Scattering hypervolume of fermions in two dimensions

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We define the three-body scattering hypervolume D_F for identical spin-polarized fermions in two dimensions by considering the wave function of three such fermions colliding at zero energy and zero orbital angular momentum. We derive the asymptotic expansions of such a wave function when three fermions are far apart or one pair and the third fermion are far apart, and D_F appears in the coefficients of such expansions. For weak-interaction potentials, we derive an approximate formula of D_F by using the Born expansion. We then study the shift of energy of three such fermions in a large periodic area due to D_F. This shift is proportional to D_F times the square of the area of the triangle formed by the momenta of the fermions. We also calculate the shifts of energy and of pressure of spin-polarized two-dimensional Fermi gases due to a nonzero D_F and the three-body recombination rate of spin-polarized ultracold atomic Fermi gases in two dimensions.

I. INTRODUCTION

In quantum mechanics, the behavior of particles colliding with low energy depends sensitively on the dimension of space. For the zero-energy s-wave collision of two particles in d-dimensional space, the wave function takes the following form:

$$\phi_{d\mathrm{D}} = \begin{cases} 1 - a_{d\mathrm{D}}/s^{d-2}, & d > 2, \\ \ln(s/a_{2\mathrm{D}}), & d = 2, \end{cases}$$

at $s > r_e$, where s is the distance between the two particles, r_e is the range of the interaction potential, and $a_{d\mathrm{D}}$ is the s-wave scattering "length" in d-dimensional space. For identical spin-polarized fermions, there are no two-body s-wave collisions. The two-body wave function for p-wave collisions also depends on the dimension of space. The two-body wave function for the zero-energy p-wave collision in two dimensions is

$$\phi^{(1\pm)}(\mathbf{s}) = \left(\frac{s}{2} - \frac{2a_p}{\pi s} \right) e^{\pm i\theta} \tag{1}$$

at $s > r_e$, where $\mathbf{s}$ is the spatial vector extending from one fermion to the other and θ is the angle from the $+x$ direction to the direction of $\mathbf{s}$, such that $s_x = s\cos\theta$ and $s_y = s\sin\theta$. a_p is the p-wave scattering length in two dimensions, although its dimension is length squared.

Two-dimensional (2D) atomic Bose gases [1–12] and 2D atomic Fermi gases [13–22] have been successfully realized in experiments. Two-dimensional Fermi gases have novel features not encountered in three dimensions. For three identical spin-polarized fermions in two dimensions with a short-range interaction fine-tuned to a p-wave resonance, there are super Efimov bound states with orbital angular momentum quantum number $L = 1$ [23–26], and their binding energies obey a universal doubly exponential scaling [23].

For three-body collisions, there are also big differences due to different spatial dimensions or different quantum statistics of the particles. The *scattering hypervolume* was defined for identical bosons [27], distinguishable particles [28–30], and spin-polarized fermions [31] in three dimensions. The scattering hypervolume is a three-body analog of the two-body s-wave or p-wave scattering length.

In this paper, we define the three-body scattering hypervolume D_F of identical spin-polarized fermions in two dimensions by studying the wave function of three such fermions colliding at zero energy and zero orbital angular momentum. We find that in two dimensions D_F has the dimension of length raised to the *sixth* power, while in three dimensions its dimension is length raised to the *eighth* power [31].

This paper is organized as follows. In Sec. II, we first review the two-body wave functions for identical spin-polarized fermions in two dimensions and then derive the asymptotic expansions of the wave function for three such fermions colliding at zero energy and zero orbital angular momentum; the parameter D_F appears as a coefficient in these expansions. In Sec. III, we derive an approximate formula for D_F for weak-interaction potentials using the Born expansion. In Sec. IV, we first consider three fermions in a large square with periodic boundary conditions and derive the shifts of their energy eigenvalues due to a nonzero D_F and then consider the dilute spin-polarized Fermi gas in two dimensions and derive the shifts of its energy and pressure due to a nonzero D_F. In Sec. V, we study the dilute spin-polarized Fermi gas in two dimensions with interaction potentials that support two-body bound states, for which we have three-body recombination processes and D_F has a nonzero imaginary part, and derive formulas for the rates of these processes in terms of the imaginary part of D_F. In Sec. VI we summarize our results and discuss the generalization of the hypervolume to three-body collisions with a higher orbital angular momentum.

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II. ASYMPTOTICS OF THE THREE-BODY WAVE FUNCTION

We consider three identical spin-polarized fermions colliding with zero kinetic energy. Their wave function Ψ satisfies the Schrödinger equation

$$\sum_{i=1}^{3}\left[-\frac{\hbar^2}{2m}\nabla_i^2 + V(s_i)\right]\Psi + V_3(s_1, s_2, s_3)\Psi = 0, \quad (2)$$

where m is the mass of each fermion, $\mathbf{r}_i$ is the position vector of the ith fermion, and $\mathbf{s}_i \equiv \mathbf{r}_j - \mathbf{r}_k$. The indices $(i, j, k) = (1, 2, 3), (2,3,1),$ or $(3,1,2)$. $V(s_i)$ is the two-body potential, and V_3 is the three-body potential. We assume that the interactions among these fermions are finite ranged and depend only on the interparticle distances. We define the Jacobi coordinates for later use:

$$\mathbf{R}_i \equiv \mathbf{r}_i - (\mathbf{r}_j + \mathbf{r}_k)/2, \quad (3a)$$

$$B \equiv \sqrt{(s_1^2 + s_2^2 + s_3^2)/2} = \sqrt{R_i^2 + \frac{3}{4}s_i^2}, \quad (3b)$$

$$\Theta_i \equiv \arctan\frac{2R_i}{\sqrt{3}s_i}. \quad (3c)$$

B is called the hyperradius, and Θ_i is the hyperangle.

Equation (2) and the translational invariance of Ψ do not uniquely determine the wave function for the zero-energy collision. We need to also specify the asymptotic behavior of Ψ when the three fermions are far apart. The leading-order term Ψ_0 in the wave function when s_1, s_2, and s_3 go to infinity simultaneously should satisfy the Laplace equation $(\nabla_1^2 + \nabla_2^2 + \nabla_3^2)\Psi_0 = 0$ and scale as B^p at large B. The most important three-body wave function for zero-energy collisions, for the purposes of understanding ultracold collisions, should be the one with the minimum value of p [31]. The reason is that the larger the value of p is, the less likely it is for the three particles to come to the range of interaction within which they can interact. One can easily show that the minimum value of p for three identical fermions in two dimensions is $p_{\min} = 2$; the corresponding Ψ_0 is

$$\Psi_0 = s_x R_y - s_y R_x, \quad (4)$$

and it takes the same form as one of the Ψ_0's in three dimensions; here and in the following, we define $\mathbf{s} \equiv \mathbf{s}_1$ and $\mathbf{R} \equiv \mathbf{R}_1$. One can verify that $\Psi_0 = s_{ix}R_{iy} - s_{iy}R_{ix}$ for any $i \in \{1, 2, 3\}$. Unlike in three dimensions, however, we have only one linearly independent three-body wave function for the zero-energy collision with $p = 2$ in two dimensions, and this wave function has *zero* total orbital angular momentum and is rotationally invariant.

A. Two-body special functions

For two-body scattering in the center-of-mass frame with collision energy $E = \hbar^2 k^2/m$ and orbital angular momentum quantum number l, the wave function can be separated as $\psi(s, \theta) = u(s)e^{\pm il\theta}/\sqrt{s}$, and the radial part $u(s)$ satisfies

$$\frac{d^2u}{ds^2} + \left[k^2 - \frac{mV(s)}{\hbar^2} - \frac{l^2 - 1/4}{s^2}\right]u = 0. \quad (5)$$

We assume a finite-range interaction such that $V(s)$ vanishes at $s > r_e$. The analytical formula for $u(s)$ at $s > r_e$ is

$$u(s) = \alpha_l\sqrt{s}[J_l(ks)\cot\delta_l(k) - Y_l(ks)], \quad (6)$$

where J_l and Y_l are the Bessel functions of the first and second kinds, respectively. α_l is an arbitrary coefficient which determines the overall amplitude of the two-body wave function. $\delta_l(k)$ is the l-wave scattering phase shift, and it satisfies the effective range expansion [32,33]:

$$k^{2l}\left[\cot\delta_l(k) - \frac{2}{\pi}\ln(k\rho_l)\right] = -\frac{1}{a_l} + \frac{1}{2}r_l k^2 + O(k^4), \quad (7)$$

where a_l is the l-wave scattering length ($l \geqslant 1$) with dimension [length]^{2l}. r_l is called the l-wave effective range for $l \neq 1$, and it has dimension [length]$^{2-2l}$. ρ_l is an arbitrary length scale.

The wave function in the l-wave channel at $s > r_e$ is

$$\psi^{(l\pm)}(\mathbf{s}) = -k^l a_l[J_l(ks)\cot\delta_l(k) - Y_l(ks)]e^{\pm il\theta}. \quad (8)$$

Here we have fixed the overall amplitude of $\psi^{(l\pm)}$ by specifying the coefficient $\alpha_l = -k^l a_l$.

At small collision energies, $k \ll 1/r_e$, the wave function can be expanded in powers of k^2 [27,29,31]:

$$\psi^{(l\pm)}(\mathbf{s}) = \phi^{(l\pm)}(\mathbf{s}) + k^2 f^{(l\pm)}(\mathbf{s}) + k^4 g^{(l\pm)}(\mathbf{s}) + \dots, \quad (9)$$

where $\phi^{(l\pm)}(\mathbf{s})$, $f^{(l\pm)}(\mathbf{s})$, $g^{(l\pm)}(\mathbf{s})$, ... are called the two-body special functions, and they satisfy

$$\widetilde{H}\phi^{(l\pm)}(\mathbf{s}) = 0, \quad (10a)$$

$$\widetilde{H}f^{(l\pm)}(\mathbf{s}) = \phi^{(l\pm)}(\mathbf{s}), \quad (10b)$$

$$\widetilde{H}g^{(l\pm)}(\mathbf{s}) = f^{(l\pm)}(\mathbf{s}), \quad \cdots, \quad (10c)$$

where $\widetilde{H}$ is defined as

$$\widetilde{H} \equiv -\nabla_{\mathbf{s}}^2 + \frac{m}{\hbar^2}V(s). \quad (11)$$

$\hbar^2\widetilde{H}/m$ is the two-body Hamiltonian for the collision of two fermions in the center-of-mass frame. The two-body special functions will appear in the expansion of the three-body wave function Ψ when two fermions are held at a fixed distance and the third fermion is far away from the other two.

$\phi^{(l\pm)}(\mathbf{s})$ is the wave function for the zero-energy collision of the two fermions in the l-wave channel. From Eqs. (8) and (9) we get

$$\phi^{(l\pm)}(\mathbf{s}) = \left[\frac{s^l}{(2l)!!} - \frac{2(2l-2)!!a_l}{\pi s^l}\right]e^{\pm il\theta}, \quad s > r_e \quad (12)$$

for $l \geqslant 1$. l must be odd for identical spin-polarized fermions due to Fermi statistics. We use symbols $p, f, h, \dots$ to represent $l = 1, 3, 5, \dots$, namely, $a_1 = a_p$, $r_1 = r_p$, $a_3 = a_f$, and so on.

For $f^{(l\pm)}(\mathbf{s})$ with $l = 1$, we find

$$f^{(1\pm)}(\mathbf{s}) = \left(-\frac{1}{16}s^3 + \frac{a_p s}{\pi}\ln\frac{s}{R_p}\right)e^{\pm i\theta}, \quad s > r_e, \quad (13)$$

where $R_p \equiv 2\rho_1 e^{\pi r_p/4+1/2-\gamma_E}$ and $\gamma_E = 0.5772\cdots$ is Euler's constant. We call R_p the p-wave effective range.

The explicit formulas for $f^{(l\pm)}$ with $l > 1$ and $g^{(l\pm)}$ are not listed for brevity because they are not used in this paper.

B. The 111 expansion and the 21 expansion

Like what we did in previous works [27,29,31], we derive two asymptotic expansions for the three-body wave function Ψ. When the three fermions are all far apart from each other, such that the pairwise distances s_1, s_2, and s_3 go to infinity simultaneously for any fixed ratio $s_1 : s_2 : s_3$, we expand Ψ in powers of $1/B$, and this expansion is called the 111 expansion. When two fermions are held at a fixed distance s but the third fermion is at large distance R away from the center of mass of the other two, we expand Ψ in powers of $1/R$, and this is called the 21 expansion. These expansions are

$$\Psi = \sum_{p=-2}^{\infty} \mathcal{T}^{(-p)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{14a}$$

$$\Psi = \sum_{q=-1}^{\infty} \mathcal{S}^{(-q)}(\mathbf{R}, \mathbf{s}), \tag{14b}$$

where $\mathbf{R} \equiv \mathbf{R}_1$, $\mathbf{s} \equiv \mathbf{s}_1$, $\mathcal{T}^{(-p)}$ scales as B^{-p}, and $\mathcal{S}^{(-q)}$ scales as R^{-q}. Here p starts from $p = -2$ because the leading order term Ψ_0 in the 111 expansion scales like B^2. Because $\Psi_0 \propto R^1$ for any fixed $\mathbf{s}$, the leading-order term in the 21 expansion should scale like R^1, so q starts from $q = -1$.

$\mathcal{T}^{(-p)}$ satisfies the free Schrödinger equation outside of the interaction range:

$$-\left(\nabla_1^2 + \nabla_2^2 + \nabla_3^2\right)\mathcal{T}^{(-p)} = 0. \tag{15}$$

If one fermion is far away from the other two, Eq. (2) becomes

$$\left(\widetilde{H} - \tfrac{3}{4}\nabla_{\mathbf{R}}^2\right)\Psi = 0. \tag{16}$$

This leads to the following equations for $\mathcal{S}^{(-q)}$:

$$\widetilde{H}\mathcal{S}^{(1)} = 0, \quad \widetilde{H}\mathcal{S}^{(0)} = 0,$$
$$\widetilde{H}\mathcal{S}^{(-q)} = \tfrac{3}{4}\nabla_{\mathbf{R}}^2 \mathcal{S}^{(-q+2)} \quad (q \geqslant 1). \tag{17}$$

We can further expand $\mathcal{T}^{(-p)}$ as $\sum_i t^{(i,-p-i)}$ when $s \ll R$, where $t^{(i,j)}$ scales like $R^i s^j$. We can also further expand $\mathcal{S}^{(-q)}$ as $\sum_j t^{(-q,j)}$ when $s \gg r_e$. Because the three-body wave function Ψ may be expanded as $\sum_p \mathcal{T}^{(-p)}$ at $B \to \infty$ and may also be expanded as $\sum_q \mathcal{S}^{(-q)}$ at $R \to \infty$, $t^{(i,j)}$ in the above two expansions should be the same. Actually, the wave function has a double expansion $\Psi = \sum_{i,j} t^{(i,j)}$ in the region $r_e \ll s \ll R$.

We show the points at which $t^{(i,j)} \neq 0$ on the (i, j) plane in Fig. 1. $\mathcal{T}^{(-p)}$ corresponds to a straight line with slope equal to -1 and an intercept equal to $-p$. $\mathcal{S}^{(-q)}$ corresponds to the vertical line $i = -q$ in the diagram.

To derive the two expansions, we start from the leading-order term (which fixes the overall amplitude of Ψ) in the 111 expansion:

$$\mathcal{T}^{(2)} = \Psi_0 = (s_x R_y - s_y R_x). \tag{18}$$

We then first derive $\mathcal{S}^{(1)}$, then derive $\mathcal{T}^{(1)}$, then derive $\mathcal{S}^{(0)}$, then derive $\mathcal{T}^{(0)}$, and so on, all the way to $\mathcal{S}^{(-5)}$. At every step, we require the 111 expansion and the 21 expansion to be consistent in the region $r_e \ll s \ll R$. See Appendix A for the details of the derivation.

The resultant 111 expansion is

$$\Psi = (s_x R_y - s_y R_x)\left\{1 - \frac{3D_F}{2\pi^2 B^6} - \frac{4a_p}{\pi}\sum_{i=1}^{3}\frac{1}{s_i^2} + \frac{32a_p^2}{\pi^2 B^2}\sum_{i=1}^{3}\frac{1}{s_i^2} - \frac{160a_p^3}{\pi^3 B^6}\sum_{i=1}^{3}\left(\frac{R_i^2}{s_i^2} + 3\ln\frac{B^2}{\sqrt{|a_p|s_i}}\right) + O(B^{-7})\right\}, \tag{19}$$

where D_F is the three-body scattering hypervolume of identical spin-polarized fermions in two dimensions and it appears at the order of B^{-4} in the expansion of Ψ. The dimension of D_F is length raised to the sixth power.

The resultant 21 expansion is

$$\Psi = i\left(R - \frac{8a_p}{\pi R} + \frac{40a_p^2}{\pi^2 R^3} - \frac{\xi}{R^5} - \frac{1920a_p^3}{\pi^3 R^5}\ln\frac{R}{\widetilde{R}}\right)[y^{(1-)}(\widehat{\mathbf{R}})\phi^{(1+)}(\mathbf{s}) - y^{(1+)}(\widehat{\mathbf{R}})\phi^{(1-)}(\mathbf{s})]$$

$$+ i\left(-\frac{48a_p}{\pi R^3} + \frac{384a_p^2}{\pi^2 R^5}\right)[y^{(3-)}(\widehat{\mathbf{R}})\phi^{(3+)}(\mathbf{s}) - y^{(3+)}(\widehat{\mathbf{R}})\phi^{(3-)}(\mathbf{s})]$$

$$- \frac{960ia_p}{\pi R^5}[y^{(5-)}(\widehat{\mathbf{R}})\phi^{(5+)}(\mathbf{s}) - y^{(5+)}(\widehat{\mathbf{R}})\phi^{(5-)}(\mathbf{s})] + \frac{240ia_p^2}{\pi^2 R^5}[y^{(1-)}(\widehat{\mathbf{R}})f^{(1+)}(\mathbf{s}) - y^{(1+)}(\widehat{\mathbf{R}})f^{(1-)}(\mathbf{s})] + O(R^{-6}), \tag{20}$$

where

$$y^{(l\pm)}(\widehat{\mathbf{R}}) \equiv (R_x \pm iR_y)^l / R^l, \tag{21}$$

$$\widetilde{R} = |a_p|^{3/8} R_p^{1/4}, \tag{22}$$

$$\xi = \frac{3D_F}{2\pi^2} - \frac{280a_p^3}{\pi^3}. \tag{23}$$

For any finite-range potentials $V(s)$ and $V_3(s_1, s_2, s_3)$, we may solve Eq. (2) numerically and match the solution to the asymptotic expansions in Eqs. (19) and (20) at large inter-fermionic distances to determine D_F numerically. But if the potentials are sufficiently weak, we may use the Born expansion to calculate D_F, as we do in Sec. III.

For attractive potentials whose strengths are fine-tuned such that there is a three-body S-wave bound state (i.e., with total orbital angular momentum quantum number $L = 0$) with energy close to zero, we anticipate that D_F becomes large. When we tune the strength of attraction such that this shal-
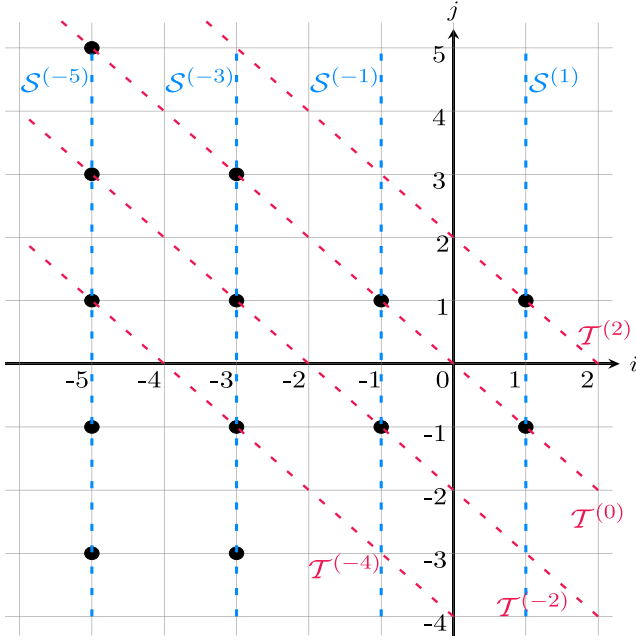
FIG. 1. Diagram of the points representing $t^{(i,j)}$ on the (i, j) plane. Each point with coordinates (i, j) represents $t^{(i,j)}$, which scales like $R^i s^j$. Dots represent those points at which $t^{(i,j)} \neq 0$. The term $\mathcal{T}^{(-p)}$ in the 111 expansion is represented by red dashed lines satisfying the equation $i + j = -p$. The term $\mathcal{S}^{(-q)}$ in the 21 expansion is represented by blue dashed lines satisfying the equation $i = -q$.

low three-body bound state barely forms (so that there is a metastable three-body state whose energy is positive and has some width due to the finite lifetime), D_F is large and negative; when we increase the strength of attraction such that this shallow three-body bound state has a small negative energy, D_F is large and positive. D_F has a pole as a function of the strength of attraction, and the pole is located at the critical strength at which the energy of this three-body bound state is zero.

III. APPROXIMATE FORMULA OF D_F FOR WEAK-INTERACTION POTENTIALS

For weak interactions, we analytically derive an approximate formula of D_F by using the Born series:

$$\Psi = \Psi_0 + \widehat{G}\mathcal{V}\Psi_0 + (\widehat{G}\mathcal{V})^2\Psi_0 + \cdots, \qquad (24)$$

where $\mathcal{V}$ is the total interaction potential and $\widehat{G}$ is the Green's operator ($\widehat{G} = -\widehat{H}_0^{-1}$, $\widehat{H}_0$ is the three-body kinetic-energy operator). Writing the interaction potential as $\mathcal{V} = V_3(s_1, s_2, s_3) + \sum_{i=1}^{3} V(s_i)$, we find

$$\widehat{G}\mathcal{V}\Psi_0 = -(s_xR_y - s_yR_x)\left(\frac{6\Lambda}{\pi B^6} + \sum_{i=1}^{3}\frac{\alpha_2}{2s_i^2} + \ldots\right), \quad (25a)$$

$$(\widehat{G}\mathcal{V})^2\Psi_0 = (s_xR_y - s_yR_x)$$

$$\times \left\{\left[\sum_{i=1}^{3}\left(\frac{\beta_2}{2s_i^2} + \frac{\alpha_2^2}{2B^2s_i^2}\right)\right] - \frac{9\alpha_2\alpha_4}{16B^6} + \cdots\right\}$$

$$+ O(VV_3) + O(V_3^2), \qquad (25b)$$

where

$$\alpha_n \equiv \frac{m}{\hbar^2}\int_0^{\infty}ds\, s^{n+1}V(s), \qquad (26a)$$

$$\beta_2 \equiv \frac{m^2}{\hbar^4}\int_0^{\infty}ds\int_0^{s}ds'\, ss'^3V(s)V(s'), \qquad (26b)$$

$$\Lambda \equiv \frac{m}{\hbar^2}\int ds_1ds_2ds_3\, s_1s_2s_3V_3(s_1, s_2, s_3)S_\Delta(s_1, s_2, s_3). \qquad (26c)$$

$S_\Delta = \sqrt{p(p - s_1)(p - s_2)(p - s_3)}$ is the area of the triangle with sides s_1, s_2, and s_3, where $p = (s_1 + s_2 + s_3)/2$. The integral on the right-hand side of Eq. (26c) is over all values of s_1, s_2, and s_3 satisfying $s_1 > 0$, $s_2 > 0$, $s_3 > 0$, $s_1 + s_2 > s_3$, $s_2 + s_3 > s_1$, and $s_3 + s_1 > s_2$ simultaneously. The details of the derivation are shown in Appendix B.

By comparing these results with the 111 expansion in Eq. (19), we find the expansions of a_p and D_F in powers of $V(s)$ and $V_3(s_1, s_2, s_3)$:

$$a_p = \frac{\pi}{8}(\alpha_2 - \beta_2) + O(V^3), \qquad (27)$$

$$D_F = 4\pi\Lambda + \frac{3\pi^2}{8}\alpha_2\alpha_4 + O(VV_3) + O(V_3^2) + O(V^3). \quad (28)$$

For any particular two-body potential $V(s)$, e.g., the square-well potential, one can calculate a_p analytically and verify that the result is consistent with Eq. (27) if V is weak. Equation (28) shows that D_F is quadratically dependent on the two-body potential V if V is weak and the three-body potential V_3 is absent. On the other hand, D_F is linearly dependent on the three-body potential V_3 if V_3 is weak and the two-body potential is absent.

If the interactions are not weak, one can solve the three-body Schrödinger equation numerically at zero energy and zero orbital angular momentum and match the resultant wave function with the asymptotic expansion in Eq. (19) or Eq. (20) to numerically extract the value of D_F.

IV. SHIFTS OF THE ENERGY AND THE PRESSURE OF IDENTICAL SPIN-POLARIZED FERMIONS

In this section, we first study the energy shift of three spin-polarized fermions caused by the scattering hypervolume D_F. We then derive the shifts of the thermodynamic properties, including the energy and the pressure, of the spin-polarized Fermi gas due to a nonzero D_F.

A. Three fermions in a large square

For the sake of simplicity, in this section we assume that the fermions have no two-body interaction or have a fine-tuned two-body interaction such that the two-body p-wave scattering length $a_p = 0$ but the three-body scattering hypervolume $D_F \neq 0$, and the 111 expansion for the zero-energy three-body wave function in Eq. (19) is simplified as

$$\Psi \simeq (s_xR_y - s_yR_x)\left(1 - \frac{3D_F}{2\pi^2B^6}\right). \qquad (29)$$

For the purpose of calculating the energy shifts due to a nonzero D_F, the true interaction potential $V(s_1) + V(s_2) + $

$V(s_3) + V_3(s_1, s_2, s_3)$, which in general has a complicated dependence on the interparticle distances, is replaced by a three-body pseudopotential V_{ps}. We use the following pseudopotential:

$$V_{ps} = \frac{\hbar^2 D_F}{6m} \left\{ \left[\nabla_s^2 \nabla_R^2 - (\nabla_s \cdot \nabla_R)^2 \right] \delta(s) \delta(R) \right\} \Lambda, \quad (30)$$

where Λ is a projection operator which, when acting on the $O(B^{-4})$ term in the three-body wave function, yields zero. The operator Λ is an analog of the operator $\frac{\partial}{\partial r} r$ in the two-body pseudopotential for s-wave collisions in Refs. [34,35]. One can check that the pseudopotential in Eq. (30) is symmetric under the interchange of the three fermions. The coefficient on the right-hand side of Eq. (30) has been chosen such that

$$-\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2 + \nabla_3^2) \Psi + V_{ps} \Psi = 0. \quad (31)$$

We now consider three fermions in a large square with area A and impose the periodic boundary conditions on the wave function. Consider an energy eigenstate in which the momenta of the fermions are $\hbar \mathbf{k}_1$, $\hbar \mathbf{k}_2$, and $\hbar \mathbf{k}_3$ in the absence of interactions. When we introduce interactions that give rise to a nonzero D_F, the energy eigenvalue of the three-body state is shifted by the following amount at first order in the perturbation:

$$\mathcal{E}_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} = \int d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 d^2 \mathbf{r}_3 \, |\Psi_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3}|^2 V_{ps}, \quad (32)$$

where $\Psi_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3}$ is the normalized unperturbed wave function and it can be written in terms of a Slater determinant:

$$\Psi_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} = \frac{1}{\sqrt{6} A^{3/2}} \begin{vmatrix} e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} & e^{i\mathbf{k}_1 \cdot \mathbf{r}_2} & e^{i\mathbf{k}_1 \cdot \mathbf{r}_3} \\ e^{i\mathbf{k}_2 \cdot \mathbf{r}_1} & e^{i\mathbf{k}_2 \cdot \mathbf{r}_2} & e^{i\mathbf{k}_2 \cdot \mathbf{r}_3} \\ e^{i\mathbf{k}_3 \cdot \mathbf{r}_1} & e^{i\mathbf{k}_3 \cdot \mathbf{r}_2} & e^{i\mathbf{k}_3 \cdot \mathbf{r}_3} \end{vmatrix}. \quad (33)$$

We get

$$\mathcal{E}_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} = \frac{\hbar^2 D_F}{3m A^2} (\mathbf{k}_1 \times \mathbf{k}_2 + \mathbf{k}_2 \times \mathbf{k}_3 + \mathbf{k}_3 \times \mathbf{k}_1)^2. \quad (34)$$

This energy shift is proportional to the square of the area of the $\mathbf{k}$-space triangle whose vertices are $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$.

B. Energy shift of many fermions and thermodynamic consequences

We generalize the energy shift to N fermions in the periodic area A. The number density of the fermions is $n = N/A$. We define the Fermi wave number $k_F = (4\pi n)^{1/2}$, the Fermi energy $\epsilon_F = \hbar^2 k_F^2 / 2m$, and the Fermi temperature $T_F = \epsilon_F / k_B$, where k_B is the Boltzmann constant.

1. Adiabatic shifts of energy and pressure in the thermodynamic limit

Starting from a many-body state at a finite temperature T, if we introduce a nonzero D_F *adiabatically*, the energy shift at first order in D_F is equal to the sum of the contributions from all the triplets of fermions, namely,

$$\Delta E = \frac{1}{6} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} \mathcal{E}_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} \, n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3}, \quad (35)$$

where $n_{\mathbf{k}} = (e^{\beta(\epsilon_{\mathbf{k}} - \mu)} + 1)^{-1}$ is the Fermi-Dirac distribution function, $\beta = 1/k_B T$, $\epsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$ is the kinetic energy of a fermion with linear momentum $\hbar \mathbf{k}$, and μ is the chemical potential. The summation over $\mathbf{k}$ can be replaced by a continuous integral $\sum_{\mathbf{k}} = A \int d^2 k / (2\pi)^2$ in the thermodynamic limit. Carrying out the integral, we get

$$\Delta E(T) = \frac{N \hbar^2 D_F}{192 \pi^2 m} k_F^8 \widetilde{T}^4 [f_2(e^{\beta\mu})]^2, \quad (36)$$

where $\widetilde{T} = T/T_F$ and the function $f_\nu(z)$ is defined as

$$f_\nu(z) \equiv -\text{Li}_\nu(-z) = \frac{2}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{2\nu-1}}{1 + e^{x^2}/z}. \quad (37)$$

The number of fermions satisfies $N = \sum_{\mathbf{k}} \frac{1}{e^{\beta(\epsilon_{\mathbf{k}} - \mu)} + 1}$, and this leads to

$$\widetilde{\mu} = \widetilde{T} \ln(e^{1/\widetilde{T}} - 1), \quad (38)$$

where $\widetilde{\mu} = \mu / \epsilon_F$.

In the low-temperature limit, $T \ll T_F$,

$$\Delta E(T) = \frac{N \hbar^2 D_F}{192 \pi^2 m} k_F^8$$
$$\times \left[\frac{1}{4} + \frac{\pi^2}{6} \widetilde{T}^2 + \frac{\pi^4}{36} \widetilde{T}^4 + O(\widetilde{T} e^{-1/\widetilde{T}}) \right]. \quad (39)$$

In the intermediate-temperature regime, $T_F \ll T \ll T_e$,

$$\Delta E(T) = \frac{N \hbar^2 D_F}{192 \pi^2 m} k_F^8 \left[\widetilde{T}^2 + \frac{\widetilde{T}}{2} + \frac{17}{144} + O(\widetilde{T}^{-1}) \right], \quad (40)$$

where $T_e = \frac{\hbar^2}{2m r_e^2 k_B}$. If T is comparable to or higher than T_e, the de Broglie wavelengths of the fermions will be comparable to or shorter than the range r_e of interparticle interaction potentials, and we can no longer use the effective parameter D_F to describe the system. See Fig. 2(a) for ΔE as a function of the initial temperature.

The pressure of the spin-polarized Fermi gas changes by the following amount due to the adiabatic introduction of D_F:

$$\Delta p = -\left(\frac{\partial \Delta E}{\partial A} \right)_{S,N} = \frac{4 \Delta E}{A}$$
$$= \frac{n \hbar^2 D_F}{48 \pi^2 m} k_F^8 \widetilde{T}^4 [f_2(e^{\beta\mu})]^2. \quad (41)$$

The subscripts S and N in Eq. (41) mean that we keep the entropy S and particle number N fixed when taking the partial derivative. See Fig. 2(b) for Δp as a function of the initial temperature.

2. Isothermal shifts of energy and pressure in the thermodynamic limit

If the interaction is introduced adiabatically, the temperature will increase (if $D_F > 0$) or decrease (if $D_F < 0$). The change in temperature is

$$\Delta T = \left(\frac{\partial \Delta E}{\partial S} \right)_{N,A}. \quad (42)$$
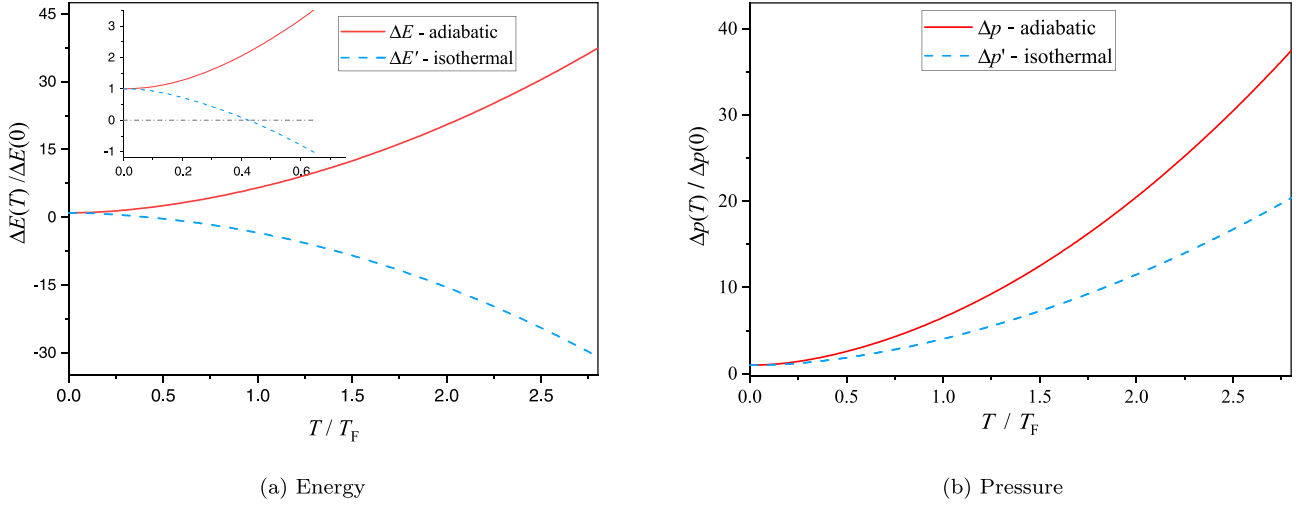
(a) Energy

(b) Pressure

FIG. 2. The shifts of (a) energy and (b) pressure caused by the adiabatic (red line) or isothermal (blue dashed line) introduction of D_F vs the temperature T. At $T \simeq 0.424 T_F$, the isothermal energy shift ΔE changes sign.

Therefore, if we introduce D_F isothermally, the energy shift $\Delta E'$ should be

$$\Delta E' = \Delta E - C\Delta T = \left(1 - T\frac{\partial}{\partial T}\right)\Delta E, \qquad (43)$$

where C is the heat capacity of the noninteracting Fermi gas at constant volume. In the low-temperature limit, $T \ll T_F$,

$$\Delta E'(T) = \frac{N\hbar^2 D_F}{192\pi^2 m}k_F^8\left[\frac{1}{4} - \frac{\pi^2}{6}\widetilde{T}^2 - \frac{\pi^4}{12}\widetilde{T}^4 + O(e^{-1/\widetilde{T}})\right]. \qquad (44)$$

In the intermediate-temperature regime, $T_F \ll T \ll T_e$,

$$\Delta E'(T) = \frac{N\hbar^2 D_F}{192\pi^2 m}k_F^8\left[-\widetilde{T}^2 + \frac{17}{144} + O(\widetilde{T}^{-1})\right]. \qquad (45)$$

According to Eqs. (44) and (45), $\Delta E'$ changes sign as we increase the temperature from $T \ll T_F$ to $T \gg T_F$. Therefore, there is a critical temperature T_c at which $\Delta E' = 0$. We find

$$T_c \simeq 0.424 T_F. \qquad (46)$$

The pressure of the spin-polarized Fermi gas changes by the following amount due to the isothermal introduction of D_F:

$$\Delta p' = \Delta p - \frac{C\Delta T}{A} = \left(1 - \frac{1}{4}T\frac{\partial}{\partial T}\right)\Delta p. \qquad (47)$$

In the low-temperature limit, $T \ll T_F$,

$$\Delta p' = \frac{n\hbar^2 D_F}{96\pi^2 m}k_F^8\left[\frac{1}{2} + \frac{\pi^2}{6}\widetilde{T}^2 + O(e^{-1/\widetilde{T}})\right]. \qquad (48)$$

In the intermediate-temperature regime, $T_F \ll T \ll T_e$,

$$\Delta p' = \frac{n\hbar^2 D_F}{96\pi^2 m}k_F^8\left[\widetilde{T}^2 + \frac{3}{4}\widetilde{T} + \frac{17}{72} + O(\widetilde{T}^{-1})\right]. \qquad (49)$$

The shifts of energy and pressure are plotted as a function of temperature in Figs. 2(a) and 2(b), respectively.

V. THE THREE-BODY RECOMBINATION RATE

If the collision of the three particles is purely elastic, D_F is a real number. But if the two-body interactions support bound states, then the three-body collisions are usually not purely elastic, and the three-body recombination may happen, which is the case for most ultracold atomic gases because most ultracold atoms have two-body bound states. In this case D_F becomes complex and acquires a *negative* imaginary part, and the three-body recombination rate constant is proportional to the imaginary part of D_F [36,37]. When the imaginary part of D_F is nonzero, the 111 expansion in Eq. (19) remains valid, but the 21 expansion in Eq. (20) should be modified by including terms describing bound pairs flying apart from the third particle.

Within a short time Δt, the probability that no recombination occurs is $\exp(-2|\mathrm{Im}E|\Delta t/\hbar) \simeq 1 - 2|\mathrm{Im}E|\Delta t/\hbar$. Then the probability of one recombination event is $2|\mathrm{Im}E|\Delta t/\hbar$. Since each recombination event causes the loss of three low-energy fermions, the change in the number of remaining low-energy fermions in the short time dt is

$$dN = -\frac{1}{6}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} 3\frac{2dt}{\hbar}|\mathrm{Im}\mathcal{E}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3}. \qquad (50)$$

This leads to

$$\frac{dn}{dt} = -L_3 n^3, \qquad (51)$$

and the coefficient L_3 is

$$L_3 = \frac{1}{2}\widetilde{T}^4[f_2(e^{\beta\mu})]^2\frac{\hbar|\mathrm{Im}D_F|}{m}k_F^4. \qquad (52)$$

L_3 depends on the density n and temperature T.

In the low-temperature limit, $T \ll T_F$,

$$L_3 \simeq \frac{1}{8}\left(1 + \frac{2\pi^2}{3}\widetilde{T}^2\right)\frac{\hbar|\mathrm{Im}D_F|}{m}k_F^4. \qquad (53)$$

In particular, at $T = 0$,

$$L_3 = \frac{\hbar |\mathrm{Im} D_F|}{8m} k_F^4, \tag{54}$$

and L_3 is proportional to n^2, so dn/dt is proportional to n^5.

In the intermediate-temperature regime, $T_F \ll T \ll T_e$, we find that

$$L_3 \simeq \frac{2m}{\hbar^3} |\mathrm{Im} D_F| (k_B T)^2. \tag{55}$$

In this case L_3 is proportional to T^2 and is approximately independent of n, so dn/dt is proportional to n^3. In Refs. [38,39] it was predicted that $L_3 \propto T^2$ for the spin-polarized Fermi gas in three or two dimensions according to the threshold law, and our Eq. (55) is consistent with this prediction. In Ref. [40] the T^2 dependence of L_3 was experimentally confirmed for a two-dimensional spin-polarized Fermi gas. One can extract the imaginary part of D_F from the experimental value of L_3 by using the formulas we have presented here.

VI. SUMMARY AND DISCUSSION

We have derived the asymptotic expansions of the three-body wave function Ψ for identical spin-polarized fermions colliding at zero energy and zero orbital angular momentum in two dimensions. The scattering hypervolume D_F appears at the order of B^{-4} in the 111 expansion of Ψ. We find that in two dimensions D_F has the dimension of length raised to the *sixth* power. In contrast, the dimension of D_F for identical spin-polarized fermions in three dimensions is length raised to the *eighth* power [31].

For weak-interaction potentials, we have derived an approximate formula for D_F by using the Born expansion. For stronger interactions, one can solve the three-body Schrödinger equation numerically at zero energy and zero orbital angular momentum and match the resultant wave function with the asymptotic expansion formulas we have derived in this paper to numerically extract the values of D_F.

We considered three fermions in a large square with periodic boundary conditions and derived the shifts of their energy eigenvalues due to a nonzero D_F and then considered the dilute spin-polarized Fermi gas in two dimensions and derived the shifts of its energy and pressure due to a nonzero D_F.

Finally, we studied the dilute spin-polarized Fermi gas in two dimensions with interaction potentials that support two-body bound states, for which we have three-body recombination processes and D_F has a nonzero imaginary part, and we derived formulas for the three-body recombination rate constant L_3 in terms of the imaginary part of D_F and the temperature and density of the Fermi gas.

We emphasize that the scattering hypervolume D_F we have studied in this work refers to the S-wave collisions of three identical spin-polarized fermions in two dimensions; that is, the total orbital angular momentum quantum number L is zero for such collisions. One can also study the P-wave collisions of these three fermions in two dimensions, with $L = 1$, and define a new three-body scattering hypervolume $D_F^{(P)}$ by studying the asymptotic expansions of the P-wave three-body wave function at zero collision energy. The P-wave three-body collisions are interesting because there is a super Efimov

effect for $L = 1$ near two-body p-wave resonances [23–26]. The behavior of $D_F^{(P)}$ near such a resonance is expected to be related to the super Efimov effect and deserves future investigation.

APPENDIX A: DERIVATION OF THE 111 EXPANSION AND THE 21 EXPANSION

We expand the three-body wave function in two forms:

$$\Psi = \sum_{p=-2}^{\infty} \mathcal{T}^{(-p)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{A1a}$$

$$\Psi = \sum_{q=-1}^{\infty} \mathcal{S}^{(-q)}(\mathbf{R}, \mathbf{s}), \tag{A1b}$$

where $\mathcal{T}^{(-p)}$ scales as B^{-p} and $\mathcal{S}^{(-q)}$ scales as R^{-q}. The hyperradius B and the vectors $\mathbf{R}$ and $\mathbf{s}$ are defined in the main text.

If $s \ll R$, we can further expand $\mathcal{T}^{(-p)}$ as

$$\mathcal{T}^{(-p)} = \sum_i t^{(i,-p-i)}, \tag{A2}$$

where $t^{(i,j)}$ scales as $R^i s^j$. If $s \gg r_e$, we can expand $\mathcal{S}^{(-q)}$ as

$$\mathcal{S}^{(-q)} = \sum_j t^{(-q,j)}. \tag{A3}$$

Because the three-body wave function Ψ may be expanded as $\sum_p \mathcal{T}^{(-p)}$ at $B \to \infty$ and may also be expanded as $\sum_q \mathcal{S}^{(-q)}$ at $R \to \infty$, $t^{(i,j)}$ in the above two expansions should be the same. In fact, the wave function has a double expansion $\Psi = \sum_{i,j} t^{(i,j)}$ in the region $r_e \ll s \ll R$.

Step 1. We start from the leading-order term in the 111 expansion:

$$\mathcal{T}^{(2)} = s_x R_y - s_y R_x = t^{(1,1)}, \tag{A4}$$

and this indicates that $\mathcal{S}^{(1)}$ is nonzero but $\mathcal{S}^{(2)}, \mathcal{S}^{(3)}, \mathcal{S}^{(4)}, \dots$ are zero. Consequently,

$$t^{(i,j)} = 0 \text{ if } i \geqslant 2. \tag{A5}$$

Expanding $\mathcal{T}^{(2)}$ at $s \ll R$, we find that

$$t^{(0,2)} = t^{(-1,3)} = t^{(-2,4)} = t^{(-3,5)} = \cdots = 0. \tag{A6}$$

Since $\mathcal{T}^{(3)}, \mathcal{T}^{(4)}, \mathcal{T}^{(5)}, \dots$ are zero, we have

$$t^{(i,j)} = 0 \text{ if } i + j \geqslant 3. \tag{A7}$$

Step 2. At $s \gg r_e$ we expand $\mathcal{S}^{(1)}$ as

$$\mathcal{S}^{(1)} = t^{(1,1)} + \sum_{j \leqslant 0} t^{(1,j)}. \tag{A8}$$

$\mathcal{S}^{(1)}$ also satisfies

$$\widetilde{H} \mathcal{S}^{(1)} = 0, \tag{A9}$$

where $\widetilde{H}$ is proportional to the two-body Hamiltonian and is defined in the main text. Therefore, $\mathcal{S}^{(1)}$ takes the following form:

$$\mathcal{S}^{(1)} = R \sum_l [c_{l+}\phi^{(l+)}(\mathbf{s}) + c_{l-}\phi^{(l-)}(\mathbf{s})]. \tag{A10}$$

Because $\phi^{(l\pm)}$ contributes a term proportional to s^l, $S^{(1)}$ contains a term scaling as $R^1 s^l$. On the other hand, the leading-order term on the right-hand side of Eq. (A8) is $t^{(1,1)}$, which scales as $R^1 s^1$. So we must have $c_{l\pm} = 0$ for $l > 1$.

Expanding Eq. (A10) at $s \gg r_e$ to the order s^1 and using Eq. (12), we get

$$t^{(1,1)} = \tfrac{1}{2}R[c_{1+}(s_x + is_y) + c_{1-}(s_x - is_y)]. \tag{A11}$$

Comparing this result with Eq. (A4), we find the coefficients $c_{1\pm}$:

$$c_{1+} = \frac{i}{R}(R_x - iR_y), \tag{A12a}$$

$$c_{1-} = \frac{-i}{R}(R_x + iR_y). \tag{A12b}$$

Therefore,

$$\mathcal{S}^{(1)} = iR[y^{(1-)}(\hat{\mathbf{R}})\phi^{(1+)}(\mathbf{s}) - y^{(1+)}(\hat{\mathbf{R}})\phi^{(1-)}(\mathbf{s})]. \tag{A13}$$

Expanding $\mathcal{S}^{(1)}$ at $s \gg r_e$, we get

$$t^{(1,0)} = 0, \tag{A14a}$$

$$t^{(1,-1)} = -\frac{4a_p}{\pi s^2}(s_x R_y - s_y R_x), \tag{A14b}$$

$$t^{(1,j)} = 0, \quad j \leqslant -2. \tag{A14c}$$

Step 3. At $s \ll R$ we expand $\mathcal{T}^{(1)}$ as

$$\mathcal{T}^{(1)} = t^{(1,0)} + t^{(0,1)} + t^{(-1,2)} + \cdots = t^{(0,1)} + t^{(-1,2)} + \cdots . \tag{A15}$$

So $\mathcal{T}^{(1)}$ goes to zero at $s \to 0$. So Eq. (15) may be written as $(\nabla_1^2 + \nabla_2^2 + \nabla_3^2)\mathcal{T}^{(1)} = 0$ for $p = -1$, and $\mathcal{T}^{(1)}$ should satisfy this partial differential equation even at $s_i = 0$. Thus, $\mathcal{T}^{(1)}$ must be a harmonic polynomial. But we do *not* have any nontrivial harmonic polynomial of degree 1 that also satisfies the fermionic antisymmetry. We are therefore forced to take

$$\mathcal{T}^{(1)} = 0. \tag{A16}$$

So

$$t^{(i,j)} = 0 \text{ if } i + j = 1. \tag{A17}$$

Step 4. At $s \gg r_e$ we expand $\mathcal{S}^{(0)}$ as

$$\mathcal{S}^{(0)} = t^{(0,2)} + t^{(0,1)} + O(s^0) = O(s^0). \tag{A18}$$

Combining this with the equation $\widetilde{H}\mathcal{S}^{(0)} = 0$, we get

$$\mathcal{S}^{(0)} = 0. \tag{A19}$$

So

$$t^{(0,j)} = 0. \tag{A20}$$

Step 5. At $s \ll R$ we expand $\mathcal{T}^{(0)}$ as

$$\mathcal{T}^{(0)} = t^{(1,-1)} + t^{(0,0)} + t^{(-1,1)} + t^{(-2,2)} + \cdots$$
$$= t^{(1,-1)} + t^{(-1,1)} + \cdots . \tag{A21}$$

$t^{(1,-1)}$ is shown in Eq. (A14b). $\mathcal{T}^{(0)}$ should satisfy the free Schrödinger equation outside of the interaction range, so $(-\nabla_s^2 - 3\nabla_R^2/4)\mathcal{T}^{(0)}$ should be equal to some Dirac δ functions that are nonzero only at $s_i = 0$. $\mathcal{T}^{(0)}$ should also be antisymmetric under the interchange of the fermions. We have

$$-\nabla_s^2 t^{(1,-1)} = 8a_p[R_y \partial_x \delta(\mathbf{s}) - R_x \partial_y \delta(\mathbf{s})],$$

so

$$\left(-\nabla_\mathbf{s}^2 - \frac{3}{4}\nabla_\mathbf{R}^2\right)\mathcal{T}_{\mathbf{s}_1}^{(0)} = 8a_p[R_y \partial_x \delta(\mathbf{s}) - R_x \partial_y \delta(\mathbf{s})], \tag{A22}$$

where $\mathcal{T}_{\mathbf{s}_1}^{(0)}$ is one term of the full $\mathcal{T}^{(0)}$. Solving the above equation, we get

$$\mathcal{T}_{\mathbf{s}_1}^{(0)} = -\frac{4a_p}{\pi s_1^2}(s_x R_y - s_y R_x). \tag{A23}$$

The full $\mathcal{T}^{(0)}$ should also be antisymmetric under the interchange of the fermions, so

$$\mathcal{T}^{(0)} = -\frac{4a_p}{\pi}(s_x R_y - s_y R_x)\left(\frac{1}{s_1^2} + \frac{1}{s_2^2} + \frac{1}{s_3^2}\right). \tag{A24}$$

If $s \ll R$, we expand $\mathcal{T}^{(0)}$ as $\sum_{n+m=0} t^{(n,m)}$ and get

$$t^{(0,0)} = 0, \tag{A25a}$$

$$t^{(-1,1)} = -\frac{8a_p}{\pi R^2}(s_x R_y - s_y R_x), \tag{A25b}$$

$$t^{(-2,2)} = 0, \tag{A25c}$$

$$t^{(-3,3)} = \frac{2a_p}{\pi R^6}(s_x R_y - s_y R_x)(R^2 - 4R_s^2)s^2, \tag{A25d}$$

$$t^{(-4,4)} = 0, \tag{A25e}$$

$$t^{(-5,5)} = -\frac{a_p}{2\pi R^{10}}(s_x R_y - s_y R_x)$$
$$\times (R^4 - 12R^2 R_s^2 + 16R_s^4)s^4, \cdots, \tag{A25f}$$

where $R_s \equiv \mathbf{R} \cdot \hat{\mathbf{s}}$.

Step 6. At $s \gg r_e$ we expand $\mathcal{S}^{(-1)}$ as

$$\mathcal{S}^{(-1)} = t^{(-1,1)} + O(s^0). \tag{A26}$$

$\mathcal{S}^{(-1)}$ satisfies the equation

$$\widetilde{H}\mathcal{S}^{(-1)} = \frac{3}{4}\nabla_\mathbf{R}^2 \mathcal{S}^{(1)} = 0. \tag{A27}$$

From the above equations we deduce

$$\mathcal{S}^{(-1)} = \frac{1}{R}[d_+\phi^{(1+)} + d_-\phi^{(1-)}]. \tag{A28}$$

$\mathcal{S}^{(-1)}$ contains only the p-wave component in order to be compatible with Eq. (A26). Expanding $\mathcal{S}^{(-1)}$ at $s \gg r_e$, we get

$$t^{(-1,1)} = \frac{1}{2R}[d_+(s_x + is_y) + d_-(s_x - is_y)]. \tag{A29}$$

Comparing this with Eq. (A25b), we find

$$d_+ = -\frac{8a_p i}{\pi R}(R_x - iR_y), \tag{A30a}$$

$$d_- = \frac{8a_p i}{\pi R}(R_x + iR_y). \tag{A30b}$$

So

$$\mathcal{S}^{(-1)} = -\frac{8a_p i}{\pi R}[y^{(1-)}(\hat{\mathbf{R}})\phi^{(1+)}(\mathbf{s}) - y^{(1+)}(\hat{\mathbf{R}})\phi^{(1-)}(\mathbf{s})]. \tag{A31}$$

Expanding $\mathcal{S}^{(-1)}$ at $s \gg r_e$, we get

$$t^{(-1,0)} = 0, \tag{A32a}$$

$$t^{(-1,-1)} = \frac{32a_p^2}{\pi^2 s^2 R^2}(s_x R_y - s_y R_x), \tag{A32b}$$

$$t^{(1,j)} = 0, \quad j \leqslant -2. \tag{A32c}$$

Step 7. At $s \ll R$ we expand $\mathcal{T}^{(-1)}$ as

$$\mathcal{T}^{(-1)} = t^{(1,-2)} + t^{(0,-1)} + t^{(-1,0)} + O(s^1) = O(s^1). \tag{A33}$$

The solution to the equation $(\nabla_1^2 + \nabla_2^2 + \nabla_3^2)\mathcal{T}^{(-1)} = 0$ that is compatible with the above expansion is

$$\mathcal{T}^{(-1)} = 0. \tag{A34}$$

Step 8. At $s \gg r_e$ we expand $\mathcal{S}^{(-2)}$ as

$$\mathcal{S}^{(-2)} = t^{(-2,4)} + t^{(-2,3)} + t^{(-2,2)} + t^{(-2,1)} + \sum_{j \leqslant 0} t^{(-2,j)}$$

$$= O(s^0). \tag{A35}$$

$\mathcal{S}^{(-2)}$ satisfies the equation

$$\widetilde{H}\mathcal{S}^{(-2)} = \tfrac{3}{4}\nabla_{\mathbf{R}}^2 \mathcal{S}^{(0)} = 0. \tag{A36}$$

So we get

$$\mathcal{S}^{(-2)} = 0. \tag{A37}$$

So

$$t^{(-2,j)} = 0. \tag{A38}$$

Step 9. At $s \ll R$ we expand $\mathcal{T}^{(-2)}$ as

$$\mathcal{T}^{(-2)} = \sum_{j=-3}^{0} t^{(-2-j,j)} + O(s^1)$$

$$= t^{(-1,-1)} + O(s^1). \tag{A39}$$

Solving the equation $(\nabla_1^2 + \nabla_2^2 + \nabla_3^2)\mathcal{T}^{(-2)} = 0$ (for only $s_i \neq 0$) and using the above expansion, we find

$$\mathcal{T}^{(-2)} = \frac{32a_p^2}{\pi^2}(s_x R_y - s_y R_x)\sum_{i=1}^{3}\frac{1}{B^2 s_i^2}. \tag{A40}$$

For $s \ll R$, we expand $\mathcal{T}^{(-2)}$ as $\sum_{n+m=-2} t^{(n,m)}$ and get

$$t^{(-2,0)} = 0, \tag{A41a}$$

$$t^{(-3,1)} = \frac{40a_p^2}{\pi^2 R^4}(s_x R_y - s_y R_x), \tag{A41b}$$

$$t^{(-4,2)} = 0, \tag{A41c}$$

$$t^{(-5,3)} = -\frac{2a_p^2}{\pi^2 R^8}(s_x R_y - s_y R_x)(23R^2 - 32R_s^2)s^2, \cdots. \tag{A41d}$$

Step 10. At $s \gg r_e$ we expand $\mathcal{S}^{(-3)}$ as

$$\mathcal{S}^{(-3)} = t^{(-3,3)} + t^{(-3,1)} + O(s^0). \tag{A42}$$

Combining this with the equation

$$\widetilde{H}\mathcal{S}^{(-3)} = \tfrac{3}{4}\nabla_{\mathbf{R}}^2 \mathcal{S}^{(-1)} = 0, \tag{A43}$$

we get

$$\mathcal{S}^{(-3)} = \frac{1}{R^3}\left[\sum_{\pm} e_{\pm}\phi^{(3\pm)} + \sum_{\pm} g_{\pm}\phi^{(1\pm)}\right]. \tag{A44}$$

Here $\mathcal{S}^{(-3)}$ contains the p-wave and f-wave components in order to be compatible with Eq. (A42). Expanding $\mathcal{S}^{(-3)}$ at $s \gg r_e$, we get

$$t^{(-3,3)} = \frac{1}{48R^3}[e_+(s_x + is_y)^3 + e_-(s_x - is_y)^3], \tag{A45a}$$

$$t^{(-3,1)} = \frac{1}{2R^3}[g_+(s_x + is_y) + g_-(s_x - is_y)]. \tag{A45b}$$

Comparing these equations with Eqs. (A25d) and (A41b), we find

$$e_+ = -\frac{48a_p i}{\pi R^3}(R_x - iR_y)^3, \tag{A46a}$$

$$e_- = \frac{48a_p i}{\pi R^3}(R_x + iR_y)^3, \tag{A46b}$$

$$g_+ = \frac{40a_p^2 i}{\pi^2 R}(R_x - iR_y), \tag{A46c}$$

$$g_- = -\frac{40a_p^2 i}{\pi^2 R}(R_x + iR_y). \tag{A46d}$$

So

$$\mathcal{S}^{(-3)} = -\frac{48a_p i}{\pi R^3}[y^{(3-)}(\hat{\mathbf{R}})\phi^{(3+)}(\mathbf{s}) - y^{(3+)}(\hat{\mathbf{R}})\phi^{(3-)}(\mathbf{s})]$$

$$+ \frac{40a_p^2 i}{\pi^2 R^3}[y^{(1-)}(\hat{\mathbf{R}})\phi^{(1+)}(\mathbf{s}) - y^{(1+)}(\hat{\mathbf{R}})\phi^{(1-)}(\mathbf{s})]. \tag{A47}$$

Expanding $\mathcal{S}^{(-3)}$ at $s \gg r_e$, we get

$$t^{(-3,0)} = 0, \tag{A48a}$$

$$t^{(-3,-1)} = -\frac{160a_p^3}{\pi^3 s^2 R^4}(s_x R_y - s_y R_x), \tag{A48b}$$

$$t^{(-3,-2)} = 0, \tag{A48c}$$

$$t^{(-3,-3)} = \frac{768a_p a_f i}{\pi^2 R^6 s^6}[(R_x - iR_y)^3(s_x + is_y)^3$$

$$- (R_x + iR_y)^3(s_x - is_y)^3], \tag{A48d}$$

$$t^{(-3,j)} = 0, \quad j \leqslant -4. \tag{A48e}$$

Step 11. At $s \ll R$ we expand $\mathcal{T}^{(-3)}$ as

$$\mathcal{T}^{(-3)} = \sum_{j=-4}^{0} t^{(-3-j,j)} + O(s^1) = O(s^1). \tag{A49}$$

Solving the equation $(\nabla_1^2 + \nabla_2^2 + \nabla_3^2)\mathcal{T}^{(-3)} = 0$ (for only $s_i \neq 0$) and using the above expansion, we find

$$\mathcal{T}^{(-3)} = 0. \tag{A50}$$

So

$$t^{(i,j)} = 0 \text{ if } i + j = -3. \tag{A51}$$

Step 12. At $s \gg r_e$ we expand $\mathcal{S}^{(-4)}$ as

$$\mathcal{S}^{(-4)} = t^{(-4,6)} + t^{(-4,5)} + \cdots + t^{(-4,1)} + O(s^0)$$
$$= O(s^0). \tag{A52}$$

$\mathcal{S}^{(-4)}$ satisfies the equation

$$\widetilde{H}\mathcal{S}^{(-4)} = \tfrac{3}{4}\nabla_{\mathbf{R}}^2 \mathcal{S}^{(-2)} = 0. \tag{A53}$$

So we get

$$\mathcal{S}^{(-4)} = 0. \tag{A54}$$

So

$$t^{(-4,j)} = 0. \tag{A55}$$

Step 13. At $s \ll R$ we expand $\mathcal{T}^{(-4)}$ as

$$\mathcal{T}^{(-4)} = \sum_{j=-5}^{0} t^{(-4-j,j)} + O(s^1)$$
$$= -\frac{160 a_p^3}{\pi^3 s^2 R^4}(s_x R_y - s_y R_x) + O(s^1). \tag{A56}$$

Solving the equation $(\nabla_1^2 + \nabla_2^2 + \nabla_3^2)\mathcal{T}^{(-4)} = 0$ (for only $s_i \neq 0$) and using the above expansion, we find

$$\mathcal{T}^{(-4)} = (s_x R_y - s_y R_x)\left[-\frac{3 D_F}{2\pi^2 B^6} \right.$$
$$\left. -\frac{160 a_p^3}{\pi^3 B^6}\sum_i \left(\frac{R_i^2}{s_i^2} + 3\ln\frac{B^2}{\sqrt{|a_p|s_i}} \right) \right], \tag{A57}$$

where D_F is a new coefficient and we call it the three-body scattering hypervolume. Its value depends on the details of the interactions.

Expanding $\mathcal{T}^{(-4)}$ at $s \ll R$ as $\sum_{i+j=-4} t^{(i,j)}$, we get

$$t^{(-4,0)} = 0, \tag{A58a}$$

$$t^{(-5,1)} = (s_x R_y - s_y R_x)\left(-\frac{3 D_F}{2\pi^2 R^6} \right.$$
$$\left. +\frac{280 a_p^3}{\pi^3 R^6} - \frac{480 a_p^3}{\pi^3 R^6}\ln\frac{R^4}{|a_p|^{3/2}s} \right), \cdots. \tag{A58b}$$

Step 14. At $s \gg r_e$ we expand $\mathcal{S}^{(-5)}$ as

$$\mathcal{S}^{(-5)} = \sum_{j \leqslant 7} t^{(-5,j)}$$
$$= t^{(-5,5)} + t^{(-5,3)} + t^{(-5,1)} + O(s^0). \tag{A59}$$

Combining this with the equation

$$\widetilde{H}\mathcal{S}^{(-5)} = \tfrac{3}{4}\nabla_{\mathbf{R}}^2 \mathcal{S}^{(-3)}$$
$$= \frac{240 a_p^2 i}{\pi^2 R^6}[(R_x - i R_y)\phi^{(1+)} - (R_x + i R_y)\phi^{(1-)}], \tag{A60}$$

we get

$$\mathcal{S}^{(-5)} = \frac{240 a_p^2 i}{\pi^2 R^5}[y^{(1-)}(\widehat{R})f^{(1+)}(s) - y^{(1+)}(\widehat{R})f^{(1-)}(s)]$$
$$-\frac{960 a_p i}{\pi R^5}[y^{(5-)}(\widehat{R})\phi^{(5+)}(s) - y^{(5+)}(\widehat{R})\phi^{(5-)}(s)]$$
$$+\frac{384 a_p^2 i}{\pi^2 R^5}[y^{(3-)}(\widehat{R})\phi^{(3+)}(s) - y^{(3+)}(\widehat{R})\phi^{(3-)}(s)]$$
$$-\frac{\zeta i}{R^5}[y^{(1-)}(\widehat{R})\phi^{(1+)}(s) - y^{(1+)}(\widehat{R})\phi^{(1-)}(s)], \tag{A61}$$

where

$$\zeta = \frac{3 D_F}{2\pi^2} - \frac{280 a_p^3}{\pi^3} + \frac{1920 a_p^3}{\pi^3}\ln\frac{R}{\widetilde{\widetilde{R}}}. \tag{A62}$$

We have thus derived the 111 expansion to the order B^{-4} and the 21 expansion to the order R^{-5}.

APPENDIX B: THE BORN EXPANSION OF THE THREE-BODY WAVE FUNCTION

For weak-interaction potentials, we can expand the three-body wave function as a Born series:

$$\Psi = \Psi_0 + \widehat{G}\mathcal{V}\Psi_0 + (\widehat{G}\mathcal{V})^2\Psi_0 + \cdots, \tag{B1}$$

where $\Psi_0 = s_x R_y - s_y R_x$ is the wave function of three free fermions, $\mathcal{V}$ is the interaction potential, and $\widehat{G} = -\widehat{H}_0^{-1}$ is the Green's operator, $\widehat{H}_0$ is the three-body kinetic-energy operator. We define $\mathcal{V} = V_3(s_1, s_2, s_3) + \sum_{i=1}^3 V(s_i)$. We assume that $V(s)$ vanishes at $s > r_e$ and that $V_3(s_1, s_2, s_3)$ vanishes if $s_1 > r_e$ or $s_2 > r_e$ or $s_3 > r_e$.

1. The first-order term

The first-order term in the Born series is

$$\Psi_1(\boldsymbol{\xi}) = \widehat{G}\mathcal{V}\Psi_0 = \widehat{G}V_3\Psi_0 + \sum_{i=1}^3 \widehat{G}V^{(i)}\Psi_0$$
$$= \frac{m}{\hbar^2}\int d^4\xi' \, \mathcal{G}(\boldsymbol{\xi} - \boldsymbol{\xi}')V_3(s_1', s_2', s_3')\Psi_0(\boldsymbol{\xi}')$$
$$+ \frac{m}{\hbar^2}\sum_{i=1}^3\int d^4\xi' \, \mathcal{G}(\boldsymbol{\xi} - \boldsymbol{\xi}')V^{(i)}(\boldsymbol{\xi}')\Psi_0(\boldsymbol{\xi}'), \tag{B2}$$

where $\boldsymbol{\xi} = (\mathbf{s}, 2\mathbf{R}/\sqrt{3})$ and $\boldsymbol{\xi}' = (\mathbf{s}', 2\mathbf{R}'/\sqrt{3})$ are four-dimensional vectors, $\mathcal{G}$ is the Green's function in four-dimensional space,

$$\mathcal{G}(\boldsymbol{\xi} - \boldsymbol{\xi}') = -\frac{1}{4\pi^2|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}, \tag{B3}$$

and $V^{(i)}(\boldsymbol{\xi}') = V(s_i')$. $\mathbf{s}_1' = \mathbf{s}'$, $\mathbf{s}_2' = -\frac{1}{2}\mathbf{s}' - \mathbf{R}'$, and $\mathbf{s}_3' = -\frac{1}{2}\mathbf{s}' + \mathbf{R}'$. We write Ψ_1 as $\widehat{G}V_3\Psi_0 + \sum_{i=1}^3 \Psi_1^{(i)}$, where

$$\Psi_1^{(i)} \equiv \widehat{G}V^{(i)}\Psi_0 = \int d^4\xi' \, \frac{-m}{4\pi^2\hbar^2|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}V^{(i)}(\boldsymbol{\xi}')\,\Psi_0(\boldsymbol{\xi}')$$
$$= \frac{4}{3}\int d^2 s'\int d^2 R' \, \frac{-m V(s')(s_x' R_y' - s_y' R_x')}{4\pi^2\hbar^2[(\mathbf{s}_i - \mathbf{s}')^2 + \frac{4}{3}(\mathbf{R}_i - \mathbf{R}')^2]}. \tag{B4}$$

Finishing this integral and taking the sum over i, we get

$$\sum_{i=1}^{3} \Psi_1^{(i)} = -\frac{1}{2}(s_x R_y - s_y R_x) \sum_{i=1}^{3} \left[\frac{\alpha_2(s_i)}{s_i^2} + \bar{\alpha}_0(s_i) \right], \tag{B5}$$

where

$$\alpha_n(s) \equiv \frac{m}{\hbar^2} \int_0^s ds' \, s'^{n+1} V(s'), \tag{B6a}$$

$$\bar{\alpha}_n(s) \equiv \frac{m}{\hbar^2} \int_s^\infty ds' \, s'^{n+1} V(s'). \tag{B6b}$$

Since $V(s)$ is a finite-range potential which vanishes at $s > r_e$, we have $\alpha_n(s) = \alpha_n \equiv m \int_0^\infty ds' s'^{n+1} V(s')/\hbar^2$ and $\bar{\alpha}_n(s) = 0$ if $s > r_e$. If all three $s_i > r_e$, $\sum_{i=1}^{3} \Psi_1^{(i)}$ is simplified as

$$\sum_{i=1}^{3} \Psi_1^{(i)} = -(s_x R_y - s_y R_x) \sum_{i=1}^{3} \frac{\alpha_2}{2s_i^2}. \tag{B7}$$

$$\widehat{G} V_3 \Psi_0 = \int d^4 \xi' \frac{-m V_3(\boldsymbol{\xi}') \Psi_0(\boldsymbol{\xi}')}{4\pi^2 \hbar^2 |\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} = \frac{4}{3} \int d^2 s' \int d^2 R' \frac{-m V_3(s_1', s_2', s_3')(s_x' R_y' - s_y' R_x')}{4\pi^2 \hbar^2 \left[(\mathbf{s} - \mathbf{s}')^2 + \frac{4}{3}(\mathbf{R} - \mathbf{R}')^2 \right]}. \tag{B8}$$

Let $s_x' = s' \cos\alpha'$, $s_y' = s' \sin\alpha'$, $R_x' = R' \cos(\alpha' + \theta')$, $R_y' = R' \sin(\alpha' + \theta')$, $s_x = s \cos\alpha$, $s_y = s \sin\alpha$, $R_x = R \cos(\alpha + \theta)$, and $R_y = R \sin(\alpha + \theta)$. We have

$$\int d^2 s' \int d^2 R' = \int_0^\infty ds' s' \int_0^\infty dR' R' \int_{-\pi}^{\pi} d\theta' \int_{-\pi}^{\pi} d\alpha'. \tag{B9}$$

Since V_3 is a finite-range potential, the integral on the right-hand side of Eq. (B8) may be expanded when s and R go to infinity for any fixed ratio s/R. Expanding this integral at large B and using the fact that V_3 is an even function of θ' and is independent of α', we get

$$\widehat{G} V_3 \Psi_0 = -\frac{3mRs \sin\theta}{\pi \hbar^2 B^6} \int_0^\infty ds' \int_0^\infty dR' \int_0^\pi d\theta' \, s'^3 R'^3 \sin^2\theta' \, V_3(s_1', s_2', s_3') + O(B^{-6})$$

$$= -\frac{6m(s_x R_y - s_y R_x)}{\pi \hbar^2 B^6} \int ds_1' ds_2' ds_3' \, s_1' s_2' s_3' V_3(s_1', s_2', s_3') S_\Delta(s_1', s_2', s_3') + O(B^{-6}). \tag{B10}$$

In the first line of Eq. (B10), $s_1' = s'$, $s_2' = \sqrt{R'^2 + \frac{1}{4}s'^2 + R's' \cos\theta'}$, and $s_3' = \sqrt{R'^2 + \frac{1}{4}s'^2 - R's' \cos\theta'}$.

2. The second-order term

The second-order term in the Born series is

$$\Psi_2 = \widehat{G} \mathcal{V} \Psi_1 = \sum_{ij} \widehat{G} V^{(i)} \widehat{G} V^{(j)} \Psi_0 + \sum_i \widehat{G} V^{(i)} \widehat{G} V_3 \Psi_0 + \sum_i \widehat{G} V_3 \widehat{G} V^{(i)} \Psi_0 + (\widehat{G} V_3)^2 \Psi_0. \tag{B11}$$

We define

$$\Psi_2^{(ij)} \equiv \widehat{G} V^{(i)} \widehat{G} V^{(j)} \Psi_0 = \int d^4 \xi' \frac{-m}{4\pi^2 \hbar^2 |\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} V^{(i)}(\boldsymbol{\xi}') \Psi_1^{(j)}(\boldsymbol{\xi}'). \tag{B12}$$

In particular,

$$\Psi_2^{(ii)} = \frac{4}{3} \int d^2 s' \int d^2 R' \frac{-m V(s')(s_x' R_y' - s_y' R_x')}{4\pi^2 \hbar^2 [(\mathbf{s}_i - \mathbf{s}')^2 + \frac{4}{3}(\mathbf{R}_i - \mathbf{R}')^2]} \left(-\frac{1}{2} \right) \left[\frac{\alpha_2(s')}{s'^2} + \bar{\alpha}_0(s') \right]. \tag{B13}$$

If all three $s_i > r_e$, we can evaluate the integral to obtain

$$\Psi_2^{(ii)} = (s_x R_y - s_y R_x) \frac{\beta_2}{2s_i^2}, \tag{B14}$$

where β_2 is defined as

$$\beta_2 \equiv \frac{m^2}{\hbar^4} \int_0^\infty ds \int_0^s ds' \, ss'^3 V(s) V(s'). \tag{B15}$$

If $j \neq i$, we have two different values of j for each i. For a given value of i,

$$
\begin{aligned}
\sum_{j \neq i} \Psi_2^{(ij)} &= \frac{4}{3} \int d^2 s' \int d^2 R' \frac{m V(s')(s'_x R'_y - s'_y R'_x)}{8\pi^2 \hbar^2 [(\mathbf{s}_i - \mathbf{s}')^2 + \frac{4}{3}(\mathbf{R}_i - \mathbf{R}')^2]} \\
&\quad \times \left[\frac{\alpha_2(|\mathbf{R}' - \frac{1}{2}\mathbf{s}'|)}{|\mathbf{R}' - \frac{1}{2}\mathbf{s}'|^2} + \bar{\alpha}_0 \left(\left| \mathbf{R}' - \frac{1}{2}\mathbf{s}' \right| \right) + \frac{\alpha_2(|\mathbf{R}' + \frac{1}{2}\mathbf{s}'|)}{|\mathbf{R}' + \frac{1}{2}\mathbf{s}'|^2} + \bar{\alpha}_0 \left(\left| \mathbf{R}' + \frac{1}{2}\mathbf{s}' \right| \right) \right] \\
&= \int d^2 s' \int d^2 R' \frac{m V(s')}{8\pi^2 \hbar^2} (s'_x R'_y - s'_y R'_x) \left[\frac{\alpha_2(R')}{R'^2} + \bar{\alpha}_0(R') \right] \\
&\quad \times \left[\frac{1}{\frac{3}{4}(\mathbf{s}_i - \mathbf{s}')^2 + \left(\mathbf{R}_i - \mathbf{R}' - \frac{1}{2}\mathbf{s}' \right)^2} + \frac{1}{\frac{3}{4}(\mathbf{s}_i - \mathbf{s}')^2 + \left(\mathbf{R}_i - \mathbf{R}' + \frac{1}{2}\mathbf{s}' \right)^2} \right].
\end{aligned}
\tag{B16}
$$

We split this integral into two parts by writing

$$
\frac{\alpha_2(R')}{R'^2} + \bar{\alpha}_0(R') = \frac{\alpha_2}{R'^2} + \left[\frac{\alpha_2(R') - \alpha_2}{R'^2} + \bar{\alpha}_0(R') \right] = f_L(R') + f_S(R'),
\tag{B17}
$$

where f_S is a short-range function and f_L is a long-range function of R':

$$
f_S(R') = \frac{\alpha_2(R') - \alpha_2}{R'^2} + \bar{\alpha}_0(R') = \bar{\alpha}_0(R') - \frac{\bar{\alpha}_2(R')}{R'^2},
\tag{B18a}
$$

$$
f_L(R') = \frac{\alpha_2}{R'^2}.
\tag{B18b}
$$

The integral containing f_S is

$$
\begin{aligned}
&\int d^2 s' \int d^2 R' \frac{m V(s')}{8\pi^2 \hbar^2} (s'_x R'_y - s'_y R'_x) f_S(R') \left[\frac{1}{\frac{3}{4}(\mathbf{s}_i - \mathbf{s}')^2 + \left(\mathbf{R}_i - \mathbf{R}' - \frac{1}{2}\mathbf{s}' \right)^2} + \frac{1}{\frac{3}{4}\left(\mathbf{s}_i - \mathbf{s}' \right)^2 + (\mathbf{R}_i - \mathbf{R}' + \frac{1}{2}\mathbf{s}')^2} \right] \\
&= \int d^2 s' \int d^2 R' \frac{m V(s')}{8\pi^2 \hbar^2} (s'_x R'_y - s'_y R'_x) \left[\bar{\alpha}_0(R') - \frac{\bar{\alpha}_2(R')}{R'^2} \right] \left[\frac{12(\mathbf{s}_i \cdot \mathbf{s}')(\mathbf{R}_i \cdot \mathbf{R}')}{\left(R_i^2 + \frac{3}{4}s_i^2 \right)^3} + \cdots \right] \\
&= -\frac{3\alpha_2 \alpha_4}{8B^6} (s_x R_y - s_y R_x) + O(B^{-6}),
\end{aligned}
\tag{B19}
$$

where we have expanded the integral to the leading-order term assuming that s and R go to infinity simultaneously. The integral containing f_L is

$$
\begin{aligned}
&\int d^2 s' \int d^2 R' \frac{m V(s')}{8\pi^2 \hbar^2} (s'_x R'_y - s'_y R'_x) f_L(R') \left[\frac{1}{\frac{3}{4}(\mathbf{s}_i - \mathbf{s}')^2 + \left(\mathbf{R}_i - \mathbf{R}' - \frac{1}{2}\mathbf{s}' \right)^2} + \frac{1}{\frac{3}{4}(\mathbf{s}_i - \mathbf{s}')^2 + \left(\mathbf{R}_i - \mathbf{R}' + \frac{1}{2}\mathbf{s}' \right)^2} \right] \\
&= (s_x R_y - s_y R_x) \left(\frac{\alpha_2^2}{2B^2 s_i^2} + \frac{3\alpha_2 \alpha_4}{16B^6} \right) + O(B^{-6}).
\end{aligned}
\tag{B20}
$$

Combining the above results, we get

$$
\sum_{j \neq i} \Psi_2^{(ij)} = (s_x R_y - s_y R_x) \left(\frac{\alpha_2^2}{2B^2 s_i^2} - \frac{3\alpha_2 \alpha_4}{16B^6} \right) + O(B^{-6})
\tag{B21}
$$

for any given value of i. Substituting Eqs. (B14) and (B21) into Eq. (B11), we get

$$
\Psi_2 = (s_x R_y - s_y R_x) \left\{ \left[\sum_{i=1}^{3} \left(\frac{\beta_2}{2s_i^2} + \frac{\alpha_2^2}{2B^2 s_i^2} \right) \right] - \frac{9\alpha_2 \alpha_4}{16B^6} \right\} + O(B^{-6}) + O(VV_3) + O(V_3^2).
\tag{B22}
$$

We have not evaluated the terms $\sum_i \widehat{G} V^{(i)} \widehat{G} V_3 \Psi_0 + \sum_i \widehat{G} V_3 \widehat{G} V^{(i)} \Psi_0$ and $(\widehat{G} V_3)^2 \Psi_0$, which are of order VV_3 and V_3^2, respectively.

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