Optimal discrimination between real and complex quantum theories

Adam Bednorz [1,*] and Josep Batle [2,†]
[1] *Faculty of Physics, University of Warsaw, ul. Pasteura 5, PL02-093 Warsaw, Poland*
[2] *CRISP-Centre de Recerca Independent de sa Pobla, C. Albéniz 12, 07420 sa Pobla, Balearic Islands, Spain*

(Received 5 June 2022; accepted 27 September 2022; published 11 October 2022)

We find the minimal number of settings to test quantum theory based on real numbers, assuming separability of the sources, modifying the recent proposal [M.-O. Renou *et al.*, Nature (London) **600**, 625 (2021)]. The test needs only three settings for observers A and C, but the ratio of complex to real maximum is smaller than in the existing proposal. We also found that two settings and two outcomes for both observers are insufficient.

DOI: 10.1103/PhysRevA.106.042207

I. INTRODUCTION

Ever since the dawn of modern science, the interplay between mathematics and physics has been explored in parallel to the development of the scientific method. Roger Bacon first described mathematics as "the door and the key to the sciences" [1]. Regarding the quantification of the physical knowledge, "The great book of nature," wrote Galileo, "is written in mathematical language" [2]. And finally, quite more recently, Eugene Wigner elaborated upon "the unreasonable effectiveness of mathematics in the natural sciences" [3]. In quantum mechanics, we encounter a debate not found in the classical realm, namely, how fundamental is the utilization of the field of complex numbers $\mathbb{C}$, as opposed to real ones $\mathbb{R}$, in the description of physical phenomena. In the same way that Born, Heisenberg, and Jordan [4,5] introduced matrices in the first complete formulation of quantum mechanics, it was the Schrödinger equation that introduced explicitly $i = \sqrt{-1}$ and therefore complex states ψ [6].

Avoiding epistemological discussions such as the wave function being an element of physical reality or not [7], it is no surprise that, at least experimentally, one requires the real and imaginary parts of the wave function [8]. Additionally, local real- or complex-valued tomography could lead to different experimental results [9]. Therefore, and to no surprise, quantum physics based on complex-valued quantities is a successful theory both qualitative and quantitatively.

Several works, notably initiated by von Neumann [10–15], elaborate on the possible ways of employing real numbers only for describing the same phenomena by doubling the concomitant complex n-dimensional Hilbert spaces to real-valued ones, that is, $|1\rangle, ...|n\rangle \to |1\rangle, ..., |n\rangle, |n+1\rangle, ..., |2n\rangle$. This is possible for any observable or density operator given, as an $n \times n$ Hermitian matrix $H = A = H_R + iH_I$, with real symmetric H_R and antisymmetric H_I, can be regarded as an equivalent to the real, symmetric problem:

$$\begin{pmatrix} H_R & -H_I \\ H_I & H_R \end{pmatrix}. \tag{1}$$

In this fashion, each state $|\psi\rangle = \sum_k (\psi_{kR} + i\psi_{kI})|k\rangle$, with real $\psi_{kR,I}$, is replaced by a doublet of $\sum_k (\psi_{kR}|k\rangle + \psi_{kI}|n+k\rangle)$, $\sum_k (\psi_{kI}|k\rangle - \psi_{kR}|n+k\rangle)$. Therefore we are left also with extra degeneracy of states, which all are not normally doubled, particularly the ground state. This is not a problem for local phenomena, but separable states consisting of several parties are doubled in each party. One can keep a single doublet only by extra entanglement in real space.

Recently, Renou *et al.* [16] developed a scheme probing this possibility, i.e., testing if the states separable in complex space need to be replaced by an entangled state in real space. It turned out that real separability imposes additional constraints on correlations, leading to an inequality, with a lower bound for real states than for complex ones. The proposal involved three observers, A, B, and C, where A and B (C and B) receive qubits from the one (second) source. Then B makes a single measurement with four outcomes, while A and C make dichotomic measurements for three and six settings, respectively. The violation of the inequality has been verified experimentally [17,18].

Here we make an amendment to this scheme, reducing the number of settings to three for observer C, and constructing the corresponding witness. We also provide the example with four settings, where the impossibility of reaching the complex maximum in real space is shown. The present contribution is divided as follows. We start with the description of the setup and notation. Then we show the example with four settings for C. The later case with three settings needs a numerical search, based on a modification of the MATLAB script published with the previous proposal [16]. Reduction to two settings and two outcomes for both observers is impossible, as we show with the partial help of a numerical search. Finally, we draw some conclusions, suggesting further possible routes of optimization.

*Adam.Bednorz@fuw.edu.pl
†batlequantum@gmail.com

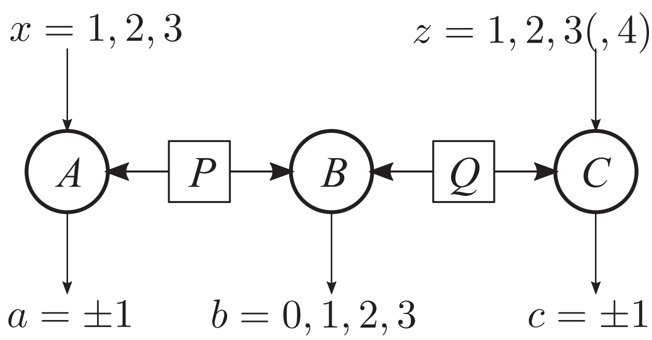

FIG. 1. The setup of the test. The separate sources P and Q generate entangled states. Central parts of these states are measured by B with four possible outcomes b, while the left and right parts are measured by observers A and C, with dichotomic outcomes a and c for settings x and z, respectively. Note that the number of settings z for C is either 3 or 4.

II. SETUP OF THE TEST

The analyzed system, as in the previous work [16], consists of three observers A, B, and C, depicted in Fig. 1. The sources P and Q are separable, which is an important assumption. The observers A and C can choose one of three settings $x = 1, 2, 3$ and $z = 1, 2, 3$ (or also 4), respectively, and make a dichotomic measurement of A_x an C_z with the outcomes $a, c = \pm 1$. The observer B has only one setting, and the outcome $b = 0, 1, 2, 3$.

In the quantum mechanics based on real numbers, the separability between P and Q takes place in real space, which leads to tighter bounds on correlations than in full complex space. The witness to distinguish the two cases reads

$$F = \sum_{x,z,b} (-1)^{s_{bx}} f_{xz} \langle A_x C_z || b \rangle, \tag{2}$$

where the correlation is expressed in terms of probabilities as

$$\langle AC || b \rangle = \sum_{ac} acp(a, b, c), \tag{3}$$

with coefficients f_{jk} and

$$s_{bx} = \begin{cases} -1 & \text{for } b \neq x, \\ +1 & \text{otherwise.} \end{cases} \tag{4}$$

The goal of our research is to find the matrix f such that the bound on F is the lowest possible in the real case in comparison with the maximum in the complex space. In the classical case, the maximum reads

$$F_c = \max_{|s_x|=|t_z|=1} \sum_{xz} f_{xz} s_x t_z. \tag{5}$$

In the quantum case the maximum F_q depends much on the actual form of the matrix f. In [16], the authors constructed f_{xz} for $x = 1, 2, 3$, $z = 1, 2, 3, 4, 5, 6$ for the combination of three Clauser-Horne-Shimony-Holt Bell inequalities [19–22].

Here we consider the case of four settings for the observer C and a family of fs, where the complex quantum maximum, Tsirelson bound [23], can be found algebraically and it is realized in the discussed setup. Namely, let us take

$$f = \begin{pmatrix} \alpha & \beta & \gamma & q \\ \gamma & \alpha & \beta & q \\ \beta & \gamma & \alpha & q \end{pmatrix}, \tag{6}$$

where

$$q = -\sqrt{3}(\alpha\beta + \beta\gamma + \gamma\alpha), \tag{7}$$

with the constraints $\alpha^2 + \beta^2 + \gamma^2 = 1$ and $q \geqslant 0$. We will show that

$$F_q = 3 + \sqrt{3}q. \tag{8}$$

Note that the following quantity is positive in both the quantum and classical case:

$$\begin{aligned} &q(\sqrt{3}C_4 - A_1 - A_2 - A_3)^2/\sqrt{3} \\ &+ (C_1 - \alpha A_1 - \beta A_2 - \gamma A_3)^2 \\ &+ (C_2 - \alpha A_2 - \beta A_3 - \gamma A_1)^2 \\ &+ (C_3 - \alpha A_3 - \beta A_1 - \gamma A_2)^2. \end{aligned} \tag{9}$$

Opening brackets and taking into account that $A^2 = C^2 = 1$ we get

$$\sum_{x,z} f_{xz} \langle A_x C_z \rangle \leqslant F_q. \tag{10}$$

This maximum holds also if B is included.

The example realizing the above maximum is constructed as follows. We recall standard conventions in Appendix A. The working space consists of four qubits, $APQC$, with $PQ \to B$. The source states read

$$\rho_L = (1 - \boldsymbol{\sigma}^A \cdot \boldsymbol{\sigma}^P)/4, \;\; \rho_R = (1 - \boldsymbol{\sigma}^C \cdot \boldsymbol{\sigma}^Q)/4, \tag{11}$$

using shorthand notation $1 \otimes 1 \equiv 1^{\otimes 2} \equiv 1$ (a tensor product of identities). The matrices σ act on the qubit in the subscript while identity on the other qubits. We assume projective measurements of A, B, C. Four measurements at B are defined by the projections

$$\begin{aligned} B_0 &= (1 - \boldsymbol{\sigma}^P \cdot \boldsymbol{\sigma}^Q)/4, \\ B_1 &= \left(1 - \sigma_1^P\sigma_1^Q + \sigma_2^P\sigma_2^Q + \sigma_3^P\sigma_3^Q\right)/4, \\ B_2 &= \left(1 - \sigma_2^P\sigma_2^Q + \sigma_3^P\sigma_3^Q + \sigma_1^P\sigma_1^Q\right)/4, \\ B_3 &= \left(1 - \sigma_3^P\sigma_3^Q + \sigma_1^P\sigma_1^Q + \sigma_2^Q\sigma_2^Q\right)/4. \end{aligned} \tag{12}$$

After the measurement at B, in the AC basis,

$$\begin{aligned} \mathrm{Tr}_{PQ} B_0 \rho_R \rho_R &= (1 - \boldsymbol{\sigma}^A \cdot \boldsymbol{\sigma}^C)/16, \\ \mathrm{Tr}_{PQ} B_1 \rho_L \rho_R &= \left(1 - \sigma_1^A\sigma_1^C + \sigma_2^A\sigma_2^C + \sigma_3^A\sigma_3^C\right)/16, \\ \mathrm{Tr}_{PQ} B_2 \rho_L \rho_R &= \left(1 - \sigma_2^A\sigma_2^C + \sigma_3^A\sigma_3^C + \sigma_1^A\sigma_1^C\right)/16, \\ \mathrm{Tr}_{PQ} B_3 \rho_L \rho_R &= \left(1 - \sigma_3^A\sigma_3^C + \sigma_1^A\sigma_1^C + \sigma_2^A\sigma_2^C\right)/16. \end{aligned} \tag{13}$$

For observables $A = \boldsymbol{a} \cdot \boldsymbol{\sigma}^A$, $C = \boldsymbol{c} \cdot \boldsymbol{\sigma}^C$ with $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{c} = 1$ we have for the measurement outcome $a \to \pm 1$ [projections $(1 \pm A)/2$], $b \to 0, 1, 2, 3$ [projections B_b], $c \to \pm 1$ [projections $(1 \pm C)/2$], and correlations

$$\begin{aligned} \langle AC || b \rangle &= \mathrm{Tr} ACB_b \rho_L \rho_R, \\ \langle AC || 0 \rangle &= -\boldsymbol{a} \cdot \boldsymbol{c}/4, \end{aligned}$$

$$\langle AC||1\rangle = (a_2c_2 + a_3c_3 - a_1c_1)/4,$$
$$\langle AC||2\rangle = (a_3c_3 + a_1c_1 - a_2c_2)/4,$$
$$\langle AC||3\rangle = (a_1c_1 + a_2c_2 - a_3c_3)/4. \quad (14)$$

We emphasize that the above formulas are valid in full complex space because of the complex matrix σ_2.

III. THE TEST OF COMPLEX VERSUS REAL QUANTUM MECHANICS

The maximum of F in the complex quantum mechanics is obtained for $(a_x)_i = \delta_{ix}$, and

$$\boldsymbol{c}_1 = -(\alpha, \beta, \gamma),\ \boldsymbol{c}_2 = -(\gamma, \alpha, \beta),$$
$$\boldsymbol{c}_3 = -(\beta, \gamma, \alpha),\ \boldsymbol{c}_4 = (1, 1, 1)/\sqrt{3}. \quad (15)$$

In this case it is also possible to prove the impossibility of achieving this maximum in the real case. To see this note, that the maximal case, saturating the inequality, in order to get zero from each of the squares in (9), must satisfy

$$\sqrt{3}C_4 = A_1 + A_2 + A_3,\ C_1 = \alpha A_1 + \beta A_2 + \gamma A_3,$$
$$C_2 = \alpha A_2 + \beta A_3 + \gamma A_1,\ C_3 = \alpha A_3 + \beta A_1 + \gamma A_2, \quad (16)$$

when acting on the state. Defining

$$C_1 = \alpha C_1' + \beta C_2' + \gamma C_3',\ C_2 = \alpha C_2' + \beta C_3' + \gamma C_1',$$
$$C_3 = \alpha C_3' + \beta C_1' + \gamma C_2', \quad (17)$$

we quickly find, using the fact $A_j^2 = 1 = C_j^2$, that acting on the state,

$$C_j' = A_j,\ A_iA_j = -A_jA_i,\ C_i'C_j' = -C_j'C_i', \quad (18)$$

for $i \neq j$. Using these identities one can repeat the reasoning in [16] to show that the initial state ρ_{APQC} is not real separable into spaces AP and QC. The actual maximum in the real case F_r can be bounded from above by the numerical code (see later). It turns out that the maximal ratio $F_q/F_r > 1$ for all choices of α, β, γ but its maximum $\simeq 1.07$ is obtained for $\beta = \gamma = -\alpha = 1/\sqrt{3}$, giving $F_q = 4$ while $F_r = 3.7367$ and $F_c = 2\sqrt{3}$. The directions C lie in the vertices of the regular tetrahedron, see Fig. 2, and our complex maximum coincides with Platonic (elegant) Bell inequalities [21,24,25].

In the case of three settings for the observer C, the problem to find the maximum is as hard as the Tsirelson bound for I_{3322} inequality [26]. Therefore we choose a family of states for a general form of f. The settings for A are the same as before. For C we take

$$(c_z)_i = -\frac{f_{iz}}{\sqrt{f_{1z}^2 + f_{2z}^2 + f_{3z}^2}}, \quad (19)$$

leading to the value

$$F_q = \sum_z \sqrt{f_{1z}^2 + f_{2z}^2 + f_{3z}^2}. \quad (20)$$

To find the real maximum F_r in the case of four and three settings for C we had to adopt the MATLAB script using the numerical technique of minimizing F with semipositive matrices of correlations of monomials of A_x and C_z up to given degree n_A and n_C [27], based on the fact that the real states

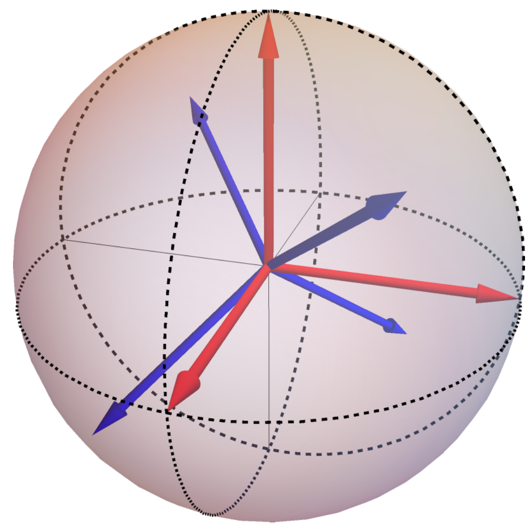

FIG. 2. The directions of the settings of A (red, light) and C (blue, dark) in the Bloch sphere, maximizing the F_q/F_r ratio for four settings of C and F defined by (2).

are separable, i.e., $\rho_{LR} = \rho_L\rho_R$ if and only if $\rho_{LR} = (\rho_L^T)_R$, i.e., the state is equal to its partial transform [28]. Note that this is a stricter criterion than in the complex case, where separable states must have a positive partial transform but not always vice versa (not if and only if) [29–31]. Such a numerical problem is convex, and a solution can be obtained using available semidefinite algorithms [32] and tools [33,34]. The script, being the modification of the original one from [16], is in the Supplemental Material [35].

As an example, take f equal to

$$\begin{pmatrix} -2 & 3 & 3 \\ 3 & -2 & 3 \\ 3 & 3 & -2 \end{pmatrix}. \quad (21)$$

Then $F_c = 12$, $F_q = 3\sqrt{22} \simeq 14.07$, while $F_r = 13.677$ (taking $n_A = n_C = 2$), so there is clearly a quantum state violating the real bound. Interestingly, we could also run $n_A = n_C = 3$ case and the value does not change (up to machine precision). Unfortunately, the ratio $F_q/F_r \simeq 1.029$ is quite demanding experimentally so we run numerical scan through a wide range of matrices f, additionally boosting the largest value by the steepest descent method. We found the maximal ratio $F_q/F_r \simeq 1.065\,94$ for f equal to

$$\begin{pmatrix} 3 & -5.009 & -4.99 \\ -5.01 & 2.6 & -5.09 \\ -5.11 & -5 & 3 \end{pmatrix}, \quad (22)$$

with $F_r = F_c = 21.607$, $F_q = 23.031\,685$. The coincidence of the real and classical maximum indicates that the constraints of the numerical method produce here the exact bound already for $n_A = n_C = 2$. We illustrated the bounds in the case of three and four settings for C in Fig. 3.

Reduction to two settings and two outcomes for A or C gives equal real and complex bounds $F_q = F_r$ for maximal entanglement. For both A and C with two settings and outcomes, we confirmed the general equality $F_q = F_r$ by a numerical survey on a sample of 40 000 random test points, see Appendix B.

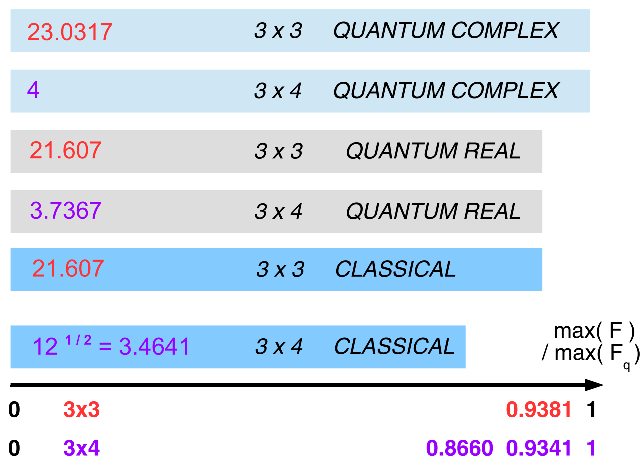

FIG. 3. The classical, quantum real, and complex bound, F_c, F_r, F_q, respectively, and their ratio in the case of tests with either 3×4 or 3×3 settings for observers $A \times C$.

Relaxing more assumptions, e.g., one observer has more settings or we allow more outcomes, seems highly nontrivial, and we cannot make any definite conclusions yet.

IV. CONCLUSIONS

We have found tests of real separability in contrast to complex separability, an alternative to those recently found in Ref. [16]. Our proposals require fewer settings for the observers, but the ratio between the complex and real bound is also lower. The numerical check takes a much shorter time. There are still some open questions: despite an extensive numerical search, we could not reduce the test to two setting for observers A or C, nor could we provide a mathematical proof of impossibility. Also, when the real maximum is higher than the classical one, a numerical search through low-dimensional real quantum systems could not find any example above the classical bound. It may probably require many more dimensions, just like the I_{3322} case [26]. One can explore possible inequalities based on more settings (six for A and C) using other Platonic inequalities [25], especially to increase the F_q/F_r ratio, but it can be demanding numerically. Another interesting problem is to check three or more outcomes and all cases where one party has only two settings, whether it is possible to rule these cases out by arguments similar to Bell tests [36]. Finally, we emphasize that such tests do have loopholes—the sources P and Q can be already entangled, and the observers could communicate during measurements, unless one uses a spacelike regime as in recent Bell tests [37–40]. The communication can be checked in every such experiment, including the already existing ones [17,18], verifying no-signaling, i.e., independence of $\langle A_x||b\rangle$, $\langle C_z||b\rangle$, and $\langle 1||b\rangle \equiv p(b)$, of z, x, and xz, respectively, but the published data are insufficient for these specific claims.

ACKNOWLEDGMENTS

J.B. acknowledges fruitful discussions with J. Rosselló, M. del Mar Batle, and R. Batle. A.B. acknowledges discussions with J. Tworzydło.

APPENDIX A: NOTATION

We use Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A1}$$

with $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$, summation convention, i.e., $X_aY_a \equiv \sum_{a=1,2,3} X_aY_a = \boldsymbol{X}\cdot\boldsymbol{Y}$, and

$$\delta_{ab} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq v \end{cases},$$
$$\epsilon_{abc} = \begin{cases} +1 & \text{for } abc = 123, 231, 312 \\ -1 & \text{for } abc = 321, 213, 132 \\ 0 & \text{otherwise} \end{cases}. \tag{A2}$$

We also use the identities

$$\epsilon_{abc}\epsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd},\ \epsilon_{abc}\epsilon_{abd} = 2\delta_{cd},$$
$$\delta_{aa} = 3,\ \delta_{ab}\delta_{bc} = \delta_{ac}. \tag{A3}$$

APPENDIX B: NO-GO WITH TWO SETTINGS AND TWO OUTCOMES

Here we prove that every test involving two settings and two outcomes for both observers will give $F_q = F_r$. First we show that $A_{1,2}$ are real matrices in some basis. The states AP and QC can be written in terms of diagonal Schmidt decomposition. By convexity we can also project A_j onto the space spun by nonzero Schmidt elements and assume $A_j^2 = 1$. We can start from a diagonal basis of A_1, where $A_1 = \mathrm{diag}(1, .., 1, -1, .., -1)$. We also adjust this basis so that

$$A_2 = \begin{pmatrix} A_+ & A' \\ A'^\dagger & A_- \end{pmatrix}, \tag{B1}$$

where $A_\pm$ are diagonal in the respective subspaces of $A_1 = \pm 1$. Since $A_2^2 = 1$, we get

$$A_+A' + A'A_- = 0. \tag{B2}$$

It means that either $(A_0)_{jk} = 0$ or $A_{+j} + A_{-k} = 0$. In the latter case, we can group subspaces $A_{+j} = -A_{-k} = a$. All off-diagonal elements of A_0 between different values a must be zero. Within the particular subspace a we can make singular value decomposition of $A'|_a$, resulting in its diagonal form (appended by zeros for nonsquare A'). By phase tuning, we end up in a real matrix. Note that the construction leads to splitting of the whole space into trivial one-dimensional spaces, where $A_1 = \pm 1$ and $A_2 = \pm 1$, or two-dimensional ones, where A_1 and A_2 are some combinations of σ_1 and σ_3. These spaces are not connected by elements of A_j.

If both A_j and the Schmidt decomposition

$$|L\rangle = \sum_j \lambda_j |jj\rangle,\ \sum_j |\lambda_j|^2 = 1, \tag{B3}$$

are real valued in the same basis as A_j, we can use the following reasoning. All monomials consisting of A_j have the form

$$m(A_1, A_2) = \cdots A_1A_2A_1A_2\cdots, \tag{B4}$$

i.e., A_1 at odd positions and A_2 at even ones, or vice versa. In any case, the monomial is a real matrix in the constructed basis. Therefore

$$\langle m(A_1,A_2)\rangle = \langle m^\dagger(A_1,A_2)\rangle^* = \langle m^T(A_1,A_2)\rangle^* = \langle m^T(A_1,A_2)\rangle, \tag{B5}$$

where the transpose is in the real basis. It is true whenever the state also has real representation. Now the monomial matrix,

$$G_b[m(A_1,A_2), m(C); m'(A_1,A_2), m'(C)] = \langle m^T(A_1,A_2)m'(A_1,A_2)m^\dagger(C)m'(C)B_b\rangle, \tag{B6}$$

is complex semidefinite in general but satisfies

$$\begin{aligned} G &= \sum_b G_b[m(A_1,A_2), m(C); m'(A_1,A_2), m'(C)] \\ &= \langle m^T(A_1,A_2)m'(A_1,A_2)m^\dagger(C)m'(C)\rangle \\ &= \langle m'^T(A_1,A_2)m(A_1,A_2)m^\dagger(C)m'(C)\rangle, \end{aligned} \tag{B7}$$

i.e., equality with the partial transpose. Taking $G_b \to (G_b + G_b^*)/2$ one obtains a real semipositive matrix. Therefore the algorithm from [16] will give the same bound for the real and complex states. For the maximally entangled states, λ_j are independent of j. The basis transformations U on A spaces can be then moved through the constant onto P.

If C has also only two settings and outcomes we reduce it to a real representation analogously. For nonmaximally entangled states, taking into account splitting A_j and C_j into two-level spaces, by convexity we can reduce the problem to a two-level space for A and C (and P and Q by Schmidt decomposition). However, for nonmaximally entangled states we cannot push U through the states, as they are not necessarily diagonal. Note that such states can reveal nonclassical features in special situations, where it is impossible with the maximally entangled ones [41]. Since the PQ space has $2\times 2 = 4$ dimensions, we can have maximally four outcomes for B. In order to deal with this situation, we have resorted to a numerical exploration. We have generated 40 000 random sets of Bell-type 36 parameters f_{bxz} for quantity

$$F = \sum_{bxz} f_{bxz}\langle A_x C_z | |b\rangle, \tag{B8}$$

with $i = 0123$, $x, z = 012$, and $A_0 = 1 = C_0$, in the hypercube $[-1,1]^{36}$, and, for each one of them, computed to the maximum value.

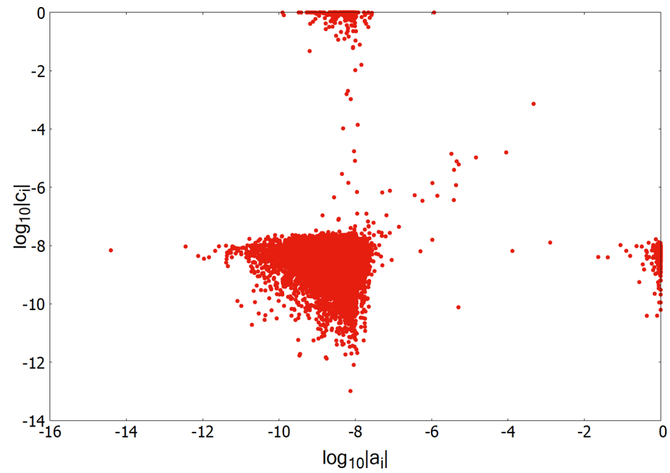

FIG. 4. Points (a_i, c_i) in the numerical test of 40 000 random points maximizing (B8), see text. Only simultaneous $a_i, c_i \sim 1$ would give a complex example, irreducible to real space.

The task is feasible, but going further to higher numbers of sets is computationally demanding. The states and observables can be then represented as follows. Diagonal and real U, V, $B_b = |b\rangle\langle b|$ where

$$|b\rangle = \sum h_{bjk}|jk\rangle, \tag{B9}$$

for the bases $|0\rangle, |1\rangle$ of P and Q being rows of the unitary 4×4 matrix H, i.e., $H_{jk} = h_{j,2j+k}$, which can be generated by the product of six matrices,

$$H^{(JK)}_{jk} = \begin{cases} \cos\theta_{JK} & \text{for } j=k=J,K \\ e^{i\psi_{JK}}\sin\theta_{JK} & \text{for } j=J, k=K \\ -e^{-i\psi_{JK}}\sin\theta_{JK} & \text{for } j=K, k=J \\ 1 & \text{for } j=k\neq J,K \\ 0 & \text{otherwise,} \end{cases} \tag{B10}$$

with $J < K$. We check if the maximal solution retrieves nonzero values for $a_{12} = -a_{22} = a_i$ and $c_{12} = -c_{22} = c_i$ (the matrices can be always put into this form by a phase shift in qubit space). The final optimal values for a_i, c_i are depicted in Fig. 4. There are two types of data, namely, the ones with either $|a_i|$ or $|c_i|$ close to unity, and the rest being very small. During our numerical survey, we have not encountered any case with both $|a_i|$ and $|c_i|$ anywhere near to 1 (our criterion has been $|a_i| > 0.01$ and $|c_i| > 0.01$). We cannot rule out entirely the possibility of finding counterexamples that refute our main claim, but our results indicate that it is unlikely.

[1] R. Bacon, *Opus Majus* (Cambridge University Press, Cambridge, England, 1615).
[2] G. Galilei, Il Saggiatore (in Italian), (Rome, 1623).; English trans. The Assayer, S. Drake and C. D. O'Malley, in The Controversy on the Comets of 1618 (University of Pennsylvania Press, 1960).
[3] E. Wigner, The unreasonable effectiveness of mathematics in the natural sciences, Commun. Pure Appl. Math. **13**, 1 (1960).
[4] M. Born and P. Jordan, Zur Quantenmechanik, Z. Phys. **34**, 858 (1925).
[5] M. Born, W. Heisenberg, and P. Jordan, Zur Quantenmechanik II, Z. Phys. **35**, 557 (1926).
[6] E. Schrödinger, An undulatory theory of the mechanics of atoms and molecules, Phys. Rev. **28**, 1049 (1926).
[7] M. F. Pusey, J. Barrett, and T. Rudolph, On the reality of the quantum state, Nat. Phys. **8**, 475 (2012).

[8] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Direct measurement of the quantum wavefunction, Nature (London) **474**, 188 (2011).

[9] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, Operational Resource Theory of Imaginarity, Phys. Rev. Lett. **126**, 090401 (2021).

[10] G. Birkhoff and J. Von Neumann, The logic of quantum mechanics, Ann. Math. **37**, 823 (1936).

[11] E. C. G. Stueckelberg, Quantum theory in real Hilbert space, Helv. Phys. Acta **33**, 727 (1960).

[12] E. C. G. Stueckelberg and M. Guenin, Quantum theory in real Hilbert space II, Helv. Phys. Acta **34**, 621 (1961).

[13] F. J. Dyson, The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics, J. Math. Phys. **3**, 1199 (1962).

[14] K. F. Pal and T. Vertesi, Efficiency of higher-dimensional Hilbert spaces for the violation of Bell inequalities, Phys. Rev. A **77**, 042105 (2008).

[15] M. McKague, M. Mosca, and N. Gisin, Simulating Quantum Systems Using Real Hilbert Spaces, Phys. Rev. Lett. **102**, 020505 (2009).

[16] M.-O. Renou, D. Trillo, M. Weilenmann, Le Phuc Thinh, A. Tavakoli, Nicolas Gisin, A. Acin and M. Navascues, Quantum theory based on real numbers can be experimentally falsified, Nature (London) **600**, 625 (2021).

[17] Zh. Li *et al.*, Testing Real Quantum Theory in an Optical Quantum Network, Phys. Rev. Lett. **128**, 040402 (2022).

[18] M. C. Chen, C. Wang, F. M. Liu, J. W. Wang, C. Ying, Z. X. Shang, Y. Wu, M. Gong, H. Deng, F. T. Liang, Q. Zhang, C. Z. Peng, X. Zhu, A. Cabello, C. Y. Lu, and J. W. Pan, Ruling Out Real-Valued Standard Formalism of Quantum Theory, Phys. Rev. Lett. **128**, 040403 (2022).

[19] J. S. Bell, On the Einstein, Podolsky, Rosen paradox, Physics (NY). **1**, 195 (1964).

[20] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).

[21] A. Acín, S. Pironio, T. Vertesi, and P. Wittek, Optimal randomness certification from one entangled bit, Phys. Rev. A **93**, 040102(R) (2016).

[22] J. Bowles, I. Supic, I. Cavalcanti, and A. Acin, Self-testing of Pauli observables for device-independent entanglement certification, Phys. Rev. A **98**, 042336 (2018).

[23] B. S. Tsirelson, Quantum generalizations of Bell's inequality, Lett. Math. Phys. **4**, 93 (1980).

[24] N. Gisin, Entanglement 25 years after quantum teleportation: Testing joint measurements in quantum networks, Entropy **21**, 325 (2019).

[25] A. Tavakoli and N. Gisin, The Platonic solids and fundamental tests of quantum mechanics, Quantum **4**, 293 (2020).

[26] K. F. Pal and T. Vertesi, Maximal violation of a bipartite three-setting, two-outcome Bell inequality using infinite-dimensional quantum systems, Phys. Rev. A **82**, 022116 (2010).

[27] S. Pironio, M. Navascues, and A. Acin, Convergent relaxations of polynomial optimization problems with noncommuting variables, SIAM J. Optim. **20**, 2157 (2010).

[28] C. M. Caves, C. A. Fuchs, and P. Rungta, Entanglement of formation of an arbitrary state of two rebits, Found. Phys. Lett. **14**, 199 (2001).

[29] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, Phys. Lett. A **232**, 333 (1997).

[30] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).

[31] D. Bruß and A. Peres, Construction of quantum states with bound entanglement, Phys. Rev. A **61**, 030301(R) (2000).

[32] L. Vandenberghe and S. Boyd, Semidefinite programming, SIAM Rev. **38**, 49 (1996).

[33] L. Vandenberghe and S. Boyd, The MOSEK optimization toolbox for MATLAB manual, Version 7.0, Revision 140 (MOSEK ApS, Denmark).

[34] J. Lofberg, Yalmip: A toolbox for modeling and optimization in MATLAB, in *Proceedings of the CACSD Conference* (Taipei, Taiwan, 2004).

[35] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.106.042207 for MATLAB code to get the bound F_r discussed in the paper.

[36] A. Fine, Hidden Variables, Joint Probability, and the Bell Inequalities, Phys. Rev. Lett. **48**, 291 (1982).

[37] B. Hensen *et al.*, Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, Nature (London) **526**, 682 (2015).

[38] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, T. Gerrits, S. Glancy, D. R. Hamel, M. S. Allman, K. J. Coakley, S. D. Dyer, C. Hodge, A. E. Lita, V. B. Verma, C. Lambrocco, E. Tortorici, A. L. Migdall, Y. Zhang, D. R. Kumor, W. H. Farr, F. Marsili, M. D. Shaw, J. A. Stern *et al.*, Strong Loophole-Free Test of Local Realism, Phys. Rev. Lett. **115**, 250402 (2015).

[39] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J.A. Larsson, C. Abellan, W. Amaya, V. Pruneri, M. W. Mitchell, J. Beyer, T. Gerrits, A. E. Lita, L. K. Shalm, S. W. Nam, T. Scheidl, R. Ursin *et al.*, Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons, Phys. Rev. Lett. **115**, 250401 (2015).

[40] W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, Event-Ready Bell Test Using Entangled Atoms Simultaneously Closing Detection and Locality Loopholes, Phys. Rev. Lett. **119**, 010402 (2017).

[41] A. Bednorz, Second-second moments and two observers testing quantum nonlocality, Ann. Phys. (Berlin) **533**, 2000456 (2021).