Nonlocality under uncertainty-disturbance relations and self-duality

Liang-Liang Sun,[1] Yong-Shun Song,[2] Sixia Yu,[1,*] and Zeng-Bing Chen[3,†]
[1]Hefei National Research Center for Physical Sciences at the Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China
[2]Department of Physics, East China University of Science and Technology, Shanghai 200237, China
[3]School of Physics, University of Nanjing, Hankou Road 22, Nanjing, Jiangsu 210093, China

(Received 25 April 2022; accepted 31 August 2022; published 15 September 2022)

We exploit the interplay between uncertainty, disturbance, nonlocality, and self-duality within the framework of generalized probabilistic theories (GPTs). We first introduce an operational uncertainty-disturbance relation that reveals the discrepancies of GPTs in uncertainty and disturbance exhibited by one single measurement. We then apply the relation to Bell's scenarios and derive upper bounds for the basic spatial Clauser-Horne-Shimony-Holt and the temporal Leggett-Garg inequalities with and without the assumption of strong self-duality, respectively. It turns out that self-duality, though being thought of as one pillar of the axiomatization of quantum theory and implying additional constraints, is insufficient to explain the quantum violations. These insights may serve as a seed for a universal understanding of these basic properties in GPTs.

DOI: 10.1103/PhysRevA.106.032213

I. INTRODUCTION

Quantum theory shows many counterintuitive features, such as intrinsic uncertainty, measurement-disturbance relation, self-duality, and nonlocality. In a wider framework of generalized probabilistic theories (GPTs) [1–7], these features may also be shared by theories having dramatically different mathematical structures [8–13]. A fundamental topic is finding physical principles that can single out quantum theory and provide an interpretation for it [2,6,7,13–19]. This topic greatly depends on how well one can understand these nonclassical features.

The investigation of this topic often starts with finding information-theoretic principles for interpreting the quantum violations of Bell's inequality [10,20–25]. It is motivated by an essential observation that, whereas quantum correlations can violate Bell's inequality, they do not reach the maximum extent allowed by the nonsignaling principle [26]. Taking the basic spatial Clauser-Horne-Shimony-Holt (CHSH) inequality [27], for example, the maximum quantum violation is known as the Tsirelson bound ($2\sqrt{2}$) [28], which is strictly less than the maximum violation (4) by the Popescu-Rohrlich (PR) box [26]. This gap can be interpreted with the restricted capability of quantum correlations in communication and computation [10,20–25]. From a different perspective, the gap can also be accounted for with the local properties exhibited by an individual subsystem, such as the uncertainty principle [29–32], local quantum mechanics [33], and self-duality [12]. While these results have provided valuable insights, we still lack a universal understanding of many aspects. For example, current efforts are commonly devoted to accounting for the quantum bounds for spatial Bell inequalities. It is not known what principle can prohibit quantum violations from reaching the maximum algebraic violation for the temporal Bell inequalities, such as the Leggett-Garg (LG) inequality [34]. Another aspect, one important conjecture that bridges the Tsirelson bound and self-duality, stating that the latter ensures the former, is based on an analysis of the polygon model [12]. It is not known what the case is for other GPTs. Still, although there is much evidence indicating a universal understanding of nonclassical properties [4,12,29–33], their interplay remains elusive.

In this paper we pursue such a universal understanding. As is known, any nonlocal GPT must exhibit intrinsic uncertainty and a measurement-disturbance relation [35]. This motivates us to introduce an uncertainty-disturbance relation (UDR). The relation states that, up to a factor referred to as an uncertainty-disturbance factor (UDF), the uncertainty of a maximal measurement is no less than its disturbance effect in a following measurement. The UDF is defined as a constant for one specific GPT, but may vary depending on theories. Thus, it can distinguish GPTs from each other and acts as a benchmark parameter. The UDR implies a strong constraint on the correlations and ensures upper bounds for the famous CHSH and temporal LG inequalities in terms of the functions of the UDF. The derivations allow us to consider the constraint of self-duality. We find that self-duality is insufficient for explaining the quantum violations. Our results thus provide an alternative perspective allowing one to universally investigate uncertainty, disturbance, strong self-duality, and violations of Bell's inequalities.

II. FRAMEWORK OF GPTS

The framework of GPTs [1,2,4–7] provides a standard background that enables one to operationally investigate physical properties without using a specific mathematical structure. It employs the essential concepts with clear

*yusixia@ustc.edu.cn
†zbchen@nju.edu.cn

operational meaning, such as state, measurement, and transformation, which are prerequisites for any theory.

State. Physical state $\mathcal{S}$ is a description of one system that can give one prediction for any potential measurement. The collection of all possible states defines one state space Ω_s, and the collection of unnormalized states $\lambda\mathcal{S}$ with $\lambda \geqslant 0$ defines a frequently useful concept of state cone Ω_s^+. It is reasonable to assume that the spaces are convex. Because of that, preparing one system in $\mathcal{S}_1$ with probability p and in a different state $\mathcal{S}_2$ with $1-p$ definitely leads to another valid mixed state $p\mathcal{S}_1 + (1-p)\mathcal{S}_2$. Conversely, the state that cannot be written as a convex combination of different states is called one pure state.

Measurement. Information of a physical state can be extracted by performing a measurement. A measurement $\mathcal{M}$ is described with a set of effects $\{e_i\}$ and each effect corresponds to an outcome. Acting an effect e_i on state $\mathcal{S}$ yields a probability $p(e_i|\mathcal{S})$, namely, the occurrence probability of the outcome i. The set of all the possible effects defines an effect space Ω_e, and the set of unnormalized effects assuming the form λe with $\lambda \geqslant 0$ defines an effect cone Ω_e^+. An effect is said to be extremal if it cannot be decomposed as a convex combination of different ones. If all the effects of a measurement are normalized extremal effects, the measurement is called a maximal measurement [6,7], where normalized means there exists at least one state such that the effect can occur with a probability of unity.

Pure states and extremal effects define the state and the effect spaces and are the building blocks of a GPT. One popular assumption about them is the logical sharpness (LS) assumption [6,7]

Logical sharpness. For each pure state there is one and only one normalized extremal effect such that one can get unity probability.

With this assumption, one can naturally define the postmeasurement state for each extremal effect as the corresponding pure state. This LS assumption ensures weak self-duality, namely, the state and the effect cones are isomorphic, with the isomorphism being defined as $\Gamma : \Omega_s^+ \to \Omega_e^+ : \sum_i p_i \cdot \mathcal{S}_A^i \to \sum_i p_i \cdot \mathcal{A}^i$, where Γ simply maps the ith pure element state $\mathcal{S}_A^i$ to the corresponding extremal effect $\mathcal{A}^i$. Strong self-duality is stronger than the weak self-duality as it requires additionally that the cones are canonically isomorphic [12,36].

Strong self-duality. A system is strongly self-dual if and only if there exists an isomorphism $T : \Omega_s^+ \to \Omega_e^+$ which is symmetric and positive semidefinite, namely, $p(e_i|T(e_j)) = p(e_j|T(e_i)) \geqslant 0$ for all effects e_i and e_j.

Letting $T = \Gamma$, the symmetry in the strong self-duality is equivalent to a symmetric condition concerning pure states and extremal effects

$$p\big(\mathcal{A}_k^i\big|\mathcal{S}_{A_l}^j\big) = p\big(\mathcal{A}_l^j\big|\mathcal{S}_{A_k}^i\big)\,\forall i, k, j, l, \tag{1}$$

where $\mathcal{A}_n^m$ specifies the mth extremal effect of a maximal measurement $\mathcal{A}_n$ and $\mathcal{S}_{A_n}^m$ the corresponding pure state. This equivalence can be readily verified by decomposing the concerned effects and states into pure element states and extremal effects. Strong self-duality is one pillar of the axiomatization of quantum theory [15,16,36]. It is found that finite-dimensional homogeneous, strongly self-dual cones are precisely the cones of positive elements of formally real Jordan algebras, which comes very close to quantum theory [15,16]. It is worth stressing that there are still other strongly self-dual models going beyond quantum theory, such as the polygon model [12].

In quantum theory, pure states and extremal effects are represented as rank-1 projectors. The strong self-duality immediately follows from the Born rule $p(\mathcal{A}_k^i|\mathcal{S}_{A_l}^j) = p(\mathcal{A}_l^j|\mathcal{S}_{A_k}^i) = |\langle A_k^i|A_l^j\rangle|^2$, where $|A_n^m\rangle$ specifies the mth eigenstate of observable $\mathcal{A}_n$. The set of $\{|\langle A_k^i|A_l^j\rangle|^2\}$ is more restrictive as the elements formulate a unistochastic matrix [37].

III. UNCERTAINTY AND DISTURBANCE

Classical theory rests on the assumptions of realism and locality, which imply constraint on correlations in terms of Bell's inequalities [38]. Besides quantum theory, a wide range of GPTs can violate Bell's inequalities and exhibit nonlocality and they all ensure the presentation of the intrinsic uncertainty and disturbance effect [35]. In this section, we first provide a UDR to capture their connection.

The UDR considers one basic measurement scenario: A system is prepared in state $\mathcal{S}$, which is subject to a sequential measurement scheme $\mathcal{A}_0 \to \mathcal{A}_1$, where the measurements are maximal. The previous measurement $\mathcal{A}_0$ gives outcomes with a distribution $\mathbf{p} := \{p(\mathcal{A}_0^i|\mathcal{S})\}$ and outputs an averaged postmeasurement state $\bar{\mathcal{S}}$, which then is subject to another following maximal measurement $\mathcal{A}_1$ and a disturbed distribution $\mathbf{q}' := \{p(\mathcal{A}_1^j|\bar{\mathcal{S}})\}$ is obtained. If not disturbed, directly measuring $\mathcal{A}_1$ on $\mathcal{S}$ yields a distribution $\mathbf{q} = \{p(\mathcal{A}_1^j|\mathcal{S})\}$.

Uncertainty. We quantify the uncertainty of $\mathcal{A}_0$ by

$$\Delta_{A_0} = \|\mathbf{p}\|_{1/2} - 1, \tag{2}$$

where $\|\mathbf{p}\|_{1/2} := [\sum_i \sqrt{p(\mathcal{A}_0^i|\mathcal{S})}]^2$ is the $\frac{1}{2}$-norm of the distribution $\mathbf{p}$ and relates to $\frac{1}{2}$ Rényi entropy via $H_{1/2}(\mathbf{p}) = \log \|\mathbf{p}\|_{1/2}$, which is a well-established uncertainty measure that has been used to investigate uncertainty principle [39] quantum randomness [40]. In addition, Δ_{A_0} is Schur concave and equivalent to $H_{1/2}(\mathbf{p})$ in the sense that $H_{1/2}(\mathbf{p}) = \log(\Delta_{A_0} + 1)$ and thus defines a legitimate uncertainty measure.

Disturbance effect. The distribution of $\mathbf{q}$ may be different from the disturbed distribution $\mathbf{q}'$. The difference quantifies the disturbance effect caused by $\mathcal{A}_0$ in $\mathcal{A}_1$. One popular disturbance measure is classical trace distance [31,41]

$$D_{A_0\to A_1} = \sum_j \big|q\big(\mathcal{A}_1^j\big|\mathcal{S}\big) - q'\big(\mathcal{A}_1^j\big|\bar{\mathcal{S}}\big)\big|. \tag{3}$$

Uncertainty and disturbance are two distinct features of one single measurement, which may relate to each other. As some instructive illustration, we first consider classical theory and quantum theory. Classically, measuring a quantity does not disturb a system's state but may show nontrivial uncertainty as the physical system can be prepared in a mixed state. In quantum theory, uncertainty is a prerequisite for the nontrivial disturbance effect. Because of that, zero uncertainty implies that the system is in the eigenstate of the measurement that would not be disturbed. Within GPTs, we capture the connection by introducing an operational UDR in terms of Δ_A and $D_{A_0\to A_1}$, where a parameter of the UDF is incorporated such that one can define a UDR for any theory allowing the definition of maximal measurement.

Uncertainty-disturbance relation. Up to a factor α, which is referred to as a UDF, one maximal measurement's uncertainty is no less than its disturbance effect

$$\Delta_{A_0} \geqslant \alpha D_{A_0\to A_1}, \tag{4}$$

where α is the maximal modification allowed such that the logic still holds:

$$\alpha = \sup\{\alpha' \geqslant 0 | \Delta_{A_0} \geqslant \alpha' D_{A_0\to A_1} \forall \mathcal{A}_0, \mathcal{A}_1, \mathcal{S}\}. \tag{5}$$

A GPT can violate the UDR for another theory having a greater UDF. The UDF thus serves as a benchmark parameter revealing the discrepancy of GPTs in terms of the relation between uncertainty and disturbance. Now we give the UDFs for some theories off the shelf.

Classical theory. In classical theory, a classical and maximal measurement introduces zero disturbance while it may exhibit nontrivial uncertainty. Therefore, $\Delta_{A_0} \geqslant 0$ and $D_{A_0\to A_1} = 0$, which implies $\alpha_c = \infty$.

Quantum theory. The UDF of quantum theory is $\alpha_{\rm qm} = 1$ (see the Appendix for the proof).

Popescu-Rohrlich box. [26] The PR-box can violate the CHSH inequality to the maximum extent. It is defined in the basic Bell scenario, where a bipartite system is distributed to spacelike separated observers, say, Alice and Bob, who can randomly measure one of $\{A_0, A_1\}$ and $\{B_0, B_1\}$, respectively, on the received particles and obtain outcome $a, b \in \{0, 1\}$ with a joint probability $p(a, b|A_\mu, B_\nu)$ $(\mu, \nu \in \{0, 1\})$. The PR box reads

$$p(a, b|A_\mu, B_\nu) = \frac{1 + (-1)^{a+b+\mu\nu}}{4}. \tag{6}$$

Without any loss of generality, since the measurements are spacelike separated and placed in some reference frame, one may assume that Bob first performs a measurement $\mathcal{B}_\nu$, obtains b with probability $p_{\nu_b} = \sum_a p(ab|A_\mu, B_\nu)$ (nonsignaling condition), and then simultaneously a conditional state, described by ω_{ν_b}, is prepared on Alice's side. By the definition of the PR box, we have

$$p\big(\mathcal{A}_\mu^a \big| \omega_{\nu_b}\big) = \delta_{a,b+\mu\nu}. \tag{7}$$

Thus, the uncertainty of measurement on Alice's side is always zero, namely, $\Delta_{A_\mu}(\omega_{\nu_b}) = 0$. However, measurement disturbance should not be zero, i.e., $D_{A_0\to A_1} > 0$. Otherwise, the nonsingling principle is violated [35]. Therefore, $\alpha_{\rm PR} = 0$, i.e., the PR-box model, is in another extremal point opposite to the classical theory.

It needs to be stressed that the PR box does not hold for the LS assumption. Because of that, for the effect $\mathcal{A}_1^a$ there are two states ω_{ν_b} with $b + \nu = a$ such that one gets a unit probability.

IV. BELL'S INEQUALITY AND UNCERTAINTY-DISTURBANCE RELATION

In this section we explore the connection between the UDR, self-duality, and nonlocality under the LS assumption. We focus on the case of $\alpha \leqslant 1$, which includes the quantum case and the ones going beyond it. We will derive the upper bounds for the CHSH and the LG inequalities.

A. Unbias assumption and strong self-duality

For a comparison with strong self-duality, we also consider a weaker assumption, namely, the unbias assumption.

Unbias assumption. Subjecting a white state $\mathcal{W} = \frac{1}{d}\sum_i \mathcal{S}_{A_0}^i$, which is prepared by randomly mixing eigenstates corresponding to an arbitrary maximal measurement $\mathcal{A}_0$, to another maximal measurement $\mathcal{A}_1$ yields a uniform distribution

$$p\big(\mathcal{A}_1^j|\mathcal{W}\big) = \frac{1}{d} \,\forall \mathcal{A}_1, \mathcal{A}_0. \tag{8}$$

This unbias assumption is strictly weaker than strong self-duality. Note that the self-duality implies the unbiased assumption $p(\mathcal{A}_1^j|\mathcal{W}) = \sum_i \frac{1}{d} p(\mathcal{A}_1^j|\mathcal{S}_{A_0}^i) = \sum_i \frac{1}{d} p(\mathcal{A}_0^i|\mathcal{S}_1^j) = \frac{1}{d}$, where we have used Eq. (1) in the second equality and the normalization condition in the third equality. It is easy to see that the "strictnesses" represented as $p(\mathcal{A}_0^i|\mathcal{S}_{A_1}^j)$ and $p(\mathcal{A}_1^j|\mathcal{S}_{A_0}^i)$ are independent under the unbias assumption, while they are equal to each other for strong self-duality.

The unbias assumption enables us to recast the UDR into an easily handled representation. We focus on the primary binary case and adhere to the notions $\gamma = p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) - p(\mathcal{A}_1^1|\mathcal{S}_{A_0}^0)$ and $\tau = p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^0) - p(\mathcal{A}_0^1|\mathcal{S}_{A_1}^0)$, with $\langle A_\mu\rangle = p(\mathcal{A}_\mu^0|\mathcal{S}) - p(\mathcal{A}_\mu^1|\mathcal{S})$. Under the unbias condition, the UDR in the scenario $\mathcal{A}_0 \to \mathcal{A}_1$ is recast in the form (see the Appendix)

$$(1 + \alpha^2\gamma^2)\langle A_0\rangle^2 + \alpha^2\langle A_1\rangle^2 - 2\gamma\alpha^2\langle A_0\rangle\langle A_1\rangle \leqslant 1. \tag{9}$$

Similarly, considering measurements in order $\mathcal{A}_1 \to \mathcal{A}_0$, we obtain a dual UDR as

$$(1 + \alpha^2\tau^2)\langle A_1\rangle^2 + \alpha^2\langle A_0\rangle^2 - 2\tau\alpha^2\langle A_1\rangle\langle A_0\rangle \leqslant 1. \tag{10}$$

For the strong self-duality condition, the two inequalities are subject to an additional constraint $\gamma = \tau$.

We note that γ and τ are determined solely by the settings of $\mathcal{A}_0$ and $\mathcal{A}_1$ and are irrelevant to the initial state. For given γ and τ, the two inequalities put constraints on the statistics of measurements $\mathcal{A}_0$ and $\mathcal{A}_1$ by bounding their expectation values $\langle A_{0,1}\rangle$ and thus capture the idea of the uncertainty relation. This implies that one can always define the uncertainty relations of the forms (9) and (10) for a GPT allowing sharp measurement and holding the LS and the unbias assumptions.

B. Violation of CHSH inequality under the UDR

The CHSH inequality deals with the simplest Bell scenario and holds by the local and realism theory. It is given as

$$I_{\rm CHSH} := \sum_{a,b,\mu,\nu=0}^{1} (-1)^{a+b+\mu\nu} p(a, b|A_\mu, B_\nu) \leqslant 2. \tag{11}$$

We can express $p(a, b|A_\mu, B_\nu) = p_{\nu_b} \cdot p(\mu_a|\nu_b)$, with $p_{\nu_b} = \sum_a p(a, b|A_\mu, B_\nu)$ the statistics on Bob's side and $p(\mu_a|\nu_b) = \frac{p(a,b|A_\mu,B_\nu)}{p_{\nu_b}}$ the statistics coming from measurement on the conditional state ω_{ν_b}. By the decomposition, the CHSH inequality can be reformulated as

$$\sum_{b,\nu} (-1)^{b+\nu} p_{\nu_b}\big[\langle A_0\rangle_{\nu_b} + (-1)^\nu \langle A_1\rangle_{\nu_b}\big] \leqslant 2, \tag{12}$$

where $\langle A_\mu\rangle_{\nu_b} = p(\mu_0|\nu_b) - p(\mu_1|\nu_b)$. Equation (9) and its dual relation imply $|\langle A_0\rangle \pm \langle A_1\rangle| \leqslant 2\sqrt{f(\alpha, \pm\gamma, \pm\tau)}$ (see the Appendix) and

$$f(\alpha,\gamma,\tau) = \frac{\alpha^2(\tau^2+\gamma^2-2)+2}{[\alpha^2(1-\gamma)^2+1][\alpha^2(\tau^2-1)+1]+[\alpha^2(1-\tau)^2+1][\alpha^2(\gamma^2-1)+1]}.$$

Then we have (see the Appendix)

$$\begin{aligned} I_{\text{CHSH}} &\leqslant \sum_{b,\nu}(-1)^{b+\nu}p_{\nu_b}\sqrt{f(\alpha,(-1)^\nu\gamma,(-1)^\nu\tau)} \\ &\leqslant 2\max_{-1\leqslant\gamma,\tau\leqslant 1}[\sqrt{f(\alpha,\gamma,\tau)}+\sqrt{f(\alpha,-\gamma,-\tau)}] \\ &:= n_u(\alpha). \end{aligned} \tag{13}$$

In the second inequality, we have performed an optimization over γ and τ and obtained an upper bound in terms of a function of α as $n_u(\alpha)$. Under the strong self-duality, i.e., $\gamma = \tau$, we have another bound as

$$\begin{aligned} I_{\text{CHSH}} &\leqslant 2\max_{\gamma}\left(\sqrt{\frac{1}{\alpha^2(1+\gamma)^2+1}}+\sqrt{\frac{1}{\alpha^2(1-\gamma)^2+1}}\right) \\ &:= n_{us}(\alpha). \end{aligned} \tag{14}$$

By numerical calculation, we find that $n_u(\alpha)$ coincides with $n_{us}(\alpha)$ for all α, and a few typical examples of $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ are given in the Appendix. We show $n_u(\alpha)$ in Fig. 1(a).

Specifically, we are also interested in the bound obtained when the measurements are maximum incompatible, i.e., $p(A_\mu^i|\mathcal{S}_{A_{\bar\mu}}^j) = \frac{1}{2}$ and $\gamma = \tau = 0$. We then have $f(\alpha,0,0) = \frac{1}{1+\alpha^2}$, which implies

$$I_{\text{CHSH}} \leqslant \frac{4}{\sqrt{\alpha^2+1}} := n_{us,\gamma=0}(\alpha). \tag{15}$$

We show $n_u(\alpha)$ and $n_{us,\gamma=0}(\alpha)$ in Fig. 1(a). Specifically, letting $\alpha = \alpha_{\text{qm}} = 1$, we have $n_u(1) \approx 2.93$ and $n_{us,\gamma=0}(1) = 2\sqrt{2}$, which is the Tsirelson bound. As $\alpha \to 0$, $n_u(\alpha)$ and $n_{us,\gamma=0}(\alpha)$ tend to the maximum violation of 4.

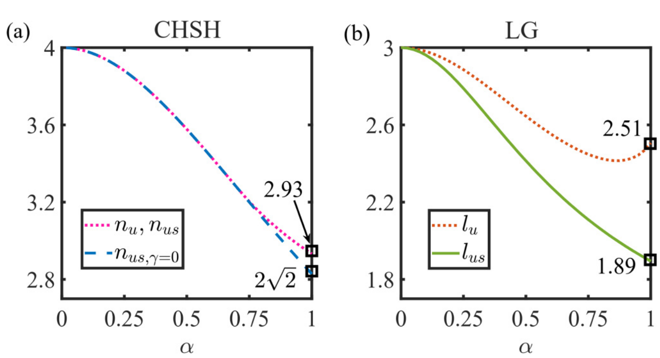

FIG. 1. The UDR implies constraints on the violations of (a) the CHSH and (b) the LG inequalities in terms of functions of the balance strength. For the CHSH inequality, n_u, which coincides with n_{us} (pink dotted line), is the upper bound under the unbias assumption and $n_{ns,\gamma=0}$ (blue dashed line) is the upper bound under the strong self-duality and the maximum incompatibility assumption. For the LG inequality l_u (orange dotted line) is the upper bound under the UDR and unbias assumption and l_{us} (green solid line) is the upper bound under the UDR and the strong self-duality assumption

Figure 1(a) shows that the UDF, though defined by local properties of a single measurement, implies strong constraint on the Bell nonlocality. We also show that the strong self-duality does not provide a tighter constraint than the unbias assumption and is insufficient to account for the Tsirelson bound.

C. Bound of the Leggett-Garg inequality under the UDR

In this section we show that the UDR can also apply to the temporal Bell scenario, which deals with correlations arising from sequential measurements performed on one system at different times. The LG inequality is the simplest temporal Bell inequality and deals with three binary measurements $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ and reads

$$\langle A_1A_2\rangle_s + \langle A_2A_3\rangle_s - \langle A_1A_3\rangle_s \leqslant 1, \tag{16}$$

where $\langle A_1A_2\rangle_s$ is the correlation between measurement $\mathcal{A}_1$ and a following measurement $\mathcal{A}_2$ on the same particle, and the other terms of the inequality are defined likewise. We note that $\langle A_1A_2\rangle_s$ is uniquely determined by γ if $\mathcal{A}_1$ and $\mathcal{A}_2$ are maximal

$$\begin{aligned} \langle A_1A_2\rangle_s &= \sum_{i=j} p(\mathcal{A}_1^i|\mathcal{S})p(\mathcal{A}_2^j|\mathcal{S}_{A_1}^i) \\ &\quad - \sum_{i\neq j} p(\mathcal{A}_1^i|\mathcal{S})p(\mathcal{A}_2^j|\mathcal{S}_{A_1}^i) \\ &= 2p(\mathcal{A}_2^0|\mathcal{S}_{A_1}^0) - 1 = \gamma_{12}, \end{aligned}$$

where we have used the normalization condition and the unbias assumption, which implies, for example, $p(\mathcal{A}_2^0|\mathcal{S}_{A_1}^0) = 1 - p(\mathcal{A}_2^1|\mathcal{S}_{A_1}^0) = p(\mathcal{A}_2^1|\mathcal{S}_{A_1}^1) = 1 - p(\mathcal{A}_2^0|\mathcal{S}_{A_1}^1)$. The temporal correlation is irrelevant to the input state. Without loss of generality, we let the initial state $\mathcal{S} = \mathcal{S}_{A_1}^0$, which leads to a zero uncertainty for measurement $\mathcal{A}_1$. By the UDR, measuring $\mathcal{A}_1$ on the state does not disturb the following measurement. Thus, $\langle A_1A_\mu\rangle_s = \langle A_\mu\rangle$ and the LG inequality can be written as

$$\langle A_2\rangle + \gamma_{23} - \langle A_3\rangle \leqslant 1. \tag{17}$$

Noting that $|\langle A_2\rangle - \langle A_3\rangle| \leqslant f(\alpha, -\gamma_{23}, -\tau_{23})$, then the LG inequality has an upper bound as

$$l_u(\alpha) := \max_{-1\leqslant\gamma_{23},\tau_{23}\leqslant 1}[2\sqrt{f(\alpha,-\gamma_{23},-\tau_{23})}+\gamma_{23}]. \tag{18}$$

Under the strong self-duality assumption, the LG inequality is upper bounded by

$$l_{us}(\alpha) := \max_{-1\leqslant\gamma_{23}\leqslant 1}[2\sqrt{f(\alpha,-\gamma_{23},-\gamma_{23})}+\gamma_{23}]. \tag{19}$$

The two upper bounds, namely, $l_u(\alpha)$ and $l_{us}(\alpha)$, are shown in Fig. 1(b). Specifically, for $\alpha = \alpha_{\rm qm} = 1$, $l_u(1) = 2.51$ and $l_{us}(1) = 1.89$, which is close to the quantum violation of the Lüders bound $\frac{3}{2}$. Here the strong self-duality, though being significantly stronger than the unbiased assumption, is insufficient for interpreting the quantum violation.

V. CONCLUSION

Searching for a deep understanding of quantum theory based on the law of physics is a key task in the field of quantum foundation. This topic relies on how well one can operationally understand the specialties of the nonclassical phenomena. In this paper we have provided an uncertainty-disturbance relation that allows us to comprehensively study uncertainty, disturbance, self-duality, and Bell's nonlocality. By the UDF, we provided a dimension that can distinguish quantum theory from other GPTs. We also showed that the UDR implies constraint on correlations in both the spatial and the temporal Bell scenarios. We identified the effects by providing the upper bounds for the famous CHSH and LG inequalities in terms of functions of the UDF, where we accounted for the contributions of strong self-duality.

ACKNOWLEDGMENTS

This work was supported by the Chinese Academy of Sciences, the National Natural Science Foundation of China under Grant No. 61125502, and the National Fundamental Research Program under Grant No. 2011CB921300.

APPENDIX

1. Proof of $\alpha_{\rm qm} = 1$

In quantum theory, the maximal measurement is a rank-1 projection measurement. Suppose that a quantum state is ρ. After a projection measurement $\mathcal{A}_0 = \{\hat{p}_i = |A_0^i\rangle\langle A_0^i|\}$, the state is brought into a completely decohered state $\bar{\rho} = \sum_i |A_0^i\rangle\langle A_0^i|\rho|A_0^i\rangle\langle A_0^i| = \sum_i p(\mathcal{A}_0^i|\rho)\cdot|A_0^i\rangle\langle A_0^i|$. Specify $\{q(\mathcal{A}_1^j|\rho)\}$ and $\{q'(\mathcal{A}_1^j|\bar{\rho})\}$ as the distributions obtained by performing $\mathcal{A}_1 := \{|A_1^j\rangle\langle A_1^j|\}$ on ρ and $\bar{\rho}$, respectively. We have

$$\begin{aligned} D_{A\to A'} &= \sum_j \left|q(\mathcal{A}_1^j|\rho) - q'(\mathcal{A}_1^j|\bar{\rho})\right| \\ &= \mathrm{Tr}|\Phi_{A_1}(\rho) - \Phi_{A_1}(\bar{\rho})| \\ &\leqslant \mathrm{Tr}|\rho - \bar{\rho}| = \mathrm{Tr}\left|\sum_{k<l}\sigma_{kl}\right| \\ &\leqslant \sum_{k<l}\mathrm{Tr}|\sigma_{kl}| = \sum_{k<l} 2\left|\langle A_0^k|\rho|A_0^l\rangle\right| \\ &\leqslant \sum_{k<l} 2\sqrt{p(A_0^k|\rho)p(A_0^l|\rho)} = \Delta_{A_0}, \end{aligned} \tag{A1}$$

where $\mathrm{Tr}|X| := \mathrm{Tr}\sqrt{X^\dagger X}$ and $\Phi_{A_1}(\cdot) := \sum_j |A_1^j\rangle\langle A_1^j|\cdot|A_1^j\rangle\langle A_1^j|$ is dephasing map with respect to measurement $\mathcal{A}_1$, $\sigma_{kl} := |A_0^k\rangle\langle A_0^k|\rho|A_0^l\rangle\langle A_0^l| + |A_0^l\rangle\langle A_0^l|\rho|A_0^k\rangle\langle A_0^k|$ for $k \neq l$, and we have used the data processing inequality in the first inequality and convexity of the trace-norm in the second inequality. Then the above proof implies that $\alpha_{\rm qm} \geqslant 1$.

To see that $\alpha_{\rm qm} = 1$ for quantum theory, we show that the UDR can be saturated by preparing ρ in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\mathcal{A}_0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and $\mathcal{A}_1 = \{|+\rangle\langle +|, |-\rangle\langle -|\}$. Then $p(A_0^0|\rho) = p(A_0^1|\rho) = \frac{1}{2}$ and we have $\{q(\mathcal{A}_1^+|\rho) = 1, q(\mathcal{A}_1^-|\rho) = 0\}$ and $\{q(\mathcal{A}_1^+|\bar{\rho}) = \frac{1}{2}, q(\mathcal{A}_1^-|\bar{\rho}) = \frac{1}{2}\}$. In addition, we have $\Delta_{A_0} = 1 = D_{A\to A'}$, meaning any UDR with $\alpha > 1$ would be violated by quantum theory. Then the proof for $\alpha_{\rm qm} = 1$ is completed.

2. Proof of Eqs. (9) and (10)

According to the LS assumption, for the ith outcome of maximal measurement $\mathcal{A}_0$, the postmeasurement state is updated into the $\mathcal{S}_{A_0}^i$. Then, after measurement $\mathcal{A}_0$, on average, the postmeasurement $\bar{\mathcal{S}}$ would be a convex combination of the eigenstates $\bar{\mathcal{S}} = \sum_i p(A_0^i|\mathcal{S})\cdot\mathcal{S}_{A_0}^i$. A following measurement $\mathcal{A}_1$ on $\bar{\mathcal{S}}$ gives a disturbed distribution as

$$q'(\mathcal{A}_1^j|\bar{\mathcal{S}}) = \sum_i p(\mathcal{A}_0^i|\mathcal{S})\cdot p(\mathcal{A}_1^j|\mathcal{S}_{A_0}^i). \tag{A2}$$

We refer to $p(\mathcal{A}_1^j|\mathcal{S}_{A_0}^i)$ and $p(\mathcal{A}_0^i|\mathcal{S}_{A_1}^j)$ as transition probabilities following the specification in quantum theory. Consider the two binary measurements $\mathcal{A}_0$ and $\mathcal{A}_1$ and their eigenstates. If without any assumption the transition probabilities are subject to the normalization condition only and there are four independent transition probabilities, namely, $p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0)$, $p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^1)$, and their duals $p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^0)$ and $p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^1)$, by the unbias condition

$$\begin{aligned} p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) + p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^1) &= 1, \\ p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^0) + p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^1) &= 1, \end{aligned}$$

namely, the independent transition probabilities reduce to $p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0)$ and $p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^0)$. Under the assumption of strongly self-duality, $p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) = p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^0)$ and the number of independent transition probabilities reduces further to one.

Let us explore the UDR under the unbias assumption. Consider the scenario $\mathcal{A}_0 \to \mathcal{A}_1$ and adhere to the notions $\gamma = p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) - p(\mathcal{A}_1^1|\mathcal{S}_{A_0}^0) = 2p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) - 1$ and $\langle A_\mu\rangle = p(\mathcal{A}_\mu^0|\mathcal{S}) - p(\mathcal{A}_\mu^1|\mathcal{S}) = 2p(\mathcal{A}_\mu^0|\mathcal{S}) - 1$. We have

$$\begin{aligned} q'(\mathcal{A}_1^0|\bar{\mathcal{S}}) &= \sum_{i=0,1} p(\mathcal{A}_0^i|\mathcal{S})\cdot p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^i) \\ &= p(\mathcal{A}_0^0|\mathcal{S})\cdot p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) + p(\mathcal{A}_0^1|\mathcal{S})p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^1) \\ &= p(\mathcal{A}_0^0|\mathcal{S})\cdot p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) \\ &\quad + [1 - p(\mathcal{A}_0^0|\mathcal{S})]\cdot[1 - p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0)] \\ &= [2p(\mathcal{A}_1^0|\mathcal{S}_{A_0}^0) - 1]\left[p(\mathcal{A}_1^0|\mathcal{S}) - \frac{1}{2}\right] + \frac{1}{2} \\ &= \frac{1}{2}\gamma\langle A_0\rangle + \frac{1}{2}. \end{aligned}$$

Then the disturbance in the main text simplifies to

$$\begin{aligned} D_{A_0\to A_1} &= \sum_{j=0,1} |q(\mathcal{A}_1^j|\mathcal{S}) - q'(\mathcal{A}_1^j|\bar{\mathcal{S}})| \\ &= 2|q(\mathcal{A}_1^0|\mathcal{S}) - q'(\mathcal{A}_1^0|\bar{\mathcal{S}})| \\ &= 2\left|p(\mathcal{A}_1^0|\mathcal{S}) - \frac{1}{2} - \frac{1}{2}\gamma\langle A_0\rangle\right| \\ &= |\langle A_1\rangle - \gamma\langle A_0\rangle|. \end{aligned}$$

Noting that $\Delta_{A_0} = 2\sqrt{p(A_0^0|\mathcal{S})p(A_0^1|\mathcal{S})} = \sqrt{1-\langle A_0\rangle^2}$, we can rewrite the UDR as

$$\sqrt{1-\langle A_0\rangle^2} \geqslant \alpha|\langle A_1\rangle - \gamma\langle A_0\rangle|.$$

Squaring the inequality, we have

$$(1+\alpha^2\gamma^2)\langle A_0\rangle^2 + \alpha^2\langle A_1\rangle^2 - 2\gamma\alpha^2\langle A_0\rangle\langle A_1\rangle \leqslant 1. \quad \text{(A3)}$$

The dual UDR for the case $\mathcal{A}_0 \to \mathcal{A}_1$ is obtained likewise

$$(1+\alpha^2\tau^2)\langle A_1\rangle^2 + \alpha^2\langle A_0\rangle^2 - 2\tau\alpha^2\langle A_1\rangle\langle A_0\rangle \leqslant 1, \quad \text{(A4)}$$

where $\tau = 2p(\mathcal{A}_0^0|\mathcal{S}_{A_1}^0) - 1$.

3. Proof of Eq. (13)

Defining $u = \langle A_0\rangle + \langle A_1\rangle$ and $v = \langle A_0\rangle - \langle A_1\rangle$, we can rewrite Eqs. (A3) and (A4) as

$$4 \geqslant u^2[\alpha^2(1-\gamma)^2+1] + v^2[\alpha^2(1+\gamma)^2+1] + 2uv[\alpha^2(\gamma^2-1)+1], \quad \text{(A5)}$$

$$4 \geqslant u^2[\alpha^2(1-\tau)^2+1] + v^2[\alpha^2(1+\tau)^2+1] - 2uv[\alpha^2(\tau^2-1)+1]. \quad \text{(A6)}$$

By eliminating uv terms we obtain

$$4 \geqslant \frac{u^2}{f(\alpha,\gamma,\tau)} + \frac{v^2}{f(\alpha,-\gamma,-\tau)}, \quad \text{(A7)}$$

with

$$f(\alpha,\gamma,\tau) = \frac{\alpha^2(\tau^2+\gamma^2-2)+2}{[\alpha^2(1-\gamma)^2+1][\alpha^2(\tau^2-1)+1] + [\alpha^2(1-\tau)^2+1][\alpha^2(\gamma^2-1)+1]}. \quad \text{(A8)}$$

Equation (A7) implies constraints on u and v, namely, $|\langle A_0\rangle \pm \langle A_1\rangle|$ as

$$|\langle A_0\rangle \pm \langle A_1\rangle| \leqslant 2\sqrt{f(\alpha,\pm\gamma,\pm\tau)}. \quad \text{(A9)}$$

In this expression, $f(\alpha,\pm\gamma,\pm\tau)$ are formulated with transition probabilities and determined solely with the choice of $\mathcal{A}_0$ and $\mathcal{A}_1$. However, the left-hand side is formulated with $\mathcal{A}_{0,1}$'s expectation values on $\mathcal{S}$. Thus, the UDR establishes constraints on $|\langle A_0\rangle \pm \langle A_1\rangle|$ in terms of transition probabilities, which can bound quantum nonlocality as

$$\begin{aligned} I_{\text{CHSH}} &\leqslant \sum_{b,\nu=0}^{1} p_{\nu_b}|\langle A_0\rangle_{\nu_b} + (-1)^\nu\langle A_1\rangle_{\nu_b}| \\ &\leqslant \sum_{b,\nu=0}^{1} p_{\nu_b} 2\sqrt{f(\alpha,(-1)^\nu\gamma,(-1)^\nu\tau)} \\ &\leqslant \max_{\gamma,\tau} 2\sqrt{f(\alpha,\gamma,\tau)} + 2\sqrt{f(\alpha,-\gamma,-\tau)} \\ &\equiv n_u(\alpha), \end{aligned} \quad \text{(A10)}$$

where the second inequality is due to applying Eq. (A9) to the state ω_{ν_b}. For a given α a typical diagram of the upper bound $n_u(\alpha)$ of I_{CHSH} as a symmetric function of γ and τ is plotted in Fig. 2, showing that the maximal values are attained in the case of $\gamma = \tau$.

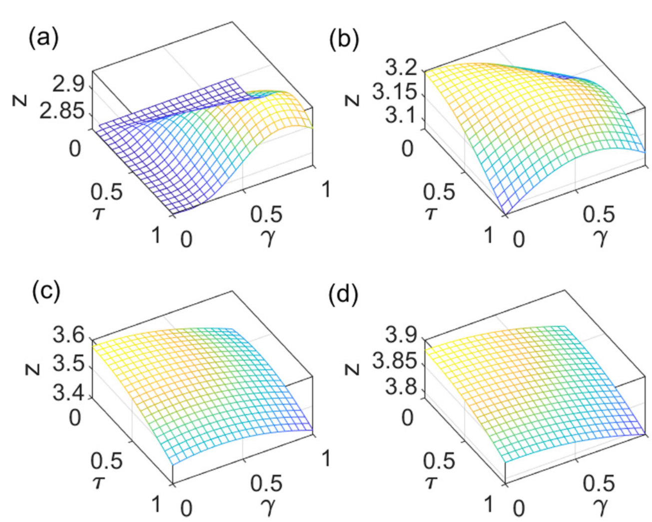

FIG. 2. Plot of $N \equiv 2\sqrt{f(\alpha,\gamma,\tau)} + 2\sqrt{f(\alpha,-\gamma,-\tau)}$ as a function of γ and τ with (a) $\alpha = 1$, (b) $\alpha = \frac{3}{4}$, (c) $\alpha = \frac{1}{2}$, and (d) $\alpha = \frac{1}{4}$. We see clearly that the maximum is always attained at $\gamma = \tau$.

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