Linearity conditions leading to complete positivity

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The reduced dynamics of an open quantum system S, interacting with its environment E, is not completely positive, in general. In this paper, we demonstrate that if the two following conditions are satisfied, simultaneously, then the reduced dynamics is completely positive: (1) the reduced dynamics of the system is linear, for arbitrary system-environment unitary evolution U; and (2) the reduced dynamics of the system is linear, for arbitrary initial state of the system ρ_S.

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I. INTRODUCTION

In the axiomatic approach to quantum operations, as legitimate maps describing the (reduced) dynamics of a quantum system S, a quantum operation $\mathcal{E}_S$ is defined as a linear trace-preserving completely positive map [1]. At first glance, requiring that $\mathcal{E}_S$ is linear seems admissible since the unitary evolution of a closed quantum system is linear, and we may expect similar property for open quantum systems as well. In addition, nonlinear evolution may lead to superluminal signaling [2].

But, instead of being trace-preserving *completely positive*, one may expect that $\mathcal{E}_S$ must be solely a trace-preserving *positive* map since the only general requirement seems to be that $\mathcal{E}_S$ must map density operators to density operators.

It seems that there are two major reasons, for the usual use of completely positive maps, instead of the positive ones, in quantum information theory [1], and in the theory of open quantum systems [3–5]. First, there exists a simple operator sum representation, for each trace-preserving completely positive (CP) map $\mathcal{E}_S$, as

$$\mathcal{E}_S(\rho_S) = \sum_i E_i\, \rho_S\, E_i^{\dagger}, \qquad \sum_i E_i^{\dagger} E_i = I_S, \tag{1}$$

where E_i are linear operators and I_S is the identity operator, on the Hilbert space of the system $\mathcal{H}_S$ [1].

Second, in the theory of open quantum systems, it is common to consider the set of initial states of the system-environment as $\mathcal{S} = \{\rho_{SE} = \rho_S \otimes \tilde{\omega}_E\}$, where ρ_S is an arbitrary state (density operator) on $\mathcal{H}_S$ and $\tilde{\omega}_E$ is a fixed state on the Hilbert space of the environment $\mathcal{H}_E$ [3–5]. Then, for such an initial set $\mathcal{S}$, it is famous that the reduced dynamics of the system is CP, for arbitrary system-environment unitary evolution U [1].

The main question of this paper is to investigate whether it is possible to obtain the result of the CP-ness of the reduced dynamics, from its positivity or even from the less restrictive condition of its linearity.

Unlike the reduced dynamics, for which, in general, its positivity is not equivalent to its CP-ness, there exists an important map for which it is so. This important map is the inverse of the partial trace over the environment and is called the *assignment map* [6,7]. It can be shown that, if there exists a positive assignment map, then there exists a CP one also, which results in the CP-ness of the reduced dynamics [8].

As we will see, in Sec. IV, with the only requirement being that the reduced dynamics is linear, for arbitrary unitary evolution of the system-environment U and arbitrary initial state of the system ρ_S, results in the positivity of the assignment map, and so the CP-ness of the reduced dynamics.

The paper is organized as follows. In the next section, we review some introductory points on the reduced dynamics of an open quantum system. The assignment map, and its role in representing the reduced dynamics as a linear map, is introduced in Sec. III. Our main results are given in Sec. IV and the paper is ended in Sec. V, with a summary of our results.

II. REDUCED DYNAMICS OF AN OPEN SYSTEM

Let us denote the set of all linear operators on $\mathcal{H}_S$ as $\mathcal{L}_S$ and the set of all density operators on $\mathcal{H}_S$ as $\mathcal{D}_S$. Now, by a Hermitian map, we mean a linear trace-preserving map on $\mathcal{L}_S$, which maps each Hermitian operator to a Hermitian operator. A Hermitian map is called positive if it maps each density operator in $\mathcal{D}_S$ to a density operator. Both the Hermitian maps and the positive ones have operator sum representations as

$$\Phi_S(\rho_S) = \sum_i e_i\, \tilde{E}_i\, \rho_S\, \tilde{E}_i^{\dagger}, \qquad \sum_i e_i\, \tilde{E}_i^{\dagger} \tilde{E}_i = I_S, \tag{2}$$

where $\tilde{E}_i$ are linear operators on $\mathcal{H}_S$, and e_i are real coefficients [9–11]. When all of the coefficients e_i in Eq. (2) are positive, we can define $E_i = \sqrt{e_i}\, \tilde{E}_i$, and Eq. (2) can be rewritten as Eq. (1). Then, the map is called CP. It is also worth noting that the CP-ness of the map $\mathcal{E}_S$, in Eq. (1), is equivalent to the positivity of the map $id_W \otimes \mathcal{E}_S$, where the *witness* W is an arbitrary (finite-dimensional) quantum system, distinct from the system S (and the environment E), and id_W is the identity

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map on $\mathcal{L}_W$ [1]. ($\mathcal{L}_W$ is the set of all linear operators on the Hilbert space of the witness $\mathcal{H}_W$.)

For the open quantum system S, interacting with its environment E, we can consider the entire system-environment as a closed quantum system, which evolves unitarily as

$$\rho'_{SE} = \mathrm{Ad}_U(\rho_{SE}) \equiv U\rho_{SE}U^{\dagger}, \tag{3}$$

where U is a unitary operator on $\mathcal{H}_S \otimes \mathcal{H}_E$. In addition, ρ_{SE} and ρ'_{SE} are the initial and final states of the system-environment, respectively. So, the reduced dynamics of the system is given by

$$\rho'_S = \mathrm{Tr}_E(\rho'_{SE}) = \mathrm{Tr}_E \circ \mathrm{Ad}_U(\rho_{SE}). \tag{4}$$

In general, the reduced dynamics of the system S cannot be represented by a map [9,12], i.e., ρ'_S cannot be given as a function of the initial state of the system $\rho_S = \mathrm{Tr}_E(\rho_{SE})$, in general. Even if the reduced dynamics of the system can be given by a map, this map is not linear, in general [13,14]. In addition, even if it is linear, it is not (completely) positive, in general, but it is Hermitian [15]. The CP-ness of the reduced dynamics has been proven, only for some restricted sets $\mathcal{S} = \{\rho_{SE}\}$ of initial states of the system-environment [16–22].

In the experimentally relevant cases, one usually deals with the factorized initial states of the system-environment, i.e., the set of initial states of the system-environment, at time $t = 0$, is as $\mathcal{S} = \{\rho_S \otimes \tilde{\omega}_E\}$, where ρ_S is an arbitrary state of the system, while $\tilde{\omega}_E$ is a fixed state of the environment [3–5]. So, the reduced dynamics is CP, as stated in the Introduction. But, even in such cases, one may encounter non-CP reduced dynamics simply by changing the initial time from $t = 0$, as illustrated in the following example.

Consider the case that the reduced dynamics is given by a master equation, which is similar to the Gorini-Kossakowski-Sudarshan–Lindblad one [23,24], but with a time-dependent generator $\mathcal{K}_S(t)$, as

$$\begin{aligned}\frac{d\sigma_S}{dt} &= \mathcal{K}_S(t)[\sigma_S] \\ &= -\frac{i}{\hbar}[H(t), \sigma_S] \\ &\quad + \sum_j \gamma_j(t)\left[A_j(t)\sigma_S A_j^{\dagger}(t) - \frac{1}{2}\{A_j^{\dagger}(t)A_j(t), \sigma_S\}\right],\end{aligned} \tag{5}$$

where $\sigma_S = \sigma_S(t) \in \mathcal{D}_S$ is the reduced state of the system S at time t. In addition, the (Hermitian) Hamiltonian operator $H(t) \in \mathcal{L}_S$, the Lindblad operators $A_j(t) \in \mathcal{L}_S$, and the real rates $\gamma_j(t)$ are all time-dependent, in general [25]. Now, if all $\gamma_j(t)$ are positive for all $t \geqslant 0$, then the reduced dynamics is CP-divisible [25]:

$$\mathcal{E}_S(t_2, 0) = \mathcal{E}_S(t_2, t_1) \circ \mathcal{E}_S(t_1, 0), \tag{6}$$

where $t_2 > t_1 > 0$, and $\mathcal{E}_S(t, s)$ is a CP map, which maps $\sigma_S(s)$ to $\sigma_S(t)$. But, if, in the *canonical* form of the generator $\mathcal{K}_S(t)$ [26], all $\gamma_j(t)$ are positive only during the time interval $[0, t_1]$, then we have

$$\mathcal{E}_S(t_2, 0) = \Phi_S(t_2, t_1) \circ \mathcal{E}_S(t_1, 0), \tag{7}$$

where, though $\mathcal{E}_S(t_1, 0)$ and $\mathcal{E}_S(t_2, 0)$ are CP, but $\Phi_S(t_2, t_1)$, i.e., the Hermitian map which maps $\sigma_S(t_1)$ to $\sigma_S(t_2)$, is non-CP, in general. So, changing the initial time, from $t = 0$ to $t = t_1$, results in the reduced dynamics of the system being given by the non-CP map $\Phi_S(t, t_1)$, for $t > t_1$.

In addition to the simplicity and experimental relevance, which were mentioned above and in the Introduction, one can give a rather general discussion leading to the CP-ness of the reduced dynamics: always, in addition to the system under study S, one can consider another quantum system, the witness W, which does not interact with S, and, during the evolution of S, it does not evolve. Now, assuming that the evolution of the witness-system is given by a local map $id_W \otimes \mathcal{E}_S$, results in the CP-ness of $\mathcal{E}_S$. Note that the initial state of the witness-system ρ_{WS} can be entangled. Now, the CP-ness of $\mathcal{E}_S$, and so the positivity of the $id_W \otimes \mathcal{E}_S$, is necessary to ensure that the final state $\rho'_{\mathrm{WS}} = id_W \otimes \mathcal{E}_S(\rho_{\mathrm{WS}})$ is a valid density operator [1]. However, one can find situations in which, though the dynamics of the witness-system is local (and the reduced state of the witness does not change, during the evolution), it cannot be written as $id_W \otimes \mathcal{E}_S$ (see, e.g., [27]). So, the reduced dynamics of the system S can be non-CP, in general, as we have seen for $\Phi_S(t, t_1)$ in the previous paragraph.

At the end of this section, we mention that the utilization of the completely positive maps for describing the reduced dynamics of the system S can be extended, at least, through the two following ways. First, consider the case that the set of initial states of the system-environment is given by $\mathcal{S} = \{\rho_{SE} = \sum_\alpha \tilde{w}_\alpha Q_\alpha \otimes \tilde{\sigma}_\alpha\}$, where the linear operators $Q_\alpha \in \mathcal{L}_S$ vary by changing ρ_{SE}, but $\tilde{\sigma}_\alpha$ are fixed density operators on $\mathcal{H}_E$ and the (positive) weights $\tilde{w}_\alpha$ are also fixed. Then the reduced dynamics of the system S, in Eq. (4), for arbitrary system-environment unitary evolution U, is given by

$$\rho'_S = \sum_\alpha \tilde{w}_\alpha \mathcal{E}_S^{(\alpha)}(Q_\alpha), \tag{8}$$

where $\mathcal{E}_S^{(\alpha)}$ is a CP map, depending on U and $\tilde{\sigma}_\alpha$ [28]. In other words, in this case, the reduced dynamics is given by a set of CP maps $\{\mathcal{E}_S^{(\alpha)}\}$ instead of only one CP map.

Second, consider the case that set of initial states of the system-environment is given by $\mathcal{S} = \{\rho_{SE} = \mathcal{E}_S \otimes id_E(\tilde{\omega}_{SE})\}$, where $\tilde{\omega}_{SE}$ is a fixed state on $\mathcal{H}_S \otimes \mathcal{H}_E$, $\mathcal{E}_S$ is an arbitrary CP map on $\mathcal{L}_S$, and id_E is the identity map on $\mathcal{L}_E$, the set of all linear operators on $\mathcal{H}_E$. Splitting a quantum experiment into the three steps of preparation, evolution, and measurement, choosing the set $\mathcal{S}$ as above means that we can only manipulate the system S, through the CP maps $\mathcal{E}_S$, during the preparation step. Now, it can be shown that, for arbitrary system-environment unitary evolution U, the final state of the system ρ'_S in Eq. (4) can be written as a completely positive map on (the Choi matrix representation [29,30] of) $\mathcal{E}_S$ [31,32]. In other words, in this case, even if ρ'_S cannot be given as a completely positive map on the initial state of the system ρ_S, but it can be given by a completely positive map on the preparation map $\mathcal{E}_S$.

III. ASSIGNMENT MAP

Consider the set $\mathcal{S} = \{\rho_{SE}\}$ of initial states of the system-environment. The set $\mathcal{S}$ includes all initial ρ_{SE} which are prepared (chosen) through the preparation step of the experiment. Obviously, in general, $\mathcal{S}$ is a subset of $\mathcal{D}$, the set of all density operators on $\mathcal{H}_S \otimes \mathcal{H}_E$.

The set of initial states of the system is given by $\mathcal{S}_S = \mathrm{Tr}_E \mathcal{S}$. Assuming that the system S is finite-dimensional of dimension d_S only a finite number m of the members of $\mathcal{S}_S$, where the integer m is $0 < m \leqslant (d_S)^2$, are linearly independent. Let us denote this linearly independent set as $\mathcal{S}'_S = \{\rho_S^{(1)}, \rho_S^{(2)}, \dots, \rho_S^{(m)}\}$. Therefore, any $\rho_S \in \mathcal{S}_S$ can be expanded as

$$\rho_S = \sum_{i=1}^{m} a_i \rho_S^{(i)}, \tag{9}$$

where a_i are real coefficients. Note that ρ_S is a Hermitian operator. So $\sum (a_i - a_i^*) \rho_S^{(i)} = 0$. Now, since all $\rho_S^{(i)} \in \mathcal{S}'_S$ are linearly independent, all a_i must be real.

In general, there may be more than one state in $\mathcal{S}$ such that tracing over the environment gives $\rho_S^{(i)}$. However, we choose only one of them and denote it as $\rho_{SE}^{(i)}$. Linear independence of $\rho_S^{(i)} \in \mathcal{S}'_S$ results in linear independence of $\rho_{SE}^{(i)}$. We denote this linearly independent set as $\mathcal{S}' = \{\rho_{SE}^{(1)}, \rho_{SE}^{(2)}, \dots, \rho_{SE}^{(m)}\}$ [33]. So, each $\rho_{SE} \in \mathcal{S}$, for which $\rho_S = \mathrm{Tr}_E(\rho_{SE})$ is expanded in Eq. (9) can be written as

$$\rho_{SE} = \sum_{i=1}^{m} a_i \rho_{SE}^{(i)} + Y(\rho_{SE}), \tag{10}$$

where a_i are the same as those in Eq. (9) and Y is a Hermitian operator on $\mathcal{H}_S \otimes \mathcal{H}_E$ such that $\mathrm{Tr}_E(Y) = 0$. In other words, Eq. (9) results that ρ_{SE} and $\sum a_i \rho_{SE}^{(i)}$ can differ with each other up to a Hermitian operator Y, for which $\mathrm{Tr}_E(Y) = 0$. In general, Y is a function of ρ_{SE}. This dependence is explicitly given in Eq. (10) by writing it as $Y(\rho_{SE})$.

The subspaces $\mathcal{V}$ and $\mathcal{V}_S$ are defined as [9]

$$\mathcal{V} = \mathrm{Span}_{\mathbb{C}}\, \mathcal{S}, \tag{11}$$

and

$$\mathcal{V}_S = \mathrm{Tr}_E \mathcal{V} = \mathrm{Span}_{\mathbb{C}}\, \mathcal{S}_S = \mathrm{Span}_{\mathbb{C}}\, \mathcal{S}'_S. \tag{12}$$

Therefore, each $X \in \mathcal{V}$ can be written as $X = \sum_l c_l\, \tau_{SE}^{(l)}$, where $\tau_{SE}^{(l)} \in \mathcal{S}$, and c_l are complex coefficients. Using Eq. (10), we can expand each $\tau_{SE}^{(l)}$ as $\tau_{SE}^{(l)} = \sum_i a_{li} \rho_{SE}^{(i)} + Y^{(l)}$. So,

$$\begin{aligned} X &= \sum_{i=1}^{m} \left(\sum_l a_{li} c_l \right) \rho_{SE}^{(i)} + \sum_l c_l\, Y^{(l)} \\ &= \sum_{i=1}^{m} d_i \rho_{SE}^{(i)} + Y(X), \end{aligned} \tag{13}$$

where $d_i = \sum_l a_{li} c_l$ are complex coefficients and the linear operator $Y(X) = \sum_l c_l\, Y^{(l)}$ is such that $\mathrm{Tr}_E[Y(X)] = 0$. Consequently, for each $x \in \mathcal{V}_S$ we have

$$x = \mathrm{Tr}_E(X) = \sum_{i=1}^{m} d_i \rho_S^{(i)}, \tag{14}$$

where the coefficients d_i are the same as those in Eq. (13). In Fig. 1, the sets $\mathcal{S}_S$ and $\mathcal{D}_S$, the subspace $\mathcal{V}_S$, and the vector space $\mathcal{L}_S$ are given in a Venn diagram.

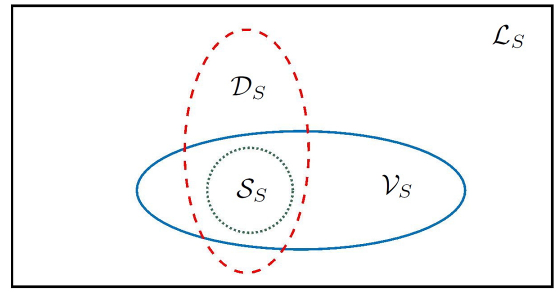

FIG. 1. The set $\mathcal{S}_S = \mathrm{Tr}_E \mathcal{S}$ (green dotted circle) is the set of initial states of the system S. The set $\mathcal{D}_S$ (red dashed ellipse) is the set of all states (density operators) on $\mathcal{H}_S$. Obviously, $\mathcal{S}_S \subseteq \mathcal{D}_S$. The subspace $\mathcal{V}_S$ (blue solid ellipse) is defined in Eq. (12), and so, $\mathcal{S}_S \subset \mathcal{V}_S$. Finally, $\mathcal{L}_S$ (black solid rectangle) is the set of all linear operators on $\mathcal{H}_S$. So, $\mathcal{D}_S \subset \mathcal{L}_S$ and $\mathcal{V}_S \subseteq \mathcal{L}_S$. When $\mathcal{S}_S = \mathcal{D}_S$, then $\mathcal{V}_S = \mathcal{L}_S$.

Now we can define the linear trace-preserving assignment map Λ_S as follows. First, we define $\Lambda_S(\rho_S^{(i)}) = \rho_{SE}^{(i)}$. Then we extend the definition of Λ_S to the entire $\mathcal{V}_S$ as a linear map. So, for any $x \in \mathcal{V}_S$, in Eq. (14), we have

$$\Lambda_S(x) = \sum_{i=1}^{m} d_i \Lambda_S\big(\rho_S^{(i)}\big) = \sum_{i=1}^{m} d_i \rho_{SE}^{(i)}. \tag{15}$$

The assignment map Λ_S maps $\mathcal{V}_S$ to (a subspace of) $\mathcal{V}$ and is Hermitian by construction. [When x is a Hermitian operator, all d_i in Eq. (14) are real. So, $\Lambda_S(x)$ is also a Hermitian operator.] Comparing Eqs. (13) and (15) shows that Λ_S does not necessarily map x to X, unless $Y(X) = 0$. In addition, note that the assignment map Λ_S in Eq. (15) is defined on the subspace $\mathcal{V}_S$. This definition can be extended to the entire $\mathcal{L}_S$ simply, i.e., one can find a Hermitian map Λ'_S on the entire $\mathcal{L}_S$ such that, for each $x \in \mathcal{V}_S$, it acts as Λ_S [8]. However, only for each $x \in \mathcal{V}_S$, but not necessarily for arbitrary $f \in \mathcal{L}_S$, we have $\mathrm{Tr}_E \circ \Lambda'_S(x) = \mathrm{Tr}_E \circ \Lambda_S(x) = x$. In other words, the *extension*Λ'_S of the assignment map Λ_S is *self-consistent* only on $\mathcal{V}_S$, not necessarily on the entire $\mathcal{L}_S$.

Now, using Eqs. (4), (9), (10), and (15), the reduced dynamics of the system for each $\rho_{SE} \in \mathcal{V}$ is given by

$$\begin{aligned} \rho'_S &= \mathrm{Tr}_E \circ \mathrm{Ad}_U(\rho_{SE}) \\ &= \sum_{i=1}^{m} a_i \mathrm{Tr}_E \circ \mathrm{Ad}_U\big(\rho_{SE}^{(i)}\big) + \mathrm{Tr}_E \circ \mathrm{Ad}_U(Y) \\ &= \mathrm{Tr}_E \circ \mathrm{Ad}_U \circ \Lambda_S(\rho_S) + \mathrm{Tr}_E \circ \mathrm{Ad}_U(Y) \\ &= \Phi_S(\rho_S) + \mathrm{Tr}_E \circ \mathrm{Ad}_U(Y), \end{aligned} \tag{16}$$

where $\Phi_S \equiv \mathrm{Tr}_E \circ \mathrm{Ad}_U \circ \Lambda_S$. The map Φ_S is a (linear) Hermitian map on $\mathcal{V}_S$ since Tr_E and Ad_U are CP [1] and the assignment map Λ_S is Hermitian on $\mathcal{V}_S$, as we saw in Eq. (15). When $\mathrm{Tr}_E \circ \mathrm{Ad}_U(Y) = 0$, the subspace $\mathcal{V}$ is called *U-consistent* [9]. The reduced dynamics of the system, for each $\rho_{SE} \in \mathcal{V}$, is given by the linear Hermitian trace-preserving map Φ_S if and only if $\mathcal{V}$ is U-consistent [9,34]. In Fig. 2 we represent when the Hermitian map Φ_S gives the reduced dynamics of the system in a commutative diagram. It is also worth noting that, in the theory of open quantum systems, one usually approximates the reduced dynamics as a

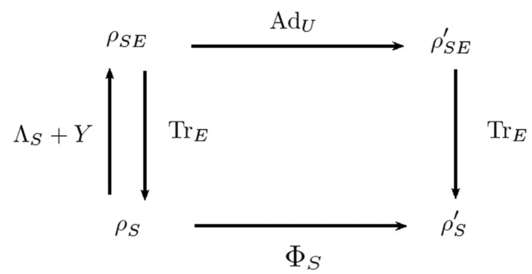

FIG. 2. The state ρ_{SE} is the initial state of the entire system-environment. The final state of the system-environment ρ'_{SE} is given in Eq. (3). Tracing over the environment E, gives the initial state of the system $\rho_S = \mathrm{Tr}_E(\rho_{SE})$, and its final state $\rho'_S = \mathrm{Tr}_E(\rho'_{SE})$. According to Eqs. (9), (10), and (15), $\Lambda_S(\rho_S) + Y$ gives ρ_{SE}. The map Φ_S is defined as $\Phi_S = \mathrm{Tr}_E \circ \mathrm{Ad}_U \circ \Lambda_S$. According to Eq. (16), Φ_S gives ρ'_S, if the U-consistency condition $\mathrm{Tr}_E \circ \mathrm{Ad}_U(Y) = 0$ is satisfied. Then, rounding the diagram clockwise, from ρ_S to ρ'_S, is equivalent to rounding it counterclockwise, through the Hermitian map Φ_S.

linear map, utilizing some simplifying assumptions (about $\mathcal{V}$) [3–5,35].

The CP-ness of Tr_E and Ad_U results in the fact that only the assignment map Λ_S determines whether Φ_S is CP or not. If Λ_S is Hermitian, then Φ_S can be either Hermitian, positive, or CP. But when the extension Λ'_S of the assignment map Λ_S is positive, then Φ_S is necessarily CP [8].

We end this section with the following point. Assuming unitary dynamics for the entire system-environment, the (non)linearity of the reduced dynamics is only a consequence of U-(in)consistency of the subspace $\mathcal{V}$. In other words, it is only a consequence of how we choose (construct) the initial set $\mathcal{S}$ and there is no fundamental reasoning behind it [34]. In addition, as discussed in [34], the nonlinearity of the reduced dynamics does not lead to superluminal signaling.

IV. MAIN RESULT

Assume that the reduced dynamics of the system for each $\rho_S \in \mathcal{S}_S$ is given by a *dynamical map* Ψ_S, i.e., the final state ρ'_S, in Eq. (4), is given by $\Psi_S(\rho_S)$. As discussed in the Introduction, in the axiomatic approach to quantum operations, postulating that the dynamical map Ψ_S is linear seems more natural than postulating it as a CP map. In addition, it can be shown simply [34] that when the map Ψ_S is linear, on the subspace $\mathcal{V}_S$, then it is equal to Φ_S in Eq. (16). Now we ask under what circumstances does only requiring that Ψ_S is linear [and so is equal to Φ_S, in Eq. (16)] result in that it is also CP? Such circumstances are given in the following Proposition.

Proposition 1. Requiring that the reduced dynamics of the system for each $\rho_S \in \mathcal{D}_S$ and for arbitrary system-environment unitary evolution U is a linear function of ρ_S results in the CP-ness of the assignment map Λ_S. Thus the reduced dynamics of the system S is CP, as Eq. (1).

Proof. First, we require that the reduced dynamics of the system, for arbitrary system-environment unitary evolution U is linear. So, the reduced dynamics is given by the map Φ_S in Eq. (16) for arbitrary U [34]. In other words, the subspace $\mathcal{V}$ in Eq. (11) is U-consistent for arbitrary U. This results in the one-to-one correspondence between the subspaces $\mathcal{V}$ and $\mathcal{V}_S = \mathrm{Tr}_E \mathcal{V}$ [9]. Hence, for each $X, Z \in \mathcal{V}$, $\mathrm{Tr}_E(X) = \mathrm{Tr}_E(Z)$ if and only if $X = Z$. It indicates that $Y(\rho_{SE})$ in Eq. (10) and so $Y(X)$ in Eq. (13) are zero. Therefore, $\Lambda_S(\rho_S) = \rho_{SE}$ and $\Lambda_S(x) = X$ where the linear assignment map Λ_S is defined in Eq. (15) and ρ_S, ρ_{SE}, X, and x are given in Eqs. (9), (10), (13), and (14), respectively.

Second, we require that the reduced dynamics of the system is linear for arbitrary initial state of the system $\rho_S \in \mathcal{D}_S$. This means that we choose the set of initial states of the system-environment $\mathcal{S}$ such that $\mathcal{S}_S = \mathcal{D}_S$. Therefore, since one can find $(d_S)^2$ linearly independent states in $\mathcal{D}_S$ (see, e.g., [36]), we have $\mathcal{V}_S = \mathrm{Span}_{\mathbb{C}}\, \mathcal{D}_S = \mathcal{L}_S$.

Note that we want to find the conditions which ensure the positivity of (the extension of) the assignment map Λ_S in Eq. (15). Requiring that, for a given U, the reduced dynamics is linear, for arbitrary initial state $\rho_S \in \mathcal{D}_S$, results that $\mathcal{S}_S = \mathcal{D}_S$ (and so $\Lambda'_S = \Lambda_S$ since $\mathcal{V}_S = \mathcal{L}_S$) and $\mathrm{Tr}_E \circ \mathrm{Ad}_U(Y) = 0$, where Y is given in Eq. (10). However, it does not necessitate that $Y = 0$. So the assignment map Λ_S, which maps ρ_S in Eq. (9) to $Z = \sum_{i=1}^{m} a_i \rho_{SE}^{(i)}$, is not necessarily positive since Z is not necessarily a positive operator. However, if we add the first requirement as well, which ensures that $Y = 0$, then we conclude that $\Lambda_S = \Lambda'_S$ is positive.

On the other hand, only assuming the first requirement though results in the positivity of Λ_S on $\mathcal{S}_S$, but it does not necessarily lead to the positivity of the extension Λ'_S of the assignment map Λ_S on the whole $\mathcal{D}_S$ ($\mathcal{L}_S$). Nevertheless, if we add the second requirement also, which states that $\mathcal{S}_S = \mathcal{D}_S$, we ensure that $\Lambda'_S = \Lambda_S$ is positive on the entire $\mathcal{D}_S$ ($\mathcal{L}_S$).

Consequently, assuming that both the first and the second requirements are satisfied *simultaneously*, results in the fact that $\Lambda'_S = \Lambda_S$ is positive on the entire $\mathcal{D}_S$. Now, it was showed that when there is a positive extension Λ'_S of the assignment map Λ_S, on the entire $\mathcal{D}_S$ ($\mathcal{L}_S$) then there exists a CP assignment map $\Lambda_S^{(\mathrm{CP})}$ also [8]. In fact, in this case, where $\mathcal{S}_S = \mathcal{D}_S$ and so $\Lambda'_S = \Lambda_S$, and, in addition, there is a one-to-one correspondence between the subspaces $\mathcal{V}$ and $\mathcal{V}_S$, there is a unique way to define (the extension of) the assignment map. So the CP assignment map $\Lambda_S^{(\mathrm{CP})}$ is the same as our positive $\Lambda_S = \Lambda'_S$, with the explicit form

$$\Lambda_S(\rho_S) = \Lambda_S^{(\mathrm{CP})}(\rho_S) = \rho_S \otimes \tilde{\omega}_E, \quad (17)$$

where $\tilde{\omega}_E$ is a fixed state on $\mathcal{H}_E$ [6,8,11]. This fact that $\tilde{\omega}_E$ is a fixed state is a consequence of assuming that the assignment map is a self-consistent positive map on the entire $\mathcal{D}_S$ ($\mathcal{L}_S$) [6,8,11]. The assignment map $\Lambda_S^{(\mathrm{CP})}$ given in Eq. (17) is, in fact, the famous Pechukas's one first introduced in [6]. Finally, the CP-ness of $\Lambda_S^{(\mathrm{CP})}$ leads to the CP-ness of the reduced dynamics $\Phi_S = \mathrm{Tr}_E \circ \mathrm{Ad}_U \circ \Lambda_S = \mathrm{Tr}_E \circ \mathrm{Ad}_U \circ \Lambda_S^{(\mathrm{CP})}$. ■

In the axiomatic approach to quantum operations it is more appropriate to postulate that the dynamical map Ψ_S is *convex-linear* instead of considering it linear. A convex-linear map is defined as follows.

Definition 1. When Ψ_S is convex-linear on $\mathcal{D}_S$ then we have $\Psi_S[p\rho_S + (1-p)\tau_S] = p\Psi_S(\rho_S) + (1-p)\Psi_S(\tau_S)$, where $\rho_S,\ \tau_S \in \mathcal{D}_S$, and $0 \leqslant p \leqslant 1$.

In the following Proposition, we refer to the convexity of the set $\mathcal{S}_S$. This property is defined as below.

Definition 2. When $\mathcal{S}_S$ is convex, if $\rho_S, \tau_S \in \mathcal{S}_S$, then also $\omega_S = p\rho_S + (1-p)\tau_S \in \mathcal{S}_S$ where $0 \leqslant p \leqslant 1$.

In Proposition 1, we saw that requiring the reduced dynamics of the system S is linear leads to its CP-ness. Now we want to go further and show that requiring the reduced dynamics is convex-linear results in the CP-ness of the reduced dynamics as well.

Proposition 1′. Requiring that the reduced dynamics of the system for each $\rho_S \in \mathcal{D}_S$ and for arbitrary system-environment unitary evolution U is a convex-linear function of ρ_S results in the CP-ness of the assignment map Λ_S, as in Eq. (17). Thus, the reduced dynamics is CP, as in Eq. (1), for arbitrary U and arbitrary $\rho_S \in \mathcal{D}_S$.

Proof. Since, as before, we have $\mathcal{S}_S = \mathcal{D}_S$, the set $\mathcal{S}_S$ is convex. Thus, we can show that the convex-linearity of the reduced dynamics results in its linearity, following a similar procedure as [34].

Note that some of the real coefficients a_i, in Eq. (9) are positive and the others are negative. Let us denote the positive ones as $a_i^{(+)}$ and the negative ones as $a_i^{(-)}$. So, from Eq. (9), we have

$$\rho_S + \sum_i |a_i^{(-)}|\rho_S^{(i)} = \sum_i a_i^{(+)}\rho_S^{(i)}. \tag{18}$$

Tracing from both sides we have $1 + \sum_i |a_i^{(-)}| = \sum_i a_i^{(+)} \equiv b$. Dividing both sides of Eq. (18) into b results in

$$\frac{1}{b}\left(\rho_S + \sum_i |a_i^{(-)}|\rho_S^{(i)}\right) = \frac{1}{b}\left(\sum_i a_i^{(+)}\rho_S^{(i)}\right) \equiv \omega_S, \tag{19}$$

where $\omega_S \in \mathcal{D}_S = \mathcal{S}_S$. Therefore, assuming that Ψ_S is convex-linear on $\mathcal{S}_S$ we have

$$\Psi_S(\omega_S) = \Psi_S\left(\frac{1}{b}\left(\rho_S + \sum_i |a_i^{(-)}|\rho_S^{(i)}\right)\right) = \Psi_S\left(\frac{1}{b}\left(\sum_i a_i^{(+)}\rho_S^{(i)}\right)\right)$$

$$\Rightarrow \quad \frac{1}{b}\left(\Psi_S(\rho_S) + \sum_i |a_i^{(-)}|\Psi_S\big(\rho_S^{(i)}\big)\right) = \frac{1}{b}\left(\sum_i a_i^{(+)}\Psi_S\big(\rho_S^{(i)}\big)\right), \tag{20}$$

which leads to

$$\Psi_S(\rho_S) = \sum_{i=1}^{m} a_i \Psi_S(\rho_S^{(i)}). \tag{21}$$

So, noting Eq. (9), we conclude that Ψ_S is linear. Hence, if Ψ_S is convex-linear, for arbitrary U and arbitrary $\rho_S \in \mathcal{D}_S$, then it is also linear for arbitrary U and arbitrary $\rho_S \in \mathcal{D}_S$. Now Proposition 1 shows that the assignment map Λ_S is CP, as Eq. in (17), and so the reduced dynamics of the system $\Psi_S = \Phi_S$ is also CP. ■

V. SUMMARY

Requiring that the reduced dynamics of the system S interacting with its environment E is (convex) linear means that (1) the reduced dynamics is (convex) linear for arbitrary system-environment evolution U, and (2) the reduced dynamics is (convex) linear for arbitrary initial state of the system $\rho_S \in \mathcal{D}_S$.

In Proposition 1 (1′), it was shown that the above requirement results in the CP-ness of the reduced dynamics. So in the axiomatic approach to quantum operations there is no need to consider the CP-ness as a distinct postulate. It is only a consequence of (convex) linearity.

In addition, when the reduced dynamics is (convex) linear for arbitrary U and arbitrary ρ_S, then the set of initial states of the system-environment is as $\mathcal{S} = \{\rho_S \otimes \tilde{\omega}_E\}$, where ρ_S is an arbitrary state of the system and $\tilde{\omega}_E$ is a fixed state of the environment. In other words, under such circumstances the assignment map is as the Pechukas's one [6], given in Eq. (17).

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