Propagating-wave approximation in two-dimensional potential scattering

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We introduce a nonperturbative approximation scheme for performing scattering calculations in two dimensions that involves neglecting the contribution of the evanescent waves to the scattering amplitude. This corresponds to replacing the interaction potential v with an associated energy-dependent nonlocal potential $\mathscr{V}_k$ that does not couple to the evanescent waves. The scattering solutions $\psi(\mathbf{r})$ of the Schrödinger equation, $(-\nabla^2+\mathscr{V}_k)\psi(\mathbf{r})=k^2\psi(\mathbf{r})$, have the remarkable property that their Fourier transform $\tilde{\psi}(\mathbf{p})$ vanishes unless $\mathbf{p}$ corresponds to the momentum of a classical particle whose magnitude equals k. We construct a transfer matrix for this class of nonlocal potentials and explore its representation in terms of the evolution operator for an effective nonunitary quantum system. We show that the above approximation reduces to the first Born approximation for weak potentials and, similar to the semiclassical approximation, becomes valid at high energies. Furthermore, we identify an infinite class of complex potentials for which this approximation scheme is exact. We also discuss the appealing practical and mathematical aspects of this scheme.

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I. INTRODUCTION

Evanescent waves are time-harmonic waves, $e^{-i\omega t}\psi(\mathbf{r})$, that undergo exponential damping or growth in regions of space where the interaction ceases to exist. They arise in space dimensions higher than one and are responsible for the major differences between the behavior of waves propagating in one dimension and those propagating in two and higher dimensions. Suppose that ψ solves the stationary Schrödinger equation,

$$\big[-\partial_x^2-\partial_y^2+v(x,y)\big]\psi(x,y)=k^2\psi(x,y),\quad (x,y)\in\mathbb{R}^2,\tag{1}$$

in two dimensions, where v is a real or complex scattering potential, k is the wave number, and we use units where $\hbar=2m=1$. In regions $\mathcal{R}$ where the potential vanishes, (1) reduces to the Helmholtz equation, $(\nabla^2+k^2)\psi=0$, whose solutions are superpositions of the plane-wave solutions,

$$e^{\pm i\sqrt{k^2-p^2}\,x}e^{ipy},\quad p\in(-k,k),\tag{2}$$

and the evanescent-wave solutions,

$$e^{\pm\sqrt{p^2-k^2}\,x}e^{ipy},\quad p\notin(-k,k).\tag{3}$$

We can express the plane-wave solutions (2) in the familiar form $e^{i\mathbf{p}\cdot\mathbf{r}}$, where $\mathbf{p}$ is the real wave vector with the x and y components $p_x:=\pm\sqrt{k^2-p^2}$ and $p_y:=p$. Therefore, we can identify $\mathbf{p}$ with the momentum of a classical particle. It is clear that this correspondence does not extend to the evanescent-wave solutions (3); they do not have a classical counterpart. The purpose of the present article is to explore the consequences of ignoring the contribution of the evanescent waves (3) to the scattering features of the potential. This corresponds to a particular "quasiclassical approximation" scheme which is not to be confused with the standard semiclassical (Wentzel-Kramers-Brillouin) approximation [1–4]. To avoid possible confusion we call it the "propagating-wave approximation."

If v is a short-range potential [5], i.e., $rv(\mathbf{r})\to 0$ as $r\to\infty$, the Schrödinger equation (1) admits scattering solutions satisfying

$$\psi(\mathbf{r})\to\frac{1}{2\pi}\left[e^{i\mathbf{k}_0\cdot\mathbf{r}}+\sqrt{\frac{i}{kr}}\,e^{ikr}\mathfrak{f}(\theta)\right]\quad\text{for}\quad r\to\infty,\tag{4}$$

where $\mathbf{r}:=x\,\mathbf{e}_x+y\,\mathbf{e}_y$, $\mathbf{e}_u$ is the unit vector along the u axis for $u\in\{x,y\}$, $\mathbf{k}_0\in\mathbb{R}^2$ is the incident wave vector, (r,θ) are the polar coordinates of $\mathbf{r}$, and $\mathfrak{f}(\theta)$ is the scattering amplitude. Note also that $|\mathbf{k}_0|=k$ and $\mathbf{k}_0\cdot\mathbf{r}:=kr\cos(\theta-\theta_0)$, where θ_0 is the incident angle which specifies the direction of $\mathbf{k}_0$ according to $\mathbf{k}_0/k=\cos\theta_0\mathbf{e}_x+\sin\theta_0\mathbf{e}_y$.

In a scattering experiment, the source of the incident wave and the detectors measuring the scattered wave are located at spatial infinities. We choose our coordinate system in such a way that they lie along the lines $x=\pm\infty$. As we illustrate in Fig. 1, the detectors may be put on both of these lines, but the source of the incident wave is either at $x=-\infty$ or $x=+\infty$.[1]

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[1]In Fig. 1 we consider an interaction potential vanishing outside the region bounded by the lines given by $x=a_\pm$ for some $a_\pm\in\mathbb{R}$. This restriction has been enforced for future use and can be lifted for a general short-range potential by letting $a_\pm$ tend to $\pm\infty$. In particular, our choice of coordinates does not restrict the nature of the interaction potential.

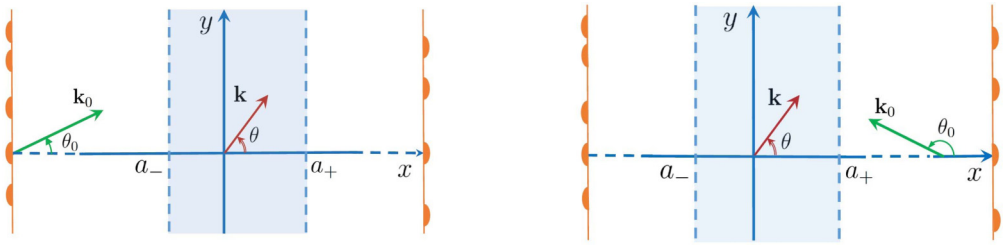

FIG. 1. Schematic view of the scattering setup for a left-incident wave (on the left) and a right-incident wave (on the right). The interaction potential vanishes outside the blue region. The orange vertical lines represent the lines given by $x = \pm\infty$ on which the detectors (depicted by orange half ellipses) are placed. $\mathbf{k}_0$ and $\mathbf{k}$ are, respectively, the incident and scattered wave vectors. For the left- and right-incident waves the incidence angle θ_0 has values of $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{3\pi}{2})$, respectively.

These correspond to left-incident and right-incident waves whose incidence angles θ_0 range over the intervals $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{3\pi}{2})$, respectively. We denote the corresponding scattering amplitudes by $\mathfrak{f}^l(\theta)$ and $\mathfrak{f}^r(\theta)$, respectively.

Let $\mathscr{F}$ be the space of complex-valued functions of a real variable, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and its inverse, i.e., for all $y, p \in \mathbb{R}$,

$$\begin{aligned}(\mathcal{F}f)(p) &:= \int_{-\infty}^{\infty} dy\, e^{-ipy} f(y),\\ (\mathcal{F}^{-1}f)(y) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ipy} f(p).\end{aligned} \tag{5}$$

Here $f \in \mathscr{F}$ is a test function, and a tilde over a function of y stands for its Fourier transform, so that $\tilde{f}(p) := (\mathcal{F}f)(p)$. Performing the Fourier transform of both sides of (1) with respect to y, we find

$$\begin{aligned}-\tilde{\psi}''(x,p) + v(x, i\partial_p)\tilde{\psi}(x,p) &= \varpi(p)^2\tilde{\psi}(x,p),\\ (x,p) &\in \mathbb{R}^2,\end{aligned} \tag{6}$$

where a prime stands for differentiation with respect to x, $\tilde{\psi}(x,p)$ marks the Fourier transform of $\psi(x,y)$ with respect to y evaluated at p, i.e., $\tilde{\psi}(x,p) := (\mathcal{F}\psi(x,\cdot))(p)$,

$$v(x, i\partial_p)f(p) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \tilde{v}(x, p-q) f(q), \tag{7}$$

and

$$\varpi(p) := \begin{cases}\sqrt{k^2 - p^2} & \text{for} \quad |p| < k,\\ i\sqrt{p^2 - k^2} & \text{for} \quad |p| \geqslant k.\end{cases} \tag{8}$$

Consider potentials v that vanish outside a region bounded by a pair of lines parallel to the y axis; that is, there is an interval $[a_-, a_+]$ of real numbers such that $v(x,y) = 0$ for $x \notin [a_-, a_+]$, as depicted in Fig. 1. Then, $\tilde{v}(x,p) = 0$ for $x \notin [a_-, a_+]$, and (6) gives

$$\big[\partial_x^2 + \varpi(p)^2\big]\tilde{\psi}(x,p) = 0 \quad \text{for} \quad x \notin [a_-, a_+].$$

Solving this equation and performing the inverse Fourier transform with respect to p, we can write ψ in the form

$$\psi = \psi_{\rm os} + \psi_{\rm ev}, \tag{9}$$

where $\psi_{\rm os}, \psi_{\rm ev} : \mathbb{R}^2 \to \mathbb{C}$ are a pair of functions satisfying

$$\psi_{\rm os}(x,y) = \begin{cases}\int_{-k}^{k} \frac{dp}{4\pi^2\varpi(p)}[A_-(p)e^{i\varpi(p)x} + B_-(p)e^{-i\varpi(p)x}]e^{ipy} & \text{for} \quad x \leqslant a_-,\\ \int_{-k}^{k} \frac{dp}{4\pi^2\varpi(p)}[A_+(p)e^{i\varpi(p)x} + B_+(p)e^{-i\varpi(p)x}]e^{ipy} & \text{for} \quad x \geqslant a_+,\end{cases} \tag{10}$$

$$\psi_{\rm ev}(x,y) = \begin{cases}\int_{|p|\geqslant k} \frac{dp}{4\pi^2\varpi(p)} C_-(p)e^{|\varpi(p)|x}e^{ipy} & \text{for} \quad x \leqslant a_-,\\ \int_{|p|\geqslant k} \frac{dp}{4\pi^2\varpi(p)} C_+(p)e^{-|\varpi(p)|x}e^{ipy} & \text{for} \quad x \geqslant a_+,\end{cases} \tag{11}$$

and $A_\pm, B_\pm, C_\pm : \mathbb{R} \to \mathbb{C}$ are functions such that

$$A_\pm(p) = B_\pm(p) = 0 \quad \text{for} \ |p| \geqslant k, \quad C_\pm(p) = 0 \quad \text{for} \ |p| < k. \tag{12}$$

In particular, $A_\pm$ and $B_\pm$ belong to

$$\mathscr{F}_k := \{\phi \in \mathscr{F} | \phi(p) = 0 \quad \text{for} \ p \notin (-k, k)\}.$$

According to (9)–(11),

$$\psi(x,y) \to \int_{-k}^{k} \frac{dp}{4\pi^2\varpi(p)}\big[A_\pm(p)e^{i\varpi(p)x} + B_\pm(p)e^{-i\varpi(p)x}\big]e^{ipy} \ \text{ for } \ x \to \pm\infty. \tag{13}$$

This relation is generally valid for bounded solutions of the Schrödinger equation (1) provided that v is a short-range potential [5]. It shows that $A_\pm$ and $B_\pm$ determine the asymptotic form of the solutions ψ of the Schrödinger equation (1). This in turn suggests that they store the information about the scattering features of the potential. To see this, we use $A_\pm^{l/r}$ and $B_\pm^{l/r}$ to denote

the coefficient functions $A_\pm$ and $B_\pm$ for the left and right incident waves. Then, as we showed in Ref. [6],[2] if we use p_0 to label the y component of the incident wave vector $\mathbf{k}_0$ so that $p_0 := k\sin\theta_0$ and notice that the y component of the scattered wave vector $\mathbf{k} := k\mathbf{r}/r$ is given by $p = k\sin\theta$, we have

$$A^l_-(p) = B^r_+(p) = 2\pi\varpi(p_0)\delta(p-p_0) = 2\pi\delta(\theta-\theta_0), \quad B^l_+(p) = A^r_-(p) = 0, \tag{14}$$

$$\mathfrak{f}^l(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} A^l_+(k\sin\theta) - 2\pi\delta(\theta-\theta_0) & \text{for} \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ B^l_-(k\sin\theta) & \text{for} \quad \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \end{cases} \tag{15}$$

$$\mathfrak{f}^r(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} A^r_+(k\sin\theta) & \text{for} \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ B^r_-(k\sin\theta) - 2\pi\delta(\theta-\theta_0) & \text{for} \quad \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases} \tag{16}$$

In analogy with one dimension [7,8], we can identify the scattering operator, also known as the S matrix, with a 2×2 matrix $\widehat{\mathbf{S}}$ satisfying

$$\widehat{\mathbf{S}}\begin{bmatrix} A_- \\ B_+ \end{bmatrix} = \begin{bmatrix} A_+ \\ B_- \end{bmatrix}. \tag{17}$$

Notice, however, that unlike its one-dimensional counterpart, the S matrix $\widehat{\mathbf{S}}$ is not a numerical matrix; it is a 2×2 matrix whose entries $\widehat{S}_{ij}$ are linear operators acting in $\mathscr{F}_k$. This makes $\widehat{\mathbf{S}}$ a linear operator acting in the space of two-component functions,

$$\mathscr{F}_k^{2\times1} := \left\{ \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \middle| \phi_\pm \in \mathscr{F}_k \right\}.$$

In view of (14)–(16), we can express the scattering amplitudes $\mathfrak{f}^{l/r}$ in terms of $\widehat{\mathbf{S}}$ [9].

The evanescent waves $\psi_{\rm ev}(x,y)$ decay as $x \to \pm\infty$, and as a result their Fourier coefficients $C_\pm(p)$ do not enter the asymptotic expression (13) for ψ and the scattering amplitudes $\mathfrak{f}^{l/r}$. This does not, however, mean that they do not affect the outcome of the scattering calculations. This is because $C_\pm$ contribute to the value of the wave function $\psi(x,y)$ on the lines $x = a_\pm$. This makes them influence the way $A^{l/r}_\pm$ and $B^{l/r}_\pm$ relate to one another. Therefore, they do contribute to the S matrix and the scattering amplitudes. Ignoring the contribution of $C_\pm$ to the scattering amplitudes would, in general, lead to errors which, depending on the behavior of the potential, may or may not be negligible. We wish to study the (propagating-wave) approximation scheme in which these errors are ignored. The outcome of this approximation turns out to be equivalent to the solution of the scattering problem for a certain nonlocal potential that does not couple to evanescent waves. As we show below, the propagating-wave approximation has certain appealing properties. For example, it produces reliable results for weak potentials and high-energy incident waves. Therefore, its domain of applicability overlaps those of the first Born approximation and the semiclassical approximation. An important advantage of the propagating-wave approximation is that it is capable of providing the exact solution of the scattering problem for certain highly nontrivial complex potentials with possible optical realizations.

[2]Ref. [6] uses $\check{A}^{l/r}_\pm$ and $\check{B}^{l/r}_\pm$ for what we call $A^{l/r}_\pm$ and $B^{l/r}_\pm$, respectively.

II. COUPLING OF POTENTIALS TO EVANESCENT WAVES

According to (10) and (11), for $x \notin [a_-, a_+]$, $\psi_{\rm os}(x,y)$ is a superposition of Fourier modes e^{ipy} with $|p| < k$, while $\psi_{\rm ev}(x,y)$ is a superposition of Fourier modes e^{ipy} with $|p| \geqslant k$. To identify $\psi_{\rm os}$ and $\psi_{\rm ev}$ as functions defined in $\mathbb{R}^2$, we demand that $\psi_{\rm os}(x,y)$ and $\psi_{\rm ev}(x,y)$ possess the same property for $x \in [a_-, a_+]$. It proves useful to use Dirac's bra-ket notation for this purpose.

For each $x \in \mathbb{R}$, let $|\psi(x)\rangle : \mathbb{R} \to \mathbb{C}$ denote the function that assigns $\psi(x,y)$ to each $y \in \mathbb{R}$ according to $\langle y|\psi(x)\rangle := \psi(x,y)$. Then we can express (1) as

$$\left[-\partial_x^2 + \widehat{p}^2 + v(x,\widehat{y})\right]|\psi(x)\rangle = k^2|\psi(x)\rangle, \quad x \in \mathbb{R}, \tag{18}$$

where $\widehat{y}$ and $\widehat{p}$ are the standard position and momentum operators,

$$\langle y|\widehat{y}|\phi\rangle = y\langle y|\phi\rangle, \quad \langle y|\widehat{p}|\phi\rangle := -i\partial_y\langle y|\phi\rangle.$$

As an operator acting in $L^2(\mathbb{R})$, $\widehat{p}$ is self-adjoint, and its spectrum coincides with $\mathbb{R}$. We use $|p\rangle$ to denote its generalized eigenfunctions, where $p \in \mathbb{R}$, so that $\langle y|p\rangle = (2\pi)^{-1/2}e^{ipy}$. We denote the orthogonal projection operators associated with $\widehat{p}$ by $|p\rangle\langle p|$, introduce the projection operator

$$\widehat{\Pi}_k := \int_{-k}^{k} dp\, |p\rangle\langle p|, \tag{19}$$

and use $\widehat{1}$ and $\widehat{0}$ to label the identity and zero operator acting in $L^2(\mathbb{R})$, respectively.

Recalling that for each $|\phi\rangle \in L^2(\mathbb{R})$, $\tilde{\phi}(p) := \int_{-\infty}^{\infty} e^{-ipy}\langle y|\phi\rangle$ is the Fourier transform of $\langle y|\phi\rangle$, we can use (19) to infer that

$$\langle y|\widehat{\Pi}_k|\phi\rangle = \frac{1}{2\pi}\int_{-k}^{k} dp\, e^{ipy}\tilde{\phi}(p). \tag{20}$$

In view of the completeness of $|p\rangle\langle p|$, we also have

$$\langle y|(\widehat{1}-\widehat{\Pi}_k)|\phi\rangle = \frac{1}{2\pi}\int_{|p|\geqslant k} dp\, e^{ipy}\tilde{\phi}(p). \tag{21}$$

Furthermore, because $\widehat{\Pi}_k^2 = \widehat{\Pi}_k$,

$$\widehat{\Pi}_k(\widehat{1}-\widehat{\Pi}_k) = (\widehat{1}-\widehat{\Pi}_k)\widehat{\Pi}_k = \widehat{0}, \quad (\widehat{1}-\widehat{\Pi}_k)^2 = \widehat{1}-\widehat{\Pi}_k. \tag{22}$$

Next, we use the projection operator $\widehat{\Pi}_k$ to identify the oscillating and evanescent parts, $\psi_{\rm os}$ and $\psi_{\rm ev}$, of the solutions of the Schrödinger equation (18) as follows:

$$|\psi_{\rm os}(x)\rangle := \widehat{\Pi}_k|\psi(x)\rangle, \quad |\psi_{\rm ev}(x)\rangle := (\widehat{1}-\widehat{\Pi}_k)|\psi(x)\rangle. \tag{23}$$

Clearly,

$$|\psi(x)\rangle = |\psi_{\rm os}(x)\rangle + |\psi_{\rm ev}(x)\rangle. \tag{24}$$

We also introduce

$$\widehat{V}_k(x) := \widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k, \quad \widehat{W}_k(x) := (\widehat{1}-\widehat{\Pi}_k)\, v(x,\widehat{y})\,(\widehat{1}-\widehat{\Pi}_k), \tag{25}$$

$$\begin{aligned}\widehat{V}_-(x) &:= \widehat{\Pi}_k\, v(x,\widehat{y})\,(\widehat{1}-\widehat{\Pi}_k),\\ \widehat{V}_+(x) &:= (\widehat{1}-\widehat{\Pi}_k)\, v(x,\widehat{y})\,\widehat{\Pi}_k.\end{aligned} \tag{26}$$

Applying $\widehat{\Pi}_k$ and $\widehat{1}-\widehat{\Pi}_k$ to both sides of (18) and using (23) and (24), we find

$$\big[-\partial_x^2+\widehat{p}^2+\widehat{V}_k(x)\big]|\psi_{\rm os}(x)\rangle + \widehat{V}_-(x)|\psi_{\rm ev}(x)\rangle = k^2|\psi_{\rm os}(x)\rangle, \tag{27}$$

$$\big[-\partial_x^2+\widehat{p}^2+\widehat{W}_k(x)\big]|\psi_{\rm ev}(x)\rangle + \widehat{V}_+(x)|\psi_{\rm os}(x)\rangle = k^2|\psi_{\rm ev}(x)\rangle. \tag{28}$$

As seen from (27) the potential couples to the evanescent part of the wave through the operator $\widehat{V}_-(x)$. With the aid of (20), (21), and (26), we can express the action of $\widehat{V}_-(x)$ on a test function $|\phi\rangle$ in the form

$$\begin{aligned}\widehat{V}_-(x)|\phi\rangle = &\int_{-k}^{k} dp \int_{-\infty}^{-k} dq\, \tilde{v}(x,p-q)\langle q|\phi\rangle\, |p\rangle\\ &+ \int_{-k}^{k} dp \int_{k}^{\infty} dq\, \tilde{v}(x,p-q)\langle q|\phi\rangle\, |p\rangle.\end{aligned} \tag{29}$$

Suppose that this quantity vanishes for all ϕ. If $\phi(q)=0$ for $q\geqslant k$, the second term on the right-hand side of (29) is zero. This shows that for all such ϕ, the first integral on the right-hand side of (29) must also vanish. This happens only if $\tilde{v}(x,p-q)=0$ for $|p|\leqslant k$ and $q\leqslant -k$.[3] It is easy to see that this condition is equivalent to $\tilde{v}(x,p')=0$ for $p'\geqslant 0$. Similarly, considering arbitrary test functions ϕ such that $\phi(q)=0$ for $q\leqslant k$, we can use $\widehat{V}_-(x)|\phi\rangle=0$ to conclude that $\tilde{v}(x,p')=0$ for $p'\leqslant 0$. This argument shows that the term $\widehat{V}_-(x)|\psi_{\rm ev}(x)\rangle$ on the left-hand side of (27) vanishes and the potential does not couple to the evanescent part of the wave provided that $\tilde{v}(x,p')=0$ for all $p'\in\mathbb{R}$. But this implies that $v(x,y)=0$ for all $y\in\mathbb{R}$, i.e., the potential vanishes. Therefore, as far as the solution of the Schrödinger equation (1) is concerned, one can never neglect the coupling of a (nonzero short-range) potential to the evanescent part of the wave. This no-go argument does not, however, imply that neglecting $\widehat{V}_-(x)|\psi_{\rm ev}(x)\rangle$ will always introduce errors in the solution of the scattering problem for the potential. This is simply because the scattering amplitudes $\mathfrak{f}^{l/r}$ are sensitive to only the asymptotic form of the scattering solutions of the Schrödinger equation, and it is, in principle, possible that the contribution of the term $\widehat{V}_-(x)|\psi_{\rm ev}(x)\rangle$ to these solutions becomes negligible or disappears altogether as $x\to\pm\infty$.

[3]Here we make use of the fact that because $\tilde{v}(x,p')$ is the Fourier transform of $v(x,y)$ with respect to y, it is a continuous function of p'.

In the propagating-wave approximation, where $\widehat{V}_-(x)|\psi_{\rm ev}(x)\rangle$ is neglected, (27) reduces to

$$\big[-\partial_x^2+\widehat{p}^2+\widehat{V}_k(x)\big]|\psi_{\rm os}(x)\rangle = k^2|\psi_{\rm os}(x)\rangle. \tag{30}$$

This is equivalent to the Schrödinger equation,

$$(\widehat{\mathbf{p}}^2+\widehat{\mathscr{V}}_k)|\psi\rangle = k^2|\psi\rangle, \tag{31}$$

for the energy-dependent nonlocal potential

$$\widehat{\mathscr{V}}_k := \widehat{\Pi}_k\, v(\widehat{x},\widehat{y})\,\widehat{\Pi}_k, \tag{32}$$

because

$$\langle x,y|\widehat{\mathscr{V}}_k|\psi\rangle := \langle y|\widehat{\Pi}_k v(x,\widehat{y})\widehat{\Pi}_k|\psi(x)\rangle = \langle y|\widehat{V}_k(x)|\psi(x)\rangle \tag{33}$$

$$= \frac{1}{4\pi^2}\int_{-k}^{k} dp \int_{-k}^{k} dq\, e^{iyp}\,\tilde{v}(x,p-q)\tilde{\psi}(x,q). \tag{34}$$

As $k\to\infty$, the right-hand side of (34) tends to $v(x,y)\langle x,y|\psi\rangle$, $\widehat{\mathscr{V}}_k\to v(\widehat{x},\widehat{y})$, and (31) coincides with the original Schrödinger equation (1). Therefore, the propagating-wave approximation is valid at high energies. In this respect it is similar to the semiclassical approximation.

Next, consider a pair of short-range potentials, v_1 and v_2, and let $\widehat{\mathscr{V}}_{1,k} := \widehat{\Pi}_k v_1(\widehat{x},\widehat{y})\widehat{\Pi}_k$ and $\widehat{\mathscr{V}}_{2,k} := \widehat{\Pi}_k v_2(\widehat{x},\widehat{y})\widehat{\Pi}_k$. Suppose that

$$\tilde{v}_1(x,p) = \tilde{v}_2(x,p) \ \text{ for } \ |p|<2k. \tag{35}$$

Then, in view of (34), $\widehat{\mathscr{V}}_{1,k}=\widehat{\mathscr{V}}_{2,k}$. This implies that the application of the propagating-wave approximation for these potentials yields identical results. For example, let

$$v_1(x,y) = g(x)\delta(y), \quad v_2(x,y) = \frac{g(x)\sin(\mathfrak{K}y)}{\pi y},$$

where $g:\mathbb{R}\to\mathbb{C}$ is a function such that $xg(x)\to 0$ as $x\to\pm\infty$ and $\mathfrak{K}\in\mathbb{R}^+$. Because

$$\tilde{v}_1(x,p) = g(x), \quad \tilde{v}_2(x,p) = \begin{cases} g(x) & \text{for} \quad |p|<\mathfrak{K},\\ 0 & \text{for} \quad |p|>\mathfrak{K},\end{cases}$$

v_1 and v_2 satisfy (35) for $\mathfrak{K}\geqslant 2k$. Therefore, the propagating-wave approximation does not distinguish between their scattering properties for wave numbers $k\leqslant\mathfrak{K}/2$. This observation becomes particularly useful for the case where $g(x)=\mathfrak{z}\delta(x)$ and $\mathfrak{z}$ is a real or complex coupling constant, i.e., when v_1 is a δ-function potential in two dimensions, because for this potential the propagating-wave approximation turns out to give the exact expression for the scattering amplitude. This is actually not an exclusive feature of the δ-function potential; there is a large class of potentials for which the propagating-wave approximation is exact. In the remainder of this article we employ the dynamical formulation of stationary scattering (DFSS) of Ref. [6] to identify these potentials and arrive at a better understanding of the propagating-wave approximation.

III. DYNAMICAL FORMULATION OF STATIONARY SCATTERING

In one dimension, there is an alternative to the S matrix, called the transfer matrix, which also stores information about

the scattering properties of the potential [10–12]. In addition, it enjoys a useful composition property which allows for the calculation of the scattering properties of a short-range potential v using the scattering properties of a finite number of its truncations v_i that add up to v and have smaller nonoverlapping supports [12].[4] This feature of the transfer matrix is the main reason for its wide range of applications [13–32]. The practical advantages of using the transfer matrix in one dimension has motivated the development of its multichannel [34–38] and higher-dimensional generalizations [39–45]. The latter involve a discretization of either the configuration or momentum-space variables along the directions normal to the principle scattering or propagation axis and yield large numerical transfer matrices whose treatment requires appropriate numerical schemes.

Reference [6] pursues a different route to obtain a higher-dimensional notion of the transfer matrix which does not require any discretization (and a corresponding approximation) scheme.[5] Its point of departure is a natural higher-dimensional extension of the definition of the transfer matrix in one dimension. Specifically, in two dimensions, it identifies the transfer matrix with a 2×2 matrix $\widehat{\mathbf{M}}$ with operator entries $\widehat{M}_{ij}:\mathscr{F}_k\to\mathscr{F}_k$ such that

$$\widehat{\mathbf{M}}\begin{bmatrix}A_-\\ B_-\end{bmatrix}=\begin{bmatrix}A_+\\ B_+\end{bmatrix}. \tag{36}$$

If we substitute $A_\pm^{l/r}$ and $B_\pm^{l/r}$ for $A_\pm$ and $B_\pm$ in this relation, respectively, and make use of (14), we arrive at

$$\widehat{M}_{22}\,B_-^l=-2\pi\widehat{M}_{21}\,\delta_{p_0},\quad \widehat{M}_{22}\,B_-^r=2\pi\delta_{p_0} \tag{37}$$

$$A_+^l=2\pi\widehat{M}_{11}\,\delta_{p_0}+\widehat{M}_{12}B_-^l,\quad A_+^r=\widehat{M}_{12}B_-^r, \tag{38}$$

where δ_{p_0} stands for the Dirac delta function centered at p_0, i.e., $\delta_{p_0}(p):=\delta(p-p_0)$. Notice that being the y component of the incident wave vector $\mathbf{k}_0$, p_0 satisfies $p_0=k\sin\theta_0$. In particular, $|p_0|<k$.

Equations (37) and (38) provide a method for calculating the scattering amplitude of the potential; solving (37) for $B_-^{l/r}$, using the result in (38) to determine $A_+^{l/r}$, and substituting $A_+^{l/r}$ and $B_-^{l/r}$ in (15) and (16), we can calculate $\mathfrak{f}^{l/r}$ [6].

A remarkable property of the transfer matrix $\widehat{\mathbf{M}}$ is that, similar to its well-known one-dimensional counterpart [47,48], it can be expressed in terms of the evolution operator for an effective nonunitary quantum system [6]. This requires the introduction of an auxiliary transfer matrix $\widehat{\mathfrak{M}}$ which satisfies

$$\widehat{\mathfrak{M}}\begin{bmatrix}A_-\\ \mathscr{B}_-\end{bmatrix}=\begin{bmatrix}\mathscr{A}_+\\ B_+\end{bmatrix}, \tag{39}$$

where

$$\mathscr{B}_-(p)=B_-(p)+C_-(p)=\begin{cases}B_-(p) & \text{for} \quad |p|<k,\\ C_-(p) & \text{for} \quad |p|\geqslant k,\end{cases} \tag{40}$$

$$\mathscr{A}_+(p)=A_+(p)+C_+(p)=\begin{cases}A_+(p) & \text{for} \quad |p|<k,\\ C_+(p) & \text{for} \quad |p|\geqslant k.\end{cases} \tag{41}$$

[4]For a discussion of the generalization of the transfer matrix for long-range potentials in one dimensions, see [33].

[5]See also Ref. [46].

According to (39), we should view $\widehat{\mathfrak{M}}$ as a linear operator acting in the space of two-component functions,

$$\mathscr{F}^{2\times1}:=\left\{\left.\begin{bmatrix}\xi_+\\ \xi_-\end{bmatrix}\right|\xi_\pm\in\mathscr{F}\right\}.$$

The auxiliary transfer matrix has two important properties [6]:

(1) It admits an expression in terms of the evolution operator $\widehat{\mathcal{U}}(x,x_0)$ for the Hamiltonian operator,

$$\widehat{\mathcal{H}}(x):=\tfrac{1}{2}\,e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\,v(x,\widehat{y})\,\widehat{\varpi}^{-1}\mathcal{K}\,e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3}, \tag{42}$$

where x plays the role of time, $\widehat{\varpi}:=\varpi(\widehat{p})$, $\widehat{p}$ is the y component of the standard momentum operator,

$$\mathcal{K}:=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}=\boldsymbol{\sigma}_3+i\boldsymbol{\sigma}_2, \tag{43}$$

and $\boldsymbol{\sigma}_j$ denote the Pauli matrices:

$$\boldsymbol{\sigma}_1:=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \boldsymbol{\sigma}_2:=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \boldsymbol{\sigma}_3:=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{44}$$

The evolution operator $\widehat{\mathcal{U}}(x,x_0)$ for the Hamiltonian (42) gives the auxiliary transfer matrix according to $\widehat{\mathfrak{M}}=\widehat{\mathcal{U}}(a_+,a_-)$. In particular, employing the Dyson series expansion of $\mathcal{U}(x,x_0)$, and noting that $\widehat{\mathcal{H}}(x)=\hat{\mathbf{0}}$ for $x\notin[a_-,a_+]$, we have

$$\widehat{\mathfrak{M}}=\widehat{\mathbf{I}}+\sum_{n=1}^{\infty}(-i)^n\int_{-\infty}^{\infty}dx_n\int_{-\infty}^{x_n}dx_{n-1}\cdots\int_{-\infty}^{x_2}dx_1\,\widehat{\mathcal{H}}(x_n)\widehat{\mathcal{H}}(x_{n-1})\cdots\widehat{\mathcal{H}}(x_1). \tag{45}$$

(2) Let $\widehat{\Pi}_k:\mathscr{F}\to\mathscr{F}$ be the following extension of the projection operator (19) to $\mathscr{F}$:

$$\big(\widehat{\Pi}_k\xi\big)(p):=\begin{cases}\xi(p) & \text{for} \quad |p|<k,\\ 0 & \text{for} \quad |p|\geqslant k,\end{cases} \tag{46}$$

and let $\widehat{\mathbf{\Pi}}_k:\mathscr{F}^{2\times1}\to\mathscr{F}^{2\times1}$ be the projection operator defined by

$$\widehat{\mathbf{\Pi}}_k\begin{bmatrix}\xi_+\\ \xi_-\end{bmatrix}:=\begin{bmatrix}\widehat{\Pi}_k\xi_+\\ \widehat{\Pi}_k\xi_-\end{bmatrix}. \tag{47}$$

Then, we can express the transfer matrix $\widehat{\mathbf{M}}$ in terms of $\widehat{\mathfrak{M}}$ and $\widehat{\mathbf{\Pi}}_k$ according to

$$\widehat{\mathbf{M}}=\widehat{\mathbf{\Pi}}_k\,\widehat{\mathfrak{M}}\,\widehat{\mathbf{\Pi}}_k. \tag{48}$$

The presence of the operator $v(x,\widehat{y})$ on the right-hand side of (42) shows that if we scale the potential as

$$v(x,y)\to\alpha\,v(x,y) \tag{49}$$

for some $\alpha\in\mathbb{R}^+$, then the effective Hamiltonian also scales by a factor of α. This, in turn, allows us to view the Dyson series (45) as a power series in the strength of the potential. Affecting the scaling transformation (49) in (45), we find a series in powers of α which we can identify as a perturbation series. Suppose that we neglect all but the first $N+1$ terms of the series in (45) and substitute the result in (48) to determine an approximate expression for the transfer matrix $\widehat{\mathbf{M}}$. If we

use this expression to solve (37) and (38) for $B_-^{l/r}$ and $A_+^{l/r}$ and insert the outcome in (15) and (16), we obtain approximate formulas for the scattering amplitudes $\mathfrak{f}^{l/r}$. If we expand these formulas in powers of α, neglect the terms of order α^{N+1} and higher, and finally set $\alpha = 1$, we recover the result of the Nth Born approximation. In particular, the first Born approximation corresponds to setting $N = 1$. In this case, (45) and (48) give

$$\widehat{\mathbf{M}} \approx \widehat{\mathbf{\Pi}}_k - i\int_{-\infty}^{\infty} dx\, \widehat{\boldsymbol{\mathcal{H}}}_k(x), \tag{50}$$

where

$$\widehat{\boldsymbol{\mathcal{H}}}_k(x) := \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x) \widehat{\mathbf{\Pi}}_k. \tag{51}$$

In view of (25), (42), (46), (47), and (51) and the fact that $[\widehat{p}, \widehat{\Pi}_k] = [\widehat{\varpi}, \widehat{\Pi}_k] = \widehat{0}$,

$$\begin{aligned}\widehat{\boldsymbol{\mathcal{H}}}_k(x) &= \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, \widehat{\Pi}_k\, v(x,\widehat{y})\, \widehat{\Pi}_k \widehat{\varpi}^{-1}\, \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3} \\ &= \tfrac{1}{2}\, e^{-i\widehat{\varpi} x \boldsymbol{\sigma}_3}\, \widehat{V}_k(x)\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi} x \boldsymbol{\sigma}_3}.\end{aligned} \tag{52}$$

Substituting this relation in the right-hand side of (50) and recalling (33), we can identify the resulting approximate expression for $\widehat{\mathbf{M}}$ as the one we would obtain if we let the nonlocal potential $\widehat{\mathscr{V}}_k$ play the role of the original potential $v(\widehat{x}, \widehat{y})$. This argument shows that the propagating-wave approximation is consistent with the first Born approximation; that is, it is a valid approximation for weak potentials.

Next, we examine the utility of DFSS in the study of the scattering properties of the nonlocal potentials $\widehat{\mathscr{V}}_k$. Then, in view of (33), $\widehat{\boldsymbol{\mathcal{H}}}(x) = \widehat{\boldsymbol{\mathcal{H}}}_k(x)$. In particular, $\widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x) \widehat{\mathbf{\Pi}}_k = \widehat{\boldsymbol{\mathcal{H}}}(x)$. This equation together with (45) and (48) shows that for the nonlocal potentials $\widehat{\mathscr{V}}_k$, the (fundamental) transfer matrix $\widehat{\mathbf{M}}$ coincides with the auxiliary transfer matrix $\widehat{\mathfrak{M}}$. Therefore, for these potentials,

$$\begin{aligned}\widehat{\mathbf{M}} = \widehat{\mathbf{I}} + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{x_n} dx_{n-1} \\ \cdots \int_{-\infty}^{x_2} dx_1\, \widehat{\boldsymbol{\mathcal{H}}}_k(x_n) \widehat{\boldsymbol{\mathcal{H}}}_k(x_{n-1}) \cdots \widehat{\boldsymbol{\mathcal{H}}}_k(x_1).\end{aligned} \tag{53}$$

IV. EXACTNESS OF THE PROPAGATING-WAVE APPROXIMATION

In Ref. [6] we showed that for potentials of the form $v(x, y) = \delta(x)\mathfrak{g}(y)$, with $\mathfrak{g} \in \mathscr{F}$, (50) holds as an exact equality.[6] Therefore, for these potentials the propagating-wave approximation provides the exact expression for the scattering amplitudes. The following result identifies an infinite class of complex potentials with this property.

Theorem 1. Let $v : \mathbb{R}^2 \to \mathbb{C}$ be a potential such that its Fourier transform with respect to y vanishes on one of the half axes given by $\pm p \leqslant 0$, i.e., either

$$\tilde{v}(x, p) = 0 \ \text{ for } \ p \leqslant 0 \tag{54}$$

or

$$\tilde{v}(x, p) = 0 \ \text{ for } \ p \geqslant 0. \tag{55}$$

[6]A simple example is the δ-function potential $v(x, y) = \mathfrak{z}\, \delta(x)\delta(y)$. Then the propagating-wave approximation is exact for v.

To prove this theorem, first we use (45) and (48) to express the fundamental transfer matrix in the form

$$\begin{aligned}\widehat{\mathbf{M}} = \widehat{\mathbf{I}} + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{x_n} dx_{n-1} \\ \cdots \int_{-\infty}^{x_2} dx_1\, \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_n) \widehat{\boldsymbol{\mathcal{H}}}(x_{n-1}) \cdots \widehat{\boldsymbol{\mathcal{H}}}(x_1) \widehat{\mathbf{\Pi}}_k.\end{aligned} \tag{56}$$

According to (42) and (52), the propagating-wave approximation is exact if we can replace the $\widehat{\boldsymbol{\mathcal{H}}}(x_j)$ in this equation with $\widehat{\boldsymbol{\mathcal{H}}}_k(x_j)$. To arrive at a more explicit description of this condition, we derive an alternative expression for the product of $\widehat{\boldsymbol{\mathcal{H}}}(x_j)$.

Consider the operators

$$\begin{aligned}&\widehat{s}(x) := \widehat{\varpi}^{-1} \sin(x\, \widehat{\varpi}), \\ &\widehat{\boldsymbol{\mathcal{L}}}(x) := e^{-ix\widehat{\varpi}\boldsymbol{\sigma}_3} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} e^{ix\widehat{\varpi}\boldsymbol{\sigma}_3} = \begin{bmatrix} 0 & 0 \\ e^{2ix\widehat{\varpi}} & 1 \end{bmatrix},\end{aligned} \tag{57}$$

which respectively act in $\mathscr{F}$ and $\mathscr{F}^{2\times 1}$. Then a straightforward usage of the properties of the Pauli matrices and Eqs. (42) and (43) allows us to show that

$$\begin{aligned}&\widehat{\boldsymbol{\mathcal{H}}}(x_n) \widehat{\boldsymbol{\mathcal{H}}}(x_{n-1}) \cdots \widehat{\boldsymbol{\mathcal{H}}}(x_1) \\ &\quad = \frac{i^n}{2}\, e^{-ix_n\widehat{\varpi}\boldsymbol{\sigma}_3} \widehat{\mathcal{H}}_n(x_n, x_{n-1}, \ldots, x_1)\, \boldsymbol{\mathcal{K}}\, e^{ix_1\widehat{\varpi}\boldsymbol{\sigma}_3} \widehat{\varpi}^{-1} \widehat{\boldsymbol{\mathcal{L}}}(x_1),\end{aligned} \tag{58}$$

where $\widehat{\mathcal{H}}_1(x_1) := v(x_1, \widehat{y})$, and for $n \geqslant 2$,

$$\begin{aligned}\widehat{\mathcal{H}}_n(x_n, x_{n-1}, \ldots, x_1) := v(x_n, \widehat{y})\, \widehat{s}(x_n - x_{n-1}) v(x_{n-1}, \widehat{y}) \\ \times \widehat{s}(x_{n-1} - x_{n-2})\, v(x_{n-2}, \widehat{y}) \cdots \\ \times v(x_2, \widehat{y})\, \widehat{s}(x_2 - x_1)\, v(x_1, \widehat{y}).\end{aligned} \tag{59}$$

Notice that $\widehat{\mathcal{H}}_n(x_n, x_{n-1}, \ldots, x_1)$ act in $\mathscr{F}$ and contain all the information about the potential.[7] Therefore, in order to establish the exactness of the propagating-wave approximation for potentials fulfilling (54) or (55), it suffices to show that

$$\begin{aligned}\widehat{\Pi}_k \widehat{\mathcal{H}}_n(x_n, x_{n-1}, \ldots, x_1) \widehat{\Pi}_k = \widehat{V}_k(x_n) \widehat{s}(x_n - x_{n-1}) \widehat{V}_k(x_{n-1}) \\ \times \widehat{s}(x_{n-1} - x_{n-2})\, \widehat{V}_k(x_{n-2}) \cdots \\ \times \widehat{V}_k(x_2) \widehat{s}(x_2 - x_1)\, \widehat{V}_k(x_1).\end{aligned} \tag{60}$$

This holds trivially for $n = 1$. We give its proof for $n \geqslant 2$ in the Appendix for potentials satisfying (54). This completes the proof of the exactness of the propagating-wave approximation for these potentials.

Next, suppose that $v(x, y)$ fulfills (55), and let $w(x, y) := v(x, -y)$. Then $\tilde{w}(x, p) = \tilde{v}(x, -p) = 0$ for $p \leqslant 0$. Therefore, the propagating-wave approximation is exact for w. Because the solution of the scattering problem for v can be easily

[7]The presence of $\widehat{\boldsymbol{\mathcal{L}}}(x_1)$ on the right-hand side of (58) is necessary for the correct identification of the domain of $\widehat{\boldsymbol{\mathcal{H}}}(x_n) \widehat{\boldsymbol{\mathcal{H}}}(x_{n-1}) \cdots \widehat{\boldsymbol{\mathcal{H}}}(x_1)$ [49]. It does not, however, play a role in the proof of Theorem 1.

mapped to that of w, this implies exactness of the propagating-wave approximation for v.[8]

In one dimension, short-range potentials whose Fourier transform vanishes on the negative or positive half axis have the remarkable property of being unidirectionally invisible for all frequencies [50–53]. Theorem 1 reveals the exactness of the propagating-wave approximation for the treatment of their two-dimensional analogs. Constructing concrete examples of the latter is quite easy; given $u, g \in \mathscr{F}$, the potential defined by $v(x,y) := g(x)\int_0^\infty dp\, e^{ipy}u(p)$ fulfills (54). We can also select g and u in such a way that v is a short-range potential. For instance, suppose that g has a compact support, i.e., there are $a_\pm \in \mathbb{R}$ such that $a_- < a_+$, $g(x) = 0$ for $x \notin [a_-, a_+]$, and $u(p) := p^\ell e^{-\beta|p|}/\ell!$, where ℓ is a positive integer and β is a positive real parameter. Then,

$$v(x,y) = \frac{g(x)}{(\beta - iy)^{\ell+1}}, \tag{61}$$

which is a short-range potential. By construction it satisfies (54), and the propagating-wave approximation provides an exact description of its scattering properties. Notice also that whenever g is a real-valued even function, this potential is $\mathcal{PT}$ symmetric.

Next, consider a potential $v(x,y)$ that does not satisfy (54). Then the propagating-wave approximation is not exact, but it may still provide a reliable approximate description of the scattering features of the potential. For example, consider a potential of the form

$$v(x,y) = g(x)\, e^{i\alpha y} e^{-\beta^2 y^2/2}, \quad g(x) := \begin{cases} \mathfrak{z} & \text{for } x \in (a_-, a_+), \\ 0 & \text{otherwise,} \end{cases} \tag{62}$$

where α and β are positive real parameters and $\mathfrak{z}$ is a real or complex coupling constant. Then $\tilde{v}(x,p) = \sqrt{2\pi}\, g(x)\beta^{-1}e^{-(p-\alpha)^2/2\beta^2}$. For $p \leqslant 0$, this implies $|\tilde{v}(x,p)| \leqslant |\mathfrak{z}|\, \beta^{-1}e^{-\alpha^2/2\beta^2}$. Therefore, if $\alpha \gg \beta \gtrapprox |\mathfrak{z}|$, $\tilde{v}(x,p) \approx 0$ for $p \leqslant 0$, and the propagating-wave approximation is expected to be reliable.

V. PRACTICAL AND MATHEMATICAL ASPECTS OF THE PROPAGATING-WAVE APPROXIMATION

At first glance there seems to be no major difference between practical aspects of the dynamical formulation of scattering for a given short-range potential v and the corresponding nonlocal potential $\widehat{\mathscr{V}}_k$. The calculation of the transfer matrix $\widetilde{\mathbf{M}}$ for both of these potentials amounts to summing up certain Dyson series. But there is a very important difference between the Dyson series expansion of the transfer matrices for v and $\widehat{\mathscr{V}}_k$. As we demonstrated in Ref. [49], this has to do with the fact that we can setup the dynamical formulation of stationary scattering for $\widehat{\mathscr{V}}_k$ in the space $L^2(-k,k)$ of square-integrable complex-valued functions defined in the interval $(-k,k)$. This already implies that one can develop effective numerical schemes for computing the transfer matrix for $\widehat{\mathscr{V}}_k$ because they need to deal only with functions with a finite domain, namely, $(-k,k)$.

A more important advantage of dealing with the nonlocal operator $\widehat{\mathscr{V}}_k$ is that it defines a normal Hilbert-Schmidt (in particular, compact) operator acting in $L^2(-k,k)$, [49]. This implies that there is an orthonormal basis of $L^2(-k,k)$ consisting of the eigenvectors of $\widehat{\mathscr{V}}_k$, the spectrum of $\widehat{\mathscr{V}}_k$ consists only of eigenvalues [54], and the nonzero eigenvalues of $\widehat{\mathscr{V}}_k$ are finitely degenerate and have zero as their accumulation point. Another benefit of dealing with $\widehat{\mathscr{V}}_k$ is that the operator $\widehat{s}(x)$ that enters the calculation of the product of the effective Hamiltonians $\widehat{\boldsymbol{\mathcal{H}}}_k(x_j)$ through (58) and (59) becomes a bounded self-adjoint operator acting in $L^2(-k,k)$. This, in turn, implies that the operators $\widehat{\mathcal{H}}_n(x_n, x_{n-1}, \ldots, x_1)$, which determine the Dyson series expansion of the transfer matrix for $\widehat{\mathscr{V}}_k$, are Hilbert-Schmidt and hence compact operators acting in $L^2(-k,k)$. A basic result of the theory of compact operators is that they can be approximated by finite-range operators [55,56], which in effect means that we can develop accurate approximate descriptions of these operators using finite numerical matrices [57,58]. This observation opens up a new research front for effective numerical treatments of the scattering problem for $\widehat{\mathscr{V}}_k$, i.e., the application of the propagating-wave approximation.

Another implication of the desirable mathematical properties of the operators $\widehat{\boldsymbol{\mathcal{H}}}_k(x_j)$ is the development of a mathematically rigorous dynamical formulation of stationary scattering for the nonlocal potentials $\widehat{\mathscr{V}}_k$ [49]. This means dealing with the notorious domain issues of the related unbounded operators and proving the existence of the transfer matrix as a densely defined operator.

VI. CONCLUDING REMARKS

Evanescent waves are at the heart of the major differences between wave propagation in one and higher dimensions. In this article, we have explored the consequences of ignoring the contribution of the evanescent waves to the scattering properties of a given short-range potential in two dimensions. This has led us to introduce a nonperturbative approximation scheme in which the potential is replaced with an energy-dependent nonlocal potential $\widehat{\mathscr{V}}_k$ that does not couple to the evanescent waves. The scattering solutions of the Schrödinger equation for this potential have purely oscillating Fourier modes. Therefore, they correspond to propagating waves.

The requirement that $\widehat{\mathscr{V}}_k$ is nonlocal stems from the fact that every local potential necessarily couples to the evanescent waves. This does not, however, mean that this coupling always affects the scattering properties of the potential; there are potentials for which the propagating-wave approximation is exact. Especially, we have shown that whenever the Fourier transform of a potential $v(x,y)$ with respect

[8] If $f_v^{l/r}(\theta_0,\theta)$ and $f_w^{l/r}(\theta_0,\theta)$ respectively denote the scattering amplitudes of the potentials v and w, then

$$f_v^l(\theta_0,\theta) = \begin{cases} f_w^l(-\theta_0,-\theta) & \text{for } \theta \in (-\frac{\pi}{2},\frac{\pi}{2}), \\ f_w^l(-\theta_0,2\pi-\theta) & \text{for } \theta \in (\frac{\pi}{2},\frac{3\pi}{2}), \end{cases}$$

$$f_v^r(\theta_0,\theta) = \begin{cases} f_w^r(2\pi-\theta_0,-\theta) & \text{for } \theta \in (-\frac{\pi}{2},\frac{\pi}{2}), \\ f_w^r(2\pi-\theta_0,2\pi-\theta) & \text{for } \theta \in (\frac{\pi}{2},\frac{3\pi}{2}). \end{cases}$$

to y vanishes on the negative or positive real axis, the propagating-wave approximation produces the exact expression for its scattering amplitudes. It is quite remarkable that enforcing the very same condition in one dimension ensures the unidirectional invisibility of the potential at all frequencies [50–52]!

The propagating-wave approximation has important practical advantages, for its numerical implementations can benefit from the fact that the functions entering the propagating-wave scattering calculations have a common finite domain, namely, the interval $(-k, k)$, and the operators appearing in the expression for the transfer matrix can be accurately approximated by matrices. These observations provide ample motivation for developing effective numerical schemes for performing the propagating-wave approximation.

Finally, we wish to note that the validity of the results we have reported in this article are not confined to short-range potentials. For example, they apply to potentials whose support lies between a pair of lines that are parallel to the y axis, such as those considered in Ref. [9]. We can also use the machinery of the dynamical formulation of stationary scattering in three dimensions [6] to arrive at the three-dimensional extensions of these results.

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APPENDIX: PROOF OF EQUATION (60) FOR $n \geqslant 2$

Suppose that v is a potential satisfying (54). Then to prove (60) for $n \geqslant 2$, we proceed as follows.

(1) We use (59) to infer that for every $m \geqslant 2$ and $x_1, x_2, \ldots, x_m \in \mathbb{R}$,

$$\widehat{\Pi}_k \widehat{\mathcal{H}}_m(x_m, x_{m-1}, \ldots, x_1)\widehat{\Pi}_k = \widehat{\Pi}_k \widehat{\mathcal{H}}_{m-1}(x_m, x_{m-1}, \ldots, x_2)\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k. \quad \text{(A1)}$$

This relation suggests that if we can verify

$$\widehat{\Pi}_k \widehat{\mathcal{H}}_{m-1}(x_m, \ldots, x_2)\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k = \widehat{\Pi}_k \widehat{\mathcal{H}}_{m-1}(x_m, \ldots, x_2)\widehat{\Pi}_k\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k, \quad \text{(A2)}$$

then by repeated use of this equation and (A1) we can prove (60).

(2) For each real number α, we introduce $\mathcal{S}_\alpha := \{\phi \in \mathscr{F} \mid \phi(p) = 0 \text{ for } p \leqslant \alpha\}$ and show that whenever (54) holds, $f(\widehat{p})v(x, \widehat{y})$ maps $\mathcal{S}_\alpha$ to $\mathcal{S}_\alpha$ for every $f \in \mathscr{F}$. To see this we observe that for all $\phi \in \mathcal{S}_\alpha$,

$$\begin{aligned}\langle p|f(\widehat{p})v(x, \widehat{y})|\phi\rangle &= f(p)v(x, i\partial_p)\phi(p)\\ &= \frac{f(p)}{2\pi}\int_{-\infty}^{\infty} dq\, \tilde{v}(x, p-q)\phi(q)\\ &= \frac{f(p)}{2\pi}\int_{\alpha}^{\infty} dq\, \tilde{v}(x, p-q)\phi(q),\end{aligned}$$

where we have made use of (7). If $p \leqslant \alpha$, $p - q \leqslant 0$, (54) implies $\tilde{v}(x, p-q) = 0$, and we obtain $\langle p|f(\widehat{p})v(x, \widehat{y})|\phi\rangle = 0$. Therefore, $f(\widehat{p})v(x, \widehat{y})\phi \in \mathcal{S}_\alpha$.

(3) We observe that for all $\phi \in \mathscr{F}$, $\widehat{\Pi}_k\phi \in \mathcal{S}_{-k}$. Moreover, because $\widehat{s}(x) = \widehat{\varpi}^{-1}\sin(x\,\widehat{\varpi})$ and $\widehat{\varpi} = \varpi(\widehat{p})$, $\widehat{s}(x)$ is a function of $\widehat{p}$. This implies that for potentials satisfying (54), $\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k\phi \in \mathcal{S}_{-k}$. By virtue of this relation and the fact that $(\widehat{1} - \widehat{\Pi}_k)\xi = 0$ for $\xi \in \mathcal{S}_{-k}$, we have $(\widehat{1} - \widehat{\Pi}_k)\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k\phi = 0$. Since ϕ is arbitrary, this is equivalent to

$$\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k = \widehat{\Pi}_k\widehat{s}(x_2-x_1)\, v(x_1, \widehat{y})\widehat{\Pi}_k.$$

This equation clearly implies (A2).

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