Adiabatic theorem independent of the quantum representations: Resolution of the Marzlin-Sanders inconsistency

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We construct the adiabatic theorem independent of the quantum representations. The quantum-representation-independent geometrical phase, adiabatic condition, and fidelity of the adiabatic approximation are derived. As an example, we apply the quantum-representation-independent adiabatic theorem to provide a thorough resolution of the Marzlin-Sanders inconsistency. We show that the Marzlin-Sanders inconsistency results from the wrong identification of the time-dependent Hamiltonian as the instantaneous energy operator and the quantum representation dependence of the Marzlin-Sanders transformation.

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I. INTRODUCTION

The adiabatic theorem [1,2] is one of the most important results in quantum mechanics, and has found interesting applications in quantum field theory [3], geometric phase [4,5], quantum control [6], and quantum computation [7]. The theorem states that if the Hamiltonian $\hat{H}(t)$ evolves slowly enough, then the system prepared in the nth eigenstate of $\hat{H}(0)$ will evolve approximately to the adiabatic state $|\psi_{\rm adi}(t)\rangle$:

$$|\psi(t)\rangle \approx |\psi_{\rm adi}(t)\rangle. \tag{1}$$

The adiabatic state $|\psi_{\rm adi}(t)\rangle$ is given by

$$|\psi_{\rm adi}(t)\rangle = \exp\left[i\alpha_n(t)\right]|n(t)\rangle. \tag{2}$$

with

$$\alpha_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt' + i\int_0^t \langle n(t')|\frac{d}{dt'}|n(t')\rangle dt'. \tag{3}$$

Here, $|n(t)\rangle$ is the nth eigenstate of $\hat{H}(t)$ with eigenvalue $E_n(t)$. The second term in (3) is the geometrical phase [4,5]. The fidelity of the adiabatic approximation is

$$|\langle\psi_{\rm adi}(t)|\psi(t)\rangle| = |\langle n(t)|\hat{U}(t,0)|n(0)\rangle|, \tag{4}$$

where $\hat{U}(t,0)$ is the time evolution operator satisfying

$$i\hbar\frac{d}{dt}\hat{U}(t,0) = \hat{H}(t)\hat{U}(t,0). \tag{5}$$

The traditional adiabatic condition which assures the validity of the adiabatic approximation is

$$\left|\frac{\hbar\langle n(t)|\frac{d}{dt}|m(t)\rangle}{E_n(t)-E_m(t)}\right| \ll 1, \quad m \neq n. \tag{6}$$

In recent years, the adiabatic theorem has been a subject of controversy. In their interesting work, Marzlin and Sanders [8] pointed out that the application of the adiabatic theorem leads to an inconsistency. The Marzlin-Sanders inconsistency can be demonstrated in the following way due to Tong *et al.* [9]: Consider a system $S^{(a)}$ described by the Hamiltonian $\hat{H}^{(a)}(t)$ and the time evolution operator $\hat{U}^{(a)}(t,0)$. The state of the system is denoted by $|\psi^{(a)}(t)\rangle$. The eigenvalues and eigenstates of $\hat{H}^{(a)}(t)$ are denoted by $E_n^{(a)}(t)$ and $|n^{(a)}(t)\rangle$, respectively. Assume that $S^{(a)}$ is prepared in $|n^{(a)}(0)\rangle$ and fulfills the adiabatic condition (6). The adiabatic theorem guarantees that the fidelity of the adiabatic approximation for $S^{(a)}$ is approximately equal to one:

$$|\langle n^{(a)}(t)|\hat{U}^{(a)}(t,0)|n^{(a)}(0)\rangle| \approx 1. \tag{7}$$

One can always construct a dual system $S^{(b)}$ via the Marzlin-Sanders transformation:

$$\hat{H}^{(b)}(t) = -\hat{U}^{(a)\dagger}(t,0)\hat{H}^{(a)}(t)\hat{U}^{(a)}(t,0). \tag{8}$$

The state of the system $S^{(b)}$ is denoted by $|\psi^{(b)}(t)\rangle$. The eigenvalues and the eigenstates of $\hat{H}^{(b)}(t)$ are denoted by $E_n^{(b)}(t)$ and $|n^{(b)}(t)\rangle$, respectively. It follows from (8) that there is a one-to-one correspondence between the eigenvalues and the eigenstates of $\hat{H}^{(a)}(t)$ and $\hat{H}^{(b)}(t)$:

$$E_n^{(b)}(t) = -E_n^{(a)}(t), \tag{9}$$

$$|n^{(b)}(t)\rangle = \hat{U}^{(a)\dagger}(t,0)|n^{(a)}(t)\rangle. \tag{10}$$

In addition, it follows from (5) and (8) that

$$\hat{U}^{(b)}(t,0) = \hat{U}^{(a)\dagger}(t,0), \tag{11}$$

where $\hat{U}^{(b)}(t,0)$ is the time evolution operator for $S^{(b)}$. Equations (5), (9), and (10) lead to

$$\left|\frac{\langle n^{(b)}(t)|\frac{d}{dt}|m^{(b)}(t)\rangle}{E_n^{(b)}(t)-E_m^{(b)}(t)}\right| = \left|\frac{\langle n^{(a)}(t)|\frac{d}{dt}|m^{(a)}(t)\rangle}{E_n^{(a)}(t)-E_m^{(a)}(t)}\right| \ll 1, \quad m \neq n, \tag{12}$$

which implies that $S^{(b)}$ also fulfills the adiabatic condition (6). The adiabatic theorem guarantees that the fidelity of the adiabatic approximation for $S^{(b)}$ is approximately equal to

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one:

$$\begin{aligned}\left|\langle\psi_{adi}^{(b)}(t)|\psi^{(b)}(t)\rangle\right| &= |\langle n^{(b)}(t)|\hat{U}^{(b)}(t,0)|n^{(b)}(0)\rangle| \\ &= |\langle n^{(a)}(t)|n^{(a)}(0)\rangle| \\ &\approx 1. \end{aligned} \tag{13}$$

In the second equals sign of (13), Eqs. (10) and (11) have been used. However, conditions (7) and (13) may not hold simultaneously. This indicates that the application of the adiabatic theorem leads to an inconsistency.

Extensive investigations [9–14] have been performed to unveil the origin of the Marzlin-Sanders inconsistency. The current consensus is that the adiabatic theorem is beyond any dispute. The inconsistency arises from the fact that the traditional adiabatic condition (6) is not sufficient for the adiabatic approximation. Many attempts [15–19] have been dedicated to the search for new adiabatic conditions. Our viewpoint is that the Marzlin-Sanders inconsistency is more subtle and a deeper understanding of its origin is required.

Motivated by the need of deeper insight into the origin of the Marzlin-Sanders inconsistency, in this paper, we construct the adiabatic theorem independent of the quantum representations. The quantum-representation-independent geometrical phase, adiabatic condition, and fidelity of the adiabatic approximation are derived. As an example, we show that this quantum-representation-independent adiabatic theorem provides a thorough resolution of the Marzlin-Sanders inconsistency.

II. QUANTUM REPRESENTATION INDEPENDENCE

We provide a concise elucidation of the quantum representation independence. An introduction of the quantum representation independence can be found in Ref. [20]. The discussion is not specific to the system made up of particles interacting with the electromagnetic field. For simplicity, we consider a particle with one degree of freedom on the x axis. Physical variables that can be measured are functions of x and $\dot{x}$. For example, the instantaneous energy of a particle is given by

$$E(x,\dot{x},t) = \tfrac{1}{2}m\dot{x}^2 + V(x,t), \tag{14}$$

where $V(x,t)$ is the potential energy. The standard Lagrangian, which is a function of x and $\dot{x}$, is given by

$$L^{(0)}(x,\dot{x},t) = \tfrac{1}{2}m\dot{x}^2 - V(x,t). \tag{15}$$

The canonical momentum with respect to the standard Lagrangian $L^{(0)}$ is

$$p_{L^{(0)}} = \frac{\partial L^{(0)}}{\partial \dot{x}} = m\dot{x}, \tag{16}$$

which coincides with the mechanical momentum π. In the Hamiltonian formalism, the dynamical variables are x and $P_{L^{(0)}}$. The value of the instantaneous energy is then a function $\varepsilon_{L^{(0)}}(x, p_{L^{(0)}}, t)$. The value of $\varepsilon_{L^{(0)}}(x, p_{L^{(0)}}, t)$ should be identical to that of $E(x,\dot{x},t)$. In view of (14) and (16), we have

$$\varepsilon_{L^{(0)}}(x, p_{L^{(0)}}, t) = \frac{p_{L^{(0)}}^2}{2m} + V(x,t). \tag{17}$$

The Hamiltonian, which is a function of x and $p_{L^{(0)}}$, is defined by

$$H_{L^{(0)}}(x, p_{L^{(0)}}, t) = \dot{x}p_{L^{(0)}} - L^{(0)}(x,\dot{x},t). \tag{18}$$

It follows from (15)–(18) that

$$H_{L^{(0)}}(x, p_{L^{(0)}}, t) = \frac{p_{L^{(0)}}^2}{2m} + V(x,t) = \varepsilon_{L^{(0)}}(x, p_{L^{(0)}}, t). \tag{19}$$

Hence the Hamiltonian associated with the standard Lagrangian $L^{(0)}$ coincides with the instantaneous energy.

One gets equivalent Lagrangians by adding to $L^{(0)}$ the derivative of functions $F^{(i)}(x,t)$ [20]:

$$\begin{aligned} L^{(i)}(x,\dot{x},t) &= L^{(0)}(x,\dot{x},t) + \frac{d}{dt}F^{(i)}(x,t) \\ &= L^{(0)}(x,\dot{x},t) + \dot{x}\frac{\partial}{\partial x}F^{(i)}(x,t) + \frac{\partial}{\partial t}F^{(i)}(x,t). \end{aligned} \tag{20}$$

The gauge transformation is a special case of this type of transformation. The transformation (20) does not affect the equation of motion. However, it does affect the canonical momentum:

$$p_{L^{(i)}} = \frac{\partial L^{(i)}}{\partial \dot{x}} = m\dot{x} + \frac{\partial}{\partial x}F^{(i)}(x,t). \tag{21}$$

For the equivalent Lagrangian $L^{(i)}$, the value of $\varepsilon_{L^{(i)}}(x, p_{L^{(i)}}, t)$ should also be identical to that of $E(x,\dot{x},t)$. Therefore we have

$$\varepsilon_{L^{(i)}}(x, p_{L^{(i)}}, t) = \varepsilon_{L^{(0)}}(x, p_{L^{(0)}}, t). \tag{22}$$

The physical predictions with regard to the energy are independent of the Lagrangian chosen. In light of (14) and (21), we have

$$\varepsilon_{L^{(i)}}(x, p_{L^{(i)}}, t) = \frac{\left(p_{L^{(i)}} - \frac{\partial F^{(i)}}{\partial x}\right)^2}{2m} + V(x,t). \tag{23}$$

The Hamiltonian associated with $L^{(i)}$ is given by

$$\begin{aligned} H_{L^{(i)}}(x, p_{L^{(i)}}, t) &= \dot{x}p_{L^{(i)}} - L^{(i)}(x,\dot{x},t) \\ &= H_{L^{(0)}}(x, p_{L^{(0)}}, t) - \frac{\partial}{\partial t}F^{(i)}(x,t). \end{aligned} \tag{24}$$

In the second equals sign of (24), Eqs. (18), (20), and (21) have been used. Hence the value of the Hamiltonian depends on the Lagrangian chosen. Variables whose values are independent of the Lagrangian chosen are called physical variables [20]. Examples of physical variables are x, π, and ε. Equations (21) and (24) show that p and H are not physical variables. In view of (19), (22), and (24), we have

$$H_{L^{(i)}}(x, p_{L^{(i)}}, t) = \varepsilon_{L^{(i)}}(x, p_{L^{(i)}}, t) - \frac{\partial}{\partial t}F^{(i)}(x,t). \tag{25}$$

Hence the Hamiltonian is in general different from the instantaneous energy.

The quantum mechanical operators $\hat{x}$ and $\hat{p}$ are independent of the Lagrangian chosen. In the position representation, they are given by $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. The quantization of other variables is implemented by replacing x by $\hat{x}$ and p by $\hat{p}$ in the functions representing the classical variables considered. If one applies the quantization rule to classical variables starting with the standard Lagrangian $L^{(0)}$, one gets the standard quantum representation in which $\hat{p}_{L^{(0)}} = -i\hbar\frac{\partial}{\partial x}$. In view of (17) and (19), the instantaneous energy and the

Hamiltonian are represented by the same operator:

$$\hat{\varepsilon}^{(0)}(\hat{x}, \hat{p}_{L^{(0)}}, t) = \hat{H}^{(0)}(\hat{x}, \hat{p}_{L^{(0)}}, t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t). \tag{26}$$

The state of the system is represented by the state vector $|\psi^{(0)}(t)\rangle$. The time evolution of $|\psi^{(0)}(t)\rangle$ is determined by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi^{(0)}(t)\rangle = \hat{H}^{(0)}(\hat{x}, \hat{p}_{L^{(0)}}, t)|\psi^{(0)}(t)\rangle. \tag{27}$$

On the other hand, if one applies the quantization rule to classical variables starting with the Lagrangian $L^{(i)}$, one gets an equivalent quantum representation in which $\hat{p}_{L^{(i)}} = -i\hbar\frac{\partial}{\partial x}$. In view of (23) and (25), the instantaneous energy and the Hamiltonian are represented by the operators

$$\hat{\varepsilon}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t) = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} - \frac{i}{\hbar}\frac{\partial F^{(i)}}{\partial x}\right)^2 + V(x,t), \tag{28}$$

$$\hat{H}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t) = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} - \frac{i}{\hbar}\frac{\partial F^{(i)}}{\partial x}\right)^2 + V(x,t) - \frac{\partial F^{(i)}}{\partial t}, \tag{29}$$

respectively. The state of the system is represented by the state vector $|\psi^{(i)}(t)\rangle$, which fulfills the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi^{(i)}(t)\rangle = \hat{H}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t)|\psi^{(i)}(t)\rangle. \tag{30}$$

An inspection of (26) and (28) shows that

$$\hat{\varepsilon}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t) = \hat{T}^{(i)}(\hat{x}, t)\hat{\varepsilon}^{(0)}(\hat{x}, \hat{p}_{L^{(0)}}, t)\hat{T}^{(i)\dagger}(\hat{x}, t), \tag{31}$$

where

$$\hat{T}^{(i)}(\hat{x}, t) = \exp\left[\frac{i}{\hbar}F^{(i)}(\hat{x}, t)\right]. \tag{32}$$

It follows from (31) that the eigenvalues $\varepsilon_n(t)$ of $\hat{\varepsilon}^{(i)}(t)$ are independent of the quantum representation chosen. In addition, the eigenstates $|n^{(0)}(t)\rangle$ of $\hat{\varepsilon}^{(0)}(t)$ and $|n^{(i)}(t)\rangle$ of $\hat{\varepsilon}^{(i)}(t)$ are related via

$$|n^{(i)}(t)\rangle = \hat{T}^{(i)}(\hat{x}, t)|n^{(0)}(t)\rangle. \tag{33}$$

We call physical operators the operators whose eigenvalues are independent of the quantum representation chosen. Physical operators transform according to (31). Operators associated with classical physical variables, such as $\hat{x}$, $\hat{\pi}$, and $\hat{\varepsilon}$, are physical operators. It is emphasized that $\hat{\varepsilon}^{(i)}(t)$ and $\hat{\varepsilon}^{(0)}(t)$ represent the same physical operator in different quantum representations. The eigenstates of physical operators are physical states. Physical states transform according to (33). It is stressed that $|n^{(i)}(t)\rangle$ and $|n^{(0)}(t)\rangle$ represent the same physical state in different quantum representations.

An inspection of (26) and (29) shows that

$$\hat{H}^{(i)} = \hat{T}^{(i)}\hat{H}^{(0)}\hat{T}^{(i)\dagger} - \frac{\partial F^{(i)}}{\partial t}. \tag{34}$$

The relation between $\hat{H}^{(i)}$ and $\hat{H}^{(0)}$ is not of the type (31). Hence the Hamiltonian does not define a physical operator. The operators $\hat{H}^{(0)}$ and $\hat{H}^{(i)}$ represent different operators. The eigenstates of the Hamiltonian are not physical states. With the aid of (26), (31), and (34), we have

$$\hat{H}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t) = \hat{\varepsilon}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t) - \frac{\partial}{\partial t}F^{(i)}(\hat{x}, t). \tag{35}$$

Hence the Hamiltonian is in general different from the instantaneous energy operator. The substitution of (35) into (30) yields the Schrödinger equation in the manifestly quantum-representation-independent form,

$$i\hbar\hat{D}_t^{(i)}|\psi^{(i)}(t)\rangle = \hat{\varepsilon}^{(i)}(\hat{x}, \hat{p}_{L^{(i)}}, t)|\psi^{(i)}(t)\rangle, \tag{36}$$

where $\hat{D}_t^{(i)} = \frac{d}{dt} - \frac{i}{\hbar}\frac{dF^{(i)}}{dt}$ constitutes a physical operator which is called the covariant time derivative. It follows from (27), (30), (32), and (34) that

$$|\psi^{(i)}(t)\rangle = \hat{T}^{(i)}(\hat{x}, t)|\psi^{(0)}(t)\rangle. \tag{37}$$

Equations (31) and (37) show that the expectation values of the physical operators are independent of the quantum representation chosen.

The time evolution operator $\hat{U}^{(0)}(t, 0)$ in the standard quantum representation is defined by

$$|\psi^{(0)}(t)\rangle = \hat{U}^{(0)}(t, 0)|\psi^{(0)}(0)\rangle. \tag{38}$$

Under a change of quantum representation, (38) becomes

$$|\psi^{(i)}(t)\rangle = \hat{U}^{(i)}(t, 0)|\psi^{(i)}(0)\rangle. \tag{39}$$

In view of (37)–(39), the time evolution operator transforms as

$$\hat{U}^{(i)}(t, 0) = \hat{T}^{(i)}(\hat{x}, t)\hat{U}^{(0)}(t, 0)\hat{T}^{(i)-1}(\hat{x}, 0). \tag{40}$$

Equations (33) and (40) guarantee that the transition amplitudes do not depend on the quantum representation chosen. We have thus verified that different quantum representations are mathematically different but physically equivalent.

III. ADIABATIC THEOREM INDEPENDENT OF THE QUANTUM REPRESENTATIONS

We are now in a position to construct the adiabatic theorem independent of the quantum representations. Consider a system with time-dependent Hamiltonian $\hat{H}^{(i)}(t)$. The state of the system can be expressed as an expansion in terms of the eigenstates of the instantaneous energy operator $\hat{\varepsilon}^{(i)}(t)$:

$$|\psi^{(i)}(t)\rangle = \sum_n C_n^{(i)}(t)\exp\left[-\frac{i}{\hbar}\int_0^t \varepsilon_n(t')dt'\right]|n^{(i)}(t)\rangle. \tag{41}$$

The expansion coefficients $C_n^{(i)}(t)$, which are interpreted as the transition amplitudes between the energy eigenstates, are independent of the quantum representation chosen in light of (33) and (37). Inserting the expansion (41) into (36) and projecting with $\langle m^{(i)}(t)|$ on this equation yields

$$\frac{d}{dt}C_n^{(i)}(t) + \langle n^{(i)}(t)|\hat{D}_t^{(i)}|n^{(i)}(t)\rangle C_n^{(i)}(t) = -\sum_{m\neq n} C_m^{(i)}(t)\exp\left\{\frac{i}{\hbar}\int_0^t [\varepsilon_n(t') - \varepsilon_m(t')]dt'\right\} \times \langle n^{(i)}(t)|\hat{D}_t^{(i)}|m^{(i)}(t)\rangle. \tag{42}$$

In arriving at (42), we have used the orthogonality of $|n^{(i)}(t)\rangle$.

Under the quantum-representation-independent adiabatic condition

$$\left|\frac{\hbar\langle n^{(i)}(t)|\hat{D}_t^{(i)}|m^{(i)}(t)\rangle}{\varepsilon_n(t)-\varepsilon_m(t)}\right| \ll 1, \quad m \neq n, \tag{43}$$

the right-hand side of (42) is negligible, and $C_n^{(i)}(t)$ is approximately given by

$$C_n^{(i)}(t) = C_n^{(i)}(0)\exp\left[i\gamma_n^{(i)}(t)\right], \tag{44}$$

where

$$\gamma_n^{(i)}(t) = i\int_0^t \langle n^{(i)}(t')|\hat{D}_{t'}^{(i)}|n^{(i)}(t')\rangle dt' \tag{45}$$

is the quantum-representation-independent geometrical phase. The adiabatic condition in (43) and the geometrical phase in (45) are quantum representation independent because $\hat{D}_t^{(i)}$ defines a physical operator and $|n^{(i)}(t)\rangle$ are physical states. It follows from (41), (44), and (45) that the system prepared in $|n^{(i)}(0)\rangle$ will evolve approximately to the adiabatic state $|\psi_{\rm adi}^{(i)}(t)\rangle$ given by

$$\left|\psi_{\rm adi}^{(i)}(t)\right\rangle = \exp\left[i\alpha_n^{(i)}(t)\right]|n^{(i)}(t)\rangle \tag{46}$$

with

$$\alpha_n^{(i)}(t) = -\frac{1}{\hbar}\int_0^t \varepsilon_n(t')dt' + i\int_0^t \langle n^{(i)}(t')|\hat{D}_{t'}^{(i)}|n^{(i)}(t')\rangle dt'. \tag{47}$$

The quantum-representation-independent fidelity of the adiabatic approximation is

$$\left|\left\langle\psi_{\rm adi}^{(i)}(t)\middle|\psi^{(i)}(t)\right\rangle\right| = |\langle n^{(i)}(t)|\hat{U}^{(i)}(t,0)|n^{(i)}(0)\rangle|. \tag{48}$$

IV. THE ORIGIN OF THE MARZLIN-SANDERS INCONSISTENCY

In the traditional adiabatic theorem, the time-dependent Hamiltonian is mistakenly identified as the instantaneous energy operator. The state of the system is expanded in terms of the eigenstates of the time-dependent Hamiltonian. Under the change of quantum representation, the state of the system transforms according to (37). Nevertheless, the eigenstates of $\hat{H}^{(i)}$ do not transform according to (33) because $\hat{H}^{(i)}$ does not define a physical operator. Hence the expansion coefficients depend on the quantum representation chosen. This results in the quantum-representation dependence of the fidelity of the adiabatic approximation.

In addition, the traditional adiabatic condition (6) depends on the quantum representation chosen because of the quantum-representation dependence of $E_n(t)$ and the fact that $\frac{d}{dt}$ does not define a physical operator. As a result, whether a system evolves adiabatically depends on the quantum representation chosen. It is unreasonable because physical results must be independent of the quantum representation chosen.

Furthermore, with the aid of (34) and (40), it is straightforward to show that the Marzlin-Sanders transformation (8) is not form invariant under the change of quantum representation. If the Marzlin-Sanders transformation (8) holds in one representation, it will not remain valid in other representations. Hence (9)–(11), which are consequences of the Marzlin-Sanders transformation (8), do not hold in all quantum representations. This in turn indicates that (12) and (13), which lead directly to the Marzlin-Sanders inconsistency, are not valid in all quantum representations. As a result, whether the Marzlin-Sanders inconsistency exists depends on the quantum representation chosen. It is not legitimate because all quantum representations are physically equivalent.

We assert that the Marzlin-Sanders inconsistency is due to the wrong identification of the time-dependent Hamiltonian as the instantaneous energy operator and the quantum-representation dependence of the Marzlin-Sanders transformation (8).

V. CONCLUSION

We construct the adiabatic theorem independent of the quantum representations. A strict distinction between the time-dependent Hamiltonian and the instantaneous energy operator is made. The instantaneous energy operator constitutes a physical operator, whereas the time-dependent Hamiltonian does not. The adiabatic theorem independent of the quantum representations leads to the quantum-representation-independent geometrical phase, adiabatic condition, and fidelity of the adiabatic approximation. As an example, we apply the adiabatic theorem independent of the quantum representations to provide a thorough resolution of the Marzlin-Sanders inconsistency. We show that the Marzlin-Sanders inconsistency can be traced to the wrong identification of the time-dependent Hamiltonian as the instantaneous energy operator and the quantum-representation dependence of the Marzlin-Sanders transformation.

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