Dynamics of two central spins immersed in spin baths

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In this article we derive the exact dynamics of a two-qubit (spin $1/2$) system interacting centrally with separate spin baths composed of qubits in a thermal state. Furthermore, each spin of the bath is coupled to every other spin of the same bath. The corresponding dynamical map is constructed. It is used to analyze the non-Markovian nature of the two qubit central spin dynamics. We further observe the evolution of quantum correlations like entanglement and discord under the influence of the environmental interaction. Moreover, we demonstrate the comparison between this exact two-qubit dynamics and the locally acting central spin model in a spin bath. This work is a stepping stone towards the realization of non-Markovian heat engines and other quantum thermal devices.

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I. INTRODUCTION

In the interactive world of quantum particles, it is almost impossible to create a physical system devoid of any external noisy influence. Quantum systems conducive for information theoretic tasks, i.e., trapped ions [1], quantum dots [2], NMR qubits [3], Josephson junctions [4], and many more, are subjected to environmental interactions. It is therefore a necessary task to investigate the dynamics of such quantum systems, which are under the influence of environmental interactions. The theory of open quantum systems [5,6], has found numerous applications in recent times in quantum information and its interface with various facets of quantum physics [7–25]. Over the past few decades, our understanding of such systems has stretched from the limitations of Markovian dynamics to the more intriguing and challenging domain of non-Markovian quantum systems [17,26–50]. Even now, it is a challenging task to construct the reduced dynamics of such a system without the Born-Markov and stationary-bath approximations [5]. One often associates a deviation from quantum dynamical semigroup evolution of a system to a non-Markovian process [49]. In a non-Markovian evolution, the timescales of the system and the environment are often not well separated, which can result in information backflow from the environment to the system [28,51]. This generally leads to recurrences of quantum properties which is important for a fundamental understanding of system dynamics.

It would be pertinent to add here that a number of techniques have been developed in recent times to tackle this problem [52]. Thus, for example, there are the embedding methods, such as the pseudomodes technique wherein the decay of an atom strongly coupled to a reservoir can be studied by considering an enlarged system that includes a set of pseudomodes [53,54]. Another relevant technique is the reaction-coordinate mapping [55–57]. Furthermore, a numerically exact hierarchical equations of motion (HEOM) method has been developed [58].

Quantum baths are generally categorized in two broad classes: (a) bosonic and (b) spin bath. Archetypal examples of bosonic baths include the Caldeira-Leggett model [10] or the spin boson model [7]. Exact quantum master equations for these types of models are common in the literature [5]. On the contrary, in the case of spin baths, we often have to rely on perturbative techniques or time-nonlocal master equations [26,59]. Although the study of these systems are extremely important in physical systems of paramount importance such as magnetic systems, quantum spin glasses, and superconducting systems [26], the theoretical modeling of such systems is still lacking in many different perspectives, especially in bipartite or multipartite systems relevant in various quantum device modeling. In this work, we attempt to lay the bedrock of such a construction from the perspective of modeling various quantum devices. Here we develop an exact reduced dynamics of a two-qubit system immersed in spin baths, each of which is interacting centrally with the system of interest. Our model Hamiltonian of the two-qubit system is inspired from a model of a quantum thermal diode [60]. In recent times, motivated by the goal of building quantum computers, a lot of effort has been devoted to developing quantum versions of various thermodynamic and electronic devices like refrigerators and heat engines [61–64], thermal diodes [60,65], transistors [66–69], quantum batteries, and so on [62,70]. Drawing the motivation from the long-term goal of realizing experimentally feasible models of quantum thermal devices, we develop the aforementioned exact reduced dynamics of a two-qubit spin system. We further analyze various thermodynamic and information-theoretic properties of the

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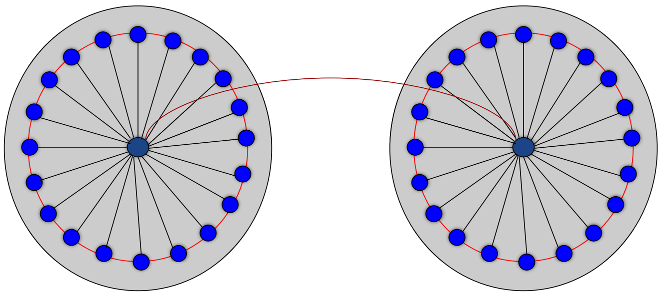

FIG. 1. Schematic diagram of coupled central spin model where each central spin interacts with individual spin baths.

system undergoing the specific open quantum evolution. We also observe the fundamentally non-Markovian behavior of the dynamical map.

The flow of the paper is as follows: In Sec. II we introduce the model. Its reduced dynamics is developed in Sec. III. The corresponding dynamical map is discussed next, along with its operator sum representation. The dynamical map is then put to use in Sec. V. This includes analyzing a witness to identify inherent non-Markovianity in the dynamics, the corresponding local and global dynamical maps, and finally the quantum correlations, including entanglement and discord, generated between the two spins by their open-system dynamics. This is followed by the conclusion.

II. THE MODEL

We present a model of two coupled qubits where each qubit is centrally coupled to different thermal spin baths (see Fig. 1). Furthermore, each spin of the bath is coupled to every other spin of the same bath.

Initially, $\rho_{SB}(0)=\rho_S(0)\otimes\rho_B(0)$, where $\rho_B(0)=e^{-H_B/K_BT}/Z$ for each central spin immersed in a spin bath. The evolution of the whole system is governed by the following Hamiltonian in our spin-bath model:

$$\begin{aligned}H &= H_{S_1}+H_{S_2}+H_{S_1S_2}+H_{B_1}+H_{B_2}+H_{S_1B_1}+H_{S_2B_2},\\ &= \frac{\hbar\omega_1}{2}\sigma_{1z}^0+\frac{\hbar\omega_2}{2}\sigma_{2z}^0+\frac{\hbar\delta}{2}\left(\sigma_{1z}^0\otimes\sigma_{2z}^0\right)\\ &\quad+\frac{\hbar\omega_a}{2M}\sum_{i=1}^{M}\left\{\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{M}\left(\sigma_{1x}^i\sigma_{1x}^j+\sigma_{1y}^i\sigma_{1y}^j\right)+\sigma_{1z}^i\right\}\\ &\quad+\frac{\hbar\omega_b}{2N}\sum_{i=1}^{N}\left\{\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\sigma_{2x}^i\sigma_{2x}^j+\sigma_{2y}^i\sigma_{2y}^j\right)+\sigma_{2z}^i\right\}\\ &\quad+\frac{\hbar\epsilon_1}{2\sqrt{M}}\sum_{i=1}^{M}\left(\sigma_{1x}^0\sigma_{1x}^i+\sigma_{1y}^0\sigma_{1y}^i\right)+\frac{\hbar\epsilon_2}{2\sqrt{N}}\\ &\quad\times\sum_{i=1}^{N}\left(\sigma_{2x}^0\sigma_{2x}^i+\sigma_{2y}^0\sigma_{2y}^i\right), \end{aligned}\tag{1}$$

where σ_{lk}^i or σ_{lk}^j ($k=x,y,z$; $l=1,2$) are Pauli matrices corresponding to ith or jth spin of the lth bath and σ_{lk}^0 ($k=x,y,z$; $l=1,2$) are same for lth central spin. ω_1 and ω_2 are the two central spin frequencies and δ corresponds to their coupling strength. Also, ω_a and ω_b are bath frequencies of two spin baths and ϵ_l ($l=1,2$) are interaction parameters of lth system bath. M and N are the respective number of atoms in two baths.

Using total spin angular-momentum operator $J_{lk'}=\frac{1}{2}\sum_{i=1}^{M\text{ or }N}\sigma_{lk'}^i$ ($k'=x,y,z,+,-$; $l=1$ or 2 corresponding to the summation upper limit M or N), we may rewrite the bath Hamiltonians as

$$H_{B_1}=\hbar\omega_a\left(\frac{J_{1+}J_{1-}}{M}-\frac{\mathbb{1}}{2}\right),\quad H_{B_2}=\hbar\omega_b\left(\frac{J_{2+}J_{2-}}{N}-\frac{\mathbb{1}}{2}\right),\tag{2}$$

and system-bath interaction Hamiltonians as

$$\begin{aligned}H_{S_1B_1}&=\frac{\hbar\epsilon_1}{\sqrt{M}}\left(\sigma_{1x}^0J_{1x}+\sigma_{1y}^0J_{1y}\right),\\ H_{S_2B_2}&=\frac{\hbar\epsilon_2}{\sqrt{N}}\left(\sigma_{2x}^0J_{2x}+\sigma_{2y}^0J_{2y}\right).\end{aligned}\tag{3}$$

Following Ref. [71], we then use the Holstein-Primakoff transformation [72] to redefine collective angular-momentum operators as

$$\begin{aligned}J_{1+}&=\sqrt{M}a^\dagger\left(1-\frac{a^\dagger a}{M}\right)^{1/2},\\ J_{1-}&=\sqrt{M}\left(1-\frac{a^\dagger a}{M}\right)^{1/2}a,\end{aligned}\tag{4}$$

for the first bath and the following for the second bath:

$$\begin{aligned}J_{2+}&=\sqrt{N}b^\dagger\left(1-\frac{b^\dagger b}{N}\right)^{1/2},\\ J_{2-}&=\sqrt{N}\left(1-\frac{b^\dagger b}{N}\right)^{1/2}b.\end{aligned}\tag{5}$$

Here a and $a^\dagger$ are the bosonic annihilation and creation operators, respectively, for the first spin bath having the property $[a,a^\dagger]=1$, and b and $b^\dagger$ represent the same for the second spin bath. After this transformation, the bath Hamiltonians appear as

$$\begin{aligned}H_{B_1}&=\hbar\omega_a\left\{a^\dagger a\left(1-\frac{a^\dagger a-1}{M}\right)-\frac{\mathbb{1}}{2}\right\},\\ H_{B_2}&=\hbar\omega_b\left\{b^\dagger b\left(1-\frac{b^\dagger b-1}{N}\right)-\frac{\mathbb{1}}{2}\right\},\end{aligned}\tag{6}$$

and the interaction Hamiltonians of respective spin baths as

$$\begin{aligned}H_{S_1B_1}&=\hbar\epsilon_1\left\{\sigma_{1+}^0\left(1-\frac{a^\dagger a}{M}\right)^{1/2}a+\sigma_{1-}^0a^\dagger\left(1-\frac{a^\dagger a}{M}\right)^{1/2}\right\},\\ H_{S_2B_2}&=\hbar\epsilon_2\left\{\sigma_{2+}^0\left(1-\frac{b^\dagger b}{N}\right)^{1/2}b+\sigma_{2-}^0b^\dagger\left(1-\frac{b^\dagger b}{N}\right)^{1/2}\right\}.\end{aligned}\tag{7}$$

It is also important to mention the limitations and the essential approximations. Here we are taking a central spin-half system interacting homogeneously with each of the bath spins with all the characteristic frequencies to be of constant value. We have taken homogeneous interactions for the sake of analytical clarity. This is not an oversimplified assumption, as these kinds of interactions exist in physical situations, of which some examples are quantum spin glasses, superconducting systems, and NMR. The more general system with in-homogeneous interaction parameters can only be handled numerically. A use is then made of the Holstein-Primakoff transformation. A Hamiltonian describing the collective behavior of N interacting spins can be mapped to a bosonic one employing this transformation, at the expense of having an infinite series in powers of the bosonic creation and annihilation operators. Truncating this series up to quadratic terms allows for obtaining analytic solutions, which become exact in the limit $N \to \infty$. In the literature, works on similar spin environments exists which make use of different methods, cf. Ref. [26]. The reason we choose Holstein-Primakoff transformation here is because of its technical advantage, both from analytical and numerical perspectives. We can work with more bath spins with fewer numerical limitations. Homogeneous interactions further enable us to modify the total Hamiltonian into a form of the nonlinear Jaynes-Cummings model. Other than considering homogeneous interactions, no further assumptions are needed for the type of system-bath model studied here. This opens up an opportunity to study more complex quantum devices and networks.

III. REDUCED DYNAMICS OF THE TWO-QUBIT CENTRAL SPIN MODEL

We derive the reduced dynamics of the system of two coupled central spins evolved under the Hamiltonian H along with the two baths by tracing over the bath degrees of freedom.

Consider the evolution of the state $|\psi(0)\rangle = |11\rangle|xy\rangle$, where two central spins are in the excited state $|1\rangle$ and $|x\rangle$ is an arbitrary state for the first bath while $|y\rangle$ is a state belonging to the second bath. Global unitary operator corresponding to the evolution under Hamiltonian H can be written as, $U(t) = \exp(-iHt/\hbar)$. After the evolution let the state at time t be $|\psi(t)\rangle = \varsigma_1(t)|11\rangle|x'y'\rangle + \varsigma_2(t)|10\rangle|x''y''\rangle + \varsigma_3(t)|01\rangle|x'''y'''\rangle$. We exclude the case when both spins of the system are flipped simultaneously; i.e., the $|11\rangle \to |00\rangle$ transition. The transitions we have considered convey the message sufficiently and hence we choose to exclude the aforementioned transition to avoid unnecessary complications in the calculations. Now we introduce three time-dependent operators $\hat{A}(t)$, $\hat{B}(t)$, and $\hat{C}(t)$ corresponding to the joint Hilbert space of the two baths such that $\hat{A}(t)|xy\rangle = \varsigma_1(t)|x'y'\rangle$, $\hat{B}(t)|xy\rangle = \varsigma_2(t)|x''y''\rangle$, and $\hat{C}(t)|xy\rangle = \varsigma_3(t)|x'''y'''\rangle$. Then we have $|\psi(t)\rangle = \hat{A}(t)|11\rangle|xy\rangle + \hat{B}(t)|10\rangle|xy\rangle + \hat{C}(t)|01\rangle|xy\rangle$.

From the time-dependent Schrödinger equation $\frac{d}{dt}|\psi(t)\rangle = -\frac{iH}{\hbar}|\psi(t)\rangle$, we can have

$$\begin{aligned}
\frac{d\hat{A}(t)}{dt} = -i\Bigg[&\frac{\omega_1+\omega_2+\delta}{2} + \omega_a\left\{a^\dagger a\left(1-\frac{a^\dagger a-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} \\
&+ \omega_b\left\{b^\dagger b\left(1-\frac{b^\dagger b-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\Bigg]\hat{A}(t) - i\epsilon_2\left(1-\frac{b^\dagger b}{N}\right)^{1/2} b\hat{B}(t) - i\epsilon_1\left(1-\frac{a^\dagger a}{M}\right)^{1/2} a\hat{C}(t),\\
\frac{d\hat{B}(t)}{dt} = -i\Bigg[&\frac{\omega_1-\omega_2-\delta}{2} + \omega_a\left\{a^\dagger a\left(1-\frac{a^\dagger a-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{b^\dagger b\left(1-\frac{b^\dagger b-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\Bigg]\hat{B}(t) - i\epsilon_2 b^\dagger\left(1-\frac{b^\dagger b}{N}\right)^{1/2}\hat{A}(t),\\
\frac{d\hat{C}(t)}{dt} = -i\Bigg[&\frac{-\omega_1+\omega_2-\delta}{2} + \omega_a\left\{a^\dagger a\left(1-\frac{a^\dagger a-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{b^\dagger b\left(1-\frac{b^\dagger b-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\Bigg]\hat{C}(t) - i\epsilon_1 a^\dagger\left(1-\frac{a^\dagger a}{M}\right)^{1/2}\hat{A}(t).
\end{aligned} \tag{8}$$

Now we substitute $\hat{A}(t) = \hat{A}_1(t)$, $\hat{B}(t) = b^\dagger\hat{B}_1(t)$, and $\hat{C}(t) = a^\dagger\hat{C}_1(t)$ and have

$$\begin{aligned}
\frac{d\hat{A}_1(t)}{dt} = -i\Bigg[&\frac{\omega_1+\omega_2+\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\Bigg]\hat{A}_1(t)\\
&- i\epsilon_2\left(1-\frac{\hat{n}}{N}\right)^{1/2}(\hat{n}+1)\hat{B}_1(t) - i\epsilon_1\left(1-\frac{\hat{m}}{M}\right)^{1/2}(\hat{m}+1)\hat{C}_1(t),\\
\frac{d\hat{B}_1(t)}{dt} = -i\Bigg[&\frac{\omega_1-\omega_2-\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{(\hat{n}+1)\left(1-\frac{\hat{n}}{N}\right)-\frac{\mathbb{1}}{2}\right\}\Bigg]\hat{B}_1(t) - i\epsilon_2\left(1-\frac{\hat{n}}{N}\right)^{1/2}\hat{A}_1(t),\\
\frac{d\hat{C}_1(t)}{dt} = -i\Bigg[&\frac{-\omega_1+\omega_2-\delta}{2} + \omega_a\left\{(\hat{m}+1)\left(1-\frac{\hat{m}}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\Bigg]\hat{C}_1(t) - i\epsilon_1\left(1-\frac{\hat{m}}{M}\right)^{1/2}\hat{A}_1(t).
\end{aligned} \tag{9}$$

Here $\hat{m} = a^\dagger a$ and $\hat{n} = b^\dagger b$ are number operators corresponding to bosonic operators of the first and second baths, respectively, which have the properties $\hat{m}|m\rangle = m|m\rangle$ and $\hat{n}|n\rangle = n|n\rangle$. Therefore, we define, $\hat{A}_1(t)|mn\rangle = A_1(m,n,t)|mn\rangle$, $\hat{B}_1(t)|mn\rangle = B_1(m,n,t)|mn\rangle$, and $\hat{C}_1(t)|mn\rangle = C_1(m,n,t)|mn\rangle$.

By tracing out bath modes, the reduced state of the system $(|11\rangle\langle 11|)$ becomes

$$\begin{aligned}\phi(|11\rangle\langle 11|) &= \mathrm{Tr}_{B_1B_2}\,[|\psi(t)\rangle\langle\psi(t)|]\\ &= \frac{1}{Z}\sum_m^M\sum_n^N\{|A_1(m,n,t)|^2|11\rangle\langle 11| + (n+1)|B_1(m,n,t)|^2|10\rangle\langle 10|\\ &\quad + (m+1)|C_1(m,n,t)|^2|01\rangle\langle 01|\}\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]\\ &\quad\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right].\end{aligned} \tag{10}$$

From the solution of Eq. (9), one can derive the values of $|A_1(m,n,t)|^2$, $|B_1(m,n,t)|^2$, and $|C_1(m,n,t)|^2$ following the steps mentioned in Appendix B. The partition function in the above equation is $Z = \sum_m^M\sum_n^N \exp[-\frac{\hbar\omega_a}{K_BT}\{m(1-\frac{m-1}{M})-\frac{1}{2}\}]\exp[-\frac{\hbar\omega_b}{K_BT}\{n(1-\frac{n-1}{N})-\frac{1}{2}\}]$.

In a similar way, we can derive the evolution of the other elements of the system's reduced state. The details are given in Appendix A. The reduced state of the system of two central spins after the global unitary evolution of the pair of joint system-bath state can be denoted

$$\rho_{S_1S_2} = \mathrm{Tr}_{B_1B_2}[e^{-iHt/\hbar}\rho_{SB}(0)e^{iHt/\hbar}] = \begin{pmatrix}\rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t)\\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t)\\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t)\\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(t)\end{pmatrix}, \tag{11}$$

where $\rho_{SB}(0)$ is the joint system-bath initial state. The elements of the density matrix are given in Appendix C.

IV. CONSTRUCTION OF THE DYNAMICAL MAP

Having constructed the density matrix of the reduced two-qubit central spin system, we now find the dynamical map of the system. To this effect, we derive the Kraus operators for the evolution of the reduced system.

Operator sum representation

An important facet of general quantum evolution represented by a completely positive trace preserving (CPTP) operation is the Kraus operator sum representation, given as $\rho(t) = \sum_i K_i(t)\rho(0)K_i^\dagger(t)$. The Kraus operators K_i can be constructed from the eigenvalues and eigenvectors of the Choi-Jamiołkowski (CJ) state of the corresponding dynamical map. The CJ state of the dynamic map $\phi(\cdot)$ acting on a d-dimensional system is given by $(\mathbb{I}_d\otimes\phi)|\psi\rangle\langle\psi|$, with $|\psi\rangle$ being the maximally entangled state in d^2 dimensions. For our particular case, the CJ matrix is given by

$$\mathcal{C}(t) = \begin{pmatrix}
|A_1|^2 & 0 & 0 & 0 & 0 & A_1J_1^* & 0 & 0 & 0 & 0 & A_1G_1^* & 0 & 0 & 0 & 0 & A_1D_1^*\\
0 & (n+1)|B_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & (m+1)|C_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & n|K_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
A_1^*J_1 & 0 & 0 & 0 & 0 & |J_1|^2 & 0 & 0 & 0 & 0 & J_1G_1^* & 0 & 0 & 0 & 0 & J_1D_1^*\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (m+1)|L_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m|I_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
A_1^*G_1 & 0 & 0 & 0 & 0 & J_1^*G_1 & 0 & 0 & 0 & 0 & |G_1|^2 & 0 & 0 & 0 & 0 & G_1D_1^*\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (n+1)|H_1|^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m|F_1|^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n|E_1|^2 & 0\\
A_1^*D_1 & 0 & 0 & 0 & 0 & J_1^*D_1 & 0 & 0 & 0 & 0 & G_1^*D_1 & 0 & 0 & 0 & 0 & |D_1|^2
\end{pmatrix}. \tag{12}$$

The symbols and abbreviations used in Eq. (12) are explained in Appendixes A and D. One can find the eigenspectrum of the matrix in Eq. (12) numerically, and from there the Kraus operators for the evolution of the state can be derived. These Kraus operators should satisfy the relation $\sum_i K_i^\dagger K_i = \mathbb{I}$, which is indeed the case here.

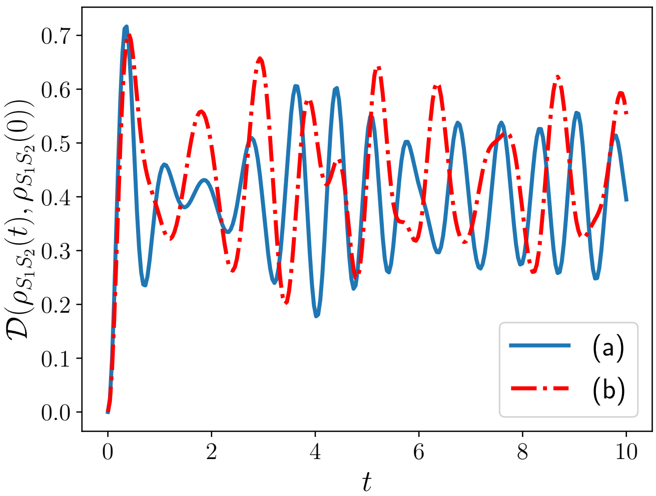

FIG. 2. Variation of trace distance, between time evolved state and initial state, with time for the reduced state of the system. In panel (a), the initial state is taken to be $|11\rangle$, and in panel (b), the initial state is taken to be $|10\rangle$. The parameters have following values: $\omega_1 = 2.0$, $\omega_2 = 1.9$, $\delta = 2.5$, $\omega_a = 1.1$, $\omega_b = 1.2$, $M = N = 100$, $T = 1$, $\epsilon_1 = 2.6$, $\epsilon_2 = 2.5$.

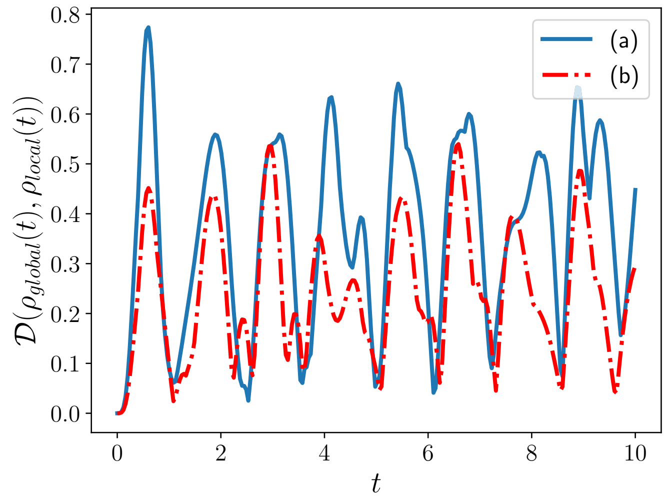

FIG. 3. Variation of trace distance for difference between local and global maps with time for the reduced state of the system. In panel (a), the initial state is taken to be $|11\rangle$, and in panel (b), the initial state is taken to be $|10\rangle$. The parameters have following values: $\omega_1 = 2.0$, $\omega_2 = 1.9$, $\delta = 5$, $\omega_a = 1.1$, $\omega_b = 1.2$, $M = N = 100$, $T = 1$, $\epsilon_1 = 2.6$, $\epsilon_2 = 2.5$.

V. ANALYSIS OF THE DYNAMICAL MAP

In this section, we use the dynamical map of the reduced state of the system to shed light on the dynamics of the two-qubit central spin model. In particular, we are interested in the study of its non-Markovian behavior from the backdrop of information backflow [51]. The study of information backflow is done by observing the time evolution of certain distance functions like trace distance or fidelity between two quantum states, which is a monotonically decreasing function under Markovian evolution. Any deviation from their monotonic behavior is thus considered as a signature of non-Markovianity. Here we study the evolution of trace distance to investigate the non-Markovian nature of the dynamics. In this context, it is interesting to observe the difference between the action of the dynamical map in their local and global avatars. Along with this, we also investigate the evolution of nonclassical correlations such as quantum entanglement and quantum discord, to determine the sustenance of such resourceful quantum properties under the dynamics in question.

A. Trace distance as a witness of non-Markovianity

The trace distance, which gives a measure of distinguishability between two quantum states, is given as $\mathcal{D}(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2||_1$, where $||(\cdot)||_1 = \mathrm{Tr}\sqrt{(\cdot)^\dagger(\cdot)}$. To this end, we calculate the trace distance between the state evolved with time through the dynamical map given above and the initial state, which is given by

$$\mathcal{D}(\rho_{S_1S_2}(t), \rho_{S_1S_2}(0)) = \tfrac{1}{2}||\rho_{S_1S_2}(t) - \rho_{S_1S_2}(0)||_1. \qquad (13)$$

In Fig. 2, we can see the variation of $\mathcal{D}(\rho_{S_1S_2}(t), \rho_{S_1S_2}(0))$ with time for different initial states. The nonmonotonic behavior of the trace distance between the time evolved state and the initial state is a witness of non-Markovianity in the system.

B. Difference between local and global dynamical maps

In this work, we have derived the global dynamical map of two central spins (Λ_{12}). Let Λ_1, Λ_2 be the local dynamical maps derived by solving the local Lindblad equations for each central spin [39], with the added proviso that the bath spins are interacting with each other. Λ_{12} is the global map constructed here. We take the same parameter values for both the global and local maps so that we can observe the difference between them from a common footing. For a particular initial state $\rho_s(0)$ we can calculate the trace distance as

$$\mathcal{D}(\rho_{\text{global}}(t), \rho_{\text{local}}(t)) = \tfrac{1}{2}||\Lambda_{12}(\rho_s(0)) - \Lambda_1 \otimes \Lambda_2(\rho_s(0))||_1. \qquad (14)$$

In Fig. 3, we can see the difference between local and global dynamical maps by setting the same values for all the parameters. It is evident from the plot that, although there is no interaction between the two baths in the dynamics we are evaluating and the two baths are acting separately with individual qubits, there is a distinct difference in the dynamical behavior of this map with the local maps acting separately with each of the qubits. Through the interaction between the qubits, bath information is passed from one environment to another and hence, despite being mutually noninteracting, the action of the bath exhibits a global trait [73]. Therefore, it is our understanding that, in those situations where baths are acting locally on a bipartite system, as presented in Eq. (1), applying local Kraus operators for each separate system-bath interaction, does not give us the complete picture.

C. Quantum correlations

Quantum correlations are a very useful resource in quantum information processing. To this end, we investigate the

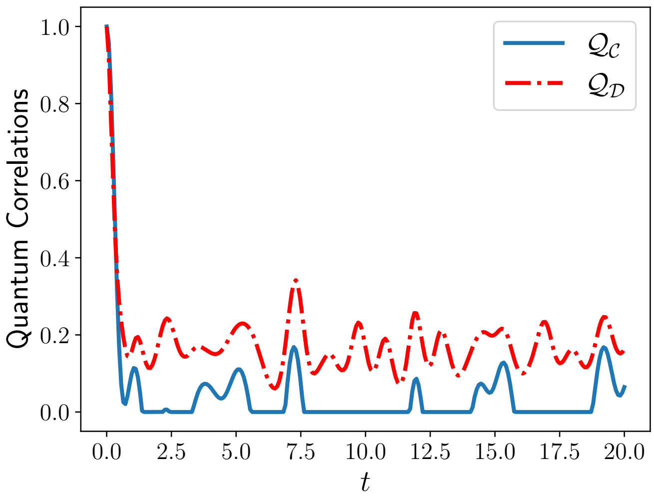

FIG. 4. Variation of concurrence $\mathcal{Q}_{\mathcal{C}}$ and quantum discord $\mathcal{Q}_{\mathcal{D}}$ with time for the reduced state of the system. The parameters have following values: $\omega_1 = 2.0$, $\omega_2 = 1.9$, $\delta = 3$, $\omega_a = 1.1$, $\omega_b = 1.2$, $M = N = 100$, $T = 1$, $\epsilon_1 = 1.3$, $\epsilon_2 = 1.25$.

quantum correlations in the reduced state of the system given in Eq. (11). Concurrence [74] is a measure of quantum entanglement in the system given by,

$$\mathcal{Q}_{\mathcal{C}} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{15}$$

where λ_i are the eigenvalues of the matrix $(\sqrt{\rho_{S_1S_2}}\tilde{\rho}_{S_1S_2}\sqrt{\rho_{S_1S_2}})^{1/2}$ in decreasing order, and $\tilde{\rho}_{S_1S_2} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

Another popular candidate for the measurement of quantum correlations is quantum discord [75]. It includes the quantum correlations due to quantum effects in the system which may not necessarily be due to quantum entanglement. For the reduced state given in Eq. (11), quantum discord can be given as

$$\mathcal{Q}_{\mathcal{D}} = S(\rho_{S_2}) - S(\rho_{S_1S_2}) + S(\rho_{S_1|S_2}), \tag{16}$$

where $S(\rho_{S_2})$ and $S(\rho_{S_1S_2})$ are the von Neumann entropy of the reduced subsystem $\mathrm{Tr}_{S_1}[\rho_{S_1S_2}]$ and the joint von Neumann entropy of the reduced system $\rho_{S_1S_2}$, respectively. $S(\rho_{S_1|S_2})$ is the quantum conditional entropy given by

$$S(\rho_{S_1|S_2}) = \min_{\{\Pi_k\}} \sum_{k=1}^{2} p_k S(\rho_{S_1|\Pi_k}), \tag{17}$$

where $\rho_{S_1|\Pi_k}$ is the state of the reduced system when measurement operator Π_k is operated on subsystem S_2, such that $\rho_{S_1|\Pi_k} = \frac{1}{p_k}\mathrm{Tr}_{S_2}(\Pi_k\rho_{S_1S_2}\Pi_k^\dagger)$. p_k is the probability associated with the measurement operators Π_k given by $p_k = \mathrm{Tr}(\Pi_k\rho_{S_1S_2}\Pi_k^\dagger)$. The generalized measurement operators Π_k in two qubits are given by $\Pi_1 = \mathbb{I}_{S_1} \otimes |u\rangle\langle u|_{S_2}$ and $\Pi_2 = \mathbb{I}_{S_1} \otimes |v\rangle\langle v|_{S_2}$, where $|u\rangle = \cos(\theta)\,|1\rangle + e^{i\phi}\sin(\theta)\,|0\rangle$ and $|v\rangle = \sin(\theta)\,|1\rangle - e^{i\phi}\cos(\theta)\,|0\rangle$. The parameters θ and ϕ vary in the range $0 \leqslant \theta \leqslant \pi/2$ and $0 \leqslant \phi \leqslant 2\pi$. We consider the two-qubit maximally entangled state, $|\psi(0)\rangle_S = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ as the initial state of the central spin system and study the variation of quantum correlations with time, depicted in Fig. 4. The profile of both the quantum discord and the concurrence is similar. There exist time intervals where entanglement is zero, but discord is nonzero. Revivals in the quantum correlations depicting the information backflow nature of non-Markovianity of the system are observed. It is also important to mention that, here we are considering finite bath spins in our numerical analysis. If we extend this to the thermodynamic limit of $M, N \to \infty$, the dynamical behavior will fall into the Markovian regime. In fact, in a previous work [39], one of the present authors has shown that, with increasing bath spins, the dynamical behavior asymptotically reaches the Markovian situation. Therefore, a natural conclusion is that non-Markovianity is a result of the finiteness of the spin environment. On the other hand, to eliminate the possible conclusion that non-Markovianity is an artifact of the truncated Holstein-Primakoff transformation, we refer to a previous work on similar spin environment treated by different methods, but also exhibiting typical non-Markovian behavior [26]. This shows that non-Markovianity is not an artifact of the method.

VI. CONCLUSION

In this article, we have derived the exact reduced dynamics of two coupled central spins where each spin is centrally coupled to different thermal baths. We develop the corresponding dynamical map by constructing the relevant Kraus operators. This helps to shed light onto the reduced dynamics of the coupled central spin model. In particular, we show evidence of the non-Markovian evolution of the system in the form of trace distance. We also calculate the difference between the local and global dynamical maps of evolution and show that there is a distinct difference in the exact dynamics of the two-qubit system interacting separately with two noninteracting environments and that of the phenomenological application of local Kraus operations of the same physical picture. It is our assertion that the latter does not give us the complete picture of the exact dynamical behavior of such interactions, where we overlook the information flow between the baths via the interaction between the qubit systems. Moreover, we have also seen the effect on the quantum correlations, in particular entanglement and quantum discord, between the central spins as the dynamics of the system evolves in time. Revival of the quantum correlations benchmarks the non-Markovian behavior in the central spin system.

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APPENDIX A: DERIVATION OF THE ELEMENTS OF THE REDUCED STATE OF THE SYSTEM

Following the procedure in Sec. III, we now define $|\chi(0)\rangle = |00\rangle|xy\rangle$ and $|\chi(t)\rangle = \hat{D}(t)|00\rangle|xy\rangle + \hat{E}(t)|01\rangle|xy\rangle + \hat{F}(t)|10\rangle|xy\rangle$. Substituting $\hat{D}(t) = \hat{D}_1(t)$, $\hat{E}(t) = b\hat{E}_1(t)$, and $\hat{F}(t) = a\hat{F}_1(t)$, we find

$$\begin{aligned}
\frac{d\hat{D}_1(t)}{dt} &= -i\left[\frac{-\omega_1-\omega_2+\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{D}_1(t)\\
&\quad - i\epsilon_2\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)^{1/2}\hat{E}_1(t) - i\epsilon_1\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)^{1/2}\hat{F}_1(t),\\
\frac{d\hat{E}_1(t)}{dt} &= -i\left[\frac{-\omega_1+\omega_2-\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{(\hat{n}-1)\left(1-\frac{\hat{n}-2}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{E}_1(t)\\
&\quad - i\epsilon_2\left(1-\frac{\hat{n}-1}{N}\right)^{1/2}\hat{D}_1(t),\\
\frac{d\hat{F}_1(t)}{dt} &= -i\left[\frac{\omega_1-\omega_2-\delta}{2} + \omega_a\left\{(\hat{m}-1)\left(1-\frac{\hat{m}-2}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{F}_1(t)\\
&\quad - i\epsilon_1\left(1-\frac{\hat{m}-1}{M}\right)^{1/2}\hat{D}_1(t).
\end{aligned} \tag{A1}$$

The evolution of the reduced state ($|00\rangle\langle 00|$) by tracing over bath modes can be found as

$$\begin{aligned}
\phi(|00\rangle\langle 00|) &= \mathrm{Tr}_{B_1B_2}\left[|\chi(t)\rangle\langle\chi(t)|\right]\\
&= \frac{1}{Z}\sum_m^M\sum_n^N\left\{|D_1(m,n,t)|^2|00\rangle\langle 00| + n|E_1(m,n,t)|^2|01\rangle\langle 01| + m|F_1(m,n,t)|^2|10\rangle\langle 10|\right\}\\
&\quad\times\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right].
\end{aligned} \tag{A2}$$

One may find the components $D_1(m,n,t)$, $E_1(m,n,t)$, and $F_1(m,n,t)$, which are the eigenvalues of the operators $\hat{D}_1(t)$, $\hat{E}_1(t)$, and $\hat{F}_1(t)$ with eigenvectors $|mn\rangle$, by following the steps described in Appendix B:

In a similar manner, we specify $|\xi(0)\rangle = |01\rangle|xy\rangle$ and $|\xi(t)\rangle = \hat{G}(t)|01\rangle|xy\rangle + \hat{H}(t)|00\rangle|xy\rangle + \hat{I}(t)|11\rangle|xy\rangle$. Now substituting with $\hat{G}(t) = \hat{G}_1(t)$, $\hat{H}(t) = b^\dagger\hat{H}_1(t)$, and $\hat{I}(t) = a\hat{I}_1(t)$, we can have

$$\begin{aligned}
\frac{d\hat{G}_1(t)}{dt} &= -i\left[\frac{-\omega_1+\omega_2-\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{G}_1(t)\\
&\quad - i\epsilon_2\left(1-\frac{\hat{n}}{N}\right)^{1/2}(\hat{n}+1)\hat{H}_1(t) - i\epsilon_1\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)^{1/2}\hat{I}_1(t),\\
\frac{d\hat{H}_1(t)}{dt} &= -i\left[\frac{-\omega_1-\omega_2+\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\}\right.\\
&\quad\left. + \omega_b\left\{(\hat{n}+1)\left(1-\frac{\hat{n}}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{H}_1(t) - i\epsilon_2\left(1-\frac{\hat{n}}{N}\right)^{1/2}\hat{G}_1(t),\\
\frac{d\hat{I}_1(t)}{dt} &= -i\left[\frac{\omega_1+\omega_2+\delta}{2} + \omega_a\left\{(\hat{m}-1)\left(1-\frac{\hat{m}-2}{M}\right)-\frac{\mathbb{1}}{2}\right\}\right.\\
&\quad\left. + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{I}_1(t) - i\epsilon_1\left(1-\frac{\hat{m}-1}{M}\right)^{1/2}\hat{G}_1(t).
\end{aligned} \tag{A3}$$

We may now express the evolution of the reduced state ($|01\rangle\langle 01|$) in the following way:

$$\begin{aligned}\phi(|01\rangle\langle 01|) &= \mathrm{Tr}_{B_1B_2}[|\xi(t)\rangle\langle\xi(t)|]\\ &= \frac{1}{Z}\sum_m^M\sum_n^N\Big\{|G_1(m,n,t)|^2|01\rangle\langle 01| + (n+1)|H_1(m,n,t)|^2|00\rangle\langle 00|\\ &\quad + m|I_1(m,n,t)|^2|11\rangle\langle 11|\Big\}\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right]. \qquad \text{(A4)}\end{aligned}$$

Following the procedure written in Appendix B, one may calculate the eigenvalues of $\hat{G}_1(t)$, $\hat{H}_1(t)$, and $\hat{I}_1(t)$, i.e., $G_1(m,n,t)$, $H_1(m,n,t)$, and $I_1(m,n,t)$ for eigenvectors $|mn\rangle$.

To determine the evolution of the fourth diagonal element ($|10\rangle\langle 10|$) of the density matrix of the system of two central spins, we assume, $|\varrho(0)\rangle\rangle = |10\rangle|xy\rangle$ and $|\varrho(t)\rangle\rangle = \hat{J}(t)|10\rangle|xy\rangle + \hat{K}(t)|11\rangle|xy\rangle + \hat{L}(t)|00\rangle|xy\rangle$. Then we replace the time-dependent operators as $\hat{J}(t) = \hat{J}_1(t)$, $\hat{K}(t) = b\hat{K}_1(t)$, and $\hat{L}(t) = a^\dagger\hat{L}_1(t)$ and we obtain

$$\begin{aligned}\frac{d\hat{J}_1(t)}{dt} &= -i\left[\frac{\omega_1-\omega_2-\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{J}_1(t)\\ &\quad - i\epsilon_2\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)^{1/2}\hat{K}_1(t) - i\epsilon_1\left(1-\frac{\hat{m}}{M}\right)^{1/2}(\hat{m}+1)\hat{L}_1(t),\\ \frac{d\hat{K}_1(t)}{dt} &= -i\left[\frac{\omega_1+\omega_2+\delta}{2} + \omega_a\left\{\hat{m}\left(1-\frac{\hat{m}-1}{M}\right)-\frac{\mathbb{1}}{2}\right\}\right.\\ &\quad \left.+\omega_b\left\{(\hat{n}-1)\left(1-\frac{\hat{n}-2}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{K}_1(t) - i\epsilon_2\left(1-\frac{\hat{n}-1}{N}\right)^{1/2}\hat{J}_1(t),\\ \frac{d\hat{L}_1(t)}{dt} &= -i\left[\frac{-\omega_1-\omega_2+\delta}{2} + \omega_a\left\{(\hat{m}+1)\left(1-\frac{\hat{m}}{M}\right)-\frac{\mathbb{1}}{2}\right\} + \omega_b\left\{\hat{n}\left(1-\frac{\hat{n}-1}{N}\right)-\frac{\mathbb{1}}{2}\right\}\right]\hat{L}_1(t) - i\epsilon_1\left(1-\frac{\hat{m}}{M}\right)^{1/2}\hat{J}_1(t).\end{aligned} \qquad \text{(A5)}$$

The reduced state now changes as

$$\begin{aligned}\phi(|10\rangle\langle 10|) &= \mathrm{Tr}_{B_1B_2}[|\varrho(t)\rangle\langle\varrho(t)|]\\ &= \frac{1}{Z}\sum_m^M\sum_n^N\{|J_1(m,n,t)|^2|10\rangle\langle 10| + n|K_1(m,n,t)|^2|11\rangle\langle 11|\\ &\quad + (m+1)|L_1(m,n,t)|^2|00\rangle\langle 00|\}\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]\\ &\quad\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \qquad \text{(A6)}\end{aligned}$$

where, by definition, $\hat{J}_1(t)|mn\rangle = J_1(m,n,t)|mn\rangle$, $\hat{K}_1(t)|mn\rangle = K_1(m,n,t)|mn\rangle$, and $\hat{L}_1(t)|mn\rangle = L_1(m,n,t)|mn\rangle$. The corresponding eigenvalues can be found from Eq. (A5) by following the approach mentioned in the Appendix B:

Now the off-diagonal components of the reduced density matrix will take the following forms:

$$\begin{aligned}\phi(|11\rangle\langle 00|) &= \mathrm{Tr}_{B_1B_2}[|\psi(t)\rangle\langle\chi(t)|]\\ &= \frac{1}{Z}\sum_m^M\sum_n^N[A_1(m,n,t)D_1^*(m,n,t)|11\rangle\langle 00|]\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]\\ &\quad\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \qquad \text{(A7)}\end{aligned}$$

$$\begin{aligned}\phi(|11\rangle\langle 01|) &= \mathrm{Tr}_{B_1B_2}[|\psi(t)\rangle\langle\xi(t)|]\\ &= \frac{1}{Z}\sum_m^M\sum_n^N[A_1(m,n,t)G_1^*(m,n,t)|11\rangle\langle 01|]\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]\\ &\quad\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \qquad \text{(A8)}\end{aligned}$$

$$\phi(|11\rangle\langle10|) = \mathrm{Tr}_{B_1B_2}\,[|\psi(t)\rangle\langle\varrho(t)|]$$
$$= \frac{1}{Z}\sum_m^M\sum_n^N [A_1(m,n,t)J_1^*(m,n,t)|11\rangle\langle10|]\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]$$
$$\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{A9}$$

$$\phi(|10\rangle\langle00|) = \mathrm{Tr}_{B_1B_2}\,[|\varrho(t)\rangle\langle\chi(t)|]$$
$$= \frac{1}{Z}\sum_m^M\sum_n^N [J_1(m,n,t)D_1^*(m,n,t)|10\rangle\langle00|]\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]$$
$$\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{A10}$$

$$\phi(|10\rangle\langle01|) = \mathrm{Tr}_{B_1B_2}\,[|\varrho(t)\rangle\langle\xi(t)|]$$
$$= \frac{1}{Z}\sum_m^M\sum_n^N [J_1(m,n,t)G_1^*(m,n,t)|10\rangle\langle01|]\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]$$
$$\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{A11}$$

$$\phi(|01\rangle\langle00|) = \mathrm{Tr}_{B_1B_2}\,[|\xi(t)\rangle\langle\chi(t)|]$$
$$= \frac{1}{Z}\sum_m^M\sum_n^N [G_1(m,n,t)D_1^*(m,n,t)|01\rangle\langle00|]\exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right]$$
$$\times\exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right]. \tag{A12}$$

From the Hermiticity of the reduced density matrix, the other off-diagonal elements can be expressed as

$$\phi(|00\rangle\langle11|) = [\phi(|11\rangle\langle00|)]^\dagger,\quad \phi(|01\rangle\langle11|) = [\phi(|11\rangle\langle01|)]^\dagger,$$
$$\phi(|10\rangle\langle11|) = [\phi(|11\rangle\langle10|)]^\dagger,\quad \phi(|00\rangle\langle10|) = [\phi(|10\rangle\langle00|)]^\dagger,$$
$$\phi(|01\rangle\langle10|) = [\phi(|10\rangle\langle01|)]^\dagger,\quad \phi(|00\rangle\langle01|) = [\phi(|01\rangle\langle00|)]^\dagger.$$

APPENDIX B: SOLUTION OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS FROM BOUNDARY CONDITIONS: MATRIX METHOD

Suppose three simultaneous linear differential equations are written as

$$\frac{dX(t)}{dt} = -iaX(t) - idY(t) - ieZ(t),$$
$$\frac{dY(t)}{dt} = -ifX(t) - ibY(t),$$
$$\frac{dZ(t)}{dt} = -igX(t) - icZ(t). \tag{B1}$$

And we have to find the solution using the given boundary conditions: $X(0)=1$, $Y(0)=0$, and $Z(0)=0$.

The problem can be solved by plugging in the well-known matrix method in this context. According to this, one can write Eq. (B1) using column and square matrices of dimension three in the following way:

$$\frac{d}{dt}\begin{pmatrix}X(t)\\Y(t)\\Z(t)\end{pmatrix} = -i\begin{pmatrix}a & d & e\\ f & b & 0\\ g & 0 & c\end{pmatrix}\begin{pmatrix}X(t)\\Y(t)\\Z(t)\end{pmatrix}. \tag{B2}$$

For simplicity, we denote

$$M = \begin{pmatrix}a & d & e\\ f & b & 0\\ g & 0 & c\end{pmatrix}.$$

Any of the three eigenvalues, say λ_j $(j=1,2,3)$ of matrix M can be found by solving the characteristic equation

$$(a-\lambda_j)(b-\lambda_j)(c-\lambda_j) - fd(c-\lambda_j) - eg(b-\lambda_j) = 0,\ \forall\, j. \tag{B3}$$

And corresponding to each λ_j, eigenvectors can be expressed as

$$\begin{pmatrix}(\lambda_j-b)(\lambda_j-c)\\ f(\lambda_j-c)\\ g(\lambda_j-b)\end{pmatrix} \equiv \begin{pmatrix}\alpha_j\\ \beta_j\\ \gamma_j\end{pmatrix},$$

where $\alpha_j = (\lambda_j-b)(\lambda_j-c)$, $\beta_j = f(\lambda_j-c)$, and $\gamma_j = g(\lambda_j-b)$, except for the case when $\lambda_j = b,\ c$. Then, according to the matrix method, the general solution of Eq. (B1) in terms of column matrix will take the form,

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \sum_{j=1}^{3} v_j \, e^{-i\lambda_j t} \begin{pmatrix} \alpha_j \\ \beta_j \\ \gamma_j \end{pmatrix}, \tag{B4}$$

where the coefficients v_j can be computed from three boundary conditions given as

$$\sum_{j=1}^{3} v_j \alpha_j = 1, \quad \sum_{j=1}^{3} v_j \beta_j = 0, \quad \sum_{j=1}^{3} v_j \gamma_j = 0. \tag{B5}$$

Now one can write from Eq. (B5):

$$\begin{aligned} v_1 &= \frac{\beta_3\gamma_2 - \beta_2\gamma_3}{\alpha_1(\beta_3\gamma_2 - \beta_2\gamma_3) + \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_2\gamma_1 - \beta_1\gamma_2)}, \\ v_2 &= \frac{\beta_1\gamma_3 - \beta_3\gamma_1}{\alpha_1(\beta_3\gamma_2 - \beta_2\gamma_3) + \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_2\gamma_1 - \beta_1\gamma_2)}, \\ v_3 &= \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\alpha_1(\beta_3\gamma_2 - \beta_2\gamma_3) + \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_2\gamma_1 - \beta_1\gamma_2)}. \end{aligned} \tag{B6}$$

We see that we are able to come up with the solution (B4) only when λ_j is calculated from the Eq. (B3). For solving the three roots of such cubic equation, we follow the procedure given below.

Let us say a cubic equation is in the form

$$a_1x^3 + b_1x^2 + c_1x + d_1 = 0. \tag{B7}$$

Then we define the following quantities:

$$\begin{aligned} \Delta &= 18a_1b_1c_1d_1 - 4b_1^3d_1 + b_1^2c_1^2 - 4a_1c_1^3 - 27a_1^2d_1^2, \\ \Delta_0 &= b_1^2 - 3a_1c_1, \\ \Delta_1 &= 2b_1^3 - 9a_1b_1c_1 + 27a_1^2d_1, \\ Q &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} = \sqrt[3]{\frac{\Delta_1 + \sqrt{-27a_1^2\Delta}}{2}}, \end{aligned}$$

where Δ is called the discriminant of the cubic Eq. (B7). Next, denoting a complex number, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i = \varphi$, we may write the roots of Eq. (B7) as,

$$x_k = -\frac{1}{3a_1}\left(b_1 + \varphi^k Q + \frac{\Delta_0}{\varphi^k Q}\right), \quad k \in \{1, 2, 3\}. \tag{B8}$$

Note that the root x_1 is always real and the roots x_2 and x_3 are complex and conjugate to each other only when $\Delta < 0$. Otherwise, all the roots are real and they become equal when $\Delta = 0$.

In our case, expanding the left-hand side of Eq. (B3) in a power series of λ_j for all j, we identify, $a_1 = 1$, $b_1 = -(a + b + c)$, $c_1 = ab + bc + ac - fd - eg$, and $d_1 = fdc + egb - abc$. As described above, λ_j $(j = 1, 2, 3)$ can be determined and the solution (B4) can be attained accordingly.

All Eqs. (9), (A1), (A3), (A5) are in the form of Eq. (B1). Hence, all the coefficients can be obtained in terms of ω_1, ω_2, δ, ω_a, ω_b, ϵ_1, ϵ_2, m, n, M, and N.

Explicitly, the coefficients, $A_1(m, n, t)$, $B_1(m, n, t)$ and $C_1(m, n, t)$ can be obtained by solving Eq. (9) which in comparison to Eq. (B1) shows that

$$\begin{aligned} X(t) &= A_1(m, n, t), \\ Y(t) &= B_1(m, n, t), \\ Z(t) &= C_1(m, n, t), \\ a &= \frac{\omega_1 + \omega_2 + \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} \\ &\quad + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}, \\ b &= \frac{\omega_1 - \omega_2 - \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} \\ &\quad + \omega_b\left\{(n+1)\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}, \\ c &= \frac{-\omega_1 + \omega_2 - \delta}{2} + \omega_a\left\{(m+1)\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} \\ &\quad + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}, \\ d &= \epsilon_2\left(1 - \frac{n}{N}\right)^{1/2}(n+1), \\ e &= \epsilon_1\left(1 - \frac{m}{M}\right)^{1/2}(m+1), \\ f &= \epsilon_2\left(1 - \frac{n}{N}\right)^{1/2}, \\ g &= \epsilon_1\left(1 - \frac{m}{M}\right)^{1/2}. \end{aligned}$$

Likewise, comparing Eq. (A1) with Eq. (B1), we get

$$\begin{aligned} X(t) &= D_1(m, n, t), \\ Y(t) &= E_1(m, n, t), \\ Z(t) &= F_1(m, n, t), \\ a &= \frac{-\omega_1 - \omega_2 + \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} \\ &\quad + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}, \\ b &= \frac{-\omega_1 + \omega_2 - \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} \\ &\quad + \omega_b\left\{(n-1)\left(1 - \frac{n-2}{N}\right) - \frac{1}{2}\right\}, \\ c &= \frac{\omega_1 - \omega_2 - \delta}{2} + \omega_a\left\{(m-1)\left(1 - \frac{m-2}{M}\right) - \frac{1}{2}\right\} \\ &\quad + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}, \\ d &= \epsilon_2 n\left(1 - \frac{n-1}{N}\right)^{1/2}, \\ e &= \epsilon_1 m\left(1 - \frac{m-1}{M}\right)^{1/2}, \\ f &= \epsilon_2\left(1 - \frac{n-1}{N}\right)^{1/2}, \end{aligned}$$

$$g = \epsilon_1 \left(1 - \frac{m-1}{M}\right)^{1/2}.$$

Correspondingly, Eq. (A3) gives

$$X(t) = G_1(m, n, t),$$
$$Y(t) = H_1(m, n, t),$$
$$Z(t) = I_1(m, n, t),$$
$$a = \frac{-\omega_1 + \omega_2 - \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\},$$
$$b = \frac{-\omega_1 - \omega_2 + \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} + \omega_b\left\{(n+1)\left(1 - \frac{n}{N}\right) - \frac{1}{2}\right\},$$
$$c = \frac{\omega_1 + \omega_2 + \delta}{2} + \omega_a\left\{(m-1)\left(1 - \frac{m-2}{M}\right) - \frac{1}{2}\right\} + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\},$$
$$d = \epsilon_2 \left(1 - \frac{n}{N}\right)^{1/2}(n+1),$$
$$e = \epsilon_1 m\left(1 - \frac{m-1}{M}\right)^{1/2},$$
$$f = \epsilon_2 \left(1 - \frac{n}{N}\right)^{1/2},$$
$$g = \epsilon_1 \left(1 - \frac{m-1}{M}\right)^{1/2}.$$

Finally, the resemblance between Eqs. (A5) and (B1) manifests as

$$X(t) = J_1(m, n, t),$$
$$Y(t) = K_1(m, n, t),$$
$$Z(t) = L_1(m, n, t),$$
$$a = \frac{\omega_1 - \omega_2 - \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\},$$
$$b = \frac{\omega_1 + \omega_2 + \delta}{2} + \omega_a\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\} + \omega_b\left\{(n-1)\left(1 - \frac{n-2}{N}\right) - \frac{1}{2}\right\},$$
$$c = \frac{-\omega_1 - \omega_2 + \delta}{2} + \omega_a\left\{(m+1)\left(1 - \frac{m}{M}\right) - \frac{1}{2}\right\} + \omega_b\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\},$$
$$d = \epsilon_2 n\left(1 - \frac{n-1}{N}\right)^{1/2},$$
$$e = \epsilon_1 \left(1 - \frac{m}{M}\right)^{1/2}(m+1),$$
$$f = \epsilon_2 \left(1 - \frac{n-1}{N}\right)^{1/2},$$
$$g = \epsilon_1 \left(1 - \frac{m}{M}\right)^{1/2}.$$

In this way, all the coefficients can be obtained in order to reveal the complete dynamical map of the two-spin system.

APPENDIX C: MATRIX ELEMENTS OF THE REDUCED STATE OF THE SYSTEM

The elements of the density matrix given in Eq. (11) are given by

$$\rho_{11} = \frac{1}{Z}\sum_m^M \sum_n^N \left[|A_1(m,n,t)|^2 \rho_{11}(0) + n|K_1(m,n,t)|^2 \rho_{22}(0) + m|I_1(m,n,t)|^2 \rho_{33}(0)\right] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}\right], \tag{C1}$$

$$\rho_{22} = \frac{1}{Z}\sum_m^M \sum_n^N [(n+1)\big[B_1(m,n,t)|^2 \rho_{11}(0) + |J_1(m,n,t)|^2 \rho_{22}(0) + m|F_1(m,n,t)|^2 \rho_{44}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}\right], \tag{C2}$$

$$\rho_{33} = \frac{1}{Z}\sum_m^M \sum_n^N \left[(m+1)|C_1(m,n,t)|^2 \rho_{11}(0) + |G_1(m,n,t)|^2 \rho_{33}(0) + n|E_1(m,n,t)|^2 \rho_{44}(0)\right] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}\right], \tag{C3}$$

$$\rho_{44} = \frac{1}{Z}\sum_m^M \sum_n^N \left[(m+1)|B_1(m,n,t)|^2 \rho_{22}(0) + |D_1(m,n,t)|^2 \rho_{44}(0) + (n+1)|H_1(m,n,t)|^2 \rho_{33}(0)\right] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1 - \frac{m-1}{M}\right) - \frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1 - \frac{n-1}{N}\right) - \frac{1}{2}\right\}\right], \tag{C4}$$

$$\rho_{12} = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}[A_1(m,n,t)J_1^*(m,n,t)\rho_{12}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{C5}$$

$$\rho_{13} = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}[A_1(m,n,t)G_1^*(m,n,t)\rho_{13}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{C6}$$

$$\rho_{14} = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}[A_1(m,n,t)D_1^*(m,n,t)\rho_{14}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{C7}$$

$$\rho_{23} = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}[J_1(m,n,t)G_1^*(m,n,t)\rho_{23}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{C8}$$

$$\rho_{24} = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}[J_1(m,n,t)D_1^*(m,n,t)\rho_{24}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{C9}$$

$$\rho_{34} = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}[G_1(m,n,t)D_1^*(m,n,t)\rho_{34}(0)] \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right]. \tag{C10}$$

APPENDIX D: ELEMENTS OF THE CHOI–JAMIOŁKOWSKI MATRIX

We have used following abbreviations in the CJ matrix given in Eq. (12):

$$|A_1|^2 = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}|A_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D1}$$

$$(n+1)|B_1|^2 = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}(n+1)|B_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D2}$$

$$(m+1)|C_1|^2 = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}(m+1)|C_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D3}$$

$$n|K_1|^2 = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}n|K_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D4}$$

$$|J_1|^2 = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}|J_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D5}$$

$$(m+1)|L_1|^2 = \frac{1}{Z}\sum_{m}^{M}\sum_{n}^{N}(m+1)|L_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D6}$$

$$m|I_1|^2 = \frac{1}{Z}\sum_m^M\sum_n^N m|I_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D7}$$

$$|G_1|^2 = \frac{1}{Z}\sum_m^M\sum_n^N |G_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D8}$$

$$(n+1)|H_1|^2 = \frac{1}{Z}\sum_m^M\sum_n^N (n+1)|H_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D9}$$

$$m|F_1|^2 = \frac{1}{Z}\sum_m^M\sum_n^N m|F_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D10}$$

$$n|E_1|^2 = \frac{1}{Z}\sum_m^M\sum_n^N n|E_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D11}$$

$$|D_1|^2 = \frac{1}{Z}\sum_m^M\sum_n^N |D_1(m,n,t)|^2 \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D12}$$

$$A_1J_1^* = \frac{1}{Z}\sum_m^M\sum_n^N A_1(m,n,t)J_1^*(m,n,t) \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D13}$$

$$A_1G_1^* = \frac{1}{Z}\sum_m^M\sum_n^N A_1(m,n,t)G_1^*(m,n,t) \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D14}$$

$$A_1D_1^* = \frac{1}{Z}\sum_m^M\sum_n^N A_1(m,n,t)D_1^*(m,n,t) \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D15}$$

$$J_1G_1^* = \frac{1}{Z}\sum_m^M\sum_n^N J_1(m,n,t)G_1^*(m,n,t) \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D16}$$

$$J_1D_1^* = \frac{1}{Z}\sum_m^M\sum_n^N J_1(m,n,t)D_1^*(m,n,t) \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right], \tag{D17}$$

$$G_1D_1^* = \frac{1}{Z}\sum_m^M\sum_n^N G_1(m,n,t)D_1^*(m,n,t) \times \exp\left[-\frac{\hbar\omega_a}{K_BT}\left\{m\left(1-\frac{m-1}{M}\right)-\frac{1}{2}\right\}\right] \times \exp\left[-\frac{\hbar\omega_b}{K_BT}\left\{n\left(1-\frac{n-1}{N}\right)-\frac{1}{2}\right\}\right]. \tag{D18}$$

The rest of the elements; that is, lower diagonal elements in Eq. (12) are Hermitian conjugates of the upper diagonal terms.

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