# Brukner-Zeilinger invariant information in the presence of conjugate symmetry

Nan Li [*] and Shunlong Luo [†]
*Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

Yuan Sun [‡]
*School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China*

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To quantify the intrinsic information content in a quantum state, Brukner and Zeilinger introduced the concept of operationally invariant information in terms of the outcome probabilities of measuring a complete set of mutually complementary observables [Č. Brukner and A. Zeilinger, Phys. Rev. Lett. **83**, 3354 (1999)]. This information quantity has basic significance and implications, and the present work is devoted to some further studies of it. We first introduce the Brukner-Zeilinger invariant information in the presence of conjugate symmetry or antisymmetry, which are motivated by considerations of fundamental issues concerning conjugate symmetry in quantum mechanics. Then we prove that both the Brukner-Zeilinger invariant information with conjugate symmetry and that with conjugate antisymmetry are convex in the quantum state, and we show that they constitute a natural decomposition of the Brukner-Zeilinger invariant information. We further relate them to the imaginarity (i.e., the usage of a complex number field) of quantum mechanics and evaluate their extreme values.

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## I. INTRODUCTION

The concepts of uncertainty and information content play a fundamental role in quantum information theory. In the conventional quantum mechanics, uncertainty is often quantified via variance and entropies, with exemplary applications in characterizing the Heisenberg uncertainty principle and correlations in composite systems [1–8]. Uncertainty and information are complementary to each other and actually constitute two sides of the same substrate.

A remarkably simple and significant measure of information is the Brukner-Zeilinger invariant information,

$$I(\rho) = \mathrm{tr}\rho^2 - \frac{1}{d}, \quad (1)$$

of a quantum state $\rho$ in a $d$-dimensional system [9–13]. This quantity has several interpretations and has found interesting applications in quantum information theory [14–18]. The purpose of this work is devoted to a further study of this fundamental quantity. We introduce some derived quantities of the Brukner-Zeilinger invariant information in the presence of conjugate symmetry and further reveal their basic properties.

As the simplest conjugation, complex conjugation is one of the most primitive ingredients in complex analysis and the foundations of Hilbert space formalism of quantum mechanics. Indeed, the very definition of the inner product (scalar product) of a complex Hilbert space relies heavily on complex conjugation, which underlines the basic notions of distance, states, and observables in the quantum realm. Various quantities in quantum theory are defined via an inner product involving complex conjugation. In general, as a kind of $Z_2$ symmetry, conjugation is intimately related to the transpose, time reversal, spin flip, parity, reflection, etc.

As is well known, Wigner characterized all reversible evolutions in quantum mechanics: Any symmetry transformation in a system Hilbert space is represented by a unitary or an antiunitary operator [19–24]. Furthermore, Wigner developed a normal form of antiunitary operators and showed that any antiunitary operator can be represented as a product form of a unitary operator and a conjugation. In this context, conjugation serves as one of the most important antiunitary operators and plays a fundamental role in quantum foundations and quantum information theory [19–26]. In particular, since conjugation is intrinsically related to time reversal, it also plays a crucial role in $\mathcal{PT}$-symmetric quantum mechanics [27–32]. Conjugation enters in the explicit formula of the entanglement of formation [33] and is also intimately related to the transpose, which is indispensable in the celebrated positive partial transpose criterion for entanglement detection [34,35]. These connections indicate that conjugation and associated quantities may be used to characterize and quantify correlations.

Motivated by the above backgrounds, it is natural and interesting to consider observables with conjugate symmetry, i.e., observables commuting with the complex conjugation relative to a fixed orthonormal basis, and study the Brukner-Zeilinger invariant information in the presence of conjugate symmetry or antisymmetry.

[*] linan@amss.ac.cn
[†] luosl@amt.ac.cn
[‡] sunyuan@njnu.edu.cn

 

The work is structured as follows. In Sec. II, we review briefly conjugate symmetry and its fundamental features. In Sec. III, we study the Brukner-Zeilinger invariant information in the presence of conjugate symmetry and its basic properties. Finally, we conclude with a summary in Sec. IV. In the Appendix, we summarize various interpretations of the Brukner-Zeilinger invariant information, which may be of independent interest.

## II. CONJUGATE SYMMETRY

In this section, we recall the conjugation operator and the decomposition of any operator into the conjugate symmetric part and the conjugate antisymmetric part, which turn out to be the projections of the operator onto the symmetric subspace and the antisymmetric subspace of the Hilbert space of observables, respectively.

Given any fixed orthonormal basis $\mathcal{B}_{\mathbb{C}} = \{|\mu\rangle : \mu = 1, \dots, d\}$ of a $d$-dimensional *complex* Hilbert space $H_{\mathbb{C}}$ with an inner (scalar) product $\langle\cdot|\cdot\rangle$ conjugate linear in the first variable and complex linear in the second variable, we may forget the complex structure and formally regard it as a $2d$-dimensional *real* Hilbert space with the orthonormal basis

$$\mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{C}} \cup i\mathcal{B}_{\mathbb{C}} = \{|\mu\rangle, i|\mu\rangle : \mu = 1, \dots, d\}.$$

To indicate the difference, we denote the corresponding real Hilbert space as $H_{\mathbb{R}}$. We emphasize that $\dim H_{\mathbb{C}} = d$ (as a complex Hilbert space) while $\dim H_{\mathbb{R}} = 2d$ (as a real Hilbert space), and as sets of vectors, $H_{\mathbb{C}}$ and $H_{\mathbb{R}}$ are the same, only the number fields are different, that is, we allow multiplication of vectors by any complex number in $H_{\mathbb{C}}$, while only multiplication by a real number is allowed in $H_{\mathbb{R}}$. The inner product in $H_{\mathbb{R}}$ is the real part of the inner product in $H_{\mathbb{C}}$. The associated conjugation $J : H_{\mathbb{R}} \to H_{\mathbb{R}}$ is a real linear isometry on $H_{\mathbb{R}}$ defined as

$$J|\mu\rangle = |\mu\rangle, \quad J(i|\mu\rangle) = -i|\mu\rangle, \quad \mu = 1, \dots, d.$$

Therefore, the matrix representation of the conjugation operator $J$ on $H_{\mathbb{R}}$ relative to the (ordered) basis $\mathcal{B}_{\mathbb{R}} = \{|1\rangle, \dots, |d\rangle, i|1\rangle, \dots, i|d\rangle\}$ is $J_{H_{\mathbb{R}}} = \mathrm{diag}(\mathbf{1}_d, -\mathbf{1}_d)$, where $\mathbf{1}_d$ denotes the $d \times d$ identity matrix. We remark that in this work the capital letters such as $J$ refer to operators and those with the subscript such as $J_{H_{\mathbb{R}}}$ and $J_{H_{\mathbb{C}}}$ refer to their corresponding matrix representations relative to the bases $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\mathbb{C}}$, respectively. Clearly, when regarded as an operator on $H_{\mathbb{C}}$, $J$ is conjugate linear (also called antilinear) in the sense that $J(c|\psi\rangle) = c^* J(|\psi\rangle)$ for any $|\psi\rangle \in H_{\mathbb{C}}$ and $c \in \mathbb{C}$. Here $c^*$ denotes the complex conjugation of $c \in \mathbb{C}$. In general,

$$J\left(\sum_{\mu=1}^{d} c_\mu |\mu\rangle\right) = \sum_{\mu=1}^{d} c_\mu^* |\mu\rangle, \quad c_\mu \in \mathbb{C}. \tag{2}$$

It is clear that $J^2 = \mathbf{1}$ and $J = J^{-1}$, where $\mathbf{1}$ denotes the identity operator on the space $H_{\mathbb{C}}$.

Given the representation of the conjugation operator $J$, we come to the decomposition of any observable $A$,

$$A = A_+ + A_-, \tag{3}$$

on the system Hilbert space $H_{\mathbb{C}}$, with

$$A_+ = \tfrac{1}{2}(A + JAJ), \quad A_- = \tfrac{1}{2}(A - JAJ),$$

satisfying

$$A_+J = JA_+ = \tfrac{1}{2}\{A, J\}, \quad A_-J = -JA_- = \tfrac{1}{2}[A, J].$$

Here $\{X, Y\} = XY + YX$ and $[X, Y] = XY - YX$ denote the anticommutator (symmetric Jordan product) and the commutator (antisymmetric Lie product) of operators, respectively. We emphasize that both $A_+$ and $A_-$ are observables on $H_{\mathbb{C}}$, while $J$, $A_+J$, and $A_-J$ are not (since they are not complex linear operators on $H_{\mathbb{C}}$). However, they are real linear operators on the real Hilbert space $H_{\mathbb{R}}$.

To establish Eq. (3), in view of the orthogonal resolution of the identity $\sum_\mu |\mu\rangle\langle\mu| = \mathbf{1}$ on $H_{\mathbb{C}}$, any linear operator $A$ on $H_{\mathbb{C}}$ can be represented as

$$A = \sum_{\mu,\nu=1}^{d} \langle\mu|A|\nu\rangle |\mu\rangle\langle\nu|,$$

which induces the matrix representation of $A$ relative to the basis $\mathcal{B}_{\mathbb{C}}$ as $A_{H_{\mathbb{C}}} = (a_{\mu\nu})$, with $a_{\mu\nu} = \langle\mu|A|\nu\rangle$. Combined with Eq. (2), one has

$$\langle\mu|JAJ|\nu\rangle = a_{\mu\nu}^*.$$

Consequently, $JAJ = \sum_{\mu\nu} a_{\mu\nu}^* |\mu\rangle\langle\nu|$, and

$$A_+ = \sum_{\mu,\nu=1}^{d} \frac{a_{\mu\nu} + a_{\mu\nu}^*}{2} |\mu\rangle\langle\nu| = \sum_{\mu,\nu=1}^{d} \mathrm{Re}(a_{\mu\nu}) |\mu\rangle\langle\nu|,$$

$$A_- = \sum_{\mu,\nu=1}^{d} \frac{a_{\mu\nu} - a_{\mu\nu}^*}{2} |\mu\rangle\langle\nu| = \sum_{\mu,\nu=1}^{d} i\mathrm{Im}(a_{\mu\nu}) |\mu\rangle\langle\nu|,$$

where $\mathrm{Re}z = (z + z^*)/2$ and $\mathrm{Im}z = (z - z^*)/(2i)$ for $z \in \mathbb{C}$. Similarly,

$$A_+ = A_+J = JA_+ = \tfrac{1}{2}\{A_+, J\},$$
$$A_- = A_-J = -JA_- = \tfrac{1}{2}[A_-, J],$$

which implies that $[A_+, J] = 0$ and $\{A_-, J\} = 0$. Hence $A_+$ is the conjugate symmetric part of $A$ in the sense that it commutes with the conjugation operator $J$, and $A_-$ is the conjugate antisymmetric part of $A$ in the sense that it anticommutes with $J$. We emphasize that $A_+$ and $A_-$ are different from the conventional real part $(A + A^\dagger)/2$ and the imaginary part $(A - A^\dagger)/2$ of $A$.

From a geometrical point of view, consider the $d^2$-dimensional Hilbert space $L(H_{\mathbb{C}})$ of observables on the $d$-dimensional system Hilbert space $H_{\mathbb{C}}$ with the Hilbert-Schmidt inner product $\langle A|B\rangle = \mathrm{tr}AB$ for observables $A$ and $B$ on $H_{\mathbb{C}}$. Noticing that for any observable $A$, $A_+$ and $A_-$ are also observables, i.e., $A_+^\dagger = A_+$, $A_-^\dagger = A_-$, and that $\langle A_+|A_-\rangle = \mathrm{tr}(A_+A_-) = 0$, we conclude that $L(H_{\mathbb{C}})$ can be decomposed as

$$L(H_{\mathbb{C}}) = L_+(H_{\mathbb{C}}) \oplus L_-(H_{\mathbb{C}}),$$

with $L_+(H_{\mathbb{C}})$ being the $d_+ = (d^2 + d)/2$-dimensional real subspace of observables with the vanishing conjugate antisymmetric part ($A_- = 0$) and $L_-(H_{\mathbb{C}})$ being the $d_- = (d^2 - d)/2$-dimensional real subspace of observables with the vanishing conjugate symmetric part ($A_+ = 0$). Thus, $A_+$ and $A_-$ are just the projections of the operator $A$ onto the real subspaces $L_+(H_\mathbb{C})$ and $L_-(H_\mathbb{C})$, respectively.

## III. BRUKNER-ZEILINGER INVARIANT INFORMATION IN THE PRESENCE OF CONJUGATE SYMMETRY OR ANTISYMMETRY

In this section, we investigate the Brukner-Zeilinger invariant information in the presence of conjugate symmetry or antisymmetry. For this aim, we first recall the interpretation of the Brukner-Zeilinger invariant information as the difference between the total variance of the maximally mixed state and that of the state $\rho$ [14].

Let $X = \{X_j : j = 1, \dots, d^2\}$ be an orthonormal basis for the real Hilbert space $L(H_\mathbb{C})$. The total variance of the set $X$ in a state $\rho$ is defined as [14]

$$V(\rho, X) = \sum_{j=1}^{d^2} V(\rho, X_j),$$

which is actually independent of the choice of the orthonormal basis $X$ and turns out to be equal to $d - \mathrm{tr}\rho^2$. For this reason, we denote by

$$V(\rho) = V(\rho, X) = d - \mathrm{tr}\rho^2 \tag{4}$$

the total variance of $\rho$. On the other hand, the total variance is just the shifted Tsallis-2 entropy in the sense that $V(\rho) = S_2(\rho) + (d-1)$, where $S_2(\rho) = 1 - \mathrm{tr}\rho^2$ is the Tsallis-2 entropy (linear entropy) of the state $\rho$ [36]. Thus, the total variance is an entropylike quantity and quantifies the uncertainty of the state $\rho$. It achieves its maximum at the maximally mixed state $\mathbf{1}/d$, i.e., $\max_\rho V(\rho) = V(\mathbf{1}/d) = d - 1/d$, and attains its minimum if and only if $\rho$ is pure, i.e., $\min_\rho V(\rho) = V(|\psi\rangle) = d - 1$, for any pure state $|\psi\rangle$.

In this context, the Brukner-Zeilinger invariant information of the state $\rho$ may be reinterpreted as the difference between the total variance of the maximally mixed state and that of the state $\rho$ [14]:

$$I(\rho) = V\left(\frac{\mathbf{1}}{d}\right) - V(\rho), \tag{5}$$

which implies a tradeoff relation between uncertainty and information,

$$V(\rho) + I(\rho) = d - \frac{1}{d}, \tag{6}$$

in view of Eqs. (1) and (4).

Based on Eq. (5) between the total variance and the Brukner-Zeilinger invariant information, before introducing the concept of the Brukner-Zeilinger invariant information in the presence of conjugate symmetry, we first discuss the total variance in the presence of conjugate symmetry.

Given an orthonormal operator basis $S = \{S_j : j = 1, \dots, d_+\}$ of the conjugate symmetric subspace $L_+(H_\mathbb{C})$ of $L(H_\mathbb{C})$ with $d_+ = (d^2 + d)/2$, the total variance of a state $\rho$ on $H_\mathbb{C}$ relative to $L_+(H_\mathbb{C})$ is defined as

$$V(\rho, S) = \sum_{j=1}^{d_+} V(\rho, S_j).$$

Now we show that $V(\rho, S)$ is independent of the choice of the basis $S$. Let $Y = \{Y_j : j = 1, \cdots, d_+\}$ be another orthonormal basis of $L_+(H_\mathbb{C})$. Suppose that

$$S_i = \sum_{j=1}^{d_+} t_{ij} Y_j, \quad i = 1, \dots, d_+,$$

with $(t_{ij})$ being a real orthogonal matrix satisfying $\sum_i t_{ij} t_{ij'} = \sum_i t_{ji} t_{j'i} = \delta_{jj'}$, $j, j' = 1, \dots, d_+$, then

$$\sum_{i=1}^{d_+} \mathrm{tr}\rho S_i^2 = \sum_{j,j'=1}^{d_+} \left( \sum_{i=1}^{d_+} t_{ij} t_{ij'} \right) \mathrm{tr}(\rho Y_j Y_{j'}) = \sum_{j=1}^{d_+} \mathrm{tr}\rho Y_j^2.$$

Similarly,

$$\sum_{i=1}^{d_+} (\mathrm{tr}\rho S_i)^2 = \sum_{j=1}^{d_+} (\mathrm{tr}\rho Y_j)^2,$$

and consequently,

$$\sum_{i=1}^{d_+} V(\rho, S_i) = \sum_{j=1}^{d_+} V(\rho, Y_j),$$

which implies that $V(\rho, S)$ is independent of the choice of the basis $S$. Thus we may denote it by $V_+(\rho) = V(\rho, S)$. Similarly, the total variance of $\rho$ relative to the conjugate antisymmetry subspace $L_-(H_\mathbb{C})$ can be defined as

$$V_-(\rho) = V(\rho, T) = \sum_{i=1}^{d_-} V(\rho, T_i),$$

with $T = \{T_i : i = 1, \dots, d_-\}$ being any orthonormal operator basis of the conjugate antisymmetric subspace $L_-(H_\mathbb{C})$ of $L(H_\mathbb{C})$ and $d_- = (d^2 - d)/2$. This quantity is also independent of the basis choice of $L_-(H_\mathbb{C})$. $V_+(\rho)$ and $V_-(\rho)$ can be interpreted as the uncertainty of the state $\rho$ relative to the conjugate symmetric subspace $L_+(H_\mathbb{C})$ and the conjugate antisymmetric subspace $L_-(H_\mathbb{C})$, respectively. It can be evaluated that

$$V_+(\rho) = \frac{d+1}{2} - \sum_{\mu,\nu} (\mathrm{Re}\rho_{\mu\nu})^2 = \frac{d+1}{2} - ||\rho_+||^2, \tag{7}$$

$$V_-(\rho) = \frac{d-1}{2} - \sum_{\mu,\nu} (\mathrm{Im}\rho_{\mu\nu})^2 = \frac{d-1}{2} - ||\rho_-||^2. \tag{8}$$

Here $\|A\|^2 = \mathrm{tr}A^\dagger A$ denotes the squared Hilbert-Schmidt norm of an operator $A$ on $H_\mathbb{C}$.

To establish the above relations, we choose the orthonormal operator basis $A = \{A_{\mu\nu} : \mu, \nu = 1, \dots, d\}$ of $L(H_\mathbb{C})$, with

$$A_{\mu\nu} = \begin{cases} |\mu\rangle\langle\mu|, & \mu = \nu, \\ \frac{1}{\sqrt{2}}(|\mu\rangle\langle\nu| + |\nu\rangle\langle\mu|), & \mu > \nu, \\ \frac{i}{\sqrt{2}}(|\mu\rangle\langle\nu| - |\nu\rangle\langle\mu|), & \mu < \nu. \end{cases}$$

It is easy to verify that $\{A_{\mu\nu} : \mu \geqslant \nu\}$ and $\{A_{\mu\nu} : \mu < \nu\}$ constitute a basis of $L_+(H_\mathbb{C})$ and that of $L_-(H_\mathbb{C})$, respectively. Consequently,

$$V_+(\rho) = \sum_{\mu\geqslant\nu} V(\rho, A_{\mu\nu}), \quad V_-(\rho) = \sum_{\mu<\nu} V(\rho, A_{\mu\nu}).$$

Direct calculations show that

$$V(\rho, A_{\mu\nu}) = \begin{cases} \rho_{\mu\mu} - \rho_{\mu\mu}^2, & \mu = \nu, \\ \frac{1}{2}(\rho_{\mu\mu} + \rho_{\nu\nu}) - 2(\mathrm{Re}\rho_{\mu\nu})^2, & \mu > \nu, \\ \frac{1}{2}(\rho_{\mu\mu} + \rho_{\nu\nu}) - 2(\mathrm{Im}\rho_{\mu\nu})^2, & \mu < \nu, \end{cases}$$

which imply Eqs. (7) and (8). Furthermore, we know that $V_+(\rho)$ and $V_-(\rho)$ achieve their maximum at $\rho = \mathbf{1}/d$. That is, the maximally mixed state is the maximal uncertainty state with respect to both $L_+(H_\mathbb{C})$ and $L_-(H_\mathbb{C})$.

With the above derivations, we naturally obtain a decomposition of the total variance $V(\rho)$ as follows.

*Proposition 1.* The total variance $V(\rho)$ of a state $\rho$ can be decomposed into the variance $V_+(\rho)$ [relative to the conjugate symmetric subspace $L_+(H_\mathbb{C})$] and the variance $V_-(\rho)$ [relative to the conjugate antisymmetric subspace $L_-(H_\mathbb{C})$], i.e.,

$$V(\rho) = V_+(\rho) + V_-(\rho). \tag{9}$$

We remark that for a given orthonormal operator basis $X = \{X_i : i = 1, \dots, d^2\}$ of the space $L(H_\mathbb{C})$, $V_+(\rho)$ and $V_-(\rho)$ can also be expressed as

$$V_+(\rho) = \sum_{i=1}^{d^2} V(\rho, X_{i+}), \quad V_-(\rho) = \sum_{i=1}^{d^2} V(\rho, X_{i-}),$$

where $X_{i+}$ and $X_{i-}$ denote the conjugate symmetric and antisymmetric parts of $X_i$, respectively.

With the above preparation, now we proceed to study Brukner-Zeilinger invariant information in the presence of conjugate symmetry. Following the relation between the total variance and the Brukner-Zeilinger invariant information in Eq. (5), it is natural to introduce

$$I_+(\rho) = V_+\left(\frac{\mathbf{1}}{d}\right) - V_+(\rho),\ I_-(\rho) = V_-\left(\frac{\mathbf{1}}{d}\right) - V_-(\rho), \tag{10}$$

which are called the Brukner-Zeilinger invariant information in the presence of conjugate symmetry and antisymmetry, respectively. It can be evaluated that

$$I_+(\rho) = \left\|\rho_+ - \frac{\mathbf{1}}{d}\right\|^2, \quad I_-(\rho) = \|\rho_-\|^2.$$

Actually, $I_-(\rho)$ turns out to coincide with the $l_2$-norm measure of imaginarity of the state $\rho$ [37].

Since $I_+(\rho)$ and $I_-(\rho)$ can also be interpreted as the information of $\rho$ relative to the conjugate symmetric subspace $L_+(H_\mathbb{C})$ and the conjugate antisymmetric subspace $L_-(H_\mathbb{C})$, respectively, similar trade-off relations between uncertainty and information as in Eq. (6) can be directly established as

$$V_+(\rho) + I_+(\rho) = V_+\left(\frac{\mathbf{1}}{d}\right) = \frac{d+1}{2} - \frac{1}{d}, \tag{11}$$

$$V_-(\rho) + I_-(\rho) = V_-\left(\frac{\mathbf{1}}{d}\right) = \frac{d-1}{2}. \tag{12}$$

*Proposition 2.* The Brukner-Zeilinger invariant information $I(\rho)$ defined by Eq. (1) can be decomposed into two parts as

$$I(\rho) = I_+(\rho) + I_-(\rho). \tag{13}$$

Moreover,

(i) $I_+(\rho)$ and $I_-(\rho)$ are convex in $\rho$.

(ii) $I_+(\rho)$ and $I_-(\rho)$ are invariant under orthogonal transformation, i.e., for any orthogonal operator $O$ on $H_\mathbb{C}$, $I_+(O\rho O^\dagger) = I_+(\rho)$ and $I_-(O\rho O^\dagger) = I_-(\rho)$.

(iii) It holds that

$$0 \leqslant I_+(\rho) = \|\rho_+\|^2 - \frac{1}{d} \leqslant 1 - \frac{1}{d},$$

$$0 \leqslant I_-(\rho) = \|\rho_-\|^2 \leqslant \frac{1}{2}.$$

Now we sketch the proof of the above properties. First, Eq. (13) follows readily from Proposition 1 and Eq. (10).

For item (i), the convexity of both $I_+(\rho)$ and $I_-(\rho)$ follows directly from the convexity of $\mathrm{tr}\rho^2$ in $\rho$ and the linearity of $\rho_+$ and $\rho_-$ in $\rho$.

Item (ii) follows from the unitary invariance of the Brukner-Zeilinger invariant information $I(\rho)$ and the fact that, for any operator $A$ and any orthogonal operator $O$, $(OAO^\dagger)_+ = OA_+O^\dagger$ and $(OAO^\dagger)_- = OA_-O^\dagger$.

For item (iii), since $||\rho_+||^2 \leqslant ||\rho||^2 \leqslant 1$ and $||(|\mu\rangle\langle\mu|)_+||^2 = 1$ for $\mu = 1, \dots, d$, we obtain $\max_\rho ||\rho_+||^2 = 1$. On the other hand, by

$$\left\|\left(\rho - \frac{\mathbf{1}}{d}\right)_+\right\|^2 = \|\rho_+\|^2 - \frac{1}{d} \geqslant 0,$$

we know that $\min_\rho ||\rho_+||^2 = 1/d$. From Eq. (11) we get

$$\frac{d-1}{2} \leqslant V_+(\rho) \leqslant \frac{d+1}{2} - \frac{1}{d}. \tag{14}$$

For the minimum of $I_-(\rho)$, by noting that $I_-(\rho)$ is just the $l_2$-norm measure of imaginarity for the state $\rho$, it is clear that the minimum of $I_-(\rho)$ is 0 and that, for any state $\rho$ satisfying $\rho_- = 0$, its Brukner-Zeilinger invariant information in the presence of conjugate antisymmetry vanishes, i.e., $I_-(\rho) = 0$. Thus, such a $\rho$ has the maximal uncertainty relative to the subspace $L_-(H_\mathbb{C})$ by Eq. (12), i.e., $V_-(\rho) = (d-1)/2$.

To evaluate the maximum of $I_-(\rho)$, we first note that, by use of the convexity of $I_-(\rho)$, the maximum of $I_-(\rho)$ is achieved by pure states. Given the orthonormal basis $\{|\mu\rangle : \mu = 1, \dots, d\}$ of the system Hilbert space, let $|\psi\rangle = \sum_{\mu=1}^d \alpha_\mu|\mu\rangle$ be an arbitrary pure state. Let $\alpha_\mu = |\alpha_\mu|e^{i\theta_\mu}$, with $\sum_\mu |\alpha_\mu|^2 = 1$, then $\rho = |\psi\rangle\langle\psi| = \sum_{\mu,\nu} \alpha_\mu\alpha_\nu^*|\mu\rangle\langle\nu|$, and

$$\begin{aligned} ||\rho_-||^2 &= \sum_{\mu,\nu=1}^d [\mathrm{Im}(\alpha_\mu\alpha_\nu^*)]^2 \\ &= \sum_{\mu,\nu=1}^d |\alpha_\mu|^2|\alpha_\nu|^2 \sin^2(\theta_\mu - \theta_\nu) \\ &= \sum_{\mu,\nu=1}^d |\alpha_\mu|^2|\alpha_\nu|^2 \frac{1 - \cos 2(\theta_\mu - \theta_\nu)}{2} \end{aligned}$$

$$= \frac{1}{2}\left(\sum_{\mu=1}^{d}|\alpha_\mu|^2\right)^2 - \frac{1}{2}\left(\sum_{\mu=1}^{d}|\alpha_\mu|^2\cos 2\theta_\mu\right)^2 - \frac{1}{2}\left(\sum_{\mu=1}^{d}|\alpha_\mu|^2\sin 2\theta_\mu\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\left|\sum_{\mu=1}^{d}|\alpha_\mu|^2 e^{2i\theta_\mu}\right|^2$$

$$= \frac{1}{2} - \frac{1}{2}\left|\sum_{\mu=1}^{d}\alpha_\mu^2\right|^2 \leqslant \frac{1}{2},$$

where in the last relation the equality holds if and only if $|\sum_{\mu=1}^{d}\alpha_\mu^2| = 0$. The states with this property have the maximal Brukner-Zeilinger invariant information with conjugate antisymmetry and thus are the minimal uncertainty states with respect to the subspace $L_-(H_\mathbb{C})$. They are also the maximally imaginary states [37]. Actually, there are infinite states satisfying $|\sum_{\mu=1}^{d}\alpha_\mu^2| = 0$, such as $|\psi\rangle = \sum_{\mu=1}^{d} e^{i\theta_\mu}|\mu\rangle/\sqrt{d}$, with $|\alpha_\mu|^2 = 1/d$ and $\theta_\mu = \mu\pi/d$ for $\mu = 1, \dots, d$.

Until now we have established the properties of $I_-(\rho)$ in item (iii). By Eq. (12) we get the upper and lower bounds for $V_-(\rho)$, i.e.,

$$\frac{d}{2} - 1 \leqslant V_-(\rho) \leqslant \frac{d-1}{2}.$$

Therefore, combining Eq. (14) we know that $V_-(\rho) \leqslant V_+(\rho)$ for any state $\rho$.

To gain a more concrete and intuitive understanding of the Brukner-Zeilinger invariant information in the presence of conjugate symmetry or antisymmetry, we consider the simple qubit case. Suppose the qubit space $H_\mathbb{C} = \mathbb{C}^2$ has the computational basis $\{|0\rangle, |1\rangle\}$ (eigenvectors of the third Pauli operator $\sigma_z$, i.e., $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$). The conjugation (relative to the computational basis) is defined as

$$J(a|0\rangle + b|1\rangle) = a^*|0\rangle + b^*|1\rangle, \quad a, b \in \mathbb{C},$$

which is an antilinear isometry on the qubit system Hilbert space $\mathbb{C}^2$. Any qubit state has the following Bloch representation, $\rho = (\mathbf{1} + \boldsymbol{r}\cdot\sigma)/2$, with the Bloch vector $\boldsymbol{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$, $|\boldsymbol{r}|^2 = r_x^2 + r_y^2 + r_z^2 \leqslant 1$, $\mathbf{1}$ being the identity operator, and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ being the vector consisting of the Pauli matrices. In this context of Bloch representation,

$$J\rho J = \tfrac{1}{2}(\mathbf{1} + r_x\sigma_x - r_y\sigma_y + r_z\sigma_z) \tag{15}$$

implements the conjugation or the transpose of the density operator, i.e., the reflection about the $xz$ plane, and transforms the Bloch vector $\boldsymbol{r} = (r_x, r_y, r_z)$ to $(r_x, -r_y, r_z)$. From straightforward calculation we obtain

$$I(\rho) = \tfrac{1}{2}\left(r_x^2 + r_y^2 + r_z^2\right),\ I_+(\rho) = \tfrac{1}{2}\left(r_x^2 + r_z^2\right),\ I_-(\rho) = \tfrac{1}{2}r_y^2,$$

which is intuitive in view of Eq. (15).

## IV. SUMMARY

Given a computational basis of a finite-dimensional complex Hilbert space, a natural conjugation arises, relative to which any observable on the system Hilbert space can be decomposed into a conjugate symmetric part and a conjugate antisymmetric part. Correspondingly, the real Hilbert space of observables can be written as the direct sum of the subspaces consisting of all observables with vanishing conjugate symmetric parts and those with vanishing conjugate antisymmetric parts, respectively. Along this line, we decompose the total variance and the Brukner-Zeilinger invariant information of a state into the conjugate symmetric parts and the conjugate antisymmetric parts. These quantities may have applications in analyzing informational aspects of quantum systems.

The Brukner-Zeilinger invariant information was proposed as an operationally invariant quantifier that reflects the intrinsic information of the underlying system [9]. It is desirable to

TABLE I. Interpretations of Brukner-Zeilinger invariant information $I(\rho)$. The various quantities are defined as follows: (i) $V(\rho) = V(\rho, X) = \sum_{j=1}^{d^2} V(\rho, X_j) = d - \mathrm{tr}\rho^2$, where $X = \{X_j;\ j = 1, \dots, d^2\}$ is any orthonormal basis for the operator Hilbert space $L(H_\mathbb{C})$ and $V(\rho, X_j) = \mathrm{tr}\rho X_j^2 - (\mathrm{tr}\rho X_j)^2$ is the conventional variance. (ii) $||A||^2 = \mathrm{tr}A^\dagger A$. (iii) $S_2(\rho) = 1 - \mathrm{tr}\rho^2$ is the Tsallis-2 entropy [36]. (iv) $I(\rho, \Pi) = V(\mathbf{1}/d, \Pi) - V(\rho, \Pi)$ and $V(\rho, \Pi) = \sum_{i=1}^{d} V(\rho, \Pi_i)$ for any von Neumann measurement $\Pi = \{\Pi_i;\ i = 1, \dots, d\}$, $U\Pi U^\dagger = \{U\Pi_i U^\dagger\!: i = 1, \dots, d\}$, with $U$ being any unitary operator. The integration is with respect to the normalized Haar measure $dU$ on the unitary group of a $d$-dimensional complex Hilbert space. (v) The index of coincidence is defined as $C(\rho, N) = \sum_{j=1}^{d^2} p_j^2 = (\mathrm{tr}\rho^2 + 1)/[d(d+1)]$, with $p_j = \mathrm{tr}\rho N_j$ for any symmetric informationally complete positive operator valued measure (SIC-POVM) $N = \{N_j\!: j = 1, \dots, d^2\}$ [15,39]. (vi) $V(\rho, N) = \sum_{j=1}^{d^2} V(\rho, N_j)$. (vii) The variance of a state $\rho$ in an observable $K$ was defined as $V_K(\rho) = \mathrm{tr}K(\rho - \mathrm{tr}\rho K)^2$ [40,41].

| Interpretation of $I(\rho)$ | Expression of $I(\rho)$ |
|---|---|
| Difference of variance | $V(\mathbf{1}/d) - V(\rho)$ |
| Hilbert-Schmidt distance | $\|\|\rho - \mathbf{1}/d\|\|^2$ |
| Average noncommutativity | $\sum_{j=1}^{d^2} \|\|[\rho, X_j]\|\|^2/(2d)$ |
| Difference of Tsallis-2 entropy | $S_2(\mathbf{1}/d) - S_2(\rho)$ |
| Integration over unitary group | $(d+1)\int I(\rho, U\Pi U^\dagger)dU$ |
| Difference of coincidence index of SIC-POVM | $d(d+1)[C(\rho, N) - C(\mathbf{1}/d, N)]$ |
| Difference of variance relative to SIC-POVM | $d(d+1)[V(\mathbf{1}/d, N) - V(\rho, N)]$ |
| Variance of state in $\mathbf{1}/d$ | $dV_{\mathbf{1}/d}(\rho)$ |
| Average variance of state | $(d+1)\int V_{U\|\phi\rangle\langle\phi\|U^\dagger}(\rho)dU$ |

investigate the operational meaning of the Brukner-Zeilinger invariant information in the presence of conjugate symmetry or antisymmetry and to characterize them as the extractable information under the condition that the available observables are limited to be conjugate symmetry and conjugate antisymmetry, respectively. This may shed light on our understanding of the concrete difference between the quantum mechanics on a real Hilbert space and the conventional quantum mechanics (on a complex Hilbert space) [38].

Since conjugation plays a fundamental role in quantum mechanics, we hope the present work may be helpful in quantitative studies of conjugation from the perspective of information and uncertainty.

Finally, the connections of this work with coherence and asymmetry relative to $Z_2$ symmetry (including parity, time reversal, conjugation, spin flip, etc.) also deserve further study.

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## APPENDIX

In view of the significance and many facets of the Brukner-Zeilinger invariant information $I(\rho) = \mathrm{tr}\rho^2 - 1/d$, we summarize its various interpretations in Table I.


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