Comment on "Scattering of light by a parity-time-symmetric dipole beyond the first Born approximation"

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In the work by Rebouças and Brandão [Phys. Rev. A **104**, 063514 (2021)] the authors compute the scattering amplitude for a $\mathcal{PT}$-symmetric double-δ-function potential in three dimensions by invoking the far-zone approximation and summing the resulting Born series. We show that the analysis of this paper suffers from a basic error. Therefore, its results are inconclusive. We give an exact closed-form expression for the scattering amplitude of this potential.

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The authors of [1] consider the scattering problem for a $\mathcal{PT}$-symmetric double-δ-function potential in three dimensions,

$$v(\mathbf{r}) := \mathfrak{z}_1\delta(\mathbf{r}-\mathbf{r}_0) + \mathfrak{z}_2\delta(\mathbf{r}+\mathbf{r}_0), \tag{1}$$

where $\mathfrak{z}_1 = \mathfrak{z}_2^* = -\alpha k^2(\sigma + i\gamma)$ and α, σ, and γ are real parameters; k is the wave number; and $\pm\mathbf{r}_0$ are the positions of the point scatterers. They substitute the Born series $u(\mathbf{r}) = \sum_{n=0}^{\infty} u_n(\mathbf{r})\alpha^n$ in the Lippmann-Schwinger equation to show that

$$u_n(\mathbf{r}) = k^2 \int_{\mathbb{R}^3} \chi(\mathbf{r}')G(|\mathbf{r}-\mathbf{r}'|)u_{n-1}(\mathbf{r}')d^3r', \quad n \geqslant 1, \tag{2}$$

where $\chi(\mathbf{r}') := v(\mathbf{r})/\alpha k^2$ and $G(x) := e^{ikx}/4\pi x$. Then they let $r := |\mathbf{r}|$, $r' := |\mathbf{r}'|$, and $\hat{s} := \mathbf{r}/r$; denote the direction of the incident wave vector by $\hat{a}$; and use $u_0(\mathbf{r}) = e^{ik\hat{a}\cdot\mathbf{r}}$ and the far-zone (FZ) approximation,

$$G(|\mathbf{r}-\mathbf{r}'|) \sim \frac{e^{ikr}}{4\pi r}e^{-ik\hat{s}\cdot\mathbf{r}'} \quad \text{for} \quad r \gg r', \tag{3}$$

in (2) to obtain the recursion relation,

$$u_n(\mathbf{r}) = \frac{k^2 e^{ikr}}{4\pi r}[(\sigma + i\gamma)u_{n-1}(\mathbf{r}_0)e^{-ik\hat{s}\cdot\mathbf{r}_0} + (\sigma - i\gamma)u_{n-1}(-\mathbf{r}_0)e^{ik\hat{s}\cdot\mathbf{r}_0}], \quad n \geqslant 1. \tag{4}$$

The results of [1] rely on the authors' solution of this relation. But as we explain below, there is a basic error in their analysis. To determine $u_1(\mathbf{r})$, they substitute (3) in (2) and set $n = 1$. Because (3) holds whenever $r \gg r'$, this gives an approximate expression $u_1^{\rm FZ}(\mathbf{r})$ for $u_1(\mathbf{r})$ which is valid in the FZ, i.e., $u_1(\mathbf{r}) \sim u_1^{\rm FZ}(\mathbf{r})$ for $r \to \infty$. Repeating the same procedure for $n = 2$, they express $u_2(\mathbf{r})$ in terms of $u_1(\pm\mathbf{r}_0)$, which they calculate by substituting $\pm\mathbf{r}_0$ for $\mathbf{r}$ in $u_1^{\rm FZ}(\mathbf{r})$, i.e., set $u_1(\pm\mathbf{r}_0) \sim u_1^{\rm FZ}(\pm\mathbf{r}_0)$. This is inadmissible because $|\mathbf{r}_0|$ does not tend to infinity. In general, the iterative solution of (4) given in the Appendix of [1] is unacceptable because this equation holds for $r \to \infty$. Therefore, one cannot use it to determine $u_n(\pm\mathbf{r}_0)$ even approximately.

There is actually no need to invoke the FZ approximation to treat this problem [2,3]. Reference [4] gives the exact solution of the scattering problem for the multi-δ-function potentials,

$$v(\mathbf{r}) := \sum_{n=1}^{N} \mathfrak{z}_n\delta(\mathbf{r}-\mathbf{a}_n), \tag{5}$$

in two dimensions, where $\mathfrak{z}_n$ are real or complex coupling constants and $\mathbf{a}_n$ are the positions of the point scatterers. The analysis of [4] has a straightforward extension to three dimensions. To see this, first, we use the notation of Ref. [1] to express the Lippmann-Schwinger equation for the potential (5) in the form

$$u(\mathbf{r}) = u_0(\mathbf{r}) - \sum_{n=1}^{N} \mathfrak{z}_n G(|\mathbf{r}-\mathbf{a}_n|)u(\mathbf{a}_n). \tag{6}$$

Setting $\mathbf{r} = \mathbf{a}_m$ in this equation, we find a system of linear equations for $u(\mathbf{a}_n)$. Because $G(0) = \infty$, the matrix of coefficients of this system has divergent diagonal entries. Therefore, we regularize $G(x)$ and perform a coupling-constant renormalization to remove the singularities. We can do this with a cutoff renormalization or dimensional regularization as outlined in [5] or other renormalization schemes [6]. In this way, we can set $\mathbf{r} = \mathbf{a}_m$ in (6) to arrive at

$$\sum_{n=1}^{N} A_{mn}X_n = e^{ik\mathbf{a}_m\cdot\hat{a}}, \tag{7}$$

where

$$A_{mn} := \begin{cases} \tilde{\mathfrak{z}}_n^{-1} + \frac{ik}{4\pi} & \text{for } m = n, \\ G(|\mathbf{a}_m - \mathbf{a}_n|) & \text{for } m \neq n, \end{cases} \tag{8}$$

$\tilde{\mathfrak{z}}_n$ are the renormalized coupling constants and $X_n := \tilde{\mathfrak{z}}_n u(\mathbf{a}_n)$. Solving (7) for X_n, substituting the result in (6), and noting that the scattering amplitude $\tilde{u}_s(\mathbf{r})$ is given by

$$u(\mathbf{r}) \to u_0(\mathbf{r}) + \tilde{u}_s(\mathbf{r})\frac{e^{ikr}}{r} \quad \text{for} \quad r \to \infty,$$

we find

$$\tilde{u}_s(\mathbf{r}) = -\frac{1}{4\pi}\sum_{m,n=1}^{N} A^{-1}_{mn}\, e^{ik(\mathbf{a}_n\cdot\hat{a}-\mathbf{a}_m\cdot\hat{s})}, \tag{9}$$

where A^{-1}_{mn} are the entries of the inverse of the $N\times N$ matrix $\mathbf{A}=[A_{mn}]$ and we have also made use of (3).

For the $\mathcal{PT}$-symmetric double-δ-function potential (1), $N=2$; $\tilde{\mathfrak{z}}_1=\tilde{\mathfrak{z}}_2^*=-k^2\tilde{\alpha}(\tilde{\sigma}+i\tilde{\gamma})$; $\tilde{\alpha}, \tilde{\sigma}$, and $\tilde{\gamma}$ are real renormalized parameters; and $\mathbf{a}_1=-\mathbf{a}_2=\mathbf{r}_0$. Substituting these relations and Eq. (8) in Eq. (9), we obtain

$$\tilde{u}_s(\mathbf{r}) = \frac{1}{2\pi D}\left[\frac{\mathfrak{s}\cos\xi_- - \mathfrak{g}\sin\xi_-}{k^2(\mathfrak{s}^2+\mathfrak{g}^2)} + \frac{e^{2ikr_0}\cos\xi_+ - 2ikr_0\cos\xi_-}{8\pi r_0}\right],$$

where

$$D := \det\mathbf{A} = \frac{2\pi - ik^3\mathfrak{s}}{2\pi k^4(\mathfrak{s}^2+\mathfrak{g}^2)} - \frac{4k^2r_0^2+e^{4ikr_0}}{64\pi^2r_0^2},$$

$\mathfrak{s}:=\tilde{\alpha}\tilde{\sigma}$, $\mathfrak{g}:=\tilde{\alpha}\tilde{\gamma}$, $r_0:=|\mathbf{r}_0|$, and $\xi_\pm := k\mathbf{r}_0\cdot(\hat{a}\pm\hat{s})$. Notice that the parameters $\mathfrak{s}$ and $\mathfrak{g}$ enter our calculations after we renormalize the bare coupling constants $\alpha\sigma$ and $\alpha\gamma$. Therefore, they may depend on other physical parameters of the problem.

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