


**Quantum Cramér-Rao bound for quantum statistical models with parameter-dependent rank**Yating Ye and Xiao-Ming Lu <sup>\*</sup>*Department of Physics, Hangzhou Dianzi University, Hangzhou 310018, China* (Received 3 November 2021; revised 12 May 2022; accepted 9 August 2022; published 23 August 2022)

Recently a widely used computation expression for quantum Fisher information was shown to be discontinuous at the parameter points where the rank of the parametric density operator changes. The quantum Cramér-Rao bound can be violated on such singular parameter points if one uses this computation expression for quantum Fisher information. We point out that the discontinuity of the computation expression of quantum Fisher information is accompanied with the unboundedness of the symmetric logarithmic derivation operators, based on which the quantum Fisher information is formally defined and the quantum Cramér-Rao bound is originally proved. We argue that the limiting version of the quantum Cramér-Rao bound still holds when the parametric density operator changes its rank by closing the potential loophole of involving an unbounded SLD operator in the proof of the bound. Moreover, we analyze a typical example of the quantum statistical models with parameter-dependent rank.

DOI: [10.1103/PhysRevA.106.022429](https://doi.org/10.1103/PhysRevA.106.022429)**I. INTRODUCTION**

The Fisher information matrix characterizes the least mean-square error of parameter estimation for a large number of samples [1,2]. For a finite number of samples, the Fisher information matrix also gives a lower bound (namely, the Cramér-Rao bound) on estimation errors when the estimator is restricted to be locally unbiased [3,4]. These properties make the Fisher information matrix play a pivotal role in classical parameter estimation theory [5–7].

For quantum parameter estimation problems, not only the classical estimators but also the quantum measurements should be taken into consideration for minimizing the estimation errors. Helstrom derived the first quantum Cramér-Rao bound (QCRB) by defining the quantum Fisher information (QFI) matrix as an analog of the classical one [8–12]. Due to measurement incompatibility caused by Heisenberg's uncertainty principle [13], the quantum estimation problems for multiple parameters are much more intricate than the classical estimation problems [14–22]. For single-parameter estimation problems, the QCRB, which is given by the QFI, gives a quite satisfactory approach to revealing the ultimate quantum limit of estimation precision.

However, recent researches disclosed a possible defect about the QCRB at the parameter points where the rank of the parametric density operator changes [23–27]. Šafránek [23] demonstrated that a widely used computation expression of QFI can be different from the Bures metric, which had been considered to be equivalent (up to a constant factor) to the QFI for all cases, and discontinuous at such singular parameter points. Later Seveso *et al.* [24] showed through examples that the QCRB based on the QFI does not hold at these limiting cases. However, it is still not clear why the QCRB can be

violated and where the loophole is in the previous proof of the QCRB. This motivated us to study in detail the QCRB in the quantum parameter estimation problems with parameter-dependent rank.

In this work, we revisit the derivation of the QCRB and analyze the potential loophole that may cause the failure of the QCRB. We point out that the symmetric logarithmic derivative (SLD) operator, through which the QCRB is derived and the QFI is defined, is implicitly assumed to be bounded in the previous derivation of the QCRB as well as in the computation of the QFI. When the value of parameter approaches the singular parameter points where the rank of density operator changes, the SLD operator may become unbounded, and some previously used computation expressions of the QFI might inconsistent with the definition of the QFI. We show that this potential loophole can be closed by proving the QCRB in a manner that the SLD operator is not involved. Moreover, we analyze in detail the QCRB in some typical examples of the quantum statistical models with parameter-dependent rank.

This paper is organized as follows. In Sec. II we give a brief review of the formal definition of the QFI and two expressions for the QFI. In Sec. III we investigate the relationships between the discontinuity of the QFI and the boundedness of the SLD operator. In Sec. IV we discuss how to prove the QCRB without resorting to the SLD operator. In Sec. V we analyze in detail the validity of the QCRB with typical examples. We summarize our results in Sec. VI.

**II. EXPRESSIONS FOR THE QFI AND THE DISCONTINUITY**

Let us start by considering the general problem of estimating a single parameter with quantum systems. Assume that the state of the quantum system, described by a density operator  $\rho_\theta$ , depends on an unknown parameter  $\theta$ . The value of  $\theta$  can be estimated by data processing outcomes obtained from a

<sup>\*</sup>lxm@hdu.edu.cn; <http://xmlu.me>.

measurement performed on the quantum system. A quantum measurement is mathematically characterized by a positive-operator-valued measure (POVM)  $\{E_x \mid E_x \geq 0, \sum_x E_x = \mathbb{1}\}$  with  $x$  denoting the outcome and  $\mathbb{1}$  being the identity operator. According to quantum mechanics, the probability of obtaining an outcome  $x$  is given by  $\text{tr}(\rho_\theta E_x)$ . The data processing is represented by an estimator  $\hat{\theta}$  that maps the measurement outcome  $x$  into the estimates for  $\theta$ . The estimation error can be assessed by the mean-square error defined by

$$\mathcal{E}_\theta := \mathbb{E}_\theta[(\hat{\theta} - \theta)^2] = \sum_x [\hat{\theta}(x) - \theta]^2 \text{tr}(\rho_\theta E_x), \quad (2.1)$$

where  $\mathbb{E}_\theta$  denotes the expectation under the probability distribution given by  $p_\theta(x) = \text{tr}(\rho_\theta E_x)$ . For all quantum measurements and all unbiased estimators  $\hat{\theta}$  that satisfy

$$\mathbb{E}_\theta[\hat{\theta}] = \sum_x \hat{\theta}(x) \text{tr}(\rho_\theta E_x) = \theta \quad (2.2)$$

for all possible true values of  $\theta$ , the estimation error obeys the QCRB (also known as the Helstrom bound):

$$\mathcal{E}_\theta \geq \frac{1}{\mathcal{F}_\theta}, \quad (2.3)$$

where  $\mathcal{F}_\theta$  is the QFI whose definition will be given in what follows.

There exist three expressions for the QFI that are often used in the previous works [8,9,11,28]. The first one is its formal definition [8,9]:

$$\mathcal{F}_\theta^{(1)} := \text{tr}(\rho_\theta L_\theta^2), \quad (2.4)$$

where  $L_\theta$  is the SLD operator defined as the Hermitian operator satisfying

$$\frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta) = \frac{d\rho_\theta}{d\theta}. \quad (2.5)$$

The QCRB  $\mathcal{E}_\theta \geq 1/\mathcal{F}_\theta^{(1)}$  was originally proved by utilizing Eq. (2.4) and Eq. (2.5).

Suppose that  $\rho_\theta = \sum_j \lambda_j |e_j\rangle\langle e_j|$  is the spectral decomposition of  $\rho_\theta$ . Note that the eigenvalues  $\lambda_j$  and the eigenvectors  $|e_j\rangle$  depend on the true value of the parameter  $\theta$ . The second form of the QFI is a widely used computation expression given by

$$\mathcal{F}_\theta^{(2)} := \sum_{j,k \in K_\theta} \frac{2}{\lambda_j + \lambda_k} \left| \left\langle e_j \left| \frac{d\rho_\theta}{d\theta} \right| e_k \right\rangle \right|^2, \quad (2.6)$$

where  $K_\theta$  is the index set  $\{k \mid \lambda_k(\theta) = 0\}$  with  $\lambda_j(\theta)$  denoting the value of  $\lambda_j$  at the parameter point  $\theta$ . This expression can be derived by solving the matrix representation of the SLD operator  $L_\theta$  with the basis constituted by the eigenvectors of  $\rho_\theta$  and then substituting it into the formal definition Eq. (2.4) of the QFI.

The third form of the QFI is given by

$$\mathcal{F}_\theta^{(3)} := 8 \lim_{\epsilon \rightarrow 0} \frac{1 - \|\sqrt{\rho_\theta} \sqrt{\rho_{\theta+\epsilon}}\|_1}{\epsilon^2}, \quad (2.7)$$

where  $\|X\|_p := [\text{tr}(|X|^p)]^{1/p}$  with  $|X| := \sqrt{X^\dagger X}$  is the Schatten- $p$  norm of an operator  $X$ . The quantity  $\|\sqrt{\rho_\theta} \sqrt{\rho_{\theta+\epsilon}}\|_1$  is known as Uhlmann's fidelity between the density

operators  $\rho_\theta$  and  $\rho_{\theta+\epsilon}$  [29–31]. This expression is deeply relevant to the Bures distance [32] between two density operators  $\rho$  and  $\sigma$ :

$$d_B(\rho, \sigma) := \sqrt{2(1 - \|\sqrt{\rho} \sqrt{\sigma}\|_1)}. \quad (2.8)$$

It can be seen that the third form  $\mathcal{F}_\theta^{(3)}$  is equivalent to the metric of the Bures distance between two density operators up to an insignificant constant factor 4,

$$\mathcal{F}_\theta^{(3)} = 4 \lim_{\epsilon \rightarrow 0} \frac{d_B(\rho_\theta, \rho_{\theta+\epsilon})^2}{\epsilon^2}. \quad (2.9)$$

In Ref. [25] Zhou and Jiang showed that the computation expression  $\mathcal{F}_\theta^{(2)}$  can be fully equivalent to a modified Bures metric:

$$\mathcal{F}_\theta^{(2)} = 4 \lim_{\epsilon \rightarrow 0} \frac{d_B(\rho_{\theta-\epsilon/2}, \rho_{\theta+\epsilon/2})^2}{\epsilon^2}. \quad (2.10)$$

### III. DISCONTINUOUS QFI AND UNBOUNDED SLD

As shown by Šafránek [23], the expressions  $\mathcal{F}_\theta^{(2)}$  and  $\mathcal{F}_\theta^{(3)}$ , given in Eqs. (2.6) and (2.7), can be different at the parameter points where the rank of  $\rho_\theta$  changes. The discrepancy occurs when a term excluded from the summation in Eq. (2.7) for  $\mathcal{F}_\theta^{(2)}$  has a finite value as the parameter tends to that specific value. At a specific value  $\theta'$ , it has been shown that [23]

$$\begin{aligned} \Delta_{\theta'} &:= \mathcal{F}_{\theta'}^{(3)} - \mathcal{F}_{\theta'}^{(2)} = \lim_{\theta \rightarrow \theta'} \sum_{k \in K_{\theta'}} \frac{1}{\lambda_k} \left( \frac{d\lambda_k}{d\theta} \right)^2 \\ &= \sum_{k \in K_{\theta'}} 2 \frac{d^2 \lambda_k}{d\theta^2} \Bigg|_{\theta=\theta'}, \end{aligned} \quad (3.1)$$

where  $\lambda_j$  are the eigenvalues of  $\rho_\theta$  and  $K_{\theta'}$  denotes the index set  $\{k \mid \lambda_k(\theta') = 0\}$ . Note that the last equality in Eq. (3.1) is due to L'Hôpital's rule in calculus [23,24]. The  $\mathcal{F}_\theta^{(2)}$ -QCRB can be violated when  $\Delta_{\theta'} > 0$ ; see Ref. [24]. In what follows we shall interpret why the  $\mathcal{F}_\theta^{(2)}$ -QCRB breaks down at the parameter point where the rank of the density operators changes.

First, when the SLD operator is bounded, the  $\mathcal{F}_\theta^{(2)}$ -QCRB always holds. This is because the original proof of  $\mathcal{F}_\theta^{(1)}$ -QCRB and the relation  $\mathcal{F}_\theta^{(2)} = \mathcal{F}_\theta^{(1)}$  are both rigorous for such a case. To see the latter, recall that the expression Eq. (2.6) for  $\mathcal{F}_\theta^{(2)}$  is a consequence of solving the SLD equation Eq. (2.5) for  $L_\theta$  through matrix representation with the basis constituted by the eigenvectors of  $\rho_\theta$ . With the spectral decomposition  $\rho_\theta = \sum_j \lambda_j |e_j\rangle\langle e_j|$ , it follows from Eq. (2.5) that

$$\frac{\lambda_j + \lambda_k}{2} \langle e_j | L_\theta | e_k \rangle = \left\langle e_j \left| \frac{d\rho_\theta}{d\theta} \right| e_k \right\rangle. \quad (3.2)$$

When  $\lambda_j + \lambda_k \neq 0$ , the above equality implies that

$$\langle e_j | L_\theta | e_k \rangle = \frac{2}{\lambda_j + \lambda_k} \left\langle e_j \left| \frac{d\rho_\theta}{d\theta} \right| e_k \right\rangle. \quad (3.3)$$

For those indices  $j$  and  $k$  such that  $\lambda_j + \lambda_k = 0$ ,  $\langle e_j | L_\theta | e_k \rangle$  is indefinite by Eq. (3.2). On the other hand, the QFI defined in Eq. (2.4) can be expressed with the spectral decomposition of

$\rho$  as

$$\mathcal{F}_\theta^{(1)} = \sum_{jk} \lambda_j |\langle e_j | L_\theta | e_k \rangle|^2 = \sum_{jk} \frac{\lambda_j + \lambda_k}{2} |\langle e_j | L_\theta | e_k \rangle|^2, \quad (3.4)$$

where the second equality is due to the symmetrization with respect to the indices  $j$  and  $k$ . For those indices  $j$  and  $k$  such that  $\lambda_j + \lambda_k = 0$ , the corresponding terms in the right-hand side of Eq. (3.4) have no contribution to the summation as long as the SLD operator is *bounded*. In such a case, combining Eqs. (3.2) and (3.4), we can see that  $\mathcal{F}_\theta^{(1)} = \mathcal{F}_\theta^{(2)}$ . Therefore, the QCRB based on  $\mathcal{F}_\theta^{(2)}$  must hold when there exists an bounded SLD operator.

It is not always possible to have a bounded SLD operator [33] for all range of the parameters, even when the QFI is finite. For example, for

$$\rho_\theta = \cos^2 \theta |0\rangle\langle 0| + \sin^2 \theta |1\rangle\langle 1| \text{ with } 0 \leq \theta \leq \pi/2, \quad (3.5)$$

the SLD operator is

$$L_\theta = -2 \tan \theta |0\rangle\langle 0| + 2 \cot \theta |1\rangle\langle 1|, \quad (3.6)$$

which is unbounded at  $\theta = 0$  and  $\theta = \pi/2$ . Meanwhile, the QFI for the above statistical model is 4 for all  $0 \leq \theta \leq \pi/2$ .

As shown by Ref. [23],  $\mathcal{F}_\theta^{(3)}$  represents the continuous version of the QFI  $\mathcal{F}_\theta^{(1)}$ , i.e.,  $\lim_{\theta \rightarrow \theta'} \mathcal{F}_\theta^{(1)} = \mathcal{F}_{\theta'}^{(3)}$ . Therefore, we can substitute Eq. (3.4) into the definition of  $\Delta_{\theta'}$  and thus get

$$\Delta_{\theta'} = \lim_{\theta \rightarrow \theta'} \sum_{j,k \in K_{\theta'}} \frac{\lambda_j + \lambda_k}{2} |\langle e_j | L_\theta | e_k \rangle|^2. \quad (3.7)$$

Since  $(\lambda_j + \lambda_k)/2 = 0$  at the parameter point  $\theta'$  for all  $j, k \in K_{\theta'}$ , we get

$$\Delta_{\theta'} > 0 \iff \lim_{\theta \rightarrow \theta'} L_\theta \text{ is unbounded}, \quad (3.8)$$

meaning that the discrepancy between  $\mathcal{F}_\theta^{(2)}$  and  $\mathcal{F}_\theta^{(3)}$  must be accompanied by an unbounded SLD operator. The divergence of the SLD operator has also been investigated through the integral representation of the SLD operator by Rezakhani, Hassani, and Alipour [26].

Strictly speaking, it is not the QFI (defined by  $\mathcal{F}_\theta^{(1)}$ ) but its computation expression  $\mathcal{F}_\theta^{(2)}$  that is discontinuous when the density operator changes its rank. At these singular parameter points,  $\mathcal{F}_\theta^{(2)}$  does not coincide with the definition of the QFI, and thus the  $\mathcal{F}^{(2)}$ -QCRB, i.e.,  $\mathcal{E}_\theta \geq 1/\mathcal{F}_\theta^{(2)}$ , can be violated.

#### IV. QCRB AT RANK-CHANGING POINTS

Even though the violation of the  $\mathcal{F}_\theta^{(2)}$  can be ascribed to the inconsistency between  $\mathcal{F}_\theta^{(2)}$  and  $\mathcal{F}_\theta^{(1)}$  when the SLD operator is unbounded, it does not mean that the limiting version of  $\mathcal{F}^{(1)}$ -QCRB, namely,

$$\mathcal{E}_{\theta'} \geq \frac{1}{\lim_{\theta \rightarrow \theta'} \mathcal{F}_\theta^{(1)}}, \quad (4.1)$$

has been rigorously proved at such singular parameter point  $\theta'$ . This is because the original proof of QCRB [8,9] did not discuss the cases of unbounded SLD operators. We have to

be cautious about whether some mathematical formulas used in the original proof of the QCRB still hold for unbounded operators; e.g., the cyclic property of the trace operation,  $\text{tr}(AB) = \text{tr}(BA)$  for two operators  $A$  and  $B$ , can be violated if unbounded operators are involved. We here argue that the limiting version of  $\mathcal{F}^{(1)}$ -QCRB holds even when the SLD operator becomes unbounded and elucidate it from two perspectives in what follows.

Our first approach to supporting Eq. (4.1) is adapting the original proof of the QCRB [8,9,34] to avoid involving the unbounded SLD. Note that the influence of the SLD operator is always imposed via the product  $L_\theta \sqrt{\rho_\theta}$ . When the SLD operator is unbounded, the product  $L_\theta \sqrt{\rho_\theta}$  may still be bounded. For example, for the density operators in Eq. (3.5), even the SLD operators is unbounded at  $\theta = 0$  and  $\theta = \pi/2$ , the product

$$L_\theta \sqrt{\rho_\theta} = -2 \sin \theta |0\rangle\langle 0| + 2 \cos \theta |1\rangle\langle 1| \quad (4.2)$$

is still bounded at  $\theta = 0$  and  $\theta = \pi/2$ . This motives us to introduce the linear operator  $Q_\theta$  that satisfies the following two conditions:

$$\frac{1}{2}(Q_\theta \sqrt{\rho_\theta} + \sqrt{\rho_\theta} Q_\theta^\dagger) = \frac{d\rho_\theta}{d\theta}, \quad (4.3)$$

$$\sqrt{\rho_\theta} Q_\theta - Q_\theta^\dagger \sqrt{\rho_\theta} = 0. \quad (4.4)$$

In fact,  $Q_\theta$  plays the role of  $L_\theta \sqrt{\rho_\theta}$ , as Eq. (4.3) corresponds to the SLD equation (2.5) and Eq. (4.4) corresponds to the hermiticity of the SLD operator. Moreover, the QFI can always be expressed as

$$\mathcal{F}_\theta^{(1)} = \|Q_\theta\|_2^2. \quad (4.5)$$

The advantage of  $Q_\theta$  over the SLD operator  $L_\theta$  is that, according to Eq. (4.5), the QFI is infinitely large if and only if  $Q_\theta$  is unbounded. Therefore, if  $Q_\theta$  is unbounded, the  $\mathcal{F}^{(1)}$ -QCRB becomes  $\mathcal{E}_\theta \geq 0$ , whose validation is trivial. So we only need to consider the cases where  $Q_\theta$  are bounded.

Now, we assume that  $Q_\theta$  is bounded henceforth and prove the QCRB without directly resorting to the SLD operator. For any POVM  $\{E_x\}$  and any estimator  $\hat{\theta}$ , let us define  $X := \sum_x \hat{\theta}(x) E_x$  and  $\Delta X := X - \text{tr}(\rho_\theta X) \mathbb{1}$ , where  $\mathbb{1}$  denotes the identity operator. The mean of the estimator is given by  $\mathbb{E}_\theta[\hat{\theta}] = \text{tr}(\rho_\theta X)$ . The variance of the estimator,  $\mathcal{V}_\theta := \mathbb{E}[\hat{\theta}^2] - \mathbb{E}[\hat{\theta}]^2$ , is bounded from below by the variance of the operator  $X$  [15, see Sec. 2.9], that is,

$$\mathcal{V}_\theta \geq \text{tr}[\rho(\Delta X)^2] = \|(\Delta X) \sqrt{\rho_\theta}\|_2^2. \quad (4.6)$$

Applying the Cauchy-Schwarz inequality, it follows that

$$\|(\Delta X) \sqrt{\rho_\theta}\|_2 \|Q_\theta\|_2 \geq |\text{tr}[\sqrt{\rho_\theta}(\Delta X)Q_\theta]|. \quad (4.7)$$

Since  $|z| \geq |\text{Re}z|$  for any complex number  $z$ , we get

$$\begin{aligned} |\text{tr}[\sqrt{\rho_\theta}(\Delta X)Q_\theta]| &\geq |\text{Re tr}[\sqrt{\rho_\theta}(\Delta X)Q_\theta]| \\ &= \left| \text{tr} \left( \Delta X \frac{Q_\theta \sqrt{\rho_\theta} + \sqrt{\rho_\theta} Q_\theta^\dagger}{2} \right) \right| \\ &= \left| \text{tr} \left( \Delta X \frac{d\rho_\theta}{d\theta} \right) \right| \\ &= \left| \frac{d \text{tr}(\rho_\theta X)}{d\theta} \right| = \left| \frac{d \mathbb{E}_\theta[\hat{\theta}]}{d\theta} \right|, \end{aligned} \quad (4.8)$$

where the equality in the second last line is due to Eq. (4.3). It then follows from Eqs. (4.6)–(4.8) that

$$\mathcal{V}_\theta \geq \left( \frac{d}{d\theta} \mathbb{E}_\theta[\hat{\theta}] \right)^2 \|Q_\theta\|_2^{-2} \geq \left( \frac{d}{d\theta} \mathbb{E}_\theta[\hat{\theta}] \right)^2 / \mathcal{F}_\theta^{(1)}. \quad (4.9)$$

For locally unbiased estimators, for which we have  $\mathcal{E}_\theta = \mathcal{V}_\theta$  and  $\frac{d}{d\theta} \mathbb{E}[\hat{\theta}] = 1$ , the inequality in Eq. (4.9) becomes  $\mathcal{E}_\theta \geq \|Q_\theta\|_2^{-2}$ .

Using the operator  $Q_\theta$  instead of the SLD operator  $L_\theta$ , we have proved that the QCRB  $\mathcal{E}_\theta \geq \|Q_\theta\|_2^{-2}$  always holds no matter whether the operator  $Q_\theta$  is bounded or not. This proof can include the special case where  $L_\theta$  is unbounded at a singular parameter point  $\theta'$  but the QFI is still finite. In such cases, the QFI can be expressed as  $\|Q_{\theta'}\|_2^2 = \lim_{\theta \rightarrow \theta'} \text{tr}(\rho_\theta L_\theta^2)$  due to the relation between  $Q_\theta$  and  $L_\theta$ . Therefore, we get Eq. (4.1).

Our second approach to supporting the limiting version of the  $\mathcal{F}^{(1)}$ -QCRB is bounding the estimation error with the Bures metric, namely, Eq. (4.10). It is known that the Bures metric satisfies  $\mathcal{F}_{\theta'}^{(3)} = \lim_{\theta \rightarrow \theta'} \mathcal{F}_\theta^{(1)}$  (e.g. see Ref. [23]), so the inequality (4.1) is equivalent to

$$\mathcal{E}_{\theta'} \geq \frac{1}{\mathcal{F}_{\theta'}^{(3)}}. \quad (4.10)$$

This inequality was recently proved by Yang, Chiribella, and Hayashi through a more fundamental relation (denoted by Yang-Chiribella-Hayashi inequality henceforth) [35]

$$\frac{1}{2}(\mathcal{E}_\theta + \mathcal{E}_{\theta+\epsilon} + \epsilon^2) \geq \frac{\epsilon^2}{4d_B(\rho_\theta, \rho_{\theta+\epsilon})^2} \quad (4.11)$$

for the estimators that are unbiased at both  $\theta$  and  $\theta + \epsilon$ . Taking the limit  $\epsilon \rightarrow 0$  of the above inequality, Eq. (4.10) can be obtained. We also give an alternative proof using the purification method in Appendix A. In addition, the inequality (4.11) can be easily generalized to include the biased estimators, as shown in Appendix B.

### V. CONCRETE EXAMPLES

Seveso *et al.* [24] argued with examples that not only the  $\mathcal{F}^{(2)}$ -QCRB but also the  $\mathcal{F}^{(3)}$ -QCRB can be violated when the parametric density operator changes its rank through examples. This conflicts with our result Eq. (4.1) and Eq. (4.10) in Sec. IV. In what follows, we shall analyze in detail the example discussed in Ref. [24] and solve the conflict.

Following Ref. [24], we first consider the example whose parametric density operator reads

$$\rho_q = (1 - q)|0\rangle\langle 0| + q|1\rangle\langle 1| \text{ with } 0 \leq q \leq 1 \quad (5.1)$$

with  $q$  being the parameter of interest. The rank of  $\rho_q$  changes at the boundary, i.e.,  $q = 0$  and  $q = 1$ . The SLD is given by

$$L_q = \frac{-1}{1 - q}|0\rangle\langle 0| + \frac{1}{q}|1\rangle\langle 1|, \quad (5.2)$$

which becomes unbounded as  $q \rightarrow 0$  or  $q \rightarrow 1$ . Meanwhile, the Bures distance is given by

$$d_B(\rho_q, \rho_{q'})^2 = 2[1 - \sqrt{(1 - q)(1 - q')} - \sqrt{qq'}]. \quad (5.3)$$

The QFI expressions are given by [24]

$$\mathcal{F}_q^{(3)} = \frac{1}{q(1 - q)}, \quad (5.4)$$

which diverges at  $q = 0$  and  $q = 1$ , and

$$\mathcal{F}_q^{(2)} = \begin{cases} 1, & \text{if } q = 0 \text{ or } 1, \\ \frac{1}{q(1 - q)}, & 0 < q < 1. \end{cases} \quad (5.5)$$

Suppose that we have  $n$  copies of the quantum system and perform on each copy the projective measurement whose POVM is given by  $E_0 = |0\rangle\langle 0|$  and  $E_1 = |1\rangle\langle 1|$ . We denote with  $x_j$  the measurement outcome on the  $j$ th system, which takes the value 0 and 1 with the probabilities  $1 - q$  and  $q$ , respectively. The statistical quantity

$$t := \sum_{j=1}^n x_j \quad (5.6)$$

is a sufficient statistic for estimating  $q$ . The probability of  $t$  is given by

$$p_q(t) = \binom{n}{t} q^t (1 - q)^{n-t}. \quad (5.7)$$

Note that

$$\frac{\partial \ln p_q(t)}{\partial q} = \frac{t}{q} - \frac{n - t}{1 - q} = \frac{t - nq}{q(1 - q)}, \quad (5.8)$$

implying that the maximum likelihood estimator for this statistical model is given by

$$\hat{q}_{\text{ML}}(t) = \frac{t}{n}. \quad (5.9)$$

The mean of  $\hat{q}_{\text{ML}}(t)$  is  $q$ , so  $\hat{q}_{\text{ML}}(t)$  is unbiased for all  $0 \leq q \leq 1$ . The variance of  $\hat{q}_{\text{ML}}(t)$  is  $q(1 - q)/n$ , so the mean-square error of the maximum likelihood estimator vanishes at  $q = 0$  and  $q = 1$ . Comparing the mean-square error of the maximum likelihood estimation with the QFI expressions Eq. (5.4) and Eq. V.5, it can be seen that the  $\mathcal{F}^{(2)}$ -QCRB is violated at  $q = 0$  and  $q = 1$  while the  $\mathcal{F}^{(3)}$ -QCRB still holds in such a scenario [24].

Furthermore, Ref. [24] considered a reparametrization of the above example, namely,

$$\rho_\vartheta = \cos^2 \vartheta |0\rangle\langle 0| + \sin^2 \vartheta |1\rangle\langle 1| \text{ with } 0 \leq \vartheta \leq \frac{\pi}{2}. \quad (5.10)$$

This is a reparametrization of  $\rho_q$  in Eq. (5.1) by substituting  $q = \sin^2 \vartheta$  therein. For this statistical model, the SLD is given by

$$L_\vartheta = -2 \tan \vartheta |0\rangle\langle 0| + 2 \cot \vartheta |1\rangle\langle 1|, \quad (5.11)$$

which becomes unbounded at  $\vartheta = 0$  and  $\vartheta = \pi/2$ . Correspondingly, the QFI are given by [23]

$$\mathcal{F}_\vartheta^{(3)} = 4 \quad \text{and} \quad \mathcal{F}_\vartheta^{(2)} = \begin{cases} 0, & \text{if } \vartheta = 0 \text{ or } \pi/2, \\ 4, & \text{otherwise.} \end{cases} \quad (5.12)$$

The optimal measurement for estimating  $\vartheta$  is still given by the POVM  $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ , as the parametric density matrix is always diagonal with the basis  $\{|0\rangle, |1\rangle\}$ . Because the maximum likelihood estimator is equivariant [36, Sec. 9.4], the

maximum likelihood estimator  $\hat{\vartheta}_{\text{ML}}$  for  $\vartheta$  can be expressed as

$$\hat{\vartheta}_{\text{ML}}(t) = \arcsin[\sqrt{\hat{q}_{\text{ML}}(t)}] = \arcsin\left(\sqrt{\frac{t}{n}}\right), \quad (5.13)$$

where  $t$  is the same as that in Eq. (5.6). This is equivalent to first estimate  $q$  in Eq. (5.1) using the maximum likelihood estimation and then calculate  $\vartheta$  via the relation  $q = \sin^2 \vartheta$ . It can be seen from Eq. (5.7) that  $t$  has a deterministic value at  $q = 0$  and  $q = 1$ , which correspond to  $\vartheta = 0$  and  $\vartheta = \pi/2$ , respectively. Consequently, the variance of  $\hat{\vartheta}_{\text{ML}}$  vanishes at these parameter points. Meanwhile, the continuous version of QCRB reads  $\mathcal{E}_\vartheta \geq 1/(4n)$ . Due to these facts, it was concluded in Ref. [24] that the continuous version of QCRB (or the  $\mathcal{F}^{(3)}$ -QCRB) is violated at  $\theta = 0$  and  $\theta = \pi/2$ . However, the continuous version of QCRB should be valid according to our analysis in the previous section. We shall solve this conflict in what follows.

Note that the QCRB is established only for *locally unbiased estimators*. Although  $\hat{q}_{\text{ML}}$  is unbiased for  $q$ ,  $\hat{\vartheta}_{\text{ML}}$ , as we will show in the following, is not locally unbiased for  $\vartheta$ . The mean of the maximum likelihood estimator is given by

$$\mathbb{E}_\vartheta[\hat{\vartheta}_{\text{ML}}(t)] = \sum_{t=0}^n \arcsin\left(\sqrt{\frac{t}{n}}\right) p_q(t) \quad (5.14)$$

with  $q = \sin^2 \vartheta$  and  $p_q(t)$  being given by Eq. (5.7). Taking the derivative of  $\mathbb{E}_\vartheta[\hat{\vartheta}_{\text{ML}}(t)]$ , we have

$$\begin{aligned} & \frac{d}{d\vartheta} \mathbb{E}_\vartheta[\hat{\vartheta}_{\text{ML}}(t)] \\ &= \sum_{t=0}^n \arcsin\left(\sqrt{\frac{t}{n}}\right) \frac{dq}{d\vartheta} \frac{\partial p_q(t)}{\partial q} \Big|_{q=\sin^2 \vartheta} \\ &= 2 \sin \vartheta \cos \vartheta \sum_{t=0}^n \arcsin\left(\sqrt{\frac{t}{n}}\right) \frac{dp_q(t)}{dq} \Big|_{q=\sin^2 \vartheta}. \end{aligned} \quad (5.15)$$

At  $\vartheta = 0$  and  $\vartheta = \pi/2$ , it can be seen that  $\frac{d}{d\vartheta} \mathbb{E}_\vartheta[\hat{\vartheta}_{\text{ML}}(t)]$  vanishes for a finite  $n$ , as  $\sin \vartheta \cos \vartheta = 0$ . Nevertheless, the locally unbiased condition requires that  $\frac{d}{d\vartheta} \mathbb{E}_\vartheta[\hat{\vartheta}_{\text{ML}}(t)] = 1$ . Therefore,  $\hat{\vartheta}$  is not locally unbiased at  $\vartheta = 0$  and  $\vartheta = \pi/2$ . Figure 1 plots the bias of the maximum likelihood estimator. It can be seen that the bias abruptly increases when the true value of  $\vartheta$  departs from 0 or  $\pi/2$ . Now, it is clear that the fact  $\hat{\vartheta}_{\text{ML}}$  has zero variance does not mean the QCRB must be violated, as  $\hat{\vartheta}_{\text{ML}}$  is not a locally unbiased estimator.

Furthermore, we use the biased version of QCRB Eq. (4.9) to verify the continuous QCRB for  $\rho_\vartheta$  at the singular parameter points  $\vartheta = 0$  and  $\vartheta = \pi/2$ . Figure 2 plots the numerical result of the rescaled variance of the maximum likelihood estimator ( $n\mathcal{V}_\vartheta$ ) and the biased versions of QCRB with a different number  $n$  of samples at different true values of  $\vartheta$ . Note that, unlike the QCRB for a locally unbiased estimator, the biased version of QCRB depends on the estimators themselves. It can be seen from Fig. 2 that the biased QCRB always holds, while the unbiased QCRB is violated near the singular parameter points  $\vartheta = 0$  and  $\vartheta = \pi/2$ .

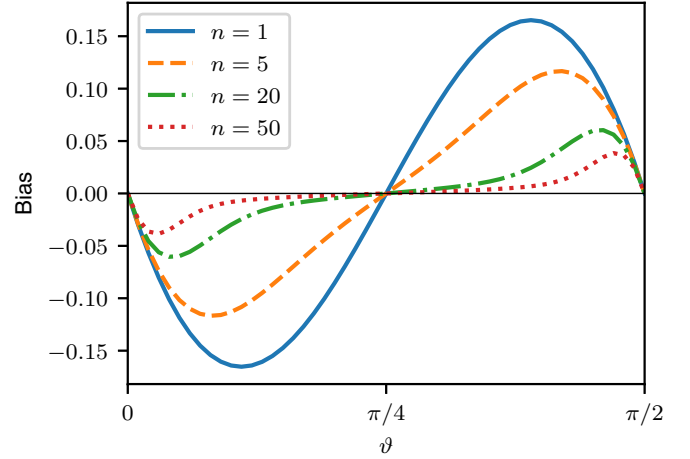


FIG. 1. The bias,  $\mathbb{E}_\vartheta[\hat{\vartheta}_{\text{ML}}] - \vartheta$ , of the maximum likelihood estimator  $\hat{\vartheta}_{\text{ML}}$  vs the true value  $\vartheta$ .  $n$  is the number of samples used for estimation.

## VI. CONCLUSION

In this work, we have studied the validity of the QCRB at the specific parameter values where the parametric density operator changes its rank. The violation of the QCRB based on the widely used computation expression of the QFI, which is discontinuous at the singular parameter points, is ascribed to the inconsistency between the computation expression of the QFI and the formal definition of the QFI as the SLD operator becomes unbounded. We have showed that the potential loophole of involving an unbounded SLD operator in the proof of the QCRB can be closed by either using the  $Q_\theta$  operator instead of the SLD operator or invoking the QCRB based on the Bures distance, e.g., the Yang-Chiribella-Hayashi inequality [35]. We have analyzed in detail the QCRB in a typical example of the quantum statistical models with parameter-dependent rank and solved the conflict with previous results

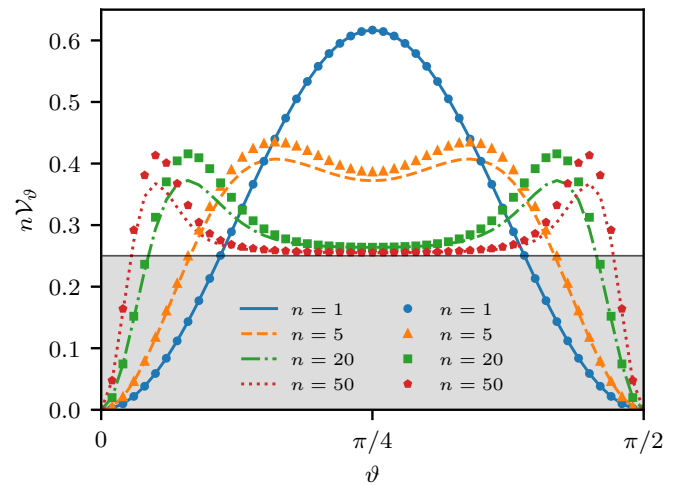


FIG. 2. Rescaled variances  $n\mathcal{V}_\vartheta$  (indicated with various markers) of the maximum likelihood estimator and corresponding biased QCRBs (indicated with various lines).  $n$  is the number of samples used for estimation. The shadow represents the region where the QCRB for a locally unbiased estimator is violated.

by showing that the maximum likelihood estimator therein is not locally unbiased. Our work consolidates the soundness of the QCRB as well as the QFI at the rank-changing points.

**ACKNOWLEDGMENT**

This work is supported by the National Natural Science Foundation of China (Grants No. 61871162, No. 11935012, and No. 11805048).

**APPENDIX A: PROOF OF THE QCRB BASED ON THE BURES DISTANCE BY PURIFICATION**

Although the  $\mathcal{F}^{(3)}$ -QCRB was already proved by Ref. [35], we here give an alternative way by utilizing the purification of mixed states. Let us start with the following proposition.

*Proposition 1.* Assume that  $A_\theta$  and  $A_{\theta'}$  are two bounded operators satisfying  $\rho_\theta = A_\theta A_\theta^\dagger$  and  $\rho_{\theta'} = A_{\theta'} A_{\theta'}^\dagger$ , respectively. For any POVM  $\{E_x\}$  and any unbiased estimator  $\hat{\theta}$ , we have

$$\mathcal{E}_\theta \geq \frac{1}{4} \beta^2 \|A_{\theta'} - A_\theta\|_2^{-2}, \tag{A1}$$

where  $\|X\|_2 = \sqrt{\text{tr}(X^\dagger X)}$  is the Schatten-2 norm of an operator  $X$  and  $\beta$  is a scalar quantity defined by

$$\beta := \theta' - \theta - \sum_x \eta_{x,\theta} \|\sqrt{E_x}(A_{\theta'} - A_\theta)\|_2^2 \tag{A2}$$

with  $\eta_{x,\theta} := \hat{\theta}(x) - \theta$  for brevity.

*Proof.* The mean-square error of estimating  $\theta$  can be expressed as

$$\mathcal{E}_\theta = \sum_x \eta_{x,\theta}^2 \|\sqrt{E_x} A_\theta\|_2^2. \tag{A3}$$

By inserting  $\sum_x E_x = \mathbb{1}$  into  $\|A_{\theta'} - A_\theta\|_2^2$ , we get

$$\|A_{\theta'} - A_\theta\|_2^2 = \sum_x \|\sqrt{E_x}(A_{\theta'} - A_\theta)\|_2^2. \tag{A4}$$

It then can be shown that

$$\begin{aligned} \mathcal{E}_\theta^{1/2} \|A_{\theta'} - A_\theta\|_2 &\geq \sum_x |\eta_{x,\theta}| \|\sqrt{E_x} A_\theta\|_2 \|\sqrt{E_x}(A_{\theta'} - A_\theta)\|_2 \\ &\geq \sum_x |\eta_{x,\theta}| |\text{tr}[A_\theta^\dagger E_x (A_{\theta'} - A_\theta)]| =: \alpha. \end{aligned} \tag{A5}$$

Here the first inequality is due to a Cauchy-Schwarz inequality  $[\sum_x f(x)^2][\sum_x g(x)^2] \geq [\sum_x f(x)g(x)]^2$  for two real functions  $f(x) = \eta_{x,\theta} \|\sqrt{E_x} A_\theta\|_2$  and  $g(x) = \|\sqrt{E_x}(A_{\theta'} - A_\theta)\|_2$ , and the second one is due to a Cauchy-Schwarz inequality  $\|X\|_2 \|Y\|_2 \geq |\text{tr}(X^\dagger Y)|$  for two bounded operators  $X = \sqrt{E_x} A_\theta$  and  $Y = \sqrt{E_x}(A_{\theta'} - A_\theta)$ . Noting that  $|z| \geq |\text{Re}z|$  for any complex number  $z$  and  $\sum_x |f_x| \geq |\sum_x f_x|$  for any series of scalars  $f_x$ , we get

$$\begin{aligned} \alpha &\geq \left| \sum_x \eta_{x,\theta} \text{Re} \text{tr}[A_\theta^\dagger E_x (A_{\theta'} - A_\theta)] \right| \\ &= \left| \sum_x \frac{\eta_{x,\theta}}{2} \text{tr}[E_x (A_{\theta'} A_\theta^\dagger + A_\theta A_{\theta'}^\dagger - 2A_\theta A_\theta^\dagger)] \right|. \end{aligned}$$

Substituting

$$\begin{aligned} A_{\theta'} A_\theta^\dagger + A_\theta A_{\theta'}^\dagger - 2A_\theta A_\theta^\dagger \\ = -(A_{\theta'} - A_\theta)(A_{\theta'} - A_\theta)^\dagger + A_{\theta'} A_{\theta'}^\dagger - A_\theta A_\theta^\dagger \end{aligned} \tag{A6}$$

into the right-hand side of the above inequality, we get

$$\begin{aligned} \alpha &\geq \frac{1}{2} \left| - \sum_x \eta_{x,\theta} \|\sqrt{E_x}(A_{\theta'} - A_\theta)\|_2^2 \right. \\ &\quad \left. + \sum_x \eta_{x,\theta} \|\sqrt{E_x} A_{\theta'}\|_2^2 - \sum_x \eta_{x,\theta} \|\sqrt{E_x} A_\theta\|_2^2 \right|. \end{aligned} \tag{A7}$$

For all unbiased estimators  $\hat{\theta}$ ,  $\sum_x \hat{\theta}(x) \text{tr}(E_x A_\theta A_\theta^\dagger) = \theta$ , which implies that

$$\sum_x \eta_{x,\theta} \|\sqrt{E_x} A_{\theta'}\|_2^2 = \theta' - \theta \tag{A8}$$

for all  $\theta$  and  $\theta'$ . Specifically for  $\theta' = \theta$ , we have

$$\sum_x \eta_{x,\theta} \|\sqrt{E_x} A_\theta\|_2^2 = 0. \tag{A9}$$

Substituting Eq. (A8) and Eq. (A9) into the right-hand side of Eq. (A7), we get  $\alpha^2 \geq \beta^2/4$ . Therefore, together with Eq. (A5), we get Eq. (A1).

The inequality (A1) can be further refined from two perspectives. First, the operator  $A$  satisfying  $\rho = AA^\dagger$  for a given  $\rho$  is not unique. All the operators  $A = \sqrt{\rho}U$  with  $U$  being an arbitrary unitary operator satisfy  $\rho = AA^\dagger$  and are called amplitudes of  $\rho$ . The inequality (A1) holds for all amplitudes of  $\rho_\theta$  and  $\rho_{\theta'}$ ; this supplies a freedom of choosing specific  $A_\theta$  and  $A_{\theta'}$  to formulate useful lower bounds based on Eq. (A1). The Bures-Uhlmann geometry for density operators tell us that [33,37]

$$\min_{A_\theta, A_{\theta'}} \|A_{\theta'} - A_\theta\|_2^2 = d_B(\rho_\theta, \rho_{\theta'})^2, \tag{A10}$$

and the minimum is attained by the amplitudes satisfying

$$A_\theta^\dagger A_{\theta'} = A_{\theta'}^\dagger A_\theta \geq 0. \tag{A11}$$

Two amplitudes satisfying the above condition are called *parallel* [38]. For parallel amplitudes  $A_\theta$  and  $A_{\theta'}$  we have

$$\mathcal{E}_\theta \geq \frac{\beta^2}{4d_B(\rho_\theta, \rho_{\theta'})^2}. \tag{A12}$$

Note that the quantity  $\beta$  defined by Eq. (A2) still depends on the POVM and estimator.

Second, the inequality Eq. (A1) holds for any two parameter points  $\theta$  and  $\theta'$  in the range of the parameter. This supplies another freedom of formulate useful lower bounds on the estimation error at  $\theta$ . Let us consider two infinitesimally neighboring parameter points by defining  $\epsilon := \theta' - \theta$  and taking the limit  $\epsilon \rightarrow 0$ . Assuming that  $A_\theta$  is differentiable so that  $A_{\theta'} \approx A_\theta + \epsilon dA_\theta/d\theta$ . Therefore, we obtain

$$\beta \approx \epsilon - \epsilon^2 \sum_x \eta_{x,\theta} \text{tr} \left( \frac{dA_\theta^\dagger}{d\theta} E_x \frac{dA_\theta}{d\theta} \right), \tag{A13}$$

and thus Eq. (A1) becomes

$$\mathcal{E}_\theta \geq \frac{1}{4} \lim_{\epsilon \rightarrow 0} \beta^2 \|A_{\theta'} - A_\theta\|_2^{-2} = \frac{1}{4} \left\| \frac{dA_\theta}{d\theta} \right\|_2^{-2}. \tag{A14}$$

Note that in Proposition 1 is established the estimator that is unbiased at both the parameter point  $\theta$  and  $\theta + \epsilon$ ; When taking the limit  $\epsilon \rightarrow 0$ , the unbiased condition at  $\theta$  and  $\theta + \epsilon$  is equivalent to the local unbiased condition.

If we consider the inequality Eq. (A1) with the parallel amplitudes for two infinitesimally neighboring parameter points, it follows that

$$\mathcal{E}_\theta \geq \left[ 4 \lim_{\epsilon \rightarrow 0} \frac{d_B(\rho_\theta, \rho_{\theta+\epsilon})^2}{\epsilon^2} \right]^{-1} = \frac{1}{\mathcal{F}_\theta^{(3)}}. \quad (\text{A15})$$

Therefore, we obtain the QCRB without resorting to the SLD operator.

The purification method is often used for approaching the QCRB for noisy quantum-enhanced metrology; e.g., see Refs. [39–41]. Here the inequalities derived in Proposition 1 can take the range of the unbiased conditions from the local infinitesimal neighborhood to any two reference points. This may be useful in further study of noisy-enhanced metrology.

### APPENDIX B: GENERALIZATION OF THE YANG-CHIRIBELLA-HAYASHI INEQUALITY TO BIASED ESTIMATORS

We here show that the original proof [35] of the Yang-Chiribella-Hayashi inequality (4.11) can be slightly changed to include biased estimators. For simplicity, we assume that the measurement outcome  $x$  is discretely valued. The derivation can be easily generalized to the continuous valued observables.

First, it can be shown from the definitions of the variance and the mean of the estimator that

$$\begin{aligned} & \mathcal{V}_\theta[\hat{\theta}] + \mathcal{V}_{\theta+\epsilon}[\hat{\theta}] + (\mathbb{E}_{\theta+\epsilon}[\hat{\theta}] - \mathbb{E}_\theta[\hat{\theta}])^2 \\ &= \sum_x (\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}])^2 [p_\theta(x) + p_{\theta+\epsilon}(x)]. \end{aligned} \quad (\text{B1})$$

The following steps are similar to that in Ref. [35]. It follows from  $p_\theta(x) + p_{\theta+\epsilon}(x) \geq [\sqrt{p_\theta(x)} + \sqrt{p_{\theta+\epsilon}(x)}]^2/2$  that

$$\begin{aligned} & \sum_x (\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}])^2 [p_\theta(x) + p_{\theta+\epsilon}(x)] \\ & \geq \frac{1}{2} \sum_x (\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}])^2 [\sqrt{p_\theta(x)} + \sqrt{p_{\theta+\epsilon}(x)}]^2. \end{aligned} \quad (\text{B2})$$

With the Bhattacharyya coefficient  $B(p_\theta, p_{\theta+\epsilon}) := \sum_x \sqrt{p_\theta(x)p_{\theta+\epsilon}(x)}$ , we have

$$2[1 - B(p_\theta, p_{\theta+\epsilon})] = \sum_x [\sqrt{p_\theta(x)} - \sqrt{p_{\theta+\epsilon}(x)}]^2. \quad (\text{B3})$$

It then follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} & \text{r.h.s. of Eq. (B2)} \times 2[1 - B(p_\theta, p_{\theta+\epsilon})] \\ & \geq \frac{1}{2} \left\{ \sum_x (\hat{\theta} - \mathbb{E}_\theta[\hat{\theta}]) [p_\theta(x) - p_{\theta+\epsilon}(x)] \right\}^2 \\ & = \frac{1}{2} (\mathbb{E}_{\theta+\epsilon}[\hat{\theta}] - \mathbb{E}_\theta[\hat{\theta}])^2. \end{aligned} \quad (\text{B4})$$

Combining Eqs. (B1), (B2), (B4), and  $B(p_\theta, p_{\theta+\epsilon}) \geq \|\sqrt{\rho_\theta}\sqrt{\rho_{\theta+\epsilon}}\|_1$  (see Ref. [42]), it can be shown that

$$\begin{aligned} & \mathcal{V}_\theta[\hat{\theta}] + \mathcal{V}_{\theta+\epsilon}[\hat{\theta}] + (\mathbb{E}_{\theta+\epsilon}[\hat{\theta}] - \mathbb{E}_\theta[\hat{\theta}])^2 \\ & \geq \frac{(\mathbb{E}_{\theta+\epsilon}[\hat{\theta}] - \mathbb{E}_\theta[\hat{\theta}])^2}{2d_B(\rho_\theta, \rho_{\theta+\epsilon})^2}. \end{aligned} \quad (\text{B5})$$

For unbiased estimators, the above inequality becomes the Yang-Chiribella-Hayashi inequality (4.11). Taking the limit  $\epsilon \rightarrow 0$  in the above inequality, we get the biased version of  $\mathcal{F}^{(3)}$ -QCRB:

$$\mathcal{V}_\theta \geq \left( \frac{d}{d\theta} \mathbb{E}_\theta[\hat{\theta}] \right)^2 / \mathcal{F}_\theta^{(3)}. \quad (\text{B6})$$

- 
- [1] R. A. Fisher, On the mathematical foundations of theoretical statistics, *Philos. Trans. R. Soc. London A* **222**, 309 (1922).
  - [2] R. A. Fisher, Theory of statistical estimation, *Math. Proc. Cambridge Philos. Soc.* **22**, 700 (1925).
  - [3] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946).
  - [4] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, *Bull. Calcutta Math. Soc.* **37**, 235 (1945).
  - [5] G. Casella and R. L. Berger, *Statistical Inference*, 2nd ed. (Duxbury Press, Pacific Grove, CA, 2002).
  - [6] A. W. van der Vaart, *Asymptotic Statistics* (Cambridge University Press, Cambridge, 1998).
  - [7] M. Hayashi, editor, *Asymptotic Theory of quantum Statistical Inference: Selected Papers* (World Scientific, Singapore, 2005).
  - [8] C. Helstrom, Minimum mean-squared error of estimates in quantum statistics, *Phys. Lett. A* **25**, 101 (1967).
  - [9] C. Helstrom, The minimum variance of estimates in quantum signal detection, *IEEE Trans. Inf. Theory* **14**, 234 (1968).
  - [10] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
  - [11] M. G. A. Paris, Quantum estimation for quantum technology, *Int. J. Quantum Inform.* **07**, 125 (2009).
  - [12] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, Quantum Fisher information matrix and multiparameter estimation, *J. Phys. A: Math. Theor.* **53**, 023001 (2020).
  - [13] W. Heisenberg, The physical content of quantum kinematics and mechanics, *Z. Phys.* **43**, 172 (1927), English translation in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, 1984), p. 62.
  - [14] H. Yuen and M. Lax, Multiple-parameter quantum estimation and measurement of nonselfadjoint observables, *IEEE Trans. Inf. Theory* **19**, 740 (1973).
  - [15] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, 2nd ed. (Edizioni della Normale, Pisa, 2011).
  - [16] X.-M. Lu and X. Wang, Incorporating Heisenberg’s Uncertainty Principle into Quantum Multiparameter Estimation, *Phys. Rev. Lett.* **126**, 120503 (2021).

- [17] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, On quantumness in multi-parameter quantum estimation, *J. Stat. Mech.* (2019) 094010.
- [18] F. Belliardo and V. Giovannetti, Incompatibility in quantum parameter estimation, *New J. Phys.* **23**, 063055 (2021).
- [19] H. Chen, Y. Chen, and H. Yuan, Information Geometry under Hierarchical Quantum Measurement, *Phys. Rev. Lett.* **128**, 250502 (2022).
- [20] I. Kull, P. A. Guérin, and F. Verstraete, Uncertainty and trade-offs in quantum multiparameter estimation, *J. Phys. A: Math. Theor.* **53**, 244001 (2020).
- [21] J. S. Sidhu, Y. Ouyang, E. T. Campbell, and P. Kok, Tight Bounds on the Simultaneous Estimation of Incompatible Parameters, *Phys. Rev. X* **11**, 011028 (2021).
- [22] J. Suzuki, Information geometrical characterization of quantum statistical models in quantum estimation theory, *Entropy* **21**, 703 (2019).
- [23] D. Šafránek, Discontinuities of the quantum Fisher information and the Bures metric, *Phys. Rev. A* **95**, 052320 (2017).
- [24] L. Seveso, F. Albarelli, M. G. Genoni, and M. G. A. Paris, On the discontinuity of the quantum Fisher information for quantum statistical models with parameter dependent rank, *J. Phys. A: Math. Theor.* **53**, 02LT01 (2020).
- [25] S. Zhou and L. Jiang, An exact correspondence between the quantum Fisher information and the Bures metric, [arXiv:1910.08473](https://arxiv.org/abs/1910.08473) (2019).
- [26] A. T. Reza khani, M. Hassani, and S. Alipour, Continuity of the quantum Fisher information, *Phys. Rev. A* **100**, 032317 (2019).
- [27] A. Z. Goldberg, J. L. Romero, Á. S. Sanz, and L. L. Sánchez-Soto, Taming Singularities of the Quantum Fisher Information, *Int. J. Quantum Inf.* **19**, 08 (2021).
- [28] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [29] A. Uhlmann, The “transition probability” in the state space of a  $*$ -algebra, *Rep. Math. Phys.* **9**, 273 (1976).
- [30] A. Uhlmann, Transition probability (fidelity) and its relatives, *Found. Phys.* **41**, 288 (2011).
- [31] R. Jozsa, Fidelity for mixed quantum states, *J. Mod. Opt.* **41**, 2315 (1994).
- [32] D. Bures, An extension of Kakutani’s theorem on infinite product measures to the tensor product of semifinite  $W^*$ -algebras, *Trans. Amer. Math. Soc.* **135**, 199 (1969).
- [33] A. Uhlmann, A gauge field governing parallel transport along mixed states, *Lett. Math. Phys.* **21**, 229 (1991).
- [34] C. Helstrom and R. Kennedy, Noncommuting observables in quantum detection and estimation theory, *IEEE Trans. Inf. Theory* **20**, 16 (1974).
- [35] Y. Yang, G. Chiribella, and M. Hayashi, Attaining the ultimate precision limit in quantum state estimation, *Commun. Math. Phys.* **368**, 223 (2019).
- [36] L. Wasserman, *All of Statistics: A Concise Course in Statistical Inference* (Springer, New York, 2010).
- [37] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2006).
- [38] A. Uhlmann and B. Crell, Geometry of state spaces, in *Entanglement and Decoherence: Foundations and Modern Trends*, edited by A. Buchleitner, C. Viviescas, and M. Tiersch (Springer, Berlin, 2009), pp. 1–60.
- [39] A. Fujiwara and H. Imai, A fibre bundle over manifolds of quantum channels and its application to quantum statistics, *J. Phys. A: Math. Theor.* **41**, 255304 (2008).
- [40] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, *Nat. Phys.* **7**, 406 (2011).
- [41] H. Yuan and C. H. F. Fung, Quantum metrology matrix, *Phys. Rev. A* **96**, 012310 (2017).
- [42] C. Fuchs and J. van de Graaf, Cryptographic distinguishability measures for quantum-mechanical states, *IEEE Trans. Inf. Theory* **45**, 1216 (1999).