Perfect teleportation with a partially entangled quantum channel

Xiang Chen,[1] Yao Shen,[2] and Fu-Lin Zhang[1,*]
[1]Department of Physics, School of Science, Tianjin University, Tianjin 300072, China
[2]School of Criminal Investigation, People's Public Security University of China, Beijing 100038, China

(Received 25 January 2021; accepted 14 September 2022; published 26 September 2022)

Quantum teleportation provides a way to transfer unknown quantum states from one system to another via an entangled state as a quantum channel without physical transmission of the object itself. The entangled channel, measurement performed by the sender (Alice), and classical information sent to the receiver (Bob) are three key ingredients in the procedure, which need to cooperate with each other. To study the relationship among the three parts, we propose a scheme for perfect teleportation of a qubit through a high-dimensional quantum channel in a pure state with two equal largest Schmidt coefficients. The scheme requires less entanglement of Alice's measurement but more classical bits than the original scheme via a Bell state. The two quantities increase with the entanglement of the quantum channel when its dimension is fixed and thereby can be regard as Alice's necessary capabilities to use the quantum channel. And the two capabilities appear complementary to each other.

DOI: 10.1103/PhysRevA.106.032430

I. INTRODUCTION

Significant differences between the quantum and classical worlds are revealed by several quantum information processes without classical counterparts [1]. One of these processes is quantum teleportation [2], in which Alice (the sender) can transfer unknown quantum states from her system to Bob (the receiver) without physical transmission of the object itself. In the simplest and original form, the task of Alice is to teleport a qubit state to Bob. They share a two-qubit Bell state as the quantum channel in advance. Alice makes a joint measurement on the state to be teleported and her qubit from the Bell state, projecting them onto one of the four Bell states. After Alice informs him of the outcome through a classical channel, Bob can perform appropriate unitary operations on his qubit to perfectly rebuild the state to be teleported.

The teleportation protocol has been extended in many branches, including probabilistic teleportation through a partially entangled pure state [3–7], controlled teleportation involving a third party as a controller [8,9], teleportation in high dimensions [10–12], and so on. These schemes play key roles in various contexts in quantum communication, including in quantum repeaters, quantum networks, and cryptographic conferences [13–17]. What these versions of teleportation have in common is that the entanglement of quantum channels is destroyed by Alice's measurement and thereby is the cost of accomplishing the task. Consequently, teleportation serves as an important example for the quantum information processes, in which entanglement plays the role of a key resource [18,19].

The entangled quantum channels, Alice's joint measurement, and the classical information sent to Bob should be regarded as three key ingredients in the procedure, which need to cooperate with each other. Measurement of entangled states, as well as their generation, is a technical challenge in laboratories [11,12] which theoretically can be implemented by using the inverse process of entanglement preparation and local measurements [1]. The primary objective of this work is to study the relationships among the three key ingredients in the procedures for teleportation. We focus on the perfect teleportation (with 100% success probability and fidelity) of a qubit, the smallest unit of quantum information.

Our first step is to propose a general scheme for perfect teleportation of a qubit by using partially entangled two-qudit (a d-dimensional quantum system with $d \geqslant 3$) states, which form continuous regions in the spaces of entanglement invariants [19–21]. In most of the protocols for perfect teleportation [2,10–12], quantum channels are limited to maximally entangled states, locating at vertices of the areas of entanglement invariants. To the best of our knowledge, Gour [22] proposed the only protocol for perfect teleportation via partially entangled two-qudit states in the literature. However, in Gour's protocol, the classical bits are fixed to be the logarithm of the total dimension of the two subsystems in Alice's hands, instead of relying on the entanglement of quantum channel. Here, we propose a general scheme for perfect teleportation of a qubit with a lower classical communication cost by using two-qudit pure states in which the two largest Schmidt coefficients are equal. These quantum channels are coherent superpositions of a set of maximally entangled states in subspaces which form a $(d-2)$-dimensional polyhedron in the space of entanglement invariants [19–21].

To measure Alice's effort in her joint measurement, we define the entanglement of measurement as the average entanglement of its basis. In general, our scheme requires less entanglement of Alice's measurement but more classical bits than the ones with Bell states. The two quantities can be regard as Alice's necessary capabilities to use the quantum channel, which increase with its entanglement when the dimension is

*Corresponding author: flzhang@tju.edu.cn

fixed. And the two abilities are complementary to each other, as the price to decrease the entanglement in the measurement is to send more classical bits to Bob.

In the next section, we explain our protocol by using the example of a three-level entangled quantum channel. The general scheme is presented in Sec. III. The relationship among the entangled channel, the entanglement of Alice's measurement, and the classical information in the task are studied in Sec. IV. In Sec. V, we give an intuitive understanding of the relationship and show the robustness of our protocol under an inhomogeneous phase noise. Finally, Sec. VI presents a summary.

II. TWO-QUTRIT QUANTUM CHANNEL

Let us first explain our basic idea to design the protocol by using the example of a two-qutrit (three-level systems) entangled quantum channel. Suppose Alice wishes to teleport to Bob the qubit state

$$|\phi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1, \tag{1}$$

with $|\alpha|^2 + |\beta|^2 = 1$, and they share a two-qutrit entangled quantum channel

$$|\Phi\rangle_{23} = a_0|00\rangle_{23} + a_1|11\rangle_{23} + a_2|22\rangle_{23}, \tag{2}$$

with $\sum_{j=0}^{2}|a_j|^2 = 1$. Without loss of generality, one can assume the Schmidt coefficients $a_{j=0,1,2}$ are real numbers and $0 \leqslant a_0 \leqslant a_1 \leqslant a_2$. Here, we set $a_1 = a_2$. Then, the entanglement entropy [23,24] of $|\Phi\rangle_{23}$ is larger than 1 and is a monotone increasing function of a_0. And in the space of entanglement invariants [21], the points of $|\Phi\rangle_{23}$ form one of the three edges of the region of arbitrary two-qutrit pure states. The two extreme cases,

$$|\Phi\rangle_{23}^{(a)} = \frac{1}{\sqrt{3}}(|00\rangle_{23} + |11\rangle_{23} + |22\rangle_{23}), \tag{3a}$$

$$|\Phi\rangle_{23}^{(b)} = \frac{1}{\sqrt{2}}(|11\rangle_{23} + |22\rangle_{23}), \tag{3b}$$

locate at two end points of the edge. The former is the maximally entangled two-qutrit state, and the latter is equivalent to the two-qubit Bell states. Based on these entanglement properties of $|\Phi\rangle_{23}$, we expect it can be adopted as the quantum channel to teleport the qubit state $|\phi\rangle_1$ perfectly.

The most crucial step for the teleportation is Alice's joint measurement on her qubit 1 and qutrit 2, which projects Bob's qutrit 3 into a state dependent on Alice's measurement result and the state $|\phi\rangle_1$. After Alice informs Bob of her measurement result through a classical channel, Bob performs a corresponding operation on his system to recover $|\phi\rangle_3 = \alpha|0\rangle_3 + \beta|1\rangle_3$. To teleport the state perfectly (with a fidelity of 1 and success probability of 1), two conditions should be satisfied as follows: (i) Alice's measurement is a projective one; (ii) the collapsed states of particle 3 are of the form $\alpha|\tilde{0}\rangle + \beta|\tilde{1}\rangle$, with $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ being two orthogonal states independent of α and β. The contrasting cases are the two schemes for probabilistic teleportation [3–7] in which Alice unambiguously discriminates nonorthogonal states or Bob performs an extracting quantum state process.

To construct the basis of Alice's measurement, we write the total tripartite state as

$$\begin{aligned}|\Psi\rangle_{123} &= |\phi\rangle_1|\Phi\rangle_{23}\\ &= \big[a_0|00\rangle\alpha|0\rangle + a_1|01\rangle\alpha|1\rangle + a_1|02\rangle\alpha|2\rangle\\ &\quad + a_0|10\rangle\beta|0\rangle + a_1|11\rangle\beta|1\rangle + a_1|12\rangle\beta|2\rangle\big]_{123}.\end{aligned} \tag{4}$$

For the two extreme cases in Eqs. (3), one can easily find a *translation strategy* from the above form in which Alice operates a measurement with the eigenstates

$$\begin{aligned}|\psi_{0\pm}^{(a)}\rangle_{12} &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{12},\\ |\psi_{1\pm}^{(a)}\rangle_{12} &= \frac{1}{\sqrt{2}}(|01\rangle \pm |12\rangle)_{12},\\ |\psi_{2\pm}^{(a)}\rangle_{12} &= \frac{1}{\sqrt{2}}(|02\rangle \pm |10\rangle)_{12}\end{aligned} \tag{5}$$

for case (3a), while she measures the entangled states

$$\begin{aligned}|\psi_{1\pm}^{(b)}\rangle_{12} &= \frac{1}{\sqrt{2}}(|01\rangle \pm |12\rangle)_{12},\\ |\psi_{2\pm}^{(b)}\rangle_{12} &= \frac{1}{\sqrt{2}}(|02\rangle \pm |11\rangle)_{12}\end{aligned} \tag{6}$$

for case (3b). The former was studied in a recent work [10], and the latter is precisely the original scheme [2] with the simple substitutions $|0\rangle \to |1\rangle$ and $|1\rangle \to |2\rangle$ in subsystem 2. In fact, to be complete, the basis for (3b) also contains two *vanished* states, $|00\rangle$ and $|10\rangle$, whose probabilities are zero in the measurement.

One may expect that the orthogonal basis of Alice's measurement, for an arbitrary $|\Phi\rangle_{23}$ with $a_1 = a_2$, is intermediate states between the two sets of extreme basis. A natural idea is to reduce the proportions of $|00\rangle$ and $|10\rangle$ in $|\psi_{0\pm}^{(a)}\rangle_{12}$ and $|\psi_{2\pm}^{(a)}\rangle_{12}$, while inserting the terms $|02\rangle$ and $|11\rangle$, respectively, as their replacement. However, this breaks the orthogonality in condition (i), mainly because it is a continuous transition from the four states to two. Thus, we suspect that the two postmeasured states of qutrit 3 corresponding to $|\psi_{2\pm}^{(b)}\rangle_{12}$ can be collapsed to four orthogonal states with the aid of the two *vanished* vectors, $|00\rangle$ and $|10\rangle$. One option is that we can replace $|\psi_{2\pm}^{(b)}\rangle_{12}$ by $|\psi_{2+}^{(b)}\rangle_{12} \pm |10\rangle_{12}$ and $|00\rangle_{12} \mp |\psi_{2-}^{(b)}\rangle_{12}$ (which are un-normalized). These can be connected smoothly with $|\psi_{j\pm}^{(a)}\rangle_{12}$ by the six orthogonal bases

$$\begin{aligned}|\psi_{0\pm}\rangle_{12} &= \frac{1}{\sqrt{2}}[|00\rangle \pm (c|11\rangle - s|02\rangle)]_{12},\\ |\psi_{1\pm}\rangle_{12} &= \frac{1}{\sqrt{2}}[|01\rangle \pm |12\rangle]_{12},\\ |\psi_{2\pm}\rangle_{12} &= \frac{1}{\sqrt{2}}[(c|02\rangle + s|11\rangle) \pm |10\rangle]_{12},\end{aligned} \tag{7}$$

with the two real numbers satisfying $c^2 + s^2 = 1$.

Now, we show that the teleportation can be accomplished perfectly by performing a joint measurement in the above basis (7) with an appropriate pair of c and s. The (un-normalized) collapsed states of qutrit 3, corresponding to Alice's measurement results $|\psi_{j\pm}\rangle_{12}$, can be derived by

$|\phi_{j\pm}\rangle_3 = {}_{12}\langle\psi_{j\pm}|\Psi\rangle_{123}$, which are

$$|\phi_{0\pm}\rangle_3 = \frac{1}{\sqrt{2}}[\alpha(a_0|0\rangle_3 \mp a_1 s|2\rangle_3) \pm \beta a_1 c|1\rangle_3],$$
$$|\phi_{1\pm}\rangle_3 = \frac{1}{\sqrt{2}}[\alpha a_1|1\rangle_3 \pm \beta a_1|2\rangle_3],$$
$$|\phi_{2\pm}\rangle_3 = \frac{1}{\sqrt{2}}[\alpha a_1 c|2\rangle_3 + \beta(a_1 s|1\rangle_3 \pm a_0|0\rangle_3)]. \quad (8)$$

It is easy to find that they fulfill condition (ii) when

$$c = \sqrt{\tfrac{1}{2}\left(1+\tfrac{a_0^2}{a_1^2}\right)}, \quad s = \sqrt{\tfrac{1}{2}\left(1-\tfrac{a_0^2}{a_1^2}\right)}. \quad (9)$$

Then, the probabilities of Alice's outcome are given by the overlap $P_{j\pm} = {}_3\langle\phi_{j\pm}|\phi_{j\pm}\rangle_3$ as

$$P_{0\pm} = P_{2\pm} = \frac{1}{4}\left(a_0^2 + a_1^2\right), \quad P_{1\pm} = \frac{1}{2}a_1^2. \quad (10)$$

Here, we omit Bob's unitary operators to transform the state on his end to $|\phi\rangle_3$, which can be directly constructed by using the forms of $|\phi_{j\pm}\rangle_3$.

III. GENERAL PROTOCOL

Now we turn to the general protocol for teleporting a qubit through the following partially entangled two-qudit state as the quantum channel:

$$|\Phi\rangle_{23} = \sum_{i=0}^{n} a_i|i\rangle_2|i\rangle_3, \quad (11)$$

where $n = d-1 = 2, 3, \dots$ and the real Schmidt coefficients $0 \leqslant a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n-1} = a_n$ and $\sum_{i=0}^{n} a_i^2 = 1$. Our only requirement for the quantum channel is that the two largest Schmidt coefficients are equal. The state $|\Phi\rangle_{23}$ is a coherent superposition, with real non-negative probabilistic amplitudes, of a set of states,

$$|\Phi\rangle_{23}^{(\tau)} = \sum_{i=\tau}^{n} \frac{1}{\sqrt{n+1-\tau}}|i\rangle_2|i\rangle_3, \quad (12)$$

with $\tau = 0, 1, \dots, n-1$, which are equivalent to $(n+1-\tau)$-dimensional maximally entangled states. The set of $|\Phi\rangle_{23}$ for a fixed n is an $(n-1)$-dimensional polyhedron, with n vertices corresponding to $|\Phi\rangle_{23}^{(\tau)}$, in the space of entanglement invariants [19–21]. Its entanglement entropy [23,24] is lower bounded by 1, and the lower bound is attained by the state with $a_{n-1} = a_n = 1/\sqrt{2}$, or, say, the Bell state $|\Phi\rangle_{23}^{(n-1)}$.

The total state in the teleportation is given by

$$|\Psi\rangle_{123} = |\phi\rangle_1|\Phi\rangle_{23} = \sum_{i=0}^{n}(a_i|0i\rangle_{12}\alpha|i\rangle_3 + a_i|1i\rangle_{12}\beta|i\rangle_3). \quad (13)$$

When $|\Phi\rangle_{23} = |\Phi\rangle_{23}^{(\tau)}$, the basis of Alice's measurement can also be chosen according to the *translation strategy* as $2(n+1-\tau)$ entangled states

$$\left|\psi_{j\pm}^{(\tau)}\right\rangle_{12} = \frac{1}{\sqrt{2}}(|0j\rangle \pm |1, j+1\rangle)_{12},$$
$$\left|\psi_{n\pm}^{(\tau)}\right\rangle_{12} = \frac{1}{\sqrt{2}}(|0n\rangle \pm |1\tau\rangle)_{12}, \quad (14)$$

with $j = \tau, \tau+1, \dots, n-1$, and 2τ *vanished* product states $|0k\rangle$ and $|1k\rangle$, with $k = 0, \dots, \tau-1$. Just like the result of the two-qutrit channel in (7), the basis for the intermediate case between $|\Phi\rangle_{23} = |\Phi\rangle_{23}^{(\tau)}$ and $|\Phi\rangle_{23} = |\Phi\rangle_{23}^{(\tau+1)}$ can be derived by a unitary transformation acting on the subspace of $\{|0n\rangle, |1, \tau+1\rangle\}$. Therefore, we surmise that the basis for a general case can be obtained by a sequence of unitary operations in order as

$$u_k = |\overline{0n}\rangle\langle 0n| + |\overline{1, k+1}\rangle\langle 1, k+1| + (\mathbb{1} - |0n\rangle\langle 0n| - |1, k+1\rangle\langle 1, k+1|) \quad (15)$$

on $|\psi_{j\pm}^{(0)}\rangle_{12}$, with $k = 0, \dots, n-2$ and $j = 0, \dots, n$. Here, the states $|\overline{0n}\rangle = c_k|0n\rangle + s_k|1, k+1\rangle$, and $|\overline{1, k+1}\rangle = c_k|1, k+1\rangle - s_k|0n\rangle$, with

$$c_k = \sqrt{\tfrac{1}{2}\left(1+\tfrac{a_k^2}{a_{k+1}^2}\right)}, \quad s_k = \sqrt{\tfrac{1}{2}\left(1-\tfrac{a_k^2}{a_{k+1}^2}\right)}.$$

That is, the bases are given by

$$|\psi_{j\pm}\rangle_{12} = u_{n-2}u_{n-3}\cdots u_3u_2u_1u_0\left|\psi_{j\pm}^{(0)}\right\rangle_{12} = |0j\rangle \pm c_j|1, j+1\rangle \mp s_j\left(\prod_{k=j+1}^{n-2} c_k|0n\rangle + \sum_{l=j+1}^{n-2}\prod_{k=j+1}^{l-1} c_k s_l|1, l+1\rangle\right), \quad (16)$$

$$|\psi_{n\pm}\rangle_{12} = u_{n-2}u_{n-3}\cdots u_0\left|\psi_{n\pm}^{(0)}\right\rangle_{12} = \left(\prod_{k=0}^{n-2} c_k|0n\rangle + \sum_{l=0}^{n-2}\prod_{k=0}^{l-1} c_k s_l|1, l+1\rangle\right) \pm |10\rangle,$$

where $j = 0, \dots, n-1$. Here, we omit their normalization coefficients $(1/\sqrt{2})$ and the subscript identifying Alice's subsystems.

One can prove that the teleportation can be accomplished perfectly by the joint measurement in the above orthonormal basis (16) by deriving the postmeasured states of qudit 3 left to Bob and showing they satisfy condition (ii). Namely, the (un-normalized) collapsed states are given by

$$|\phi_{j\pm}\rangle_3 = \alpha\left(a_j|j\rangle \mp s_j\prod_{k=j+1}^{n-2} c_k a_n|n\rangle\right) \pm \beta\left(c_j a_{j+1}|j+1\rangle - s_j\sum_{l=j+1}^{n-2}\prod_{k=j+1}^{l-1} c_k s_l a_{l+1}|l+1\rangle\right),$$

$$|\phi_{n\pm}\rangle_3 = \alpha\left(\prod_{k=0}^{n-2} c_k a_n|n\rangle\right) + \beta\left(\sum_{l=0}^{n-2}\prod_{k=0}^{l-1} c_k s_l a_{l+1}|l+1\rangle \pm a_0|0\rangle\right), \quad (17)$$

where $j = 0, \ldots, n-1$. And the corresponding probabilities of Alice's outcome are

$$P_{j\pm} = \frac{1}{2}\left(a_j^2 + s_j^2 \prod_{k=j+1}^{n-2} c_k^2 a_n^2\right), \quad P_{n\pm} = \frac{1}{2}\prod_{k=0}^{n-2} c_k^2 a_n^2. \tag{18}$$

A direct calculation shows the probability amplitudes in the above results satisfying

$$a_j^2 + s_j^2 \prod_{k=j+1}^{n-2} c_k^2 a_n^2 = c_j^2 a_{j+1}^2 + s_j^2 \sum_{l=j+1}^{n-2} \prod_{k=j+1}^{l-1} c_k^2 s_l^2 a_{l+1}^2,$$
$$\prod_{k=0}^{n-2} c_k^2 a_n^2 = \sum_{l=0}^{n-2} \prod_{k=0}^{l-1} c_k^2 s_l^2 a_{l+1}^2 + a_0^2.$$

Therefore, they are of the form $\alpha|\tilde{0}\rangle + \beta|\tilde{1}\rangle$. Consequently, according to the classical information from Alice, Bob can transform these states to $|\phi\rangle_3$ perfectly using appropriate unitary operations, which are independent of the state being teleported.

IV. ENTANGLEMENT AND CLASSICAL INFORMATION

The generation and measurement of entangled states of high-dimensional systems are two technical challenges in laboratories [11,12]. The process of sending classical information to Bob, the preparation of the quantum channel, and Alice's joint measurement can be regarded as three key ingredients consuming resources in the procedure. Our general scheme provides a continuous region to explore these resources in the perfect teleportation of a qubit, which consists of bipartite states whose two largest Schmidt coefficients are equal.

To measure these resources, we adopt the following three quantities analytically expressed in terms of the Schmidt coefficients. First, the entanglement entropy [23] of the quantum channel is given by

$$\mathcal{E}(|\Phi\rangle_{23}) = -\sum_{i=0}^{n} a_i^2 \log_2 a_i^2. \tag{19}$$

Second, the entanglement of Alice's joint measurement is defined by the average of the basis

$$\mathcal{E}_{12} = \sum_{j=0}^{n} [P_{j+}\mathcal{E}(|\psi_{j+}\rangle_{23}) + P_{j-}\mathcal{E}(|\psi_{j-}\rangle_{23})]. \tag{20}$$

It is a direct generalization of the definition in the work of Li *et al.* [3], in which their four bases have the same entanglement degree. The entanglement entropy of the qubit-qudit states (16) can be expressed as a monotone increasing function of their concurrences [24], which are

$$\mathcal{C}(|\psi_{j\pm}\rangle_{23}) = \sqrt{1 - s_j^4 \prod_{k=j+1}^{n-2} c_k^4},$$
$$\mathcal{C}(|\psi_{n\pm}\rangle_{23}) = \prod_{k=0}^{n-2} c_k \sqrt{2 - \prod_{l=0}^{n-2} c_l^2}, \tag{21}$$

with $j = 0, \ldots, n-1$. Third, the classical bits sent to Bob are given by the Shannon entropy of the distribution (18) as

$$\mathcal{H}_{12} = -\sum_{j=0}^{n} (P_{j+} \log_2 P_{j+} + P_{j-} \log_2 P_{j-}). \tag{22}$$

Below we list the results of two series of one-parameter quantum channels to show the properties of the three quantities more clearly.

Case I. The first $(n-1)$ Schmidt coefficients are equal. Let $x = a_0/a_n \in [0, 1]$; one can find that only u_{n-2} is nontrivial with the parameters $c_{n-2} = \sqrt{(1+x^2)/2}$ and $s_{n-2} = \sqrt{(1-x^2)/2}$, while the other unitary operations $u_k = \mathbb{1}$. Two pairs of the concurrence may be lower than 1:

$$\mathcal{C}(|\psi_{(n-2)\pm}\rangle_{23}) = \mathcal{C}(|\psi_{n\pm}\rangle_{23}) = \frac{1}{2}\sqrt{3 + 2x^2 - x^4}. \tag{23}$$

Their corresponding outcome probabilities are

$$P_{(n-2)\pm} = P_{n\pm} = \frac{1+x^2}{4[2+(n-1)x^2]}. \tag{24}$$

The other $(n-1)$ bases are equivalent to two-qubit Bell states with a concurrence of 1, whose outcome probabilities are given by

$$P_{k\pm} = \frac{x^2}{2[2+(n-1)x^2]}, \quad P_{(n-1)\pm} = \frac{1}{2[2+(n-1)x^2]},$$

with $k = 0, \ldots, n-3$.

Case II. The first $(n-2)$ Schmidt coefficients are zero. Since the results for $n = 2$ are already incorporated into case I, we show only the nontrivial quantities for $n \geqslant 3$ here. In addition, we adopt the convention that $c_k = 1$ and $s_k = 0$ when $a_k = a_{k+1} = 0$. Without special explanation, the unitary operations u_k default to $\mathbb{1}$, the values of the concurrence are 1, and the outcome probabilities are zero. We set $y = a_{n-2}/a_n \in [0, 1]$. The parameters in unitary operators u_{n-3} and u_{n-2} are $c_{n-3} = s_{n-3} = \sqrt{1/2}$, $c_{n-2} = \sqrt{(1+y^2)/2}$, and $s_{n-2} = \sqrt{(1-y^2)/2}$. There are three pairs of concurrences depending on the value of y,

$$\mathcal{C}(|\psi_{(n-3)\pm}\rangle_{23}) = \sqrt{1 - \frac{1}{16}(1+y^2)^2},$$
$$\mathcal{C}(|\psi_{(n-2)\pm}\rangle_{23}) = \frac{1}{2}\sqrt{3 + 2y^2 - y^4},$$
$$\mathcal{C}(|\psi_{(n-2)\pm}\rangle_{23}) = \frac{1}{4}\sqrt{7 + 6y^2 - y^4}, \tag{25}$$

and four pairs of nonzero probabilities,

$$P_{(n-3)\pm} = P_{n\pm} = \frac{1+y^2}{8(2+y^2)},$$
$$P_{(n-2)\pm} = \frac{1+y^2}{4(2+y^2)}, \quad P_{(n-1)\pm} = \frac{1}{2(2+y^2)}. \tag{26}$$

Figure 1 shows the regions of the entanglement degrees for Alice's measurement and the quantum channel, along with the curves for the two one-parameter cases, for a fixed dimension. In general, the entanglement of the quantum channels is above the two-qubit Bell states with an entanglement entropy of 1, which is a necessary condition for perfect teleportation

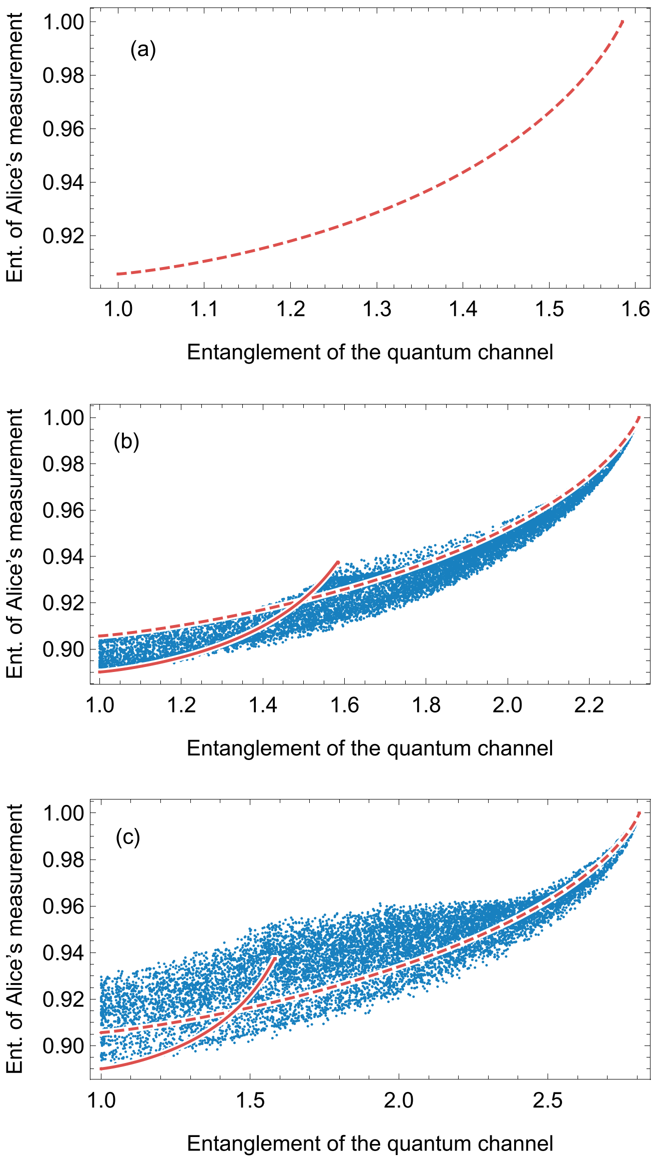

FIG. 1. Entanglement of Alice's measurement vs entanglement of quantum channels. Dashed curves show the values for case I, and solid ones show values for case II, accompanied by 10 000 random quantum channels with for each value of n: (a) $n = 2$, (b) $n = 4$, and (c) $n = 6$. The results for $n = 2$ overlap a single line.

of a qubit according to Theorem 1 in [22], while Alice's measurements are below them. In this sense, the *entanglement matching* [3] does not appear in our scheme. On the other hand, the entanglement of Alice's measurement has an overall upward trend as the entanglement of a quantum channel increases for a fixed n. It reaches the maximum of 1 when the channel is maximally entangled. Hence, the entanglement of Alice's measurement can be considered Alice's ability to use the quantum channel, and a stronger ability is required for a more entangled quantum channel.

For a fixed n, the minimal entanglement of Alice's measurement occurs at the limit of case II with $y \to 0$ and $a_{n-1} = a_n \to \sqrt{1/2}$. The minimums can be derived directly from the above analytic expressions as $\mathcal{E}_{12} = \frac{1}{2}H(\frac{3}{4}) + \frac{1}{2} \approx 0.906$ when $n = 2$ and $\mathcal{E}_{12} = \frac{1}{4}H(\frac{3}{4}) + \frac{1}{4}H(\frac{15}{16}) + \frac{1}{2} \approx 0.890$ when $n = 3, 4, \dots,$ where $H(t) = -\frac{1}{2}(1 - \sqrt{1-t})\log_2 \frac{1}{2}(1 - \sqrt{1-t}) - \frac{1}{2}(1 + \sqrt{1-t})\log_2 \frac{1}{2}(1 + \sqrt{1-t})$. Although the minimum is independent of the dimension of the channel when $n \geqslant 3$, the minimal entanglement in Alice's single measurement can be found to decrease with n. Namely, it is the entanglement of $|\psi_{n\pm}\rangle_{23}$ in the limit of $a_k/a_{k+1} \to 0$ and $a_k \to 0$ and equals $H[2^{-(n-3)} - 2^{-(2n-4)}]$. This indicates that, for a large n, one can design a protocol to detect such a small amount of entanglement to teleport a qubit exactly. However, the successful probability also decreases quickly with n as $P_{n+} + P_{n-} = 2^{-(n-1)}$.

One can also notice that the entanglement of Alice's measurement corresponding to a quantum channel $|\Phi\rangle_{23}^{(\tau)}$ is smaller than that of an $(n+1-\tau)$-dimensional maximally entangled state. The reason can be found in the example of two-qutrit channels, where we superpose $|\psi_{2\pm}^{(b)}\rangle_{12}$ with the two *vanished* vectors, $|00\rangle$ and $|10\rangle$. This reduces the entanglement of Alice's measurement while two outcomes are added, which increases the classical information sent to Bob. That is, the price to decrease the entanglement of Alice's measurement is to send more classical bits to Bob. This conclusion can be confirmed by comparing Figs. 1 and 2.

Obviously, the classical information also has an overall upward trend with the entanglement of the quantum channel. The minimum of classical bits can be found at the limit of case I with $x \to 0$, which is $5/2$ and independent of n.

For a fixed n, at the left end points of the curves for the two cases, which correspond to the same quantum channel but different measurements, Alice measures more entanglement in case I than in case II, but she sends more classical bits to Bob in case II. In addition, the classical bits in case I increase faster than in case II, while for the behaviors of entanglement the opposite is true. In conclusion, the classical information sent to Bob is also a necessary resource to use the quantum channel, which appears to be complementary to the entanglement of Alice's measurement.

V. DISCUSSION

We provide some qualitative discussion of the above results in this part. For simplicity, we show only the formulas for the two-qutrit channel, although they can be directly extended to the general case.

One can perform the unitary operation u_0 with $n = 2$ in (15) on the maximally entangled two-qutrit states [11] and obtain a set of bases as

$$|\psi_{0j}\rangle_{12} = \frac{1}{\sqrt{3}}[|00\rangle + \omega^j(c|11\rangle - s|02\rangle) + \omega^{2j}|22\rangle]_{12},$$
$$|\psi_{1j}\rangle_{12} = \frac{1}{\sqrt{3}}[|10\rangle + \omega^j|21\rangle + \omega^{2j}(c|02\rangle + s|11\rangle)]_{12},$$
$$|\psi_{2j}\rangle_{12} = \frac{1}{\sqrt{3}}[|21\rangle + \omega^j|01\rangle + \omega^{2j}|12\rangle]_{12}, \tag{27}$$

where $\omega = \exp(i\frac{2\pi}{3})$, $j = 0, 1, 2$, and c and s are defined in (9). Measurement on such a basis can teleport a qutrit state

$$|\phi\rangle_1^{(3)} = \alpha|0\rangle_1 + \beta|1\rangle_1 + \gamma|2\rangle_1, \tag{28}$$

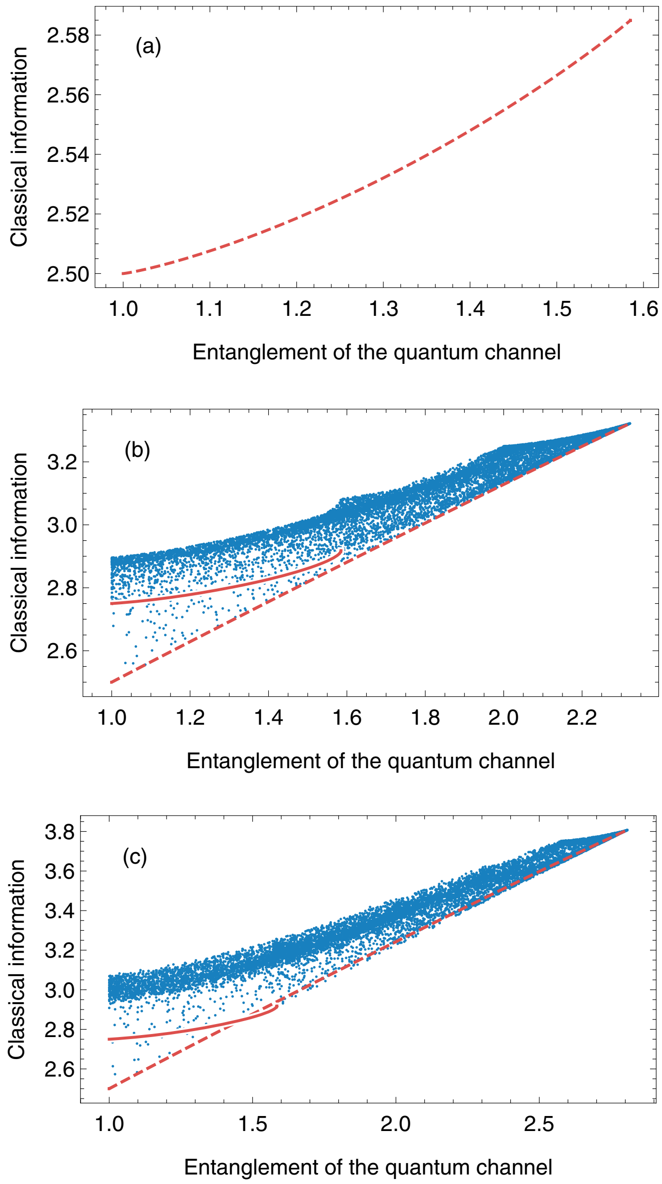

FIG. 2. Classical bits sent to Bob vs entanglement of quantum channels with the same parameters as Fig. 1.

with a state-dependent fidelity, via the two-qutrit channel studied in Sec. II. Namely, the measurement collapses the three-qutrit state $|\phi\rangle_1^{(3)}|\Phi\rangle_{23}$ into

$$|\phi_{0j}\rangle_3 \propto [\alpha(a_0|0\rangle - \omega^j a_1 s|2\rangle) + \beta a_1 c\omega^j|1\rangle + \gamma a_1\omega^{2j}|2\rangle]_3,$$
$$|\phi_{1j}\rangle_3 \propto [\alpha c a_1\omega^{2j}|2\rangle + \beta(a_0|0\rangle + a_1 s\omega^{2j}|1\rangle) + \gamma a_1\omega^j|1\rangle]_3,$$
$$|\phi_{2j}\rangle_3 \propto [\gamma\alpha a_0|0\rangle + \alpha a_1\omega^j|1\rangle + \beta a_1\omega^{2j}|2\rangle]_3.$$

In each collapsed state, the vectors multiplied by the coefficients α and β are orthogonal and equal in magnitude. When $\gamma = 0$, according to Alice's outcome, Bob can rebuild the state $|\phi\rangle_1^{(3)}$ with a fidelity of 1 by performing appropriate unitary operations on qutrit 3. These unitary operations are similar to those in Sec. II, with $\pm$ being replaced by $\omega^{0,1,2}$. However, when $\gamma \neq 0$, the same operations bring him only a state with a fidelity less than 1, which is dependent on the initial state, quantum channel, and Alice's outcome. We derive the average fidelity over Alice's outcomes and the initial states under the Haar measure [25] as

$$\langle\mathcal{F}\rangle = \frac{7}{3} + \frac{5}{2}a_1^2 + a_0 a_1 - \frac{5}{3(1-a_1^2)}. \tag{29}$$

It increases from 1/4 to 1 as a_0 increases from 0 to $1/\sqrt{3}$.

The perfect protocol studied in Sec. II can be regarded as an improved version of the above imperfect teleportation with the aid of *a priori* knowledge of $\gamma = 0$. The prior knowledge reduces the entanglement of Alice's measurement and classical bits; for example, the nine maximally entangled two-qutrit bases are replaced by the six maximally entangled states (5) in subspaces when $a_0 = 1/\sqrt{3}$.

The behaviors of Alice's capabilities in the perfect protocol can be understood by using the relation corresponding to the imperfect teleportation. Taking the standard teleportation of a qutrit [11] as a reference, the unitary operator u_0, creating the basis (27), from the maximally entangled states concentrates the fidelity into the subspace of (α, β) and simultaneously decreases the entanglement of measurement and classical information. Similarly, the same unitary operator also decreases the two quantities in the perfect protocol as it transforms $|\psi_{i\pm}^{(a)}\rangle_{12}$ in (5) into $|\psi_{0\pm}\rangle_{12}$ in (7). This leads to a smaller entanglement of measurement than the Bell-state measurement, while the classical bits are greater than 2 as the price of increasing the dimension. The increasing of the entanglement of $|\Phi\rangle_{23}$ enhances its ability to teleport the state $|\phi\rangle_1^{(3)}$; therefore, the unitary operator u_0 which ensures the fidelity in the subspace of (α, β) becomes *smaller*. Correspondingly, this increases the entanglement of Alice's measurement and classical bits in the perfect protocol.

It is also necessary to compare our protocol in Secs. II and III and the standard teleportation of a qubit via a Bell state under the influence of environmental noise. We now show our protocol is more robust than the standard one under some specific noise channels. Namely, we assume that qutrit 3 passed through an inhomogeneous phase noise when it was sent to Bob, which brings each ket a random phase as $|j\rangle \to e^{i\theta_j}|j\rangle$, with $j = 0, 1, 2$. Suppose $\langle e^{i\theta_j}\rangle = 1 - q_j$ and $q_j \in [0, 1]$. One can derive the fidelity of teleportation under the noise by calculating the overlaps between Bob's collapsed states and their ideal forms in (8) and then its average value $\langle\mathcal{F}\rangle$ over the initial states. When $q_{j'\neq j} = 0$, the average fidelity can be expressed as $\langle\mathcal{F}\rangle = 1 - (1 - q_j)f_j$, and

$$f_0 = \frac{1 + 2a_0^2 - 7a_0^4}{3(1 + a_0^2)}, \quad f_1 = f_2 = \frac{2a_0^2(3 - 5a_0^2)}{3(1 + a_0^2)}. \tag{30}$$

In the same one-sided channel, the fidelity of the standard scheme has a similar form with $f_0 = f_1 = 1/3$ and $f_2 = 0$. Here, $f_2 = 0$ is due simply to the absence of $|2\rangle$ in the standard scheme. These amounts of f_i quantify the responses of fidelities to the random phases, which are shown in Fig. 3. When the random phase occurs in $|1\rangle$, our two-qutrit protocol is more robust than the standard teleportation via a Bell state. When it occurs in $|0\rangle$, the former performs better than the latter when $a_0^2 > 1/7$, while the latter is more robust when $a_0^2 < 1/7$.

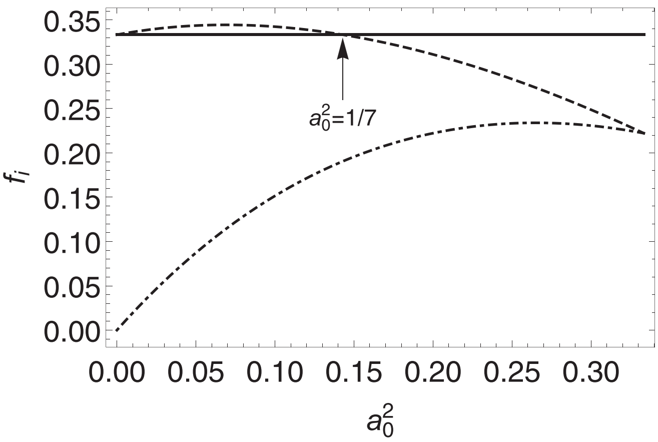

FIG. 3. The responses of fidelities to the random phases vs a_0^2. The dashed and dot-dashed curves show f_0 and f_1 in (30), respectively, and the solid line is for the value of $1/3$.

VI. SUMMARY

We presented a scheme for perfect teleportation of a qubit by using a two-qudit pure state in which the two largest Schmidt coefficients are equal. For a fixed dimension d, the quantum channels, no longer confined to the maximally entangled states, form a continuous area in the space of entanglement invariants. To play the role of a quantum channel, the entangled state requires an appropriate joint measurement by Alice and enough classical bits being sent to Bob. Our scheme requires less entanglement of Alice's measurement but more classical bits than the standard teleportation via a Bell state. The two quantities increase with the entanglement of the quantum channel and thereby can be regard as Alice's necessary capabilities to use the channel. The two capabilities appears complementary to each other. For a fixed amount of entanglement of the quantum channel, the entanglement in Alice's measurement can be partially replaced by classical bits sent to Bob. We also provided an intuitive understanding of these behaviors by using imperfect teleportation of a qutrit, in which the fidelity in a subspace is ensured to be 1. Under some specific noise channels, our protocol is more robust than the standard teleportation.

It would be interesting to consider the following open questions or extensions. First, can our scheme be generalized to the teleportation of high-dimensional states? We conjecture that a d'-level state can be teleported perfectly by using a two-qudit ($d \geqslant d'$) pure state in which the d' largest Schmidt coefficients are equal. However, we find that the joint measurement for high-dimensional teleportation may not be constructed by a simple generalization of (16), and therefore, the development of new methods to design the protocol is still necessary. Second, while we focused here on the teleportation between two participants, hybridizing the present ideas to controlled teleportation would be interesting. Third, implementing the protocol experimentally is a natural direction. Besides the generation of high-dimensional quantum channels, both Alice's measurement and Bob's operations are challenges in the laboratory. The techniques developed in recent experiments [11,12] open the possibility of implementing the process in an optical system.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (Grants No. 11675119, No. 11575125, and No. 11105097) and the Fundamental Research Funds for the Central Universities, Grant No. 2020JKF306.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[3] W.-L. Li, C.-F. Li, and G.-C. Guo, Phys. Rev. A **61**, 034301 (2000).
[4] K. Banaszek, Phys. Rev. A **62**, 024301 (2000).
[5] L. Roa, A. Delgado, and I. Fuentes-Guridi, Phys. Rev. A **68**, 022310 (2003).
[6] F. Verstraete and H. Verschelde, Phys. Rev. Lett. **90**, 097901 (2003).
[7] L. Roa and C. Groiseau, Phys. Rev. A **91**, 012344 (2015).
[8] A. Karlsson and M. Bourennane, Phys. Rev. A **58**, 4394 (1998).
[9] X.-H. Li and S. Ghose, Phys. Rev. A **90**, 052305 (2014).
[10] Y. Huang and W. Yang, Chin. J. Electron. **29**, 228 (2020).
[11] Y.-H. Luo, H.-S. Zhong, M. Erhard, X.-L. Wang, L.-C. Peng, M. Krenn, X. Jiang, L. Li, N.-L. Liu, C.-Y. Lu, A. Zeilinger, and J.-W. Pan, Phys. Rev. Lett. **123**, 070505 (2019).
[12] X.-M. Hu, C. Zhang, B.-H. Liu, Y. Cai, X.-J. Ye, Y. Guo, W.-B. Xing, C.-X. Huang, Y.-F. Huang, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. **125**, 230501 (2020).
[13] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Rev. Mod. Phys. **83**, 33 (2011).
[14] E. Biham, B. Huttner, and T. Mor, Phys. Rev. A **54**, 2651 (1996).
[15] S. Bose, V. Vedral, and P. L. Knight, Phys. Rev. A **57**, 822 (1998).
[16] P. Townsend, Nature (London) **385**, 47 (1997).
[17] B. Aoun and M. Tarifi, arXiv:quant-ph/0401076.
[18] N. Brunner, N. Gisin, and V. Scarani, New J. Phys. **7**, 88 (2005).
[19] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[20] S. Albeverio and S. M. Fei, J. Opt. B **3**, 223 (2001).
[21] R.-J. Gu, F.-L. Zhang, S.-M. Fei, and J.-L. Chen, Int. J. Quantum Inf. **09**, 1499 (2011).
[22] G. Gour, Phys. Rev. A **70**, 042301 (2004).
[23] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[24] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[25] K. Zyczkowski and M. Kus, J. Phys. A **27**, 4235 (1994).