in Eq. (17) and using the modular Hamiltonian, we can express it as

$$C_E(\Psi_{AB}) = \text{tr}[\rho_A(\log_2\rho_A)^2] - [\text{tr}(\rho_A\log_2\rho_A)]^2$$
$$= p(1-p)\left[\log_2\left(\frac{p}{1-p}\right)\right]^2. \tag{22}$$

We can define the rate of capacity of entanglement as

$$\frac{dC_E}{dt} = \frac{dC_E}{dp}\frac{dp}{dt},$$

where

$$\frac{dC_E}{dp} = (1-2p)\left(\log_2 \frac{p}{1-p}\right)^2 + 2\log_2 \frac{p}{1-p},$$

which diverges for $p \to \{0\} \cup \{1\}$.

Let Γ_C denote the rate of capacity of entanglement, i.e., $\Gamma_C := \frac{dC_E(t)}{dt}$. From the earlier result, using the transformed Hamiltonian, we have

$$\Gamma_C = 2\sqrt{p(1-p)}\left[(1-2p)\left(\log_2 \frac{p}{1-p}\right)^2 + 2\log_2 \frac{p}{1-p}\right]|h(H,\phi,\chi)|. \quad (23)$$

Thus, it will not diverge with this form for $p = 0$ or 1.

It should be clear that local terms corresponding to $\vec{\alpha}, \vec{\beta}$ in Eq. (9) give no contribution to $h_{\max}$ with the given Schmidt-decomposed form of the bipartite state. Trying to determine $h_{\max}$ in terms of $\mu_{1,2,3}$, we get

$$h(H,\phi,\chi) = \sum_{k=1}^{3} \mu_k \langle\phi|\sigma_k^A|\phi^\perp\rangle\langle\chi|\sigma_k^B|\chi^\perp\rangle. \quad (24)$$

The maximum is reached when $|\chi\rangle = |\phi^\perp\rangle$. Further utilizing completeness condition $|\phi\rangle\langle\phi^\perp| + |\chi\rangle\langle\chi^\perp| = I$, we get the expression

$$h(H,\phi) = \sum_{k=1}^{3} \mu_k - \sum_{k=1}^{3} \mu_k \langle\phi|\sigma_k|\phi\rangle^2. \quad (25)$$

It can be further inferred from $\mu_1 \geqslant \mu_2 \geqslant \mu_3$ that the maximum value is reached when $|\phi\rangle = |0\rangle$ or $|1\rangle$, which gives us

$$h_{\max} = \mu_1 + \mu_2. \quad (26)$$

Thus, the state that provides the maximum rate of capacity of entanglement and the corresponding rate are given by

$$|\Psi_E\rangle = \sqrt{p}|01\rangle + i\sqrt{1-p}|10\rangle, \quad (27)$$

$$\Gamma_{C\max} = \left.\frac{dC_E}{dt}\right|_{\max} = 2(\mu_1+\mu_2)\sqrt{p(1-p)} \times \left[(1-2p)\left(\log_2 \frac{p}{1-p}\right)^2 + 2\log_2 \frac{p}{1-p}\right]. \quad (28)$$

The maximum rate $\Gamma_{C\max}$ is obtained here for $p_0 \simeq 0.0045$, which maximizes $f(p)$ to $f(p_0) \simeq 1.2108$ for the corresponding $|\Psi_{\max}\rangle$. The capacity of entanglement for this maximum rate is $C_E(p_0) \simeq 0.1306$.

It has been shown that if we can allow local operations which can entangle each qubit with local ancilla, that can increase the $\Gamma_{\max}$ for certain kinds of Hamiltonian [11]. We shall begin by generalizing the formulas for multilevel systems which contain the ancillas and the qubits. Consider a state $|\Psi\rangle_{AB}$ with the Schmidt decomposition $|\Psi\rangle_{AB} = \sum_{n=1}^{N} \sqrt{\lambda_n}|\phi_n\rangle|\chi_n\rangle$. Again, the capacity of entanglement only depends on the Schmidt coefficients $\lambda_n \geqslant 0$. Using the definition of capacity of entanglement rate in Eq. (16), we have

$$\tilde{\Gamma}_C = \frac{dC_E}{dt} = \sum_{n=1}^{N} \frac{\partial C_E}{\partial\lambda_n}\frac{d\lambda_n}{dt} = \frac{1}{N}\sum_{n,m=1}^{N}\left[\frac{\partial C_E}{\partial\lambda_n} - \frac{\partial C_E}{\partial\lambda_m}\right]\frac{d\lambda_n}{dt}. \quad (29)$$

Using the Schrödinger equation, we find

$$\frac{d\lambda_n}{dt} = 2\sum_{m=1}^{N}\sqrt{\lambda_n\lambda_m}\,\mathrm{Im}[\langle\phi_n,\chi_n|H|\phi_m,\chi_m\rangle]. \quad (30)$$

Now, let us consider one such example where adding ancillas allows one to increase the capacity of entanglement more efficiently. Let us consider the case in which the ancillas are also qubits. Letting $\lambda_1 = p$ and $\lambda_2 = \lambda_3 = \lambda_4 = (1-p)/3$, Eq. (29) simplifies to

$$\tilde{\Gamma} = \tilde{f}(p)\tilde{h}(H,\phi_n,\chi_n), \quad (31)$$

where

$$\tilde{f}(p) = 2\sqrt{p(1-p)/3}\left[(1-2p)\log^2\frac{3p}{1-p} + 2\log_2\frac{3p}{1-p}\right], \quad (32)$$

$$\tilde{h}(H,\phi_n,\chi_n) = \sum_{n=2}^{4}\mathrm{Im}[\langle\phi_n,\chi_n|H|\phi_m,\chi_m\rangle]. \quad (33)$$

We have a freedom to choose the phase of states $|\phi_n\rangle$ such that all terms add with the same sign, thus allowing us to replace the imaginary parts of the above terms by their absolute values, i.e., $\tilde{f}(p)$ by $|\tilde{f}(p)|$. We find that $\tilde{p}_0 \simeq 0.6036$, corresponding to the capacity of entanglement $C_E(\tilde{p}_0) \simeq 0.5523$ maximizing $\tilde{f}(p)$ to $|\tilde{f}(p_0)| \simeq 1.4459$. Further, proceeding to maximize $\tilde{h}$, we obtain that the maximum value is $\tilde{h}_{\max} = \mu_1 + \mu_2 + \mu_3$, which occurs when $|\phi_n\rangle$ and $|\chi_n\rangle$ are both orthogonal, maximally entangled states between the qubit and the ancilla.

Upon comparing the cases in which ancillas are used to those in which they are not used, we can either have $|\tilde{f}(\tilde{p}_0)| \geqslant |f(p_0)|$ or $\tilde{h}_{\max} \geqslant h_{\max}$. For the case when $\mu_3 \neq 0$, we can use ancillas to increase the maximum rate of capacity of entanglement $\Gamma_{\max}$ as well as Γ for a given capacity of entanglement of the state $|\Psi\rangle$.

IV. BOUND ON RATE OF ENTANGLEMENT

In this section, we will show that the capacity of entanglement plays an important role in providing an upper bound for the entanglement rate for the nonlocal Hamiltonian. Specifically, we will show that the entanglement rate is upper bounded by the speed of transportation of the bipartite state and the time average of the square root of the capacity of entanglement. Also, this sets a quantum speed limit on the entanglement production and degradation for pure bipartite states. Thus, the capacity of entanglement has a physical meaning in deciding how much time a bipartite states takes to produce a certain amount of entanglement.

Let us consider a bipartite system initially in a pure state. Let $|\Psi(0)\rangle_{AB}$ denote the initial state of the system. We consider the dynamics generated by a nonlocal Hamiltonian H_{AB}. The time-evolved state at later time t is given by $|\Psi(t)\rangle_{AB} = U_{AB}(t)|\Psi(0)\rangle_{AB}$, where $U_{AB}(t) = e^{-iH_{AB}t}$ with $\hbar = 1$.

Now, we apply the Heisenberg-Robertson uncertainty relation [27] for two noncommuting operators K_A and H_{AB}. This leads to

$$\tfrac{1}{2}|\langle\Psi(t)|[K_A \otimes I_B, H_{AB}]|\Psi(t)\rangle| \leqslant \Delta K_A \Delta H_{AB}. \tag{34}$$

Recall that the evolution of the average of any self-adjoint operator O is given by

$$i\hbar\frac{d\langle O\rangle}{dt} = \langle[O, H]\rangle. \tag{35}$$

Using Eq. (35) (for $O = K_A$) in Eq. (34), we then obtain

$$\frac{\hbar}{2}\left|\frac{d\langle K_A\rangle}{dt}\right| \leqslant \Delta K_A \Delta H_{AB}. \tag{36}$$

Let $\Gamma(t)$ denote the rate of entanglement. Recall that the average of the modular Hamiltonian is the entanglement entropy S_{EE}. In terms of the entanglement rate $\Gamma(t)$, the above equation can be written as

$$|\Gamma(t)| \leqslant \frac{2}{\hbar}\Delta K_A \Delta H_{AB}. \tag{37}$$

The square of the standard deviation of the modular Hamiltonian is the capacity of entanglement; therefore, in terms of the capacity of entanglement, we can write the above bound as

$$|\Gamma(t)| \leqslant \frac{2}{\hbar}\sqrt{C_E(t)}\Delta H_{AB}. \tag{38}$$

To interpret the above equation, first note that $\frac{2}{\hbar}\Delta H_{AB}$ is simply the speed of transportation of the bipartite pure entangled state on the projective Hilbert space of the composite system. If we use the Fubini-Study metric for two nearby states [28–30], then the infinitesimal distance between two nearby states is defined as

$$dS^2 = 4[1 - |\langle\Psi(t)|\Psi(t+dt)\rangle|^2] = \frac{4}{\hbar^2}\Delta H_{AB}^2 dt^2. \tag{39}$$

Therefore, the speed of transportation as measured by the Fubini-Study metric is given by $V = \frac{dS}{dt} = \frac{2}{\hbar}\Delta H_{AB}$. Thus, the entanglement rate is upper bounded by the speed of quantum evolution [31] and the square root of the capacity of entanglement, i.e., $|\Gamma(t)| \leqslant \sqrt{C_E(t)}V$.

It was shown in Ref. [32] that for the ancilla unassisted case, the entanglement rate is upper bounded by $c\|H\| \log_2 d$, where $d = \min(\dim\mathcal{H}_A, \dim\mathcal{H}_B)$, c is the constant between 0 and 1, and $\|H\|$ is the operator norm of the Hamiltonian which corresponds to $p = \infty$ of the Schatten p-norm of H, which is defined as $\|H\|_p = [\mathrm{tr}(\sqrt{H^\dagger H})^p]^{\frac{1}{p}}$. Now, using the fact that the maximum value of capacity of entanglement is proportional to $S_{\max}(\rho_A)^2$ [13], where $S_{\max}(\rho_A)$ is the maximum value of the von Neumann entropy of the subsystem which is upper bounded by $\log_2 d_A$, where d_A is the dimension of Hilbert space of subsystem A, and $\Delta H \leqslant \|H\|$, a similar bound on the entanglement rate can be obtained from Eq. (38). Thus, the bound on the entanglement rate given in Eq. (38) is stronger than the previously known bound.

The bound on the entanglement rate can be used to provide a quantum speed limit for the creation or degradation of entanglement. The notion of quantum speed limit (QSL) decides how fast a quantum state evolves in time from an initial state to a final state [33]. Even though it was discovered by Mandelstam and Tamm [34], over the last decade, there have been active explorations to generalize the notion of quantum speed limit for mixed states [35,36] and of resources that a quantum system might possess [37]. Recently, the notion of a generalized quantum speed limit has been defined in Ref. [38]. In addition, the quantum speed limit for observables has been defined and it was shown that the QSL for state evolution is a special case of the QSL for the observable [39]. For a quantum system evolving under a given dynamics, there exist fundamental limitations on the speed for entropy $S(\rho)$, maximal information $I(\rho)$, and quantum coherence $C(\rho)$ [40], as well as on other quantum correlations such as entanglement, quantum mutual information, and Bell-Clauser-Horne-Shimony-Holt correlation [41]. Below, we provide a speed limit bound for the entanglement entropy which can be applied for the scenario where entanglement can be generated or degraded, based on the capacity of entanglement. Our bound highlights the nontrivial role played by the capacity of entanglement in deciding the QSL.

The speed limit for entanglement entropy can be calculated from Eq. (38) by taking the absolute value on both of the sides and integrating over time. Thus, we have

$$\int_0^T \left|\frac{dS_{EE}(t)}{dt}\right| dt \leqslant \int_0^T \frac{2}{\hbar}\sqrt{C_E(t)}\Delta H dt. \tag{40}$$

For the time-independent Hamiltonian, we obtain the following bound for the quantum speed limit for entanglement:

$$T \geqslant T_{\rm QSL}^E := \frac{\hbar|S_{EE}(T) - S_{EE}(0)|}{2\Delta H \frac{1}{T}\int_0^T \sqrt{C_E(t)}dt}. \tag{41}$$

In the case of time-dependent Hamiltonian $H(t)$, we can apply the Cauchy-Schwarz inequality in Eq. (40) and obtain the following inequality:

$$\int_0^T \left|\frac{dS_{EE}(t)}{dt}\right| dt \leqslant \sqrt{\int_0^T \frac{2}{\hbar}\sqrt{C_E(t)}dt}\sqrt{\int_0^T \frac{2}{\hbar}\Delta H_t dt}. \tag{42}$$

From the above inequality, we get the bound for the speed limit for the entanglement entropy change as given by

$$T \geqslant T_{\rm QSL}^E := \frac{\hbar|S_{EE}(T) - S_{EE}(0)|}{2\Delta\bar{H}\sqrt{\frac{1}{T}\int_0^T \sqrt{C_E(t)}dt}}, \tag{43}$$

where $\Delta\bar{H} = \frac{1}{T}\int_0^T \sqrt{\langle H(t)^2\rangle - \langle H(t)\rangle^2}\, dt$ is the time-averaged fluctuation in the Hamiltonian. In both bounds (time-dependent and time-independent Hamiltonian), it is clear that the evolution speed for entanglement generation (or degradation) is a function of the capacity of entanglement C_E. Thus, we can say that C_E controls how much time a system may take to produce a certain amount of entanglement. Now, one may ask, how tight is the QSL bound for the entanglement generation of degradation? Here, we illustrate with a specific example that the quantum speed limit for the creation of

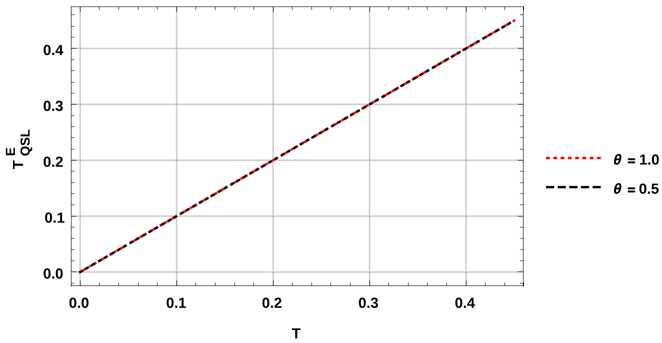

FIG. 2. $T^E_{\rm QSL}$ vs T with $p=1$ for $\theta=0.5$ and 1.0, which shows that our speed limit bound is tight.

entanglement is actually tight. Consider the initial state at $t=0$ as given in Eq. (12). The time evolution of the state is given by Eq. (13). Estimation of the speed limit bound on the entanglement entropy in Eq. (41) for the considered state would need the following quantities:

$$C_E(t)=\frac{[1-\eta(t)^2]\tanh^{-1}[\eta(t)]^2}{\ln^2(2)}, \tag{44}$$

where $\eta(t)=(1-2p)\cos(2\theta t)$, and

$$\Delta H=\theta(1-2p), \tag{45}$$

$$\begin{aligned}S_{EE}=&-\frac{\log_2\left[\left(p-\frac{1}{2}\right)\cos(2\theta t)+\frac{1}{2}\right]}{2}\\&+\frac{\log_2\left[\left(\frac{1}{2}-p\right)\cos(2\theta t)+\frac{1}{2}\right]}{2}\\&+\frac{(1-2p)\cos(2\theta t)\tanh^{-1}[(1-2p)\cos(2\theta t)]}{\ln(2)}.\end{aligned} \tag{46}$$

The plot in Fig. 2 for $T^E_{\rm QSL}$ vs $T\in[0,0.45]$ is shown under unitary dynamics through a general two-qubit nonlocal Hamiltonian H^+_{AB}, beginning with an initial state of the system $|\Psi(0)\rangle=|0\rangle|0\rangle$ [taking $p=1$ in Eq. (13)]. Our example shows that for the case of $\theta=\mu_1-\mu_2=0.5$ and 1.0, the QSL for the entanglement creation is indeed tight and attainable.

V. CAPACITY OF ENTANGLEMENT FOR SELF-INVERSE HAMILTONIAN

In this section, we will explore the dynamics of the capacity of entanglement for the self-inverse Hamiltonian. Such Hamiltonians are simpler to handle and provide many interesting insights. The rate of capacity of entanglement for the self-inverse Hamiltonian has been addressed. It was found that the inclusion of the ancilla system leads to the enhancement of the entanglement capability in Ref. [11], but for the Ising Hamiltonian $H_{\rm Ising}=\sigma_z\otimes\sigma_z$, it was shown that the entanglement capability is ancilla independent [42]. This independence on ancillas of entanglement capabilities turns out to be a consequence of the self-inverse property of the Hamiltonian, $H_{\rm Ising}=H^{-1}_{\rm Ising}$. This result was generalized to all Hamiltonian evolutions of the kind [43]

$$H_{AB}=X_A\otimes X_B, \tag{47}$$

such that $X_i=X_i^{-1}\in\mathcal{H}_i$ for $i\in\{A,B\}$. As a consequence of the self-inverse property of the Hamiltonian, we have the time-evolution operator ($\hbar=1$),

$$U(t)=e^{-iHt}=\cos t\,\mathcal{I}_A\otimes\mathcal{I}_B-i\sin t\,X_A\otimes X_B. \tag{48}$$

Let $|\Psi(0)\rangle_{AB}$ be the initial state of the bipartite system AB, which can be expressed in the Schmidt decomposition as follows:

$$|\Psi(0)\rangle_{AB}=\sum_n\sqrt{\lambda_n}|\psi_n\rangle_A\otimes|\phi_n\rangle_B. \tag{49}$$

Let $\rho_{AB}(t)$ denote the density operator at time t. The time evolution of $\rho_{AB}(t)$ is governed by the Liouville–von Neumann equation given as

$$\frac{d\rho_{AB}(t)}{dt}=-i[H_{AB},\rho_{AB}(t)], \tag{50}$$

where H_{AB} is the nonlocal Hamiltonian of the composite system. The dynamics of the reduce density operator ρ_B (or ρ_A) can be obtained from the above equation by tracing out A (or B), which is given by

$$\frac{d\rho_B}{dt}=-i\,{\rm tr}_A[H_{AB},\rho_{AB}(t)]. \tag{51}$$

Now, first we will calculate an upper bound on the rate of capacity of entanglement for unitary evolution and then we will address the case of the self-inverse Hamiltonian. To calculate an upper bound on C_E, first we differentiate both sides of Eq. (6) with respect to time, which leads to

$$\begin{aligned}\frac{dC_E(t)}{dt}&=\frac{d}{dt}\left(\langle K_A^2\rangle-\langle K_A\rangle^2\right)\\&=\frac{d}{dt}\{{\rm tr}[\rho_A(-\log_2\rho_A)^2]\}-\frac{d}{dt}[-{\rm tr}(\rho_A\log_2\rho_A)]^2\\&=\frac{d}{dt}\{{\rm tr}[\rho_A(-\log_2\rho_A)^2]\}-2S(\rho_A)\frac{d}{dt}S(\rho_A)\\&=\frac{d}{dt}\{{\rm tr}[\rho_A(-\log_2\rho_A)^2]\}-2S(\rho_A)\Gamma(t),\end{aligned} \tag{52}$$

where $\Gamma(t)$ is the rate of entanglement. Let $\mathcal{B}(\mathcal{H})_+$ denote the subset of positive semidefinite operators acting on $\mathcal{H}$. Now, we use the fact that the logarithm of an operator $A\in\mathcal{B}(\mathcal{H})_+$ can be represented by

$$\log_2 A=\int_0^\infty ds\left(\frac{1}{(s+1)\mathcal{I}}-\frac{1}{(s\mathcal{I}+A)}\right), \tag{53}$$

where $\mathcal{I}$ is the identity operator. We use the above equation to compute the first terms on the right-hand side of Eq. (52). This can be expressed as

$$\begin{aligned}\frac{d}{dt}\{\mathrm{tr}[\rho_A(-\log_2\rho_A)^2]\} &= \mathrm{tr}\left\{\frac{d}{dt}[\rho_A(-\log_2\rho_A)^2]\right\}\\ &= \mathrm{tr}\left\{[\dot{\rho}_A(\log_2\rho_A)^2] + \rho_A\frac{d}{dt}(\log_2\rho_A)^2\right\}\\ &= \mathrm{tr}\{[\dot{\rho}_A(\log_2\rho_A)^2]\} + \mathrm{tr}\left\{\rho_A\frac{d}{dt}\left[\int_0^\infty ds\left(\frac{1}{(s+1)\mathcal{I}} - \frac{1}{(s\mathcal{I}+\rho_A)}\right)\right](\log_2\rho_A)\right\}\\ &\quad + \mathrm{tr}\left\{\rho_A(\log_2\rho_A)\frac{d}{dt}\left[\int_0^\infty ds\left(\frac{1}{(s+1)\mathcal{I}} - \frac{1}{(s\mathcal{I}+\rho_A)}\right)\right]\right\}\\ &= \mathrm{tr}\{[\dot{\rho}_A(\log_2\rho_A)^2]\} + \mathrm{tr}\left\{\rho_A\left[\int_0^\infty ds\left(\frac{1}{(s\mathcal{I}+\rho_A)}\dot{\rho}_A\frac{1}{(s\mathcal{I}+\rho_A)}\right)\right](\log_2\rho_A)\right\}\\ &\quad + \mathrm{tr}\left\{\rho_A(\log_2\rho_A)\left[\int_0^\infty ds\left(\frac{1}{(s\mathcal{I}+\rho_A)}\dot{\rho}_A\frac{1}{(s\mathcal{I}+\rho_A)}\right)\right]\right\}\\ &= \mathrm{tr}\{[\dot{\rho}_A(\log_2\rho_A)^2]\} + 2\,\mathrm{tr}\,(\dot{\rho}_A\log_2\rho_A). \end{aligned} \tag{54}$$

The second term on the right-hand side of above equation is the rate of the entropy [44], so we rewrite Eq. (52) as

$$\frac{dC_E(t)}{dt} = \mathrm{tr}[\dot{\rho}_A(-\log_2\rho_A)^2] + 2\,\mathrm{tr}(\dot{\rho}_A\log_2\rho_A)[1+S(\rho_A)]. \tag{55}$$

Now, we consider the case where ρ is full rank; then, the first term of the above equation can be simplified as

$$\begin{aligned}\mathrm{tr}[\dot{\rho}(\log_2\rho)^2] &= \sum_i\langle i|\dot{\rho}|i\rangle(\log_2\lambda_i)^2\\ &\leqslant k_{\max}^2\sum_i\langle i|\dot{\rho}|i\rangle\\ &= k_{\max}^2\,\mathrm{tr}[\dot{\rho}_A] = 0,\end{aligned} \tag{56}$$

where $k_{\max}$ is the maximum of the eigenvalues of the modular Hamiltonian. We then obtain an upper bound on the capacity of entanglement as

$$\begin{aligned}|\Gamma_C| &\leqslant |2\,\mathrm{tr}(\dot{\rho}_A\log_2\rho_A)(1+\log_2 d_A)|\\ &= |2\Gamma(t)(1+\log_2 d_A)|.\end{aligned} \tag{57}$$

Using Eq. (38), we can give an upper bound on the rate of the capacity of entanglement as given by

$$|\Gamma_C| \leqslant 2\sqrt{C_E}V(1+\log_2 d_A). \tag{58}$$

Thus, we can see that the rate of entanglement depends on the evolution speed of the bipartite quantum state, which is given by $V = \frac{2}{\hbar}\Delta H_{AB}$.

For the ancilla unassisted case, the entanglement rate $\Gamma(t)$ is upper bounded by $c||H||\log_2 d$ (see Ref. [32]). Then, the upper bound on the rate of the capacity of entanglement Γ_C becomes

$$|\Gamma_C| \leqslant 2c||H||\log_2 d(1+\log_2 d_A). \tag{59}$$

Now we will find the upper bound on Γ_C for self-inverse Hamiltonians. The maximum entanglement rate $\Gamma(t)$ for the self-inverse Hamiltonian $H = X_A\otimes X_B$ was calculated in Ref. [43]. It is given by $\Gamma(t)\leqslant\beta$, where

$$\beta = 2\max_{x\in[0,1]}\sqrt{x(1-x)}\log_2\frac{x}{1-x}. \tag{60}$$

Therefore, the bound on Γ_C can be expressed as

$$|\Gamma_C| \leqslant 2\beta(1+\log_2 d_A). \tag{61}$$

This bound is independent of the details of the initial state, but uses the self-inverse nature of the nonlocal Hamiltonian.

VI. CAPACITY OF ENTANGLEMENT FOR MIXED STATES

In the previous section, we used the definition of C_E for pure states. Here, we generalize the definition for the case of mixed states in such a way that it reduces to the previous definition for pure states. For this, we use the relative entropy of entanglement since it reduces to the entanglement entropy for pure states. The relative entropy of entanglement is defined in Ref. [45] and further expanded for arbitrary dimensions in Ref. [46]. This is given by

$$E_R(\rho_{AB}) = \min_{\sigma_{AB}\in\mathrm{SEP}} S(\rho||\sigma), \tag{62}$$

where SEP is set of all separable or positive partial transpose (PPT) states and $S(\rho||\sigma) = \mathrm{tr}(\rho\log_2\rho - \rho\log_2\sigma)$. Operationally, the relative entropy of entanglement quantifies the extent to which a given mixed entangled state can be distinguished from the closest state which is either separable or has a positive partial transpose (PPT). Also, this is an entanglement monotone and is asymptotically continuous.

In the following, we shall denote the state in SEP for which the the minimum is attained for a given ρ_{AB} as ρ_{AB}^*. Then, we can write $E_R(\rho_{AB})$ as

$$E_R(\rho_{AB}) = \min_{\sigma_{AB}\in\mathrm{SEP}} S(\rho_{AB}||\sigma_{AB}) = S(\rho_{AB}||\rho_{AB}^*). \tag{63}$$

Now, we claim that the capacity of entanglement for mixed states is given by

$$\begin{aligned}C_E(\rho_{AB}) &= \mathrm{tr}[\rho_{AB}(\log_2\rho_{AB} - \log_2\rho_{AB}^*)^2]\\ &\quad - \mathrm{tr}[\rho_{AB}(\log_2\rho_{AB} - \log_2\rho_{AB}^*)]^2.\end{aligned} \tag{64}$$

We will now show that this agrees with the definition of capacity of entanglement for pure states. The relative entropy of entanglement is given by

$$E_R(\rho_{AB}) = \mathrm{tr}[\rho_{AB}(\log_2 \rho_{AB} - \log_2 \rho^*_{AB})] = \langle \log_2 \rho_{AB} - \log_2 \rho^*_{AB}\rangle. \tag{65}$$

For a pure state, the density operator ρ_{AB} is given by

$$\rho_{AB} = |\Psi\rangle_{AB}\langle\Psi| = \sum_{i,j}\sqrt{p_i p_j}|\phi_i\rangle\langle\phi_j| \otimes |\chi_i\rangle\langle\chi_j|_{AB}. \tag{66}$$

The expression for ρ^*_{AB} for ρ_{AB} is known [47] and given as follows:

$$\rho^*_{AB} = \sum_k p_k|\phi_k\rangle\langle\phi_k| \otimes |\chi_k\rangle\langle\chi_k|_{AB}. \tag{67}$$

The first term of Eq. (64) is given by

$$\begin{aligned}\langle(\log_2 \rho_{AB} - \log_2 \rho^*_{AB})^2\rangle &= {}_{AB}\langle\Psi|[(\log_2 |\Psi\rangle_{AB}\langle\Psi|)^2 + (\log_2 \rho^*_{AB})^2 \\ &\quad - (\log_2 |\Psi\rangle_{AB}\langle\Psi| \log_2 \rho^*_{AB} \\ &\quad + \log_2 \rho^*_{AB} \log_2 |\Psi\rangle_{AB}\langle\Psi|)]|\Psi\rangle_{AB}.\end{aligned} \tag{68}$$

Defining $A_\Psi = |\Psi\rangle_{AB}\langle\Psi| - \mathcal{I}$, we get

$${}_{AB}\langle\Psi|(\log_2 |\Psi\rangle_{AB}\langle\Psi|) = {}_{AB}\langle\Psi|\left[A_\Psi - \frac{(A_\Psi)^2}{2} + \cdots\right] = 0.$$

This leads to

$${}_{AB}\langle\Psi|(\log_2 |\Psi\rangle_{AB}\langle\Psi|)^2|\Psi\rangle_{AB} = 0, \tag{69}$$

where the only surviving term in Eq. (68) is ${}_{AB}\langle\Psi|(\log_2 \rho^*_{AB})^2|\Psi\rangle_{AB}$.

Now, we have

$$\begin{aligned}(\log_2 \rho^*_{AB})^2 &= \sum_k (\log_2 p_k)^2|\phi_k\rangle_A\langle\phi_k| \otimes |\chi_k\rangle_B\langle\chi_k|, \\ \langle(\log_2 \rho^*_{AB})^2\rangle &= \sum_{i,j,k}\sqrt{p_i p_j}(\log_2 p_k)^2\delta_{ik}\delta_{jk} \\ &= \sum_k p_k(\log_2 p_k)^2 = \langle(\log_2 \rho_A)^2\rangle.\end{aligned}$$

The second term of Eq. (68) is equal to $E(\rho_{AB})^2$ for pure states. Thus, for $\rho_{AB} = |\Psi\rangle\langle\Psi|_{AB}$, we have

$$C_E = \langle(\log_2 \rho_A)^2\rangle - \langle\log_2 \rho_A\rangle^2, \tag{70}$$

which agrees with the expression for the capacity of entanglement for the pure bipartite states.

It may be worth noting that the capacity of entanglement for a mixed state can also be expressed as the variance of the shifted modular Hamiltonian for the joint system. Upon defining the modular Hamiltonian for the composite state ρ_{AB} and ρ^*_{AB} as $K_{AB} = -\log_2 \rho_{AB}$ and $K^*_{AB} = -\log_2 \rho^*_{AB}$, we have

$$\begin{aligned}C_E &= \mathrm{tr}[\rho_{AB}(K_{AB} - K^*_{AB})^2] - \mathrm{tr}[\rho_{AB}(K_{AB} - K^*_{AB})]^2 \\ &= \langle(K_{AB} - K^*_{AB})^2\rangle - \langle K_{AB} - K^*_{AB}\rangle^2 \\ &= \langle\tilde{K}^2_{AB}\rangle - \langle\tilde{K}_{AB}\rangle^2,\end{aligned} \tag{71}$$

where $\tilde{K}_{AB} = K_{AB} - K^*_{AB}$ is the shifted modular Hamiltonian for the composite system. This provides another meaning for the capacity of entanglement for the mixed state.

Now, we illustrate the capacity of entanglement for a mixed state using the above definition. For general mixed entangled states, it is not always easy to find the closest separable state. However, for those cases where we know the closest separable state, we can compute the capacity of entanglement.

Let us consider a mixed entangled state as given by

$$\rho_{AB} = \lambda|\phi^+\rangle\langle\phi^+| + (1-\lambda)|01\rangle\langle 01|, \tag{72}$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is one of the four Bell states. The corresponding closest separable state which minimizes quantum relative entropy with ρ_{AB} [47] is given by

$$\begin{aligned}\rho^*_{AB} = {}& \frac{\lambda}{2}\left(1-\frac{\lambda}{2}\right)(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| \\ &+ |11\rangle\langle 11|) + \left(1-\frac{\lambda}{2}\right)^2|01\rangle\langle 01| + \frac{\lambda^2}{4}|10\rangle\langle 10|.\end{aligned} \tag{73}$$

The expression for the relative entropy of entanglement for this example is given by

$$E_R(\lambda) = (\lambda-2)\ln\left(1-\frac{\lambda}{2}\right) + (1-\lambda)\ln(1-\lambda). \tag{74}$$

Consider another example of a mixed state,

$$\rho_{AB} = \lambda|\phi^+\rangle\langle\phi^+| + (1-\lambda)|00\rangle\langle 00|. \tag{75}$$

The closest separable state minimizing relative entropy for this case is of the form [47]

$$\rho^*_{AB} = \left(1-\frac{\lambda}{2}\right)|00\rangle\langle 00| + \frac{\lambda}{2}|11\rangle\langle 11|. \tag{76}$$

The relative entropy of entanglement in this case can be analytically found and given as

$$E_R(\lambda) = s_+\ln(s_+) + s_-\ln(s_-) - 2\left(1-\frac{\lambda}{2}\right)\ln\left(1-\frac{\lambda}{2}\right), \tag{77}$$

where

$$s_\pm = \frac{1 \pm \sqrt{1-2\lambda\left(1-\frac{\lambda}{2}\right)}}{2}.$$

The detailed expressions for the capacity of entanglement for ρ_{AB} in Eq. (72) and Eq. (75) are very complicated. For the purpose of illustration, we have provided numerical plots for the same. From the behavior of the plots in Figs. 3

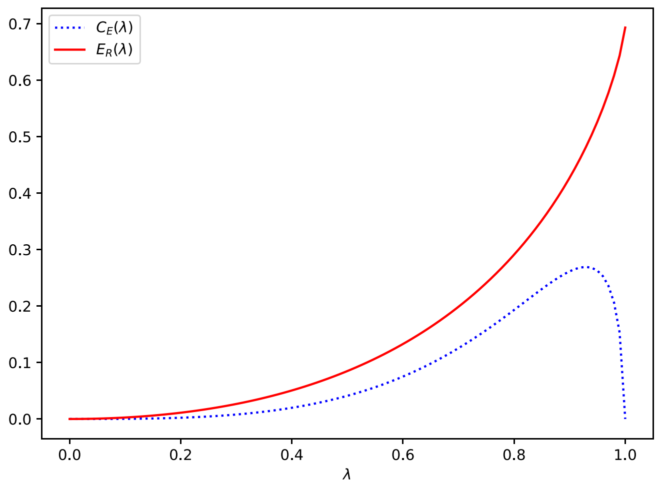

FIG. 3. Plot for capacity of entanglement (C_E) and relative entropy of entanglement (E_R) vs $\lambda \in [0, 1]$ for ρ_{AB} in Eq. (72).

and 4, it can be inferred that for $\lambda \in \{0, 1\}$, the cases where all nonzero eigenvalues of the state are the same and thus the state becomes either pure or maximally mixed, and for such flat states, the capacity of entanglement vanishes.

It will be interesting to see if we can generalize all the results obtained for a pure bipartite state to a mixed state case. However, we leave these detailed investigations for the mixed state case for future work. Before we conclude, we will show that the quantum speed limit for entanglement creation and degradation for the mixed state ρ_{AB} under unitary evolution can, in fact, be generalized. Consider the mixed state ρ_{AB} which undergoes a unitary evolution, i.e.,

$$\rho_{AB}(0) \to \rho_{AB}(t) = U_{AB}(t)\rho_{AB}(0)U_{AB}^{\dagger}(t), \tag{78}$$

where $U_{AB}(t) = e^{-\frac{i}{\hbar}H_{AB}}$ and H_{AB} is the time-independent Hamiltonian. The equation of motion for the average of the

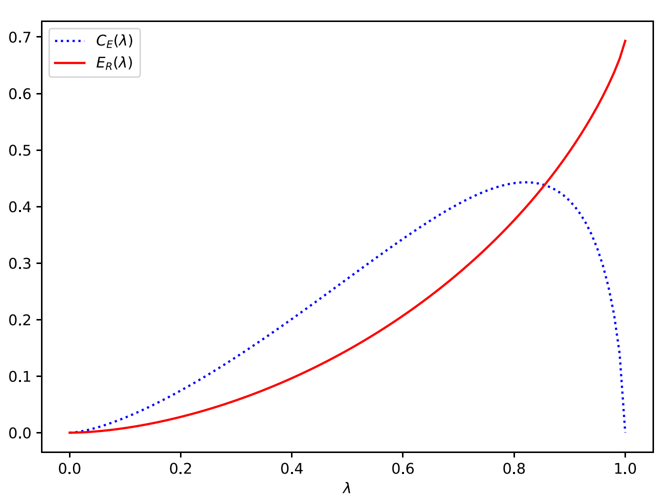

FIG. 4. Plot for capacity of entanglement (C_E) and relative entropy of entanglement (E_R) vs $\lambda \in [0, 1]$ for ρ_{AB} in Eq. (75).

observable acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$i\hbar\frac{d}{dt}\,\mathrm{tr}(\rho_{AB}O_{AB}) = \mathrm{tr}(\rho_{AB}[O_{AB}, H_{AB}]). \tag{79}$$

Now, if we take $O_{AB} = \tilde{K}_{AB}$ as the shifted modular Hamiltonian for the bipartite system, then, using the above equation of motion and the Robertson uncertainty relation, we have

$$\left|\frac{dE_R(t)}{dt}\right| \leqslant \frac{2}{\hbar}\Delta\tilde{K}_{AB}\Delta H_{AB}, \tag{80}$$

where $E_R(t) = E_R[\rho_{AB}(t)] = S[\rho_{AB}(t)||\rho_{AB}^{*}(t)]$ is the relative entropy of entanglement for the time-evolved state and $\frac{2}{\hbar}\Delta H_{AB}$ is the speed of the bipartite mixed state under unitary evolution with $\Delta H_{AB}^2 = \mathrm{tr}(\rho_{AB}H_{AB}^2) - [\mathrm{tr}(\rho_{AB}H_{AB})]^2$. Since the capacity of entanglement for the mixed state based on the relative entropy of entanglement is $C_E(\rho_{AB}) = \Delta\tilde{K}_{AB}^2$, we have the quantum speed limit for the mixed state as given by

$$T \geqslant T_{\mathrm{QSL}}^{E} := \frac{\hbar|E_R(T) - E_R(0)|}{2\Delta H_{AB}\frac{1}{T}\int_0^T \sqrt{C_E(\rho_{AB})dt}}. \tag{81}$$

The above formula will reduce to the quantum speed limit for the pure state case as given in (42). It may be noted that the derivation for the quantum speed limit for the mixed state holds as long as the closest separable state is differentiable. Thus, the minimal time for the entanglement creation for the mixed state does depend on the speed of the bipartite mixed state as well as the capacity of entanglement. This result will have important implications in understanding the dynamics of entanglement for mixed states.

VII. CONCLUSIONS

Undoubtedly, the study of quantum entanglement for bipartite and multipartite states has been one of the prime areas of research over the last several decades. Even though the dynamics of entanglement for nonlocal Hamiltonians was addressed earlier, the question of the dynamics of the capacity of entanglement has not been addressed. The notion of the capacity of entanglement is a very useful quantity and this can be regarded as the quantum information theoretic counterpart of the heat capacity. For any bipartite pure state, the capacity of entanglement is the variance of the modular Hamiltonian in the reduced state of any subsystem. In this paper, we have studied the dynamics of the capacity of entanglement under a nonlocal Hamiltonian. Our results answer a very pertinent question regarding the capacity of entanglement that the system can possess when it evolves in time under a nonlocal Hamiltonian. The capacity of entanglement has another meaning in deciding the upper bound for the entanglement rate. We have shown that the quantum speed limit for creating the entanglement is not only governed by the fluctuation in the nonlocal Hamiltonian, i.e., the speed of transportation of the bipartite state, but also depends inversely on the time average of the square root of the capacity of entanglement. In addition, we have discussed the capacity of entanglement for a self-inverse Hamiltonian and found an upper bound for this case on the rate of capacity of entanglement. We have also generalized the notion of the capacity of entanglement for bipartite mixed states based on the relative entropy of entanglement, which reduces to a known form for the pure states case. We have provided two simple examples for the capacity of entanglement for mixed states. Towards the end, we have generalized the quantum speed limit for the creation of entanglement for mixed states. The minimal time for the creation of entanglement for the mixed states depends on the speed of quantum evolution and the entanglement capacity. In the future, it will be worthwhile to explore this notion of the capacity of entanglement for mixed states and multipartite systems, which will have useful applications in other areas of physics.

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