High-temperature virial expansion of contour Green's functions

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High-temperature virial expansion is a powerful tool in equilibrium statistical mechanics. In this paper we generalize this method to the study of nonequilibrium dynamics based on contour Green's functions and present the diagrammatic rules to calculate the virial expansion coefficients. As an application of our theory, we recover the dynamics of a Bose gas quenched from noninteracting to finite interaction and obtain analytically the long-time limit of the momentum distribution function of the gas, consistent with previous numerical results.

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I. INTRODUCTION

Understanding the dynamics of quantum systems far from equilibrium is one of the most pressing problems in theoretical many-body physics. Although, for example, the contour Green's function introduced by Schwinger [1] and Keldysh [2] provides a general theoretical frame to solve this problem, a full description of nonequilibrium dynamics is still mathematically challenging. Most contour-Green's-function-based methods presently available rely to a certain extent on a perturbation expansion, which either requires a weak coupling or breaks down at long times of the evolution [3]. When dealing with quantum many-body dynamics far from equilibrium or strongly coupled systems, refined methods beyond straightforward perturbation expansion are demanding.

In this paper we introduce high-temperature virial expansion into the contour-Green's-function method. Virial expansion uses fugacity $z = e^{\beta\mu}$ as an expansion parameter [4], where β and μ are the inverse temperature and chemical potential, respectively. Virial expansion is expected to be valid when the thermal wavelength $\lambda = \sqrt{2\pi\beta/m}$ is short compared to the average interparticle distance and long compared to the range of interactions. As a powerful nonperturbative method, virial expansion has been widely used in studying dilute atomic gases [5–23], in particular strongly interacting systems, and is consistent with experimental results [24–31]. However, most of its applications are limited to equilibrium systems. Our method combines virial expansion with the contour Green's function and can be applied to strongly interacting and far-from-equilibrium systems. As an application of our theory, we consider the dynamics of momentum distribution when a Bose gas is quenched from noninteracting to finite interactions. This problem was recently studied by Sun *et al.* using an operator-based virial expansion method [32]. We recover their results, and moreover, compared with the operator-based approach, our contour-Green's-function-based method is expected to be easier for extension to more general and complicated situations.

II. GENERAL FORMALISM

We consider a system initially in equilibrium at a given inverse temperature β and chemical potential μ. Without loss of generality, we assume that the system is governed by a time-independent Hamiltonian $\hat{H}_0$ when $t < 0$, and the Hamiltonian acquires some time dependence and changes to $\hat{H}(t)$ after $t = 0$.

We define the oriented contour γ which goes from $t = 0$ to $+\infty$ and then back to 0 and finally to $t = -i\beta$, as illustrated in Fig. 1(a). The contour γ consists of three paths: a forward branch γ_-, a backward branch γ_+, and a vertical track γ_M. A generic point z of γ can lie on γ_-, γ_+, or γ_M. We define by $z = t_-$ the point of γ lying on the branch γ_- with value t and by $z = t_+$ the point of γ lying on the branch γ_+ with value t.

The Hamiltonian with arguments on γ_- and γ_+ is defined according to $\hat{H}(z = t_\pm) \equiv \hat{H}(t)$, while the Hamiltonian with arguments on γ_M is defined as $\hat{H}(z) = \hat{H}_M \equiv \hat{H}_0 - \mu\hat{N}$. Correspondingly, the contour Green's function is defined as [3]

$$G(\mathbf{x}, z; \mathbf{x}', z') \equiv -i\frac{\mathrm{Tr}[e^{-\beta\hat{H}_M}\mathcal{T}\{\hat{\psi}_H(\mathbf{x}, z)\hat{\psi}_H^\dagger(\mathbf{x}', z')\}]}{\mathrm{Tr}(e^{-\beta\hat{H}_M})},$$

where the subscript H denotes Heisenberg picture and $\mathcal{T}$ represents the contour ordering operator which orders along the contour γ, moving operators with "later" contour arguments to the left. By introducing the contour integral, the contour Green's function can be rewritten as

$$G(\mathbf{x}, z; \mathbf{x}', z') \equiv -i\frac{\mathrm{Tr}\,(\mathcal{T}\{\exp[-i\int_\gamma d\bar{z}\hat{H}(\bar{z})]\hat{\psi}(\mathbf{x}, z)\hat{\psi}^\dagger(\mathbf{x}', z')\})}{\mathrm{Tr}\,(\mathcal{T}\{\exp[-i\int_\gamma d\bar{z}\hat{H}(\bar{z})]\})}.$$

Similar to the case of the equilibrium Green's function, we can utilize Wick's theorem to expand a full contour Green's function into a free contour Green's function, which has the

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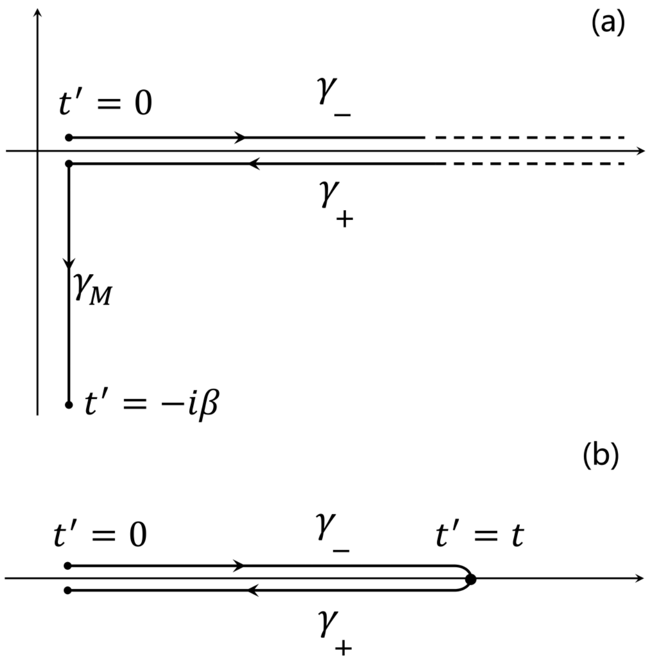

FIG. 1. (a) Oriented contour γ in the complex time plane. The contour consists of a forward and a backward branch along the real axis between 0 and $+\infty$ and a vertical track going from 0 to $i\beta$. The branches along the real axis are displaced from the real axis only for graphical purposes. (b) For the calculation of the dynamics of momentum distribution of a quenched Bose gas, the contour γ can be shrunken to a forward branch going from 0 to t and a backward branch going from t to 0 and does not contain the vertical branch.

form

$$G^0(k;z_1,z_2) = -i\left[\theta(z_1,z_2) \pm \frac{1}{e^{\beta(\varepsilon_k-\mu)} \mp 1}\right]e^{-i\varepsilon_k(z_1-z_2)},$$

where $\varepsilon_k = k^2/2m$, with m representing the mass of the constituting atom, the upper (lower) sign refers to bosons (fermions), and $\theta(z_1,z_2)$ represents the Heaviside function on the contour, i.e., $\theta(z_1,z_2)=1$ if z_1 is later than z_2 on the contour and zero otherwise. Here the free contour Green's function corresponds to a noninteracting Hamiltonian $\hat{H}_0 = \sum_i \mathbf{p}_i^2/2m$.

In the high-temperature limit, the fugacity $z = e^{\beta\mu}$ is smaller than 1 and the free Green's function can be expanded in powers of the fugacity

$$\begin{aligned} G^0(k;z_1,z_2) &= -i\left[\theta(z_1,z_2) + \sum_{n=1}(\pm)^n e^{-n\beta\varepsilon_k} z^n\right]e^{-i\varepsilon_k(z_1-z_2)} \\ &= \sum_{n\geqslant 0} G^{(0,n)}(\mathbf{p},\tau)z^n, \end{aligned} \tag{1}$$

with

$$G^{(0,0)}(k;z_1,z_2) = -i\theta(z_1,z_2)e^{-i\varepsilon_k(z_1-z_2)},$$

$$G^{(0,n)}(k;z_1,z_2) = -i(\pm)^n e^{-n\beta\varepsilon_{\mathbf{p}}}e^{-i\varepsilon_k(z_1-z_2)}, \quad n\geqslant 1.$$

Obviously, $G^{(0,0)}$ is a retarded function, while $G^{(0,n)}$, for $n \geqslant 1$, is not.

Feynman diagrams are utilized to calculate the virial expansion of the full contour Green's function. Diagrammatically, since $G^{(0,0)}$ is a retarded function, we represent it as a line with an arrow following contour time order. On the other hand, $G^{(0,n)}$, for $n \geqslant 1$, is not retarded and we represent it as an n-times slashed line, which can be oriented in either direction. Other diagrammatic rules are the same as usual, except that we need to use $G^{(0,n)}$ instead of G^0. Different from diagrammatic rules for virial expansion of equilibrium systems previously given in Ref. [8], we are working in real time rather than in imaginary time; nonetheless, we adopt diagrammatical symbols similar to those in Ref. [8].

For convenience of calculations, we would convert contour integrals into standard real-time integrals and products of functions on the contour into products of functions with real-time arguments. Thus we define four components of the contour Green's function as the following functions on the real time axis [3]:

$$\begin{aligned} G^>(t,t') &\equiv G(t_+,t'_-), \\ G^<(t,t') &\equiv G(t_-,t'_+), \\ G^T(t,t') &\equiv G(t_-,t'_-), \\ G^{\overline{T}}(t,t') &\equiv G(t_+,t'_+). \end{aligned} \tag{2}$$

As for the corresponding free contour Green's functions $G^{>,0}(k;t,t')$, $G^{<,0}(k;t,t')$, $G^{T,0}(k;t,t')$, and $G^{\overline{T},0}(k,t,t')$, virial expansion of them can be readily obtained from Eqs. (1) and (2). To illustrate our method, in next section we consider the time evolution of a Bose gas when quenched from noninteracting to finite interaction.

III. QUENCH DYNAMICS OF BOSE GASES

In the following we consider a Bose gas initially in the equilibrium state of inverse temperature β and chemical potential μ, characterized by a noninteracting Hamiltonian $\hat{H}_0$. Starting from $t=0$, the interaction, described by an s-wave scattering length a_s, is switched on.

What we want to calculate is the density distribution function, which is related to the Green's function via the relation

$$n(\mathbf{k},t) = iG^T(\mathbf{k},t-0^+,t).$$

When $G^T(\mathbf{k},t-0^+,t)$ is expanded into free contour Green's functions, all the information we need is in the interval of $(0,t)$ and the interaction vanishes on the γ_M branch; thus, for our purpose it is sufficient to use a shrunken contour depicted in Fig. 1(b). The density $n(\mathbf{k},t)$ can be expanded in powers of fugacity

$$n(\mathbf{k},t) = \delta n^{(1)}(\mathbf{k},t)z + \delta n^{(2)}(\mathbf{k},t)z^2 + \cdots.$$

To calculate the coefficient $\delta n^{(p)}$, we need to find all the Feynman diagrams for $G^T(\mathbf{k},t-0^+,t)$ of order z^p, which contain one $G^{(0,p)}$, or one $G^{(0,p-1)}$ and one $G^{(0,1)}$, or one $G^{(0,p-2)}$ and one $G^{(0,2)}$, or one $G^{(0,p-2)}$ and two $G^{(0,1)}$, etc.

The first-order coefficient can be obtained from Fig. 2(a) as

$$\begin{aligned} \delta n^{(1)}(\mathbf{k},t) &= iG^{T,(0,1)}(\mathbf{k},t-0^+,t) \\ &= e^{-\beta\varepsilon_{\mathbf{k}}}. \end{aligned} \tag{3}$$

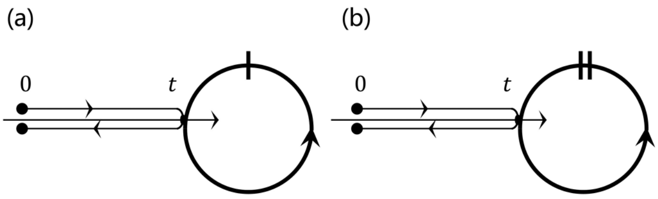

FIG. 2. (a) Lowest-order diagram contributing to $\delta n^{(1)}(\mathbf{k}, t)$. (b). Second-order diagram contributing to $\delta n^{(2,a)}(\mathbf{k}, t)$. Here and in the following figures, thin lines represent time contours, while thick lines represent Feynman diagrams.

In order to calculate the second-order coefficient $\delta n^{(2)}(\mathbf{k}, t)$, we consider all the diagrams with one $G^{(0,2)}$ or two $G^{(0,1)}$. The diagram of Fig. 2(b) gives the contribution with one $G^{(0,2)}$, that is,

$$\begin{aligned}\delta n^{(2,a)}(\mathbf{k}, t) &= iG^{T,(0,2)}(\mathbf{k}, t-0^+, t)\\ &= e^{-2\beta\varepsilon_{\mathbf{k}}},\end{aligned} \tag{4}$$

while the diagrams of Figs. 3 and 4 give the contribution with two $G^{(0,1)}$, which are denoted by $\delta n^{(2,b)}(\mathbf{k}, t)$ and $\delta n^{(2,c)}(\mathbf{k}, t)$, respectively, and will be calculated in the following two sections, respectively.

A. Contribution $\delta n^{(2,b)}(\mathbf{k}, t)$

We now consider the contribution of the diagrams in Fig. 3, where Figs. 3(a) and 3(b) correspond to contributions from branch γ_+ and branch γ_-, respectively. Both diagrams contain the effect of interactions through the two-body T matrix. We first calculate the contribution of Fig. 3(a), that is,

$$\begin{aligned}\delta n^{(2,b,1)}(\mathbf{k}, t) = \sum_{\mathbf{P}} \int_{\mathcal{D}} dt_1 dt_2 [iG^{\overline{T},(0,1)}(\mathbf{P}-\mathbf{k}, t_1 - t_2)]\\ \times [iT_2(\mathbf{P}, t_2 - t_1)][iG^{\overline{T},(0,1)}(\mathbf{k}, t - t_2)]\\ \times [iG^{\overline{T},(0,0)}(\mathbf{k}, t_1 - t)],\end{aligned}$$

where T_2 denotes the two-body T matrix and is given by the sum of ladder diagrams. The time integration is in the time domain $\{0 < t_2 < t_1 < t\}$, corresponding to the domain for time differences $\{\tau_1 > 0, \tau_2 > 0, t - \tau_1 - \tau_2 > 0\}$, where $\tau_1 = t - t_1$ and $\tau_2 = t_1 - t_2$; thus we have

$$\begin{aligned}\delta n^{(2,b,1)}(\mathbf{k}, t) = i\sum_{\mathbf{P}} \int d\tau_1 d\tau_2 \Theta(\tau_1)\Theta(\tau_2)\Theta(t-\tau_1-\tau_2)\\ \times T_2(\mathbf{P}, -\tau_2) e^{-(i\tau_2+\beta)\varepsilon_{\mathbf{P}-\mathbf{k}}} e^{-[i(\tau_1+\tau_2)+\beta]\varepsilon_{\mathbf{k}}} e^{i\tau_1\varepsilon_{\mathbf{k}}}.\end{aligned}$$

The integration on τ_1 is easily done, resulting in

$$\begin{aligned}&\delta n^{(2,b,1)}(\mathbf{k}, t)\\ &= i\sum_{\mathbf{P}} \int d\tau_2 \Theta(\tau_2)\Theta(t-\tau_2)(t-\tau_2) T_2(\mathbf{P}, -\tau_2)\\ &\times e^{-(i\tau_2+\beta)(\varepsilon_{\mathbf{k}}+\varepsilon_{\mathbf{P}-\mathbf{k}})}.\end{aligned} \tag{5}$$

On the other hand, the contribution of the graph of Fig. 3(b) is

$$\begin{aligned}\delta n^{(2,b,2)}(\mathbf{k}, t) &= \sum_{\mathbf{P}} \int_{\mathcal{D}} dt_1 dt_2 [iG^{T,(0,1)}(\mathbf{P}-\mathbf{k}, t_2 - t_1)]\\ &\quad\times[-iT_2(\mathbf{P}, t_1 - t_2)][iG^{T,(0,1)}(\mathbf{k}, t_2 - t)]\\ &\quad\times[iG^{T,(0,0)}(\mathbf{k}, t - t_1)]\\ &= [\delta n^{(2,b,1)}(\mathbf{k}, t)]^*.\end{aligned} \tag{6}$$

Combining Eqs. (5) and (6) and using the convolution theorem for Fourier transforms, we have

$$\begin{aligned}&\delta n^{(2,b)}(\mathbf{k}, t)\\ &\equiv \delta n^{(2,b,1)}(\mathbf{k}, t) + \delta n^{(2,b,2)}(\mathbf{k}, t)\\ &= -i\sum_{\mathbf{P}} e^{-\beta(\varepsilon_{\mathbf{k}}+\varepsilon_{\mathbf{P}-\mathbf{k}})} \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, t_2\left(s - \frac{P^2}{4m} - i0^+\right)\\ &\times \frac{e^{it(s-i0^+-\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{P}-\mathbf{k}})}}{(s-i0^+-\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{P}-\mathbf{k}})^2} + \text{c.c.},\end{aligned} \tag{7}$$

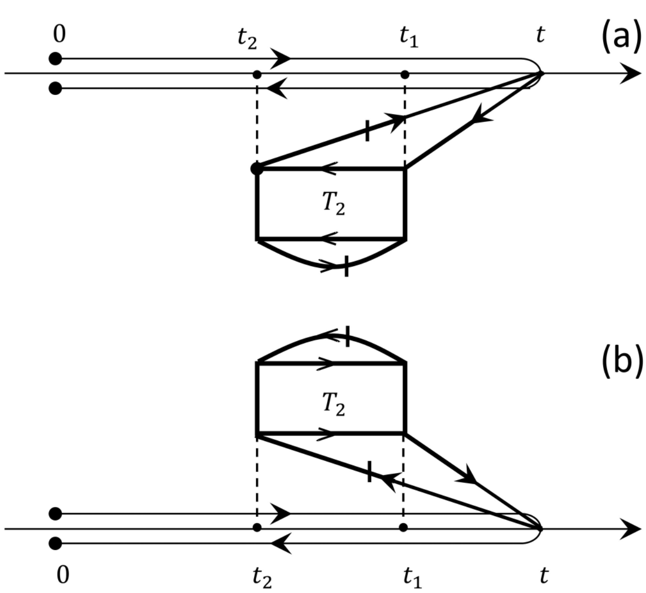

FIG. 3. Second-order diagram contributing to $\delta n^{(2,b)}(\mathbf{k}, t)$.

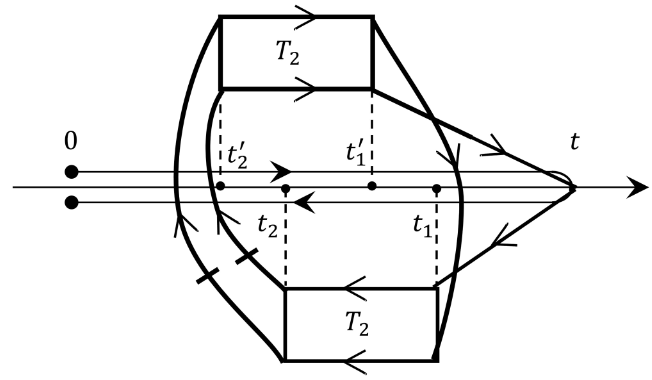

FIG. 4. Second-order diagram contributing to $\delta n^{(2,c)}(\mathbf{k}, t)$.

where $t_2(s)$ for two identical bosons reads [33]

$$t_2(s) = \frac{8\pi}{m}\frac{1}{a^{-1} - \sqrt{-ms}}.$$

Letting $s' = s - P^2/4m$ and $\mathbf{q} = \mathbf{P}/2 - \mathbf{k}$, Eq. (7) is rewritten as

$$\delta n^{(2,b)}(\mathbf{k}, t) = -8i \sum_{\mathbf{q}} e^{-\beta(\mathbf{k}^2/m + 2\mathbf{q}^2/m + 2\mathbf{q}\cdot\mathbf{k}/m)} e^{-it\varepsilon_{\mathbf{q}}} \frac{1}{2\pi}\int_{-\infty}^{\infty} ds' t_2(s' - i0^+)\frac{e^{it(s'-i0^+)}}{(s' - i0^+ - \varepsilon_{\mathbf{q}})^2} + \text{c.c.} \tag{8}$$

We choose the branch cut of the function $\sqrt{-s'}$ along the positive imaginary axis and deform the above integral on s' along the positive imaginary axis, which results in two extra terms according to the residue theorem. The two terms come from the pole of $t_2(s)$ at $-|E_b| = -1/ma^2$ (if $a > 0$) and the pole $s' = \varepsilon_{\mathbf{q}}$, respectively. Thus we have

$$\begin{aligned}\delta n^{(2,b)}(\mathbf{k}, t)\\ &= -\frac{32a}{\pi\beta k}\int_0^\infty dq\, q\, e^{-\beta(k^2/m + 2q^2/m)} \sinh\left(\frac{2\beta kq}{m}\right)\left\{\frac{[1 - 2\Theta(a)\cos(E_b + \varepsilon_q)t]/E_b + (\varepsilon_q/E_b + 1)(\varepsilon_q/E_b)^{1/2} t}{(\varepsilon_q/E_b + 1)^2}\right.\\ &\left. - \int_0^\infty ds\left[e^{-(s+i\varepsilon_q)t}\left(\frac{1}{1 - e^{i3\pi/4}\sqrt{s/E_b}} - \frac{1}{1 - e^{-i\pi/4}\sqrt{s/E_b}}\right)\frac{1}{(is - \varepsilon_q)^2} + \text{c.c.}\right]\right\}.\end{aligned} \tag{9}$$

B. Contribution $\delta n^{(2,c)}(\mathbf{k}, t)$

We now calculate the contribution of Fig. 4, which is

$$\begin{aligned}\delta n^{(2,c)}(\mathbf{k}, t) = \frac{1}{2}\sum_{\mathbf{P}}\sum_{\mathbf{q}}\int_{\mathcal{D}} dt_1 dt_2 dt_1' dt_2' [iG^{>,(0,0)}(\mathbf{P} - \mathbf{k}, t_1 - t_1')][-iT_2(\mathbf{P}, t_1' - t_2')][iG^{T,(0,0)}(\mathbf{k}, t - t_1')]\\ \times [iG^{\overline{T},(0,0)}(\mathbf{k}, t_1 - t)][iG^{<,(0,1)}(\mathbf{P} - \mathbf{q}, t_2' - t_2)][iT_2(\mathbf{P}, t_2 - t_1)][iG^{<,(0,1)}(\mathbf{q}, t_2' - t_2)],\end{aligned}$$

where the domain of time integration is $\{0 < t_2 < t_1 < t, 0 < t_2' < t_1' < t\}$. Performing the integration on t_1, t_2, t_1', t_2', the remaining integral is of the form

$$\delta n^{(2,c)}(\mathbf{k}, t) = \frac{1}{2}\sum_{\mathbf{P}}\sum_{\mathbf{q}} e^{-\beta(\varepsilon_{\mathbf{q}} + \varepsilon_{\mathbf{P-q}})} H(\mathbf{P}, \mathbf{k}, t),$$

with

$$H(\mathbf{P}, \mathbf{k}, t) = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, t_2\left(s - \frac{P^2}{4m} - i0^+\right)\frac{1}{(s - i0^+ - \varepsilon_{\mathbf{q}} - \varepsilon_{\mathbf{P-q}})(s - i0^+ - \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{P-k}})} e^{i(s-i0^+)t}\right|^2.$$

Letting $\mathbf{p} = \mathbf{P}/2 - \mathbf{q}$, $\mathbf{K} = \mathbf{P}/2 - \mathbf{k}$, and $s' = s - P^2/4m$, $\delta n^{(2,c)}(\mathbf{k}, t)$ can be simplified as

$$\delta n^{(2,c)}(\mathbf{k}, t) = \frac{m}{2\pi^4\beta k}\int_0^\infty K\, dK \int_0^\infty dp\, p^2 e^{-\beta(\mathbf{p}^2 + K^2 + k^2)/m}\sinh(2\beta kK/m) H(K, p, t)$$

and $H(P, k, t)$ can be rewritten as an integral on s',

$$H(K, p, t) = \left|\frac{4}{m}\int_{-\infty}^{\infty} ds' \frac{1}{a^{-1} - \sqrt{-m(s' - i0^+)}}\frac{1}{(s' - i0^+ - p^2/m)(s' - i0^+ - K^2/m)} e^{i(s'-i0^+)t}\right|^2.$$

Similar to the derivation of Eq. (9), the above integral on s' changes into

$$\begin{aligned}H(K, p, t) = 16a^6\left|2\pi\left[\frac{2ie^{-itE_b}}{(a^2K^2 + 1)(a^2p^2 + 1)}\Theta(a) - \frac{e^{itK^2/m}}{(aK + i)(a^2K^2 - a^2p^2)} - \frac{e^{itp^2/m}}{(ap + i)(a^2p^2 - a^2K^2)}\right]\right.\\ \left. + i\int_0^\infty ds' \frac{E_b e^{-s't}}{(is' - K^2/m)(is' - p^2/m)}\left[\frac{1}{1 - (s'/E_b)^{1/2}e^{i3\pi/4}} - \frac{1}{1 - (s'/E_b)^{1/2}e^{-i\pi/4}}\right]\right|^2.\end{aligned} \tag{10}$$

C. Long-time limit of $n(\mathbf{k}, t)$

In this section we consider the long-time limit of the density distribution function $n(\mathbf{k}, t)$. Obviously, $\delta n^{(1)}(\mathbf{k}, t)$ and $\delta n^{(2,a)}(\mathbf{k}, t)$ are essentially time independent, so in the following we focus on the $\delta n^{(2,b)}(\mathbf{k}, t)$ and $\delta n^{(2,c)}(\mathbf{k}, t)$.

For $\delta n^{(2,b)}(\mathbf{k})$, the integration on s in Eq. (9) can be seen as a Laplacian transformation of a function of s, which should vanish in the limit of $t \to \infty$. Moreover, the term proportional to $\cos(E_b + \varepsilon_q)t$ also vanishes in the limit of $t \to \infty$ according to the Riemann-Lebesgue lemma. Thus we have

$$\delta n^{(2,b)}(\mathbf{k},t)|_{t\to\infty} \to -\frac{32a}{\pi E_b \beta k}\int_0^\infty dq\, q\, e^{-\beta(k^2/m+2q^2/m)} \times \sinh\left(\frac{2\beta kq}{m}\right)\frac{1+(a^2q^2+1)qaE_bt}{(a^2q^2+1)^2}.$$

For $\delta n^{(2,c)}(\mathbf{k},t)$, when we expand $H(K,p,t)$ in Eq. (10), the terms with integration on s and the terms proportional to $e^{it(K^2/m-E_b)}$ and $e^{it(p^2/m-E_b)}$ vanish in the limit $t\to\infty$ for the same reason as above, which results in

$$\begin{aligned}&\delta n^{(2,c)}(\mathbf{k},t)|_{t\to\infty}\\ &\to \frac{32m}{\pi^2\beta k}\int_0^\infty K\,dK\int_0^\infty dp\,p^2 e^{-\beta(p^2+K^2+k^2)/m}\\ &\quad\times\frac{\sinh(2\beta kK/m)}{(a^2K^2+1)(a^2p^2+1)}\\ &\quad\times[I_0(K,q)+I_1(K,q,t)+I_2(K,q,t)],\end{aligned} \tag{11}$$

with

$$I_0(K,q)=\frac{4a^6\Theta(a)}{(a^2K^2+1)(a^2q^2+1)}+\frac{a^4}{(K+q)^2},$$

$$\begin{aligned}I_1(K,q,t)&=\frac{2a^3\sin[t(K^2-p^2)/m]}{(K^2-p^2)(K+p)},\\ I_2(K,q,t)&=\frac{2a^2(a^2Kp+1)\{1-\cos[t(K^2-p^2)/m]\}}{(K^2-p^2)^2}.\end{aligned} \tag{12}$$

First we consider $I_1(K,q,t)$. When $t\to\infty$, we have

$$\begin{aligned}I_1(K,q,t)|_{t\to\infty}&\to\frac{2\pi a^3 t\delta(K^2-q^2)}{m(K+q)}\\ &=\frac{\pi a^3}{2q^2}\delta(K-q).\end{aligned} \tag{13}$$

Next we consider an integral associated with $I_2(K,q,t)$ in Eq. (11), i.e.,

$$L\equiv\int_0^\infty dp\,p^2\frac{e^{-\beta p^2/m}}{a^2p^2+1}I_2(K,q,t).$$

Letting $Q=p^2$, the above equation is rewritten as

$$\begin{aligned}L&=\int_0^\infty dQ\frac{a^2(a^2QK+\sqrt{Q})}{a^2Q+1}\frac{e^{-\beta Q/m}\{1-\cos[t(K^2-Q)/m]\}}{(K^2-Q)^2}\\ &=\mathrm{Re}\left\{\int_0^\infty dQ\frac{a^2(a^2QK+\sqrt{Q})}{a^2Q+1}\frac{e^{-\beta Q/m}[1-e^{it(Q-K^2)/m}+it(Q-K^2)/m]}{(Q-K^2)^2}\right\}.\end{aligned}$$

According to residue theorem, the above integral can be deformed to along the positive imaginary axis, resulting in

$$L=\mathrm{Re}\left\{i\int_0^\infty dQ\frac{a^2(ia^2QK+\sqrt{Q}e^{i\pi/4})}{ia^2Q+1}\frac{e^{-i\beta Q/m}[1-e^{-t(Q+iK^2)/m}-t(Q+iK^2)/m]}{(iQ-K^2)^2}\right\}.$$

When $t\to\infty$, the terms proportional to $e^{-Qt/m}$ are discarded and we are left with

$$L|_{t\to\infty}\to\mathrm{Re}\left\{i\int_0^\infty dQ\frac{a^2(ia^2QK+\sqrt{Q}e^{i\pi/4})}{ia^2Q+1}\frac{e^{-i\beta Q/m}[1-t(Q+iK^2)/m]}{(iQ-K^2)^2}\right\}=J(K,\beta)+t\pi Ke^{-\beta K^2/m}, \tag{14}$$

with

$$\begin{aligned}J(K,\beta)&=a^2\frac{\pi e^{-\beta K^2/m}\left[\frac{2\beta K^2(a^2K^2+1)}{m}+(a^2K^2-1)\right]\mathrm{erfi}\left(\sqrt{\frac{\beta K^2}{m}}\right)+2a^2K^2e^{-\beta K^2/m}\left(\frac{a^2\beta K^2K^2}{m}+\frac{\beta K^2}{m}-1\right)\mathrm{Ei}\left(\frac{\beta K^2}{m}\right)}{2K(a^2K^2+1)^2}\\ &\quad-a^2\frac{2(1+a^2K^2)\sqrt{\frac{\pi\beta K^2}{m}}+2aK\{a^3K^3+e^{\beta E_b}[\pi\,\mathrm{erfc}(\beta E_b)-aK\,\mathrm{Ei}(-\beta E_b)]+aK\}}{2K(a^2K^2+1)^2},\end{aligned}$$

where erfc, erfi, and Ei represent the complementary error function, imaginary error function, and exponential integral function, respectively. With Eqs. (9) and (11)–(14) we have

$$\begin{aligned}f(k)\equiv&\,\delta n^{(2,b)}(k)+\delta n^{(2,c)}(k)\\ =&-\frac{16ma^3}{\pi\beta k}e^{-\beta k^2/m}\int_0^\infty K\,dK\,e^{-2\beta K^2/m}\frac{\sinh(2\beta kK/m)}{(a^2K^2+1)^2}+\frac{32m}{\pi^2\beta k}e^{-\beta k^2/m}\int_0^\infty K\,dK\,e^{-\beta K^2/m}\frac{\sinh(2\beta kK/m)}{a^2K^2+1}J(K,\beta)\\ &+\frac{32m}{\pi^2\beta k}e^{-\beta k^2/m}\int_0^\infty K\,dK\int_0^\infty dp\,p^2e^{-\beta(\mathbf{p}^2+K^2)/m}\frac{\sinh(2\beta kK/m)}{(a^2K^2+1)(a^2p^2+1)}I_0(K,q)\\ =&\frac{32m}{\pi^2\beta k}e^{-\beta k^2/m}\int_0^\infty K\,dK\,e^{-\beta K^2/m}\frac{\sinh(2\beta kK/m)}{a^2K^2+1}\left\{J(K,\beta)-\frac{\pi a^3}{2}\frac{e^{-\beta K^2/m}}{a^2K^2+1}+\int_0^\infty dp\,p^2e^{-\beta p^2}\frac{I_0(K,q)}{a^2p^2+1}\right\}.\end{aligned}$$

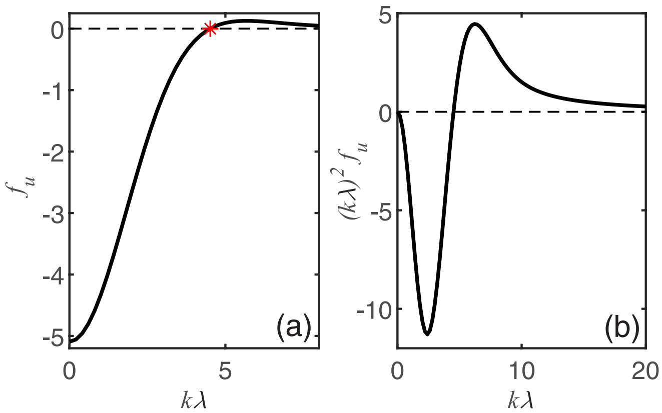

FIG. 5. (a) Plot of f_u as a function of $k\lambda$. The star indicates the zero point of f_u, which is at $k^* \approx 4.514\lambda^{-1}$. (b) Plot of $(k\lambda)^2 f_u$ as a function of $k\lambda$, which is proportional to the variation of occupation in a momentum shell with radius k long after quench up to second order.

In unitary limit, $f(k)$ has a simplified form

$$f_u(k) = \frac{32e^{-(k\lambda)^2/2\pi}}{\pi(k\lambda)} \int_0^\infty dp \sinh[(k\lambda)p/\pi] \, e^{-p^2/2\pi}\left[e^{-p^2/2\pi}\operatorname{erfi}\left(\frac{p}{\sqrt{2\pi}}\right) - \frac{\sqrt{2}}{p}\right]. \tag{15}$$

Combing Eqs. (3), (4), and (15), we obtain the long-time limit of the momentum density distribution function in the unitary limit up to order of z^2,

$$n(k,t)|_{t\to\infty} \approx e^{-\beta\varepsilon_{\mathbf{k}}}z + [e^{-2\beta\varepsilon_{\mathbf{k}}} + f_u(k)]z^2.$$

Compared with the second-order virial expansion of the initial distribution function $n(k,t)|_{t=0} \approx e^{-\beta\varepsilon_{\mathbf{k}}}z + e^{-2\beta\varepsilon_{\mathbf{k}}}z^2$, it is obvious that $f_u(k)$ represents the change of momentum density distribution up to second order and it depends on $k\lambda$ in a nonmonotonic way, as shown in Fig. 5. From Fig. 5 it can be seen that $f_u(k)$ has a zero point at $k^* \approx 4.514\lambda^{-1}$, where $n(k^*,t)|_{t\to\infty} = n(k^*,t)|_{t=0}$. Furthermore, for $k < k^*$, $n(k)$ decreases long after quench, while for $k > k^*$, $n(k)$ increases. In Ref. [32] the authors numerically obtained the limit density distribution function; here we give the analytical expression.

IV. CONCLUSION

We have developed a contour-Green's-function-based virial expansion method, which is applicable to strong-coupling and far-from-equilibrium situations. In addition, we present corresponding diagrammatic rules, and analytical results could be obtained by computing a small set of diagrams linked with few-body problems. Our approach can in principle solve more general nonequilibrium problems other than quench dynamics and can incorporate high-order contribution as well, although it may be harder to tackle numerically.

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