Recycled entanglement detection by arbitrarily many sequential and independent pairs of observers

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Two spatially separated observers, Alice and Bob, share a bipartite two-qubit entangled state and perform measurements to witness entanglement. After their measurements, they pass their qubits to a subsequent pair of observers who try to perform the same task independently, and so on. Here we ask: what is the maximum number of such pairs that can perform this task successfully? It has previously been conjectured that not more than one pair of observers can detect Clauser-Horne-Shimony-Holt "Bell-nonlocal" correlations in this setup. We prove that, on the contrary, entanglement can be witnessed by *arbitrarily* many pairs of observers. The dissimilar nature between entanglement and Bell-nonlocal correlations is therefore uncovered in a rather radical way, when considering sequentially acting pairs of observers. We prove this statement to be true when the initial shared state is any pure entangled state, a class of Bell-nonlocal mixed states, or a class of Bell-local entangled states.

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I. INTRODUCTION

Quantum entanglement [1,2] is among the most crucial resources in quantum information processing and communication. It plays a key role in numerous applications, such as quantum cryptography [3,4], quantum dense coding [5], quantum teleportation [6], entanglement swapping [7,8], and device-independent tasks like key distribution [9–11], randomness amplification [12], randomness expansion [13–15], etc. It is therefore useful to learn about different techniques of characterizing, quantifying, and utilizing entanglement. A valuable tool to analyze and characterize entanglement exploits a class of functionals called *entanglement witnesses* (EWs) [1,2,16]. Given an entangled state, there always exists an EW [2,17–20], and for example, Bell inequalities [21,22], such as the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [23], that detect the so-called "Bell-nonlocal" (or "nonlocal") states are also EWs.

In Ref. [24], the authors introduced a scenario where, given an initial bipartite entangled state, a single observer, Alice, owns one subsystem while the other subsystem is passed among multiple sequential and independent observers, Bobs. The task is to obtain a sequential violation of the CHSH Bell inequality. Initially, an unbounded number of recycled nonlocal correlations was detected only for "biased" measurement strategies [24]. Later the same was attained for a certain family of nonlocal states using a measurement strategy [25] that is unbiased. However, the question whether the same holds for all nonlocal states remains open. Nonlocal states are certainly entangled, and thus this result also answers the question about arbitrary recyclability of entanglement for Bell-nonlocal states. Recently, it was also observed that it is possible to witness and recycle entanglement an arbitrary number of times using a CHSH Bell-local entangled state [26]. For other theoretical works in the direction of recyclability of entangled and nonlocal correlations, see Refs. [27–40]. For experimental works, see Refs. [41–45].

Studying the sequential detection scenario of quantum resources is certainly of fundamental significance. It tries to answer about the fundamental limits on recycling of these resources. The fact that till now only some of the entangled quantum states are shown to facilitate an arbitrary-times sequential detection of certain quantum resources raises the interesting open problem about the situation for the rest of the entangled states [25,26]. Also, gaining information about entanglement or Bell-nonlocal correlations of a quantum state causes disturbance in the state, so that the sequential detection of these resources can tell us about the trade-off between information gain and state disturbance for the resources. Along with being fundamentally interesting, such scenarios of sequential detection of quantum resources can have potential applications in quantum technologies and can also be useful in situations when there is significant restrictions on quantum state preparation [25].

In the present work, we consider the scenario in which both the subsystems of a bipartite quantum state are passed on to multiple pairs of independent observers [37–40] (see Fig. 1). It has been conjectured in Ref. [37], with analytical and numerical evidence, that recycling of nonlocal correlations, via violation of the CHSH Bell inequality, is impossible in this scenario. That is, not more than one pair of Alice and Bob can share a CHSH Bell-nonlocal state. This is intriguing, since a single Alice can share CHSH Bell-nonlocal correlations with an arbitrary number of sequential and independent Bobs [25], but allowing multiple Alices and Bobs seems to restrict such shareability. It is therefore interesting to ask whether entanglement correlations also have such limitations. Contrary to the case of Bell nonlocality, we show that entanglement can be

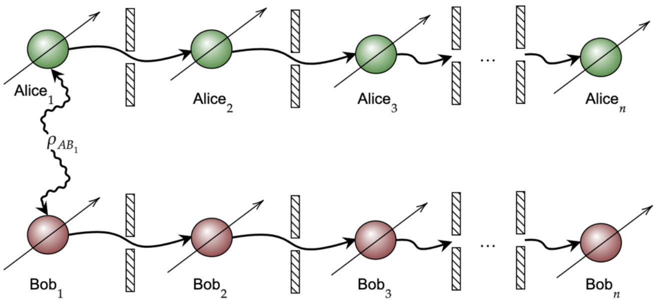

FIG. 1. Schematic of sequential entanglement detection by independent pairs of observers. Let Alice$_1$ and Bob$_1$ share a bipartite quantum system in the state ρ_{AB_1}, with one subsystem in possession of Alice$_1$ and the other in control of Bob$_1$. Their task is to detect entanglement and then pass on their subsystems to Alice$_2$ and Bob$_2$, respectively. Now, Alice$_2$ and Bob$_2$ act on their subsystems to detect entanglement and pass the quantum system to another subsequent pair, and this continues till the entanglement content of the shared state vanishes. The question is: how many such pairs of observers can detect entanglement acting sequentially and independently?

detected and recycled in this scenario an arbitrary number of times. Therefore, where only one pair could detect CHSH Bell nonlocality in shared two-qubit states, an arbitrary number of pairs can detect entanglement in at least specific instances of the same. Thus, our result uncovers an interesting face of the distinct behavior between entanglement correlation and Bell-nonlocal correlation of two-qubit quantum states, within the realm of sequential witnessing. We provide a measurement strategy with which an unbounded number of detections of entanglement can be achieved for any pure entangled two-qubit state and for a class of mixed states. We also show that such a property can be unleashed by a class of states which are CHSH Bell-local and whose entanglement content vanishes in the limit of the number of observing pairs growing unboundedly. This observation also hints at the possibility that the whole set of entangled states maybe used to generate arbitrary sequences of entangled correlations in these sequential entanglement detection scenarios.

II. ENTANGLEMENT WITNESSES AND UNSHARP MEASUREMENTS

Entanglement witnesses [1,2,16] use expectation values of Hermitian operators which separate the set of separable states from some of the entangled states. This method of entanglement detection utilizes the Hahn-Banach theorem [46,47], which says that, corresponding to any element falling outside a closed and convex set of a normed linear space, there always exists a functional on that space which "separates" the element with the closed and convex set. An entanglement witness operator is thus an operator W, such that $\langle W\rangle_{\rho_s} \geqslant 0$, for all separable states, ρ_s, and there exists at least one entangled state, ρ_e, for which $\langle W\rangle_{\rho_e} < 0$, where $\langle W\rangle_\rho$ represents the expectation value of the Hermitian operator W with respect to a state ρ. For example, entanglement of the bipartite state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ can be detected by the Hermitian operator $W_{\psi^+} = |\phi^-\rangle\langle\phi^-|^T$, where $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, T represents partial transposition of operators, i.e., transposition with respect to any of the local parties (but a fixed one), and $|0\rangle$ and $|1\rangle$ represent the eigenstates of the Pauli operator σ_z with eigenvalues 1 and -1, respectively. The expectation values required for witnessing entangled states can also be computed locally, since Hermitian operators acting on a joint Hilbert space can be decomposed in Hermitian operators acting on the local Hilbert spaces of the joint Hilbert space. For example, the entanglement witness operator W_{ψ^+}, corresponding to $|\psi^+\rangle$, can be decomposed as [20]

$$W_{\psi^+} = \frac{1}{4}[\mathbb{I}_4 + \sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y], \tag{1}$$

where $\mathbb{I}_d$ represents the identity operator acting on the d-dimensional Hilbert space $\mathbb{C}^d$, and σ_x, σ_y, and σ_z are the Pauli matrices.

While the expectation value of W_{ψ^+} will recognize the entanglement in $|\psi^+\rangle$, the projective measurements involved to evaluate the expectation value will destroy the entanglement present in the state. However, it is possible to detect entanglement as well as preserve some amount of entanglement at the same time, if one performs an unsharp version of the required projective measurements [24,25]. Consider an observable, P, acting on the qubit space, and let the state of the qubit be ρ. Corresponding to the projection measurement with the projection operators P_0 and P_1, with $P_0 + P_1 = \mathbb{I}_2$ and $P = P_0 - P_1$, we can define an unsharp version of the measurement, with the sharpness parameter λ, which consists of the operators,

$$E_0^\lambda = \frac{1}{2}(\mathbb{I}_2 + \lambda P), \quad E_1^\lambda = \frac{1}{2}(\mathbb{I}_2 - \lambda P), \tag{2}$$

where $0 \leqslant \lambda \leqslant 1$. Note that, for $\lambda = 1$, the positive operator-valued measure $\{E_0^\lambda, E_1^\lambda\}$ reduces to the projective measurement $\{P_0, P_1\}$, whereas for $\lambda = 0$, measurement operators are identity. Therefore, disturbance to the state, due to the measurement, is the greatest when $\lambda = 1$, whereas $\lambda = 0$ leads to a trivial measurement in the sense that the state is totally unaffected. Notice that the unsharp version of the measurement will lead to an expectation value of P multiplied by the sharpness parameter, since $E_0^\lambda - E_1^\lambda = \lambda P$. The postmeasurement state is given by the von Neumann-Lüder's rule [48] as

$$\rho \to \sqrt{E_0^\lambda}\rho\sqrt{E_0^\lambda} + \sqrt{E_1^\lambda}\rho\sqrt{E_1^\lambda}. \tag{3}$$

III. SCENARIO

We consider a generalized version of the sequential scenario presented in Ref. [24]. It involves a bipartite entangled state that is shared between a pair of observers, namely, the first Alice-Bob pair (AB_1). The pair performs their tasks in spatially separated labs. To witness entanglement, the first Alice (A_1) and the first Bob (B_1) each performs one of three local measurements at random on their qubit. Both the post-measurement subsystems are then passed to A_2 and B_2 who run the same task on their respective qubits independently and with no prior knowledge of the outcomes attained by their previous observers. This process continues until the state reaches a pair who fail to detect entanglement. The ultimate aim of this task is to maximize the number of pairs that can witness entanglement independently. We prove that using the weak measurement strategy presented below, one can achieve an arbitrarily long sequence of Alice-Bob pairs who can successfully detect entanglement for any pure entangled state, a class of mixed entangled states, and certain weakly entangled (again, mixed) states.

A. Adopted measurement strategy and entanglement witness

Let the kth Alice-Bob (AB_k) pair share the state ρ_{AB_k}, and let the witness operator used by the pair be given as follows:

$$W_k = \tfrac{1}{4}[\mathbb{I}_4 + \sigma_z \otimes \sigma_z - \lambda_k\sigma_x \otimes \lambda_k\sigma_x - \lambda_k\sigma_y \otimes \lambda_k\sigma_y], \quad (4)$$

where λ_k is the sharpness parameter. Therefore, each observer has three measurement settings, viz., σ_x, σ_y, and σ_z. For ease of notation, we denote the Pauli matrices σ_x, σ_y, and σ_z as σ_1, σ_2, and σ_3. The sharpness parameter corresponding to any of the two kth local observers, for the setting σ_i, is denoted as $\lambda_k^{(i)}$, where $i = 1, 2$, and 3. One can see from Eq. (4) that

$$\lambda_k^{(3)} = 1 \quad \text{and} \quad \lambda_k^{(1)} = \lambda_k^{(2)} = \lambda_k.$$

Note that $\langle W_k\rangle \geqslant 0$ for all separable states as $0 \leqslant \lambda_k^{(i)} \leqslant 1$ [26]. Let $\{\mathcal{A}_0^{\lambda_k^{(i)}}, \mathcal{A}_1^{\lambda_k^{(i)}}\}$ and $\{\mathcal{B}_0^{\lambda_k^{(i)}}, \mathcal{B}_1^{\lambda_k^{(i)}}\}$ denote the measurements performed by the observers A_k and B_k, respectively, and let their forms be taken as given in Eq. (2). Since it is assumed that all the measurement settings applied by each Alice-Bob pair are equally probable, i.e., unbiased, the state at the hand of the kth pair in the sequence is given by

$$\begin{aligned}\rho_{AB_k} &= \frac{1}{9}\sum_{i,j=1}^{3}\sum_{a,b=0}^{1}\sqrt{\mathcal{A}_a^{\lambda_{k-1}^{(i)}}} \otimes \sqrt{\mathcal{B}_b^{\lambda_{k-1}^{(j)}}}\rho_{AB_{k-1}}\sqrt{\mathcal{A}_a^{\lambda_{k-1}^{(i)}}} \otimes \sqrt{\mathcal{B}_b^{\lambda_{k-1}^{(j)}}}\\ &= \frac{1}{36}\sum_{i,j=1}^{3}\big[\big(1+\Lambda_{k-1}^i\big)\big(1+\Lambda_{k-1}^j\big)\rho_{AB_{k-1}} + \big(1+\Lambda_{k-1}^i\big)\big(1-\Lambda_{k-1}^j\big)\mathbb{I}_2\otimes\sigma_j\rho_{AB_{k-1}}\mathbb{I}_2\otimes\sigma_j\\ &\quad + \big(1-\Lambda_{k-1}^i\big)\big(1+\Lambda_{k-1}^j\big)\sigma_i\otimes\mathbb{I}_2\rho_{AB_{k-1}}\sigma_i\otimes\mathbb{I}_2 + \big(1-\Lambda_{k-1}^i\big)\big(1-\Lambda_{k-1}^j\big)\sigma_i\otimes\sigma_j\rho_{AB_{k-1}}\sigma_i\otimes\sigma_j\big],\end{aligned} \quad (5)$$

where $\Lambda_k^i = \sqrt{1-\lambda_k^{(i)2}}$, so that $\Lambda_k^i = \sqrt{1-\lambda_k^2}$ for $i = 1$ and 2, and is denoted as Λ_k. Now, the expectation values of each term present in the witness operator W_k with respect to ρ_{AB_k} can be expressed in terms of their expectation values with respect to the state ρ_{AB_1}, i.e.,

$$\mathrm{Tr}\,[\sigma_z\otimes\sigma_z\rho_{AB_k}] = \mathrm{Tr}[\sigma_z\otimes\sigma_z\rho_{AB_1}]\Pi_{l=1}^{k-1}\frac{(1+2\Lambda_l)^2}{9},$$
$$\mathrm{Tr}\,[\sigma_x\otimes\sigma_x\rho_{AB_k}] = \mathrm{Tr}[\sigma_x\otimes\sigma_x\rho_{AB_1}]\Pi_{l=1}^{k-1}\frac{(1+\Lambda_l)^2}{9}. \quad (6)$$

Note that $\mathrm{Tr}[\sigma_x\otimes\sigma_x\rho_{AB_k}] = \mathrm{Tr}[\sigma_y\otimes\sigma_y\rho_{AB_k}]$ for each $k \in \mathbb{N}$.

IV. SEQUENTIAL ENTANGLEMENT DETECTION ARBITRARILY MANY TIMES

In this section, we study the considered scenario with the first pair of observers in the sequence sharing a state chosen from different classes of states.

A. Maximally entangled state

Let the pair AB_1 share a maximally entangled state, i.e., $\rho_{AB_1} = |\psi^+\rangle\langle\psi^+|$. Therefore, the first pair will observe entanglement if $\mathrm{Tr}[\rho_{AB_1}W_1] < 0$,

$$\Rightarrow \lambda_1^2 > 0. \quad (7)$$

And the pair AB_k will able to witness the entanglement if $\mathrm{Tr}[\rho_{AB_k}W_k] < 0$,

$$\Rightarrow \lambda_k^2 > \frac{1-\Pi_{l=1}^{k-1}\frac{(1+2\Lambda_l)^2}{9}}{2\Pi_{l=1}^{k-1}\frac{(1+\Lambda_l)^2}{9}}. \quad (8)$$

Therefore, let us define the sequence λ_k^2 for $k \in \mathbb{N}$ as

$$\lambda_k^2 = \begin{cases}(1+\epsilon)\dfrac{1-\Pi_{l=1}^{k-1}\frac{(1+2\Lambda_l)^2}{9}}{2\Pi_{l=1}^{k-1}\frac{(1+\Lambda_l)^2}{9}}, & \text{if } \lambda_{k-1}^2 \in (0,1),\\ \infty, & \text{otherwise},\end{cases} \quad (9)$$

with $0 < \lambda_1^2 < 1$ and where $\epsilon > 0$. Note that $\lambda_k^2 \in (0, 1)$ implies that the pair AB_k will be able to witness the entanglement, whereas $\lambda_k^2 = \infty$ implies that the pair AB_k will not be able to witness entanglement. Now for $\lambda_k^2 \in (0, 1)$,

$$\frac{\lambda_{k+1}^2}{\lambda_k^2} = \frac{9}{(1+\Lambda_k)^2}\,\frac{1-\Pi_{l=1}^{k}\frac{(1+2\Lambda_l)^2}{9}}{1-\Pi_{l=1}^{k-1}\frac{(1+2\Lambda_l)^2}{9}} > 1, \quad (10)$$

since $\Lambda_k = \sqrt{1-\lambda_k^2} \in (0, 1)$. Therefore, the sequence λ_k^2 in Eq. (9) is positive and increasing. The next important observation about this sequence is that, as $\lambda_1^2 \to 0$, we have $\lambda_k^2 \to 0$ for all $k \in \mathbb{N}$. This proves that an arbitrarily long sequence of Alice-Bob pairs will be able to witness entanglement starting from a maximally entangled state.

Interestingly, one can check that the state received by each pair during the process is a CHSH Bell-nonlocal state when the initial state is maximally entangled. Therefore, an arbitrary number of pairs can share Bell nonlocality in the process of witnessing entanglement. However, the measurement settings required to witness such Bell nonlocality are different from the measurement settings that are required to witness the entanglement present in the state. Thus, this observation definitely does not overturn the conjecture made in Ref. [37].

B. Pure entangled states and a class of mixed entangled states

Any pure entangled state, up to a local unitary, can be written as $|\psi_\alpha\rangle = \sqrt{\alpha}|01\rangle + \sqrt{1-\alpha}|10\rangle$ for $\alpha \in (0, \frac{1}{2}]$ and, in Hilbert-Schmidt decomposition, can be expressed as

$$\frac{1}{4}[\mathbb{I}_4 - \sigma_z \otimes \sigma_z + 2\sqrt{\alpha(1-\alpha)}\sigma_x \otimes \sigma_x + 2\sqrt{\alpha(1-\alpha)}\sigma_y \otimes \sigma_y]. \tag{11}$$

Consider now a mixed entangled state shared by AB_1, given by

$$\rho_{AB_1} = p_1|\psi_\alpha\rangle\langle\psi_\alpha| + p_2|01\rangle\langle 01| + p_3|10\rangle\langle 10|, \tag{12}$$

where $p_1 > 0$, $p_2, p_3 \geqslant 0$, and $p_1 + p_2 + p_3 = 1$. In this case, using the same measurement strategy and witness operator, the sequence of sharpness parameters for the pair of observers will come out to be

$$\lambda_k^2 = \begin{cases} (1+\epsilon)\frac{1-\Pi_{l=1}^{k-1}\frac{(1+2\Lambda_l)^2}{9}}{4p_1\sqrt{\alpha(1-\alpha)}\Pi_{l=1}^{k-1}\frac{(1+\Lambda_l)^2}{9}}, & \text{if } \lambda_{k-1}^2 \in (0,1), \\ \infty, & \text{otherwise,} \end{cases} \tag{13}$$

with $0 < \lambda_1^2 < 1$ and where $\epsilon > 0$. This sequence of sharpness parameters is also positive, increasing, and goes to zero as $\lambda_1^2 \to 0$. Therefore, an unbounded number of entanglement detections are possible by sequential Alice-Bob pairs, where the first sequential pair may share any pure entangled state or a particular class of mixed entangled states, viz., those given by Eq. (12). Note that the unbounded sequential detection of these entangled states can be shown, in a similar fashion, even for the scenario where a single Alice shares the state with multiple Bobs, acting sequentially. However, it is already known that these states can produce an unbounded chain of nonlocal states [25], which in turn imply an unbounded chain of entanglement detection, so a separate analysis is not needed.

V. ARBITRARY SEQUENCE OF OBSERVERS STARTING FROM A CHSH BELL-LOCAL ENTANGLED STATE

In the scenario where multiple Bobs could witness and recycle entanglement with a single Alice, an arbitrarily long sequential entanglement detection is possible even with CHSH Bell-local entangled states [26]. The CHSH Bell-local entangled states are defined as the states which do not violate any CHSH Bell inequality. While a CHSH Bell-local state does not have a Bell-nonlocal correlation, nevertheless, it still is a resource and gives quantum advantage in some tasks. Therefore, it is potentially important to investigate whether recycling of states is possible only with Bell-nonlocal entangled states or if even Bell-local states can allow such a feat.

In this section, we investigate the scenario of multiple pairs of observers detecting and recycling entangled states with the initial pair sharing the state,

$$\rho_{AB_1} = \frac{1}{4}[\mathbb{I}_2 \otimes \mathbb{I}_2 - \cos\theta\sigma_z \otimes \sigma_z + \alpha\sin\theta\sigma_x \otimes \sigma_x + \alpha\sin\theta\sigma_y \otimes \sigma_y], \tag{14}$$

where $\theta \in (0, \frac{\pi}{4}]$ and $\frac{1-\cos\theta}{2\sin\theta} < \alpha \leqslant 1$. Note that these states are CHSH Bell-local states; i.e., they do not violate any CHSH Bell inequality, whereas the class of states given in Eq. (12) are CHSH Bell-nonlocal. The kth observer pair, AB_k, uses the same witness operator as given in Eq. (4). Therefore, the sequence of the sharpness parameter after applying the condition $\mathrm{Tr}[\rho_{AB_k}W_k] < 0$ appears as follows:

$$\lambda_k^2 = \begin{cases} (1+\epsilon)\frac{1-\cos\theta\prod_{l=1}^{k-1}\frac{(1+2\Lambda_l)^2}{9}}{2\alpha\sin\theta\prod_{l=1}^{k-1}\frac{(1+\Lambda_l)^2}{9}}, & \text{if } \lambda_{k-1}^2 \in (0,1), \\ \infty, & \text{otherwise,} \end{cases} \tag{15}$$

with $\lambda_1^2 > \frac{1-\cos\theta}{2\alpha\sin\theta}$ and where $\epsilon > 0$. We define another sequence for $\theta \in (0, \frac{\pi}{4}]$, which upper bounds the sequence in Eq. (15):

$$\gamma_k^2 = \begin{cases} (1+\epsilon)\frac{1-\left(1-\frac{\theta^2}{2}\right)\prod_{l=1}^{k-1}\left(1-\frac{2\gamma_k^2}{3}\right)^2}{\theta\prod_{l=1}^{k-1}\left(\frac{2-\gamma_k^2}{3}\right)^2}, & \text{if } \gamma_{k-1}^2 \in (0,1), \\ \infty, & \text{otherwise,} \end{cases} \tag{16}$$

with $\gamma_1^2 = (1+\epsilon)\frac{\theta}{2\alpha} > \lambda_1^2$. It is easy to show that $\gamma_k^2 > \lambda_k^2$ for all k such that λ_k^2 is finite, since $\sqrt{1-x^2} > 1 - x^2$ for $0 < x < 1$, $\cos\theta > (1 - \frac{\theta^2}{2})$, and $\sin\theta > \frac{\theta}{2}$ for $\theta \in (0, \frac{\pi}{4}]$.

Again, similar to the sequences given in Eqs. (9) and (13), this sequence, γ_k^2, is also strictly positive and increasing. Also, for $\theta \to 0$ and for $k \in \mathbb{N}$, $\gamma_k^2 \to 0$. This implies that $\lambda_k^2 \to 0$ as $\theta \to 0$. Therefore, detection and recycling of entanglement is possible when $\theta \to 0$, an arbitrary number of times. It is important to notice that as $\theta \to 0$, entanglement of the state ρ_{AB_1}, given in Eq. (14), tends to zero. Therefore, for the class of states in Eq. (14), an arbitrary number of pairs of observers detecting entanglement becomes possible only in the situation, for our measurement strategy, when the initial state has an entanglement that becomes vanishingly small.

For any finite value of entanglement in the initial shared state in Eq. (14), the number of pairs of sequential observers is finite. However, as we decrease the value of entanglement of the initial state, the sequence of pairs of observers becomes longer and longer, and in the limit of the entanglement becoming infinitesimally small, the sequence is infinitely long. Given any finite number, n_0, however large, of the sequence of detections, there always exists a state with some finite (nonvanishing) entanglement and a measurement strategy that uses the state as the initial state to detect entanglement n_0 times. This can be shown by noting the fact that the sequence λ_k is positive, increasing, and goes to zero when θ goes to zero.

VI. CONCLUSION

Sequential detection of correlations like Bell nonlocality and entanglement are fundamentally important as it tells us about the ultimate limits on the recyclability of such resources. These studies form an interesting way to understand and analyze the information gain versus state disturbance trade-off, since detection of any correlation causes disturbance in the underlying state. Such a scenario can also come in handy in quantum technologies where state preparation is costly.

The first sequential scenario considered in the literature had a single observer detecting CHSH Bell nonlocality with a spatially separated set of observers acting sequentially and independently [24]. It was subsequently shown that such a sequence of detections can be arbitrarily long, for both Bell nonlocality and entanglement [25,26]. While this shows that both the correlations (Bell nonlocality and entanglement) behave similarly in this scenario, there remained an open question on their behavior in the scenario where both the spatially separated observers can act sequentially on their parts of the bipartite system. The importance of this question stems, in particular, from the fact that there exist differences between detecting Bell nonlocality and entanglement. Detecting entanglement requires a larger set of assumptions than when detecting Bell nonlocality, e.g., quantum mechanics is accepted in the former whereas the latter does not require such an assumption. It is also known that there exist pairs of measurements on qubit pairs that can be used to detect entanglement if the appropriate expectation value exceeds $\sqrt{2}$, whereas one has to exceed 2 for the same in order to violate the CHSH inequality [49,50].

In this article, we analyzed the scenario of detection of entanglement of a two-qubit system by sequential and independent *pairs* of observers. We found that an arbitrary number of such pairs of observers can witness the entanglement. This result is potentially of importance, given that it has been conjectured that not more than a single pair of observers can detect CHSH Bell-nonlocal correlations in the same scenario [37–39]. Assuming that the conjecture in the literature is true, our finding demonstrates a stark distinction between "entanglement nonlocality" and "Bell nonlocality," with respect to sequential witnessing by pairs of observers.

We identified classes of quantum states which can produce arbitrarily long sequences of entangled correlations. Specifically, all pure entangled states and a certain class of CHSH Bell-nonlocal mixed entangled states are helpful in completing the task. We also showed that one can succeed in such tasks even with a class of CHSH Bell-local entangled states.

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