Spatial search via an interpolated memoryless walk

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The defining feature of memoryless quantum walks is that they operate on the vertex space of a graph and therefore can be used to produce search algorithms with minimal memory. We present a memoryless walk that can find a unique marked vertex on a two-dimensional lattice. Our walk is based on the construction proposed by Falk, which tessellates the lattice with squares of size 2×2. Our walk uses minimal memory, $O(\sqrt{N \log N})$ applications of the walk operator, and outputs the marked vertex with vanishing error probability. To accomplish this, we apply a self-loop to the marked vertex—a technique we adapt from interpolated walks. We prove that with our explicit choice of self-loop weight, this forces the action of the walk asymptotically into a single rotational space. We characterize this space and as a result show that our memoryless walk produces the marked vertex with a success probability asymptotically approaching one.

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I. INTRODUCTION

Search problems are one of the foundational applications of quantum algorithms and are one of the situations in which quantum algorithms are proven to provide a speedup over their classical counterparts. For example, Grover's search algorithm [1] can find a single element in an N-element database with $\Theta(\sqrt{N})$ quantum queries, while any classical algorithm requires $\Omega(N)$ queries for the same task.

In spatial search, the goal is to find a "marked" element on a graph, with the restriction that in a single time step, amplitude can only be moved between adjacent vertices on the graph. Such restrictions can emerge from some underlying physical structure [2] or from the computational cost of moving from one vertex to another [3]. Quantum spatial search was first considered by Benioff [4], who showed that direct application of Grover's search on the two-dimensional lattice does not yield a speedup over classical algorithms. A near-quadratic speedup was then discovered using a divide-and-conquer quantum algorithm [2] and quantum walks [5–7].

Quantum walks are the quantum counterparts of classical random walks. Memoryless, or coinless, walks are a less commonly studied type of quantum walk but have several advantages over other models. As the name suggests, they operate directly on the vertex space of the graph. This differentiates them from coined or Szegedy quantum walks, which use extra registers to encode the previous location of the walker. Memoryless walks can therefore be used to produce search algorithms with optimal memory requirements. Due to the limited size of near-term quantum computers, this is one of the main motivations for their study.

Memoryless walks have a simple structure, alternating two or more noncommuting reflections that can be derived from tessellations of the underlying graph. This construction results in a discrete-time quantum walk with minimum memory. A proposal for implementing memoryless walks using superconducting microwave resonators is given by Ref. [8], and an implementation of memoryless walks on IBM quantum computers is in Ref. [9].

Staggered walks are a class of memoryless walks and were defined by Portugal *et al.* in Ref. [10]. Staggered walks are based on graph tessellations into cliques, which enforces the spatial search constraint and gives a method for constructing memoryless walks on general graphs. Reference [10] shows that any quantum walk on a graph G in the standard Szegedy model [11] can be converted to a staggered walk on the underlying line graph of G. The conversion preserves the asymptotic cost, the success probability, and the space requirement. Staggered walks have also been used to derive relationships among memoryless, coined, Szegedy, and continuous quantum walks [12–17].

A memoryless walk on the line was given by Patel *et al.* in Ref. [18], who give and analyze a walk with Hamiltonians, and they note that their walk operator resembles the staggered fermion formalism. In Ref. [19], Patel *et al.* present numerical simulations on the extension of this walk to the N-vertex grid. Their results show that it finds a unique marked vertex in $\Theta(\sqrt{N \log N})$ applications of the walk operator with success probability $\Theta(\frac{1}{\log N})$, and that by using an ancilla qubit, the success probability can be improved to $\Theta(1)$.

Falk [20] gives a memoryless walk on the two-dimensional lattice, constructing a discrete walk operator by reflecting about two alternating tessellations of the lattice. The walk by Falk was analyzed by Ambainis *et al.* [21], who proved that it finds a unique marked vertex on the lattice using $\Theta(\sqrt{N \log N})$ applications of the walk operator with success probability $\Theta(\frac{1}{\log N})$. Portugal and Fernandes [22] give a memoryless walk with Hamiltonians that also finds a unique marked vertex on the two-dimensional lattice with $\Theta(\sqrt{N \log N})$ applications of the walk operator and success probability $\Theta(\frac{1}{\log N})$.

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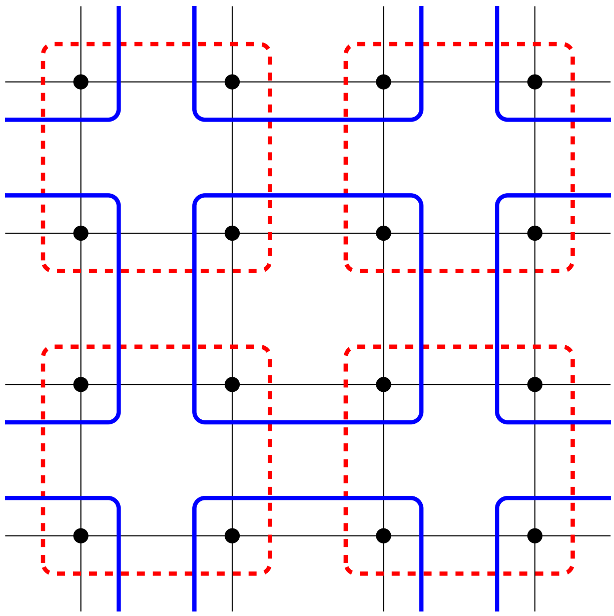

FIG. 1. Staggered tessellations of the two-dimensional lattice into squares of size 2×2. The dashed red lines represent the states $|a_{ij}\rangle$ and solid blue lines represent the states $|b_{ij}\rangle$, as defined in (1) and (2). The construction based on alternating reflections about this tessellation structure was originally proposed in Ref. [20]

The success probability of Refs. [18,20,22] is subconstant. The success probability can be improved to $\Theta(1)$ by applying amplitude amplification [23], but that would increase the number of steps by a factor of $\Theta(\sqrt{\log N})$ and introduce additional operators in an implementation. Two alternative methods [21,22] for increasing the success probability are the postprocessing local neighborhood search in Ref. [24] and Tulsi's proposal of adding an ancilla qubit to the lattice [19,25].

In this paper, we give a memoryless walk that finds the marked vertex in $\Theta(\sqrt{N \log N})$ steps with vanishing error probability. Our walk uses minimal memory and preserves the simple structure given by alternating tessellations of the lattice. To do this, we augment the basic memoryless construction by showing how the interpolated walks of Krovi *et al.* [26] can be adapted to the vertex space of a graph. We define our memoryless walk using the tessellations depicted in Fig. 1, which divides the lattice into squares of size 2×2. This is the same tessellation structure analysed in Ref. [21], who proved that using Falk's construction, the maximum success probability of the corresponding memoryless walk scales with $\Theta(\frac{1}{\log N})$. By using a self-loop to force the action of the walk into a single two-dimensional subspace, we modify their walk so that the maximum success probability asymptotically approaches one.

Classically, the interpolated version of a random walk is constructed by adding weighted self-loop edges to marked vertices. Applying Szegedy's isometry [11] to interpolated walks produces quantum walks that can find a unique marked vertex on any graph [26]. This approach has recently been used to solve the spatial search problem on any graph with a quadratic speedup over classical random walks, even for the case of multiple marked vertices. Solutions of this form have been obtained for both the discrete [27,28] and continuous [29] models using quantum walks that operate on the edge space of a graph. Quantum interpolated walks can be simulated by controlled quantum walks [30], preserving the asymptotic cost, the success probability, and the space requirement.

To extend the approach of Ref. [26] to the memoryless setting, we introduce a state corresponding to a self-loop on the marked vertex. Rather than reflecting about the marked vertex, we reflect about an interpolation between the self-loop state and the marked vertex. The result is a memoryless walk parametrized by the weight of the self-loop, s. We prove in our analysis that with our explicit choice of weight, this forces the evolution of the initial state into a single rotational subspace of our walk operator. As a result, our walk achieves a success probability $1 - O(\frac{1}{\log N})$ with $\Theta(\sqrt{N \log N})$ steps and minimal memory.

Most of our analysis considers the eigenvector of an operator with the smallest positive eigenphase. We use the term "slowest eigenvector" to refer to this eigenvector and "slowest rotational subspace" to refer to the subspace spanned by this eigenvector and its conjugate.

To prove our main result, we present a set of techniques for analyzing memoryless walks. We analyze an operator W, whose spectra are completely known, composed with a two-dimensional rotation F. As part of our proof, we determine the asymptotic behavior of both the smallest positive eigenphase of WF and its associated eigenvector. The result is a precise asymptotic description of the slowest rotational subspace of our walk operator. The techniques we use to obtain this description are general enough that they could be applied in other contexts as well.

In this paper, we consider the case where a single vertex on the lattice is marked. A natural question is to consider whether our walk can be extended to handle multiple marked vertices. We address this question in Sec. VII, where we show that there are certain configurations of marked vertices for which the straightforward extension of our walk cannot be used to find. We also propose an idea for avoiding these configurations with constant probability, which could be useful in designing memory-optimal quantum walks for the multiple marked case, both on the lattice and on different types of graphs.

We begin by defining our walk in Sec. II. An overview of the paper layout and proof structure is given in Sec. II A.

II. WALK CONSTRUCTION AND MAIN RESULT

We consider the task of finding a unique marked vertex on a two-dimensional lattice with n_r rows and n_c columns, where both n_r and n_c are even. The lattice boundaries are those of a torus, so there are edges between vertices $(i, n_c - 1)$ and $(i, 0)$ for $0 \leqslant i < n_r$, and between vertices $(n_r - 1, j)$ and $(0, j)$ for $0 \leqslant j < n_c$. The total number of vertices is given by $N = n_r \times n_c$. After Lemma 7, we restrict to the case where $n_r = n_c = \sqrt{N}$.

We construct a memoryless quantum walk with a self-loop. Our walk operates on a space of dimension $N + 1$, which is optimal for this task. The vertex at position (i, j) is represented by the quantum state $|i, j\rangle$. We introduce a state

$|\circlearrowleft\rangle$ corresponding to a self-loop on the marked vertex. Our approach and terminology are inspired by the interpolated walks of [26], which add self-loop edges to marked vertices in Szegedy-style walks.

Our walk applies two alternating reflections about the faces of the graph. Here, we follow the construction used in Ref. [21]. For $0 \leqslant i < \frac{n_r}{2}$, $0 \leqslant j < \frac{n_c}{2}$, define

$$|a_{ij}\rangle = \frac{1}{2}\sum_{i',j'=0}^{1} |2i+i', 2j+j'\rangle, \tag{1}$$

$$|b_{ij}\rangle = \frac{1}{2}\sum_{i',j'=0}^{1} |2i+1+i', 2j+1+j'\rangle. \tag{2}$$

The sets $\{|a_{ij}\rangle\}_{i,j}$ and $\{|b_{ij}\rangle\}_{i,j}$ each specify a partition of the lattice into 2×2 squares, positioned at even and odd indices, respectively. These sets form the tessellations depicted in Fig. 1.

Define projections onto the even and odd partitions as

$$\Pi_{\mathsf{e}} = \sum_{i=0}^{\frac{n_r}{2}-1}\sum_{j=0}^{\frac{n_c}{2}-1} |a_{ij}\rangle\langle a_{ij}|, \qquad \Pi_{\mathsf{o}} = \sum_{i=0}^{\frac{n_r}{2}-1}\sum_{j=0}^{\frac{n_c}{2}-1} |b_{ij}\rangle\langle b_{ij}|,$$

and let

$$\mathsf{A} = 2\Pi_{\mathsf{e}} + 2|\circlearrowleft\rangle\langle\circlearrowleft| - \mathsf{I},$$
$$\mathsf{B} = 2\Pi_{\mathsf{o}} + 2|\circlearrowleft\rangle\langle\circlearrowleft| - \mathsf{I}.$$

In Ref. [21], these reflections are alternated with a reflection about the marked state. This is the standard way of adding finding behavior to memoryless walks, where an operator composed of two or more noncommuting reflections is alternated with a reflection of the marked vertices. In our walk, we replace the reflection of the marked vertex with a reflection of an interpolated state with parameter $0 \leqslant s \leqslant 1$. Letting $|g\rangle$ denote the marked vertex, we define the interpolated state to be $|\tilde{g}\rangle = \sqrt{s}|g\rangle + \sqrt{1-s}|\circlearrowleft\rangle$. Our input-dependent reflection is then

$$\tilde{\mathsf{G}} = \mathsf{I} - 2|\tilde{g}\rangle\langle\tilde{g}|. \tag{3}$$

By selecting an appropriate value for s, we are able to force the action of the walk asymptotically into a single two-dimensional subspace. We set $s = 1 - \frac{1}{N+1}$, which is close to the value $s = 1 - \frac{1}{N}$ used by Ref. [26] for interpolated walks. This choice gives a particularly nice form for the initial state of our walk, as shown in (22).

Given a fixed value for s, we define a single step of our walk to be the operator

$$\mathsf{U} = \mathsf{B}\tilde{\mathsf{G}}\mathsf{A}\tilde{\mathsf{G}}. \tag{4}$$

We apply the walk to the initial state $|\pi\rangle$, which is defined by $\langle i, j|\pi\rangle = \frac{1}{\sqrt{N}}$ for all vertices (i, j), and with $\langle\circlearrowleft|\pi\rangle = 0$. This allows us to present our main result.

Theorem 1 (Main result). Fix $s = 1 - \frac{1}{N+1}$ and suppose $n_r = n_c$. Then there exists a constant $c > 0$ such that after $c\sqrt{N}\log N$ applications of U to $|\pi\rangle$, measuring the state will produce $|\circlearrowleft\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{\log N})$.

We remark that given the state $|\circlearrowleft\rangle$, one can obtain the marked state $|g\rangle$ through amplitude amplification [23]. This can be done by alternating the reflection $\tilde{\mathsf{G}}$ with a reflection about either $|g\rangle$ or $|\circlearrowleft\rangle$. After $\lfloor\frac{\pi}{2}[\arcsin(\frac{1}{\sqrt{N+1}})]^{-1}\rfloor \in \Theta(\sqrt{N})$ steps of amplification, measuring the resulting state will produce $|g\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{N})$. Note that both the error probability and the query complexity for obtaining $|g\rangle$ from $|\circlearrowleft\rangle$ are dominated by the cost of finding the state $|\circlearrowleft\rangle$ as in Theorem 1.

A. Proof strategy

Our proof of Theorem 1 is based on the analysis of an intermediate walk operator, which we define below.

To simplify the analysis, we also introduce a change of basis. This will allow us to compute necessary properties of the walk spectra, as the eigenvectors of BA factor into product states under this basis change. Let CZ be an operator acting on the pair of least significant bits in a tensor product space, with the action defined by $|i, j\rangle \mapsto -|i, j\rangle$ if both i and j are even, and being the identity otherwise. Let CZ act trivially on $|\circlearrowleft\rangle$. For any operator X, let $\mathsf{X}_z = \mathsf{CZ}\circ\mathsf{X}\circ\mathsf{CZ}$, and let $|\pi_z\rangle = \mathsf{CZ}|\pi\rangle$.

We define

$$\mathsf{W} = (\mathsf{BA})_z, \tag{5}$$

$$\mathsf{F} = (\mathsf{A}\tilde{\mathsf{G}}\mathsf{A}\tilde{\mathsf{G}})_z, \tag{6}$$

so that

$$\mathsf{U} = (\mathsf{BA})(\mathsf{A}\tilde{\mathsf{G}}\mathsf{A}\tilde{\mathsf{G}}) = (\mathsf{WF})_z. \tag{7}$$

This is analogous to the decomposition of U used by [21] in their analysis. Note that W is input independent, derived only from the graph tessellations, and we therefore have a complete description of the operator. Meanwhile, F depends on both the self-loop weight s and the marked vertex, so adjusting the self-loop weight allows us to adjust its spectra. We show in Sec. III that F is a two-dimensional rotation, and therefore can be decomposed as the product of two reflections, which we write as $\mathsf{F} = \mathsf{F}_1\mathsf{F}_2$.

Our intermediate walk consists of W composed with only F_1. This choice of intermediate walk has the advantage that it consists of a real operator whose spectra are completely known, composed with a one-dimensional reflection. This allows us to apply existing results about operators of this type, including the eigenvector analysis of Ref. [3] and the flip-flop theorem from Ref. [30]. An overview of these results is given in Appendix A. Applying these results allows us to asymptotically determine the eigenphases and eigenvectors of WF_1.

Our proof is based on a tight characterization of the slowest rotational subspace of the intermediate walk WF_1. This characterization is developed in Sec. V, where we prove asymptotic properties of the rotational angle and the spanning vectors. To our knowledge, this is the first case of the flip-flop theorem being used to derive properties of a subspace in this way. We show that with our choice of self-loop, the action of WF_1 on $|\pi_z\rangle$ can be reduced asymptotically to a Grover-like rotation in the slowest two-dimensional rotational subspace of

WF_1. This key property is what allows our main algorithm to achieve a success probability asymptotically close to 1.

To prove our main result, we relate the slowest rotational subspaces of WF_1 and WF in Sec. VI. We show that composing WF_1 with the reflection F_2 does not significantly alter the slowest rotational subspace, and therefore that WF has the same asymptotic behavior as WF_1 when applied to $|\pi_z\rangle$. The proof of Theorem 1 follows from the basis-change relationship between U and WF given in (7).

With our approach, we are able to derive precise statements about the behavior of an operator composed with a two-dimensional rotation. This addresses a more general challenge in analyzing quantum algorithms and may have applications outside of memoryless walks.

III. DECOMPOSITION OF F

In this section, we derive the exact form of the rotation F. We show that it can be decomposed into two one-dimensional reflections, F_1 and F_2, which we compose sequentially with W in Secs. V and VI.

Without loss of generality, assume $|g\rangle = |0,0\rangle$ is the marked vertex. Consider the three-dimensional subspace spanned by $|g\rangle$, $|\circlearrowleft\rangle$, and $|a_{00}\rangle$, the even-indexed square containing $|0,0\rangle$. The operator F only acts nontrivially in this subspace, which is spanned by $|\circlearrowleft\rangle$ and the two orthonormal states

$$|+\rangle = \frac{1}{\sqrt{3}}(|g\rangle + |a_{00}\rangle), \tag{8}$$

$$|-\rangle = (|g\rangle - |a_{00}\rangle). \tag{9}$$

Let $0 \leqslant \eta \leqslant \frac{\pi}{3}$ be such that

$$\sin^2(\eta) = \tfrac{3}{4}s. \tag{10}$$

Set $\lambda = e^{i4\eta}$. Then $\mathsf{F} = \mathsf{I} + (\lambda - 1)|f_+\rangle\langle f_+| + (\lambda^{-1} - 1)|f_-\rangle\langle f_-|$, where the two nontrivial eigenvectors are

$$|f_+\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle - i\frac{1}{\sqrt{4-3s}}(\sqrt{s}|-\rangle - 2\sqrt{1-s}|\circlearrowleft\rangle)\right],$$

$$|f_-\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + i\frac{1}{\sqrt{4-3s}}(\sqrt{s}|-\rangle - 2\sqrt{1-s}|\circlearrowleft\rangle)\right].$$

Proof. Observe that F is a real-valued operator that only acts nontrivially on the span of $|+\rangle$, $|-\rangle$, and $|\circlearrowleft\rangle$. Therefore, any complex eigenvalues of F must come in conjugate pairs, and F can have at most three nontrivial eigenvectors. Using the observation that $(\mathsf{A\tilde{G}})_z$ is real valued, any (-1) eigenspace of F would necessarily have even dimension. Thus, F must have either two or zero nontrivial eigenvectors.

Define $|f_+{}^{un}\rangle = \sqrt{2(4-3s)}|f_+\rangle$ and compute

$$\begin{aligned}(\mathsf{A\tilde{G}})_z|f_+{}^{un}\rangle &= \left[-(2-3s)\frac{\sqrt{4-3s}}{2} - \imath\sqrt{3s}(4-3s)\right]|+\rangle \\ &\quad + \left[-\frac{\sqrt{4-3s}}{2}\sqrt{3s} + \imath\sqrt{s}(2-3s)\right]|-\rangle \\ &\quad + \left[\sqrt{3s(1-s)(4-3s)} - \imath\sqrt{1-s}(2-3s)\right]|\circlearrowleft\rangle \\ &= -\frac{1}{2}((2-3s) + \imath\sqrt{3s}\sqrt{4-3s})|f_+{}^{un}\rangle \\ &= -e^{\imath 2\eta}|f_+{}^{un}\rangle.\end{aligned}$$

This shows that $|f_+\rangle$ is an eigenvector of $\mathsf{F} = (\mathsf{A\tilde{G}})_z(\mathsf{A\tilde{G}})_z$ with eigenvalue $(-e^{\imath 2\eta})^2 = \lambda$. It follows that the entrywise conjugate of $|f_+\rangle$, given by $|f_-\rangle$, must be an eigenvector of F with eigenvalue λ^{-1}. ■

Lemma 2. Lemma III shows that F is a rotation by 4η of a single two-dimensional space, spanned by $|f_+\rangle$ and $|f_-\rangle$. Therefore, F can be decomposed as the product of two one-dimensional reflections. We choose these reflections as follows.

Fact 3. Define

$$|f_1\rangle = \frac{1}{\sqrt{4-3s}}(\sqrt{s}|-\rangle - 2\sqrt{1-s}|\circlearrowleft\rangle), \tag{11}$$

$$|f_2\rangle = \sin(2\eta)|+\rangle + \cos(2\eta)|f_1\rangle, \tag{12}$$

and let

$$\mathsf{F}_1 = \mathsf{I} - 2|f_1\rangle\langle f_1|,$$

$$\mathsf{F}_2 = \mathsf{I} - 2|f_2\rangle\langle f_2|.$$

Then $\mathsf{F} = \mathsf{F}_1\mathsf{F}_2$.

IV. STRUCTURE OF W

We give exact formulas for the eigenvectors and eigenphases of W, as well as a decomposition of the $(N+1)$-dimensional domain into subspaces that are invariant under W. These properties are required for the detailed analysis of WF_1 and WF in later sections.

A. Spectra of W

Recall that $|\circlearrowleft\rangle$ is trivially a $(+1)$ eigenvector of W. The remaining N eigenvectors of W can be indexed by k and l as follows.

For $0 \leqslant k < n_r/2$, let $\tilde{k} = \frac{2\pi k}{n_r}$, and for $0 \leqslant l < n_c/2$, let $\tilde{l} = \frac{2\pi l}{n_c}$. Let $\epsilon_k = \operatorname{sign}(\cos\tilde{k})$ and $\epsilon_l = \operatorname{sign}(\cos\tilde{l})$. Define the sign of zero to be $+1$. This case occurs when n_r or n_c is divisible by 4, since $\cos(\tilde{k}) = 0$ when $k = n_r/4$ and $\cos(\tilde{l}) = 0$ when $l = n_c/4$.

Define

$$p_{kl} = \sqrt{1 - \cos^2\tilde{k}\cos^2\tilde{l}}, \tag{13}$$

$$\theta_{kl} = \epsilon_k\epsilon_l \operatorname{acos}\big(1 - 2p_{kl}^2\big), \tag{14}$$

and

$$r_{kl}^{\pm} = \sqrt{2\left(1 \pm \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right)} = \sqrt{1 + \frac{\sin\tilde{l}}{p_{kl}}} \pm \epsilon_l\sqrt{1 - \frac{\sin\tilde{l}}{p_{kl}}},$$

$$c_{kl}^{\pm} = \sqrt{2\left(1 \pm \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right)} = \sqrt{1 + \frac{\sin\tilde{k}}{p_{kl}}} \pm \epsilon_k\sqrt{1 - \frac{\sin\tilde{k}}{p_{kl}}}.$$

If $k = l = 0$, then $p_{00} = 0$ and the division by zero is ill defined. In this case, we define

$$r_{00}^+ = r_{00}^- = c_{00}^+ = c_{00}^- = \sqrt{2}.$$

For each $0 \leqslant k < n_r/2$ and $0 \leqslant l < n_c/2$, there is an eigenvector $|w_{kl}\rangle$ of W with eigenvalue $e^{\imath\theta_{kl}}$. This $|w_{kl}\rangle$ is the product state

$$|w_{kl}\rangle = |u_{kl}\rangle \otimes |v_{kl}\rangle, \tag{15}$$

where the factors are given by the normalized states

$$|u_{kl}\rangle = \sqrt{2}\big|\phi_r^k\big\rangle \circ (|1_{n_r/2}\rangle \otimes |r_{kl}\rangle),$$
$$|v_{kl}\rangle = \sqrt{2}\big|\phi_c^l\big\rangle \circ (|1_{n_c/2}\rangle \otimes |c_{kl}\rangle),$$

$$|r_{kl}\rangle = \frac{1}{2}\begin{bmatrix} r_{kl}^- \\ r_{kl}^+ \end{bmatrix},$$
$$|c_{kl}\rangle = \frac{1}{2}\begin{bmatrix} c_{kl}^- \\ c_{kl}^+ \end{bmatrix}.$$

Here, $\circ$ denotes entry-wise multiplication and $|1_n\rangle$ is the all-ones vector of dimension n. The Fourier states are $|\phi_r^k\rangle = \frac{1}{\sqrt{n_r}}\sum_{i=0}^{n_r-1}\omega_{n_r}^{ik}|i\rangle$ and $|\phi_c^k\rangle = \frac{1}{\sqrt{n_c}}\sum_{i=0}^{n_c-1}\omega_{n_c}^{ik}|i\rangle$, where $\omega_n = e^{2\pi\imath/n}$ denotes the nth root of unity.

Let

$$\big|r_{kl}^1\big\rangle = \mathsf{XZ}|r_{kl}\rangle,$$
$$\big|c_{kl}^1\big\rangle = \mathsf{XZ}|c_{kl}\rangle.$$

Then replacing $|r_{kl}\rangle$ and $|c_{kl}\rangle$ with $|r_{kl}^1\rangle$ and $|c_{kl}^1\rangle$ in the above construction yields an eigenvector with eigenphase $-\theta_{kl}$, and replacing either one of the two yields an eigenvector with eigenvalue 1. Let $|u_{kl}^1\rangle$ and $|v_{kl}^1\rangle$ be defined with $|r_{kl}^1\rangle$ and $|c_{kl}^1\rangle$, respectively. We denote $|u_{kl}^0\rangle = |u_{kl}\rangle$, $|v_{kl}^0\rangle = |v_{kl}\rangle$. Let $|w_{kl}^B\rangle$ be defined accordingly for $B \in \{00, 01, 10, 11\}$.

By this definition, the N eigenvectors $\{|w_{kl}^B\rangle\}$ of W constitute an orthonormal basis for the lattice, where the eigenvector $|w_{kl}^B\rangle$ has eigenphase $\theta_{kl}, 0, 0, -\theta_{kl}$ for $B = 00, 01, 10, 11$.

Fact 4. Both $|\pi\rangle$ and $|\pi_z\rangle$ are $(+1)$ eigenvectors of W.

Proof. Based on (14), we note that $\theta_{00} = 0$. The fact follows from the observation that

$$|\pi\rangle = \big|w_{00}^{00}\big\rangle,$$
$$|\pi_z\rangle = \tfrac{1}{2}\big(\big|w_{00}^{00}\big\rangle + \big|w_{00}^{01}\big\rangle + \big|w_{00}^{10}\big\rangle - \big|w_{00}^{11}\big\rangle\big).$$

■

B. Invariant subspaces

We partition the domain of W into subspaces W_{kl}. The number of subspaces depends on the parity of $\frac{n_c}{2}$. If $\frac{n_c}{2}$ is odd, there are $\frac{(n_r+2)(n_c-2)}{8} + 1$ invariant subspaces, and if $\frac{n_c}{2}$ is even, there are $\frac{(n_r+2)n_c}{8} + \lfloor\frac{n_r}{4}\rfloor - 1$ invariant subspaces. Each W_{kl} is spanned by a set of eigenvectors of W with eigenphase θ_{kl}, and is therefore invariant under the action of W. The projection onto subspace W_{kl} is denoted Π_{kl}.

First, observe that $\theta_{00} = 0$, so the eigenvectors $|w_{00}^{00}\rangle$ and $|w_{00}^{11}\rangle$ are both $(+1)$ eigenvectors. The $(+1)$ eigenspace of W therefore has dimension $\frac{N}{2} + 2 + 1$, where the last dimension comes from the self-loop state. We denote the $(+1)$ eigenspace as W_{00} and its associated projection as Π_{00}.

For any $0 \leqslant k < n_r/2$ and $0 \leqslant l < n_c/2$, define

$$k' = \begin{cases} \frac{n_r}{2} - k & \text{if } 0 < k < \frac{n_r}{2} \\ 0 & \text{if } k = 0 \end{cases}$$

and

$$l' = \begin{cases} \frac{n_c}{2} - l & \text{if } 0 < l < \frac{n_c}{2} \\ 0 & \text{if } l = 0. \end{cases}$$

Note that $k = k'$ exactly when $k = 0$ or $k = \frac{n_r}{4}$, and similarly for $l = l'$.

For $0 < k < \frac{n_r}{2}$, $k \neq \frac{n_r}{4}$ and $0 < l < \frac{n_c}{4}$, define

$$\mathsf{W}_{kl} = \operatorname{span}\big\{\big|w_{kl}^{00}\big\rangle, \big|w_{k'l'}^{00}\big\rangle, \big|w_{k'l}^{11}\big\rangle, \big|w_{kl'}^{11}\big\rangle\big\}.$$

Each of these subspaces has an associated eigenphase $\theta_{kl} \neq 0, \pi$. When n_r or n_c is a multiple of 4, W also has a (-1) eigenspace. In this case, there are subspaces with eigenphase π given by

$$\mathsf{W}_{kl} = \operatorname{span}\big\{\big|w_{kl}^{00}\big\rangle, \big|w_{k'l'}^{00}\big\rangle, \big|w_{k'l}^{11}\big\rangle, \big|w_{kl'}^{11}\big\rangle\big\}$$

for $k = \frac{n_r}{4}$, $0 < l < \frac{n_c}{4}$ and $l = \frac{n_c}{4}$, $0 < k \leqslant \frac{n_r}{4}$. These four vectors will be distinct unless both $k = \frac{n_r}{4}$ and $l = \frac{n_c}{4}$, in which case $\dim(\mathsf{W}_{kl}) = 2$.

For $k = 0$ and $0 < l < \frac{n_c}{2}$, the corresponding invariant subspace is

$$\mathsf{W}_{kl} = \operatorname{span}\big\{\big|w_{kl}^{00}\big\rangle, \big|w_{kl'}^{11}\big\rangle\big\},$$

and similarly for $l = 0$ and $0 < k < \frac{n_r}{2}$,

$$\mathsf{W}_{kl} = \operatorname{span}\big\{\big|w_{kl}^{00}\big\rangle, \big|w_{k'l}^{11}\big\rangle\big\}.$$

Partitioning the domain in this way allows us to closely analyze the behavior of $|+\rangle$ and $|-\rangle$ under the action of W.

Lemma 5. The following statements hold:

$$\Pi_{kl}|+\rangle \perp \Pi_{kl}|-\rangle \qquad \text{for all subspaces } \mathsf{W}_{kl}, \tag{16}$$

$$\|\Pi_{kl}|+\rangle\|^2 = \frac{2}{3}\frac{\dim(\mathsf{W}_{kl})}{N} \qquad \text{for all } k, l \text{ not both } 0, \tag{17}$$

$$\|\Pi_{kl}|-\rangle\|^2 = 2\frac{\dim(\mathsf{W}_{kl})}{N} \qquad \text{for all } k, l \text{ not both } 0, \tag{18}$$

$$\|\Pi_{00}|+\rangle\|^2 = \frac{2}{3}\frac{N+2}{N}, \tag{19}$$

$$\|\Pi_{00}|-\rangle\|^2 = \frac{4}{N}. \tag{20}$$

■

Proof. See Appendix B.

V. REDUCTION TO THE SLOWEST SUBSPACE

Our memoryless walk, given in Theorem 1, achieves a success probability asymptotically close to 1. This is possible because our choice of s reduces the walk asymptotically to a rotation in a single two-dimensional subspace. As we show, this subspace is exactly the slowest rotational subspace of the applied walk operator. We prove this by giving a tight description of the smallest positive eigenvalue of the walk operator and its associated eigenvector. We also show that the two-dimensional rotation induced by the walk maps the initial state to the desired state, $|\circlearrowleft\rangle$.

We consider two walk operators. The first consists of the real operator W composed with the one-dimensional reflection F_1. The second consists of W composed with the two-dimensional rotation F. We discuss the first operator in this section, and then use the results to derive properties of the second operator in Sec. VI.

First, we prove three key lemmas about the slowest rotational subspace of WF_1. Lemma 8 gives a tight bound on the smallest positive eigenphase of WF_1, and Lemmas 10 and 11 characterize the asymptotic behavior of its associated eigenvector. These three lemmas show that the action of WF_1 on $|\pi_z\rangle$ can be reduced to a rotation in the slowest rotational subspace. These results are the basis of the methods used in Sec. VI to prove our main result.

The intermediate operator WF_1 is fundamentally a tool for analysis. However, it is interesting to note that WF_1 can be used directly to find the marked state. The proof of the corollary below follows from a similar argument to our proof of the main theorem.

Corollary 6. Fix $s = 1 - \frac{1}{N+1}$ and suppose $n_r = n_c$. Then there exists a constant $c > 0$ such that after $c\sqrt{N}\log N$ applications of WF_1 to $|\pi_z\rangle$, measuring the state will produce $|\circlearrowleft\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{\log N})$.

Comparing Corollary 6 with Theorem 1 shows that most of the finding behavior of the memoryless walk comes from the first reflection F_1. In Sec. VI, we show that composing WF_1 with the second reflection F_2 only changes the behavior of the walk slightly, leading to our proof of Theorem 1.

Our proofs of both Corollary 6 and our main theorem rely on the following observation.

Lemma 7. The (unnormalized) vector

$$|\mathsf{U}_0\rangle = |\pi_z\rangle - \sqrt{\frac{s}{(1-s)N}}|\circlearrowleft\rangle \tag{21}$$

is a $(+1)$ eigenvector for each of W, F_1, and F_2.

Proof. Both $|\pi_z\rangle$ and $|\circlearrowleft\rangle$ are $(+1)$ eigenvectors of W, so $|\mathsf{U}_0\rangle$ is a $(+1)$ eigenvector of W. To show $|\mathsf{U}_0\rangle$ is a $(+1)$ eigenvector of F_1, we compute

$$\langle \mathsf{U}_0|f_1\rangle = \frac{2\sqrt{s}}{\sqrt{N(4-3s)}} + \frac{\sqrt{s}}{\sqrt{4-3s}}\langle\pi_z|-\rangle = 0.$$

Finally, $|+\rangle$ is orthogonal to both $|\pi_z\rangle$ and $|\circlearrowleft\rangle$, so $|\mathsf{U}_0\rangle$ is orthogonal to $|f_2\rangle$. Therefore, $|\mathsf{U}_0\rangle$ is also a $(+1)$ eigenvector of F_2. ∎

For the rest of the paper, we assume a square lattice, so $n_r = n_c = \sqrt{N}$. We also fix $s = 1 - \frac{1}{N+1}$. This choice of self-loop weight means $|\mathsf{U}_0\rangle = |\pi_z\rangle - |\circlearrowleft\rangle$, so we can decompose the initial state $|\pi_z\rangle$ as

$$|\pi_z\rangle = \tfrac{1}{2}|\mathsf{U}_0\rangle + \tfrac{1}{2}(|\pi_z\rangle + |\circlearrowleft\rangle). \tag{22}$$

We show in Sec. V B that for our chosen s, $|\pi_z\rangle + |\circlearrowleft\rangle$ lies asymptotically in the slowest rotational subspace of WF_1. Therefore, WF_1 can be used to apply a negative phase to this portion of the initial state. This rotates the state $|\pi_z\rangle$ to the state $\frac{1}{2}|\mathsf{U}_0\rangle - \frac{1}{2}(|\pi_z\rangle + |\circlearrowleft\rangle) = -|\circlearrowleft\rangle$, as we make precise in our proof of Corollary 6.

When $s = 1 - \frac{1}{N+1}$, note that the vector $|f_1\rangle$ has the form

$$|f_1\rangle = \sqrt{\frac{N}{N+4}}|-\rangle - \frac{2}{\sqrt{N+4}}|\circlearrowleft\rangle. \tag{23}$$

A. Smallest eigenphase of WF_1

We choose the operator WF_1 as our intermediate step in the analysis of WF because it is the composition of a well-characterized real operator with a one-dimensional reflection. This allows us to apply results from the literature about operators of this type. Here, we show how these results can be used to obtain a tight bound on the smallest positive eigenphase of WF_1.

An overview of the applied results is given in Appendix A.

Lemma 8. The smallest positive eigenphase φ_1 of WF_1 satisfies $\varphi_1 \in \Theta(\frac{1}{\sqrt{N\log N}})$.

Proof. First, we note that the smallest positive eigenvalue of W is given by

$$\theta_{10} = \operatorname{acos}\left(2\cos^2\left(\frac{2\pi}{\sqrt{N}}\right) - 1\right) = \frac{4\pi}{\sqrt{N}}. \tag{24}$$

We know by Lemma 5 and the decomposition in (23) that $|f_1\rangle$ satisfies

$$\|\Pi_{00}|f_1\rangle\|^2 = \frac{8}{N+4},$$
$$\|\Pi_{kl}|f_1\rangle\|^2 = \frac{2\dim(\mathsf{W}_{kl})}{N+4} \qquad \text{for all } k, l \text{ not both } 0.$$

In particular, $|f_1\rangle$ overlaps all eigenspaces of W, so by Theorem 17, $\varphi_1 < \theta_{10} = \frac{4\pi}{\sqrt{N}}$.

By Lemma 16, φ_1 must also satisfy the equation

$$\|\Pi_{00}|f_1\rangle\|^2 \cot\left(\frac{\varphi_1}{2}\right) + \sum_{kl\neq 00}\|\Pi_{kl}|f_1\rangle\|^2 \cot\left(\frac{\varphi_1 - \theta_{kl}}{2}\right) = 0, \tag{25}$$

where the sum is taken over all invariant subspaces of W except the $(+1)$ eigenspace, W_{00}. Note that the (-1) eigenspace is included this sum. Because $\varphi_1 > 0$ and $\varphi_1 \in o(1)$, we have

$$\|\Pi_{00}|f_1\rangle\|^2 \cot\left(\frac{\varphi_1}{2}\right) \in \Theta\left(\frac{1}{\varphi_1 N}\right).$$

Thus, for (25) to hold,

$$\sum_{kl\neq 00}\|\Pi_{kl}|f_1\rangle\|^2 \cot\left(\frac{\theta_{kl} - \varphi_1}{2}\right) \in \Theta\left(\frac{1}{\varphi_1 N}\right). \tag{26}$$

We argue that there cannot be a solution $\varphi_1 \in \Theta(\frac{1}{\sqrt{N}})$. By Fact 19, we know that for such a φ_1, the sum in (26) has order $\Omega(\frac{\log N}{\sqrt{N}})$. Therefore, it must be the case that $\varphi_1 \in o(\frac{1}{\sqrt{N}})$.

By Fact 19, we also know that if $\varphi_1 \in o(\frac{1}{\sqrt{N}})$, then the sum in (26) has order $\Theta(\varphi_1 \log N)$. Due to the requirement $\varphi_1 \log N \in \Theta(\frac{1}{\varphi_1 N})$, the only possible solution is $\varphi_1 \in \Theta(\frac{1}{\sqrt{N\log N}})$, as stated. ∎

B. Slowest eigenvector of WF_1

In this section, we use a constraint-solving approach to analyze the eigenvector of WF_1 associated with eigenphase

φ_1. By determining its asymptotic behavior, we show that the $|\pi_z\rangle + |\circlearrowleft\rangle$ component of (22) lies in the span of this eigenvector and its conjugate. This shows that WF_1 can be applied to rotate the initial state $|\pi_z\rangle$ to the target state $|\circlearrowleft\rangle$.

Define $|-\rangle^\perp = |-\rangle + \frac{2}{\sqrt{N}}|\pi_z\rangle$ to be the (unnormalized) component of $|-\rangle$ that is orthogonal to $|\pi_z\rangle$. Note that by Lemma 5, this vector is orthogonal to $(+1)$ eigenspace of W.

Let $|\zeta\rangle$ be an unnormalized eigenvector of WF_1 with eigenphase $\alpha \neq 0, \pi$ and $\langle\zeta|\pi_z\rangle \neq 0$, scaled such that $\langle\zeta|\pi_z\rangle = \frac{1}{2}$. Because $|\zeta\rangle$ is perpendicular to $|\mathsf{U}_0\rangle$, this implies $\langle\zeta|\circlearrowleft\rangle = \frac{1}{2}$. We decompose $|\zeta\rangle$ as

$$|\zeta\rangle = a|-\rangle^\perp + \tfrac{1}{2}|\pi_z\rangle + \tfrac{1}{2}|\circlearrowleft\rangle + |\psi\rangle, \tag{27}$$

where $|\psi\rangle$ is an unnormalized vector orthogonal to $|-\rangle^\perp$. By analyzing the asymptotic behavior of a and $|\psi\rangle$, we show that the real part of $|\zeta\rangle$ tends to $\frac{1}{2}(|\pi_z\rangle + |\circlearrowleft\rangle)$ when $\alpha = \varphi_1$.

Note that any eigenvector of W with eigenphase θ_{kl} that is orthogonal to $|f_1\rangle$ is also an eigenvector of WF_1 with eigenphase θ_{kl}. Therefore, $\Pi_{kl}|\psi\rangle$ is some scalar multiple of $\Pi_{kl}|-\rangle^\perp$ for each k, l. We determine this scalar factor in Lemma 9.

We further decompose both $|-\rangle^\perp$ and $|\psi\rangle$ into the invariant subspaces of W. Both $|-\rangle^\perp$ and $|\psi\rangle$ are orthogonal to the $(+1)$-eigenspace W_{00}, so we write the decomposition as

$$|-\rangle^\perp = \sum_{kl\neq 00} m_{kl}|-_{kl}\rangle, \tag{28}$$

$$|\psi\rangle = \sum_{kl\neq 00} |\psi_{kl}\rangle, \tag{29}$$

where the vectors $|-_{kl}\rangle$ are normalized for all k, l. We know by Lemma 5 that $m_{kl} = \sqrt{\frac{2\dim(\mathsf{W}_{kl})}{N}}$. The vectors $|\psi_{kl}\rangle$ in the decomposition of $|\psi\rangle$ are unnormalized.

Lemma 9. The following equations must be satisfied for all k, l not both 0:

$$\frac{8a(N-4)}{\sqrt{N}(N+4)} - \frac{16}{N+4} = e^{\imath\alpha} - 1, \tag{30}$$

$$\langle -_{kl}|\psi_{kl}\rangle = m_{kl}\left[a - \frac{\sqrt{N}}{4}(e^{\imath\alpha} - 1)\left(\frac{1}{1 - e^{\imath(\alpha-\theta_{kl})}}\right)\right]. \tag{31}$$

Proof. By definition, $|\zeta\rangle$ is an eigenvector of WF_1 with eigenphase α. We obtain the lemma by expanding the equation $\mathsf{WF}_1|\zeta\rangle = e^{\imath\alpha}|\zeta\rangle$ and solving for constraints.

Observe that

$$\langle f_1|\zeta\rangle = \frac{1}{\sqrt{N+4}}\left(\frac{a(N-4)}{\sqrt{N}} - 2\right).$$

Using this property, we compute

$$\begin{aligned}\mathsf{WF}_1|\zeta\rangle &= \mathsf{W}|\zeta\rangle - 2\mathsf{W}\langle f_1|\zeta\rangle|f_1\rangle \\ &= \gamma_-\mathsf{W}|-\rangle^\perp + \gamma_*(|\pi_z\rangle + |\circlearrowleft\rangle) + \mathsf{W}|\psi\rangle,\end{aligned}$$

where

$$\gamma_- = a - \frac{2}{N+4}(a(N-4) - 2\sqrt{N}),$$

$$\gamma_* = \frac{1}{2} + \frac{4}{\sqrt{N}(N+4)}(a(N-4) - 2\sqrt{N}).$$

Setting $\mathsf{WF}_1|\zeta\rangle = e^{\imath\alpha}|\zeta\rangle$ and comparing coefficients on $|\pi_z\rangle$, we get

$$\gamma_* = \tfrac{1}{2}e^{\imath\alpha},$$

which can be expanded to give (30).

To get (31), we first solve $\mathsf{WF}_1|\zeta\rangle = e^{\imath\alpha}|\zeta\rangle$ on the subspace W_{kl} to get

$$\langle -_{kl}|\psi_{kl}\rangle = m_{kl}\left(\frac{ae^{\imath\alpha} - \gamma_- e^{\imath\theta_{kl}}}{e^{\imath\theta_{kl}} - e^{\imath\alpha}}\right). \tag{32}$$

Next, we use (30) to rewrite γ_- as

$$\gamma_- = a - \frac{\sqrt{N}}{4}(e^{\imath\alpha} - 1).$$

Substituting this expression for γ_- into (32) produces (31). ■

Now, fix $|\zeta\rangle$ to be the eigenvector with eigenphase φ_1. By Lemma 16, we know that $\langle\zeta|\pi_z\rangle \neq 0$, so the constraints given in Lemma 9 apply. These constraints, together with the bound on φ_1 from Lemma 8, define asymptotic bounds on a and the real part of $|\psi\rangle$. We use this to show that as N increases, the real part of $|\zeta\rangle$ converges to $\frac{1}{2}|\pi_z\rangle + |\circlearrowleft\rangle$.

Let $|\overline{\zeta}\rangle$ denote the entrywise conjugate of $|\zeta\rangle$. Then $|\zeta\rangle$ and $|\overline{\zeta}\rangle$ span the slowest rotational subspace of WF_1. In this way, the following lemmas provide a close description of the spanning eigenvectors for the slowest rotational subspace of WF_1.

Lemma 10. Let $\alpha = \varphi_1$. Then $|a| \in \Theta(\frac{1}{\sqrt{\log N}})$.

Proof. By Lemma 8, we know that $|e^{\imath\varphi_1} - 1| \in \Theta(\frac{1}{\sqrt{N\log N}})$. Applying this to (30) produces the stated bound. ■

Lemma 11. Let $\alpha = \varphi_1$. Let $|\psi\rangle = \mathrm{Re}(|\psi\rangle) + \imath\,\mathrm{Im}(|\psi\rangle)$, where both $\mathrm{Re}(|\psi\rangle)$ and $\mathrm{Im}(|\psi\rangle)$ are vectors with real entries. Then $\|\mathrm{Re}(|\psi\rangle)\| \in O(\frac{1}{\sqrt{\log N}})$.

Proof. From (31), we know that

$$\begin{aligned}|\psi\rangle &= \sum_{kl\neq 00} m_{kl}\left[a - \frac{\sqrt{N}}{4}(e^{i\alpha} - 1)\left(\frac{1}{1 - e^{i(\alpha-\theta_{kl})}}\right)\right]|-_{kl}\rangle \\ &= a|-\rangle^\perp + \rho|v\rangle,\end{aligned}$$

where we define

$$\rho = -\frac{\sqrt{N}}{4}(e^{i\alpha} - 1),$$

$$|v\rangle = \sum_{kl\neq 00} m_{kl}\left(\frac{1}{1 - e^{i(\alpha-\theta_{kl})}}\right)|-_{kl}\rangle.$$

We know from Lemma 10 that $\|\ a|-\rangle^\perp\| \in \Theta(\frac{1}{\sqrt{\log N}})$. Thus, it remains to consider $\rho|v\rangle$.

Examining ρ shows that for $\alpha \in \Theta(\frac{1}{\sqrt{N\log N}})$,

$$|\mathrm{Re}(\rho)| \in \Theta\left(\frac{1}{\sqrt{N}\log N}\right), \quad |\mathrm{Im}(\rho)| \in \Theta\left(\frac{1}{\sqrt{\log N}}\right).$$

Therefore, we can prove the lemma by showing that $\|\mathrm{Re}(|v\rangle)\| \in O(\sqrt{N}\log N)$ and $\|\mathrm{Im}(|v\rangle)\| \in O(1)$. Observe that for all k, l,

$$\mathrm{Re}\left(\frac{1}{1-e^{i(\alpha-\theta_{kl})}}\right)=\frac{1}{2},$$
$$\mathrm{Im}\left(\frac{1}{1-e^{i(\alpha-\theta_{kl})}}\right)=-\frac{1}{2}\cot\left(\frac{\theta_{kl}-\alpha}{2}\right).$$

Using this property, we split the coefficients of $|v\rangle$ into their real and imaginary parts, giving

$$|v\rangle=\frac{1}{2}\sum_{kl\neq 00}m_{kl}|-_{kl}\rangle-\frac{\iota}{2}\sum_{kl\neq 00}m_{kl}\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)|-_{kl}\rangle$$
$$=\frac{1}{2}|-\rangle^{\perp}-\frac{\iota}{2}|v'\rangle,$$

where

$$|v'\rangle=\sum_{kl\neq 00}m_{kl}\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)|-_{kl}\rangle.$$

Note that $|-\rangle^{\perp}$ is real valued and has norm $\Theta(1)$.

We now bound the real and imaginary parts of $|v'\rangle$. Recall that for each subspace W_{kl} with eigenphase $0<\theta_{kl}<\pi$, there is a corresponding subspace with eigenphase $-\theta_{kl}$, which we denote $\overline{\mathsf{W}_{kl}}$. Because $|-\rangle^{\perp}$ is real valued, it must be the case that the normalized projection of $|-\rangle^{\perp}$ onto $\overline{\mathsf{W}_{kl}}$ is $|\overline{-_{kl}}\rangle$, the entrywise conjugate of $|-_{kl}\rangle$. Using this property, we decompose $|v'\rangle$ as

$$|v'\rangle=\sum_{0<\theta_{kl}<\pi}m_{kl}\left[\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)|-_{kl}\rangle-\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)|\overline{-_{kl}}\rangle\right]$$
$$+\sum_{\theta_{kl}=\pi}m_{kl}\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)|-_{kl}\rangle$$
$$=|v_1\rangle+|v_2\rangle+|v_3\rangle,$$

where

$$|v_1\rangle=\sum_{0<\theta_{kl}<\pi}m_{kl}\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)|-_{kl}\rangle-\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)|\overline{-_{kl}}\rangle\right]$$
$$|v_2\rangle=\sum_{0<\theta_{kl}<\pi}m_{kl}\left[\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)-\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)\right]|-_{kl}\rangle,$$
$$|v_3\rangle=\sum_{\theta_{kl}=\pi}m_{kl}\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)|-_{kl}\rangle.$$

Note that the sums are taken over the invariant subspaces of W whose eigenphases lie in the indicated range. We bound the norms of these three components individually. First, observe that

$$|v_3\rangle=\tan\left(\frac{\alpha}{2}\right)\sum_{\theta_{kl}=\pi}m_{kl}|-_{kl}\rangle,$$

where $\sum_{\theta_{kl}=\pi}m_{kl}|-_{kl}\rangle$ is the projection of $|-\rangle$ onto the (-1) eigenspace of W. Therefore, $|v_3\rangle$ must be entirely real valued, with norm $\||v_3\rangle\|\in O(\frac{1}{\sqrt{N\log N}})$ by Lemma 8.

The vector $|v_1\rangle$ is entirely imaginary valued, with norm

$$\||v_1\rangle\|^2=\sum_{0<\theta_{kl}<\pi}{m_{kl}}^2\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)^2+\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)^2\right]$$
$$\leqslant\frac{16}{N}\sum_{0<\theta_{kl}<\pi}\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)^2$$
$$=\frac{16}{N}\sum_{0<\theta_{kl}<\pi}\left(\frac{\cot\left(\frac{\theta_{kl}}{2}\right)\cot\left(\frac{\alpha}{2}\right)-1}{\cot\left(\frac{\alpha}{2}\right)+\cot\left(\frac{\theta_{kl}}{2}\right)}\right)^2$$
$$\leqslant\frac{16}{N}\cot^2\left(\frac{\alpha}{2}\right)\sum_{0<\theta_{kl}<\pi}\left(\frac{\cot\left(\frac{\theta_{kl}}{2}\right)}{\cot\left(\frac{\alpha}{2}\right)+\cot\left(\frac{\theta_{kl}}{2}\right)}\right)^2.$$

There are $O(N)$ terms in the final sum, each of which is at most 1, so $\||v_1\rangle\|^2\in O(N\log N)$.

Finally, the vector $|v_2\rangle$ has both real and imaginary parts, and has norm

$$\||v_2\rangle\|^2$$
$$=\sum_{0<\theta_{kl}<\pi}{m_{kl}}^2\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)-\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2$$
$$=\frac{2}{N}\sum_{0<\theta_{kl}<\pi}\dim(\mathsf{W}_{kl})\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)-\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2$$
$$=\frac{1}{N}\sum_{kl\neq 00}\dim(\mathsf{W}_{kl})\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)-\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2.$$

By Fact 20, this implies $\||v_2\rangle\|^2\in O(\frac{1}{\log N})$.

Combining the bounds on $|v_1\rangle$, $|v_2\rangle$, and $|v_3\rangle$, we bound the norm of the real and imaginary parts of $|v'\rangle$. Thus,

$$\|\mathrm{Re}(|v'\rangle)\|\leqslant\||v_3\rangle\|+\||v_2\rangle\|\in O\left(\frac{1}{\sqrt{\log N}}\right),$$
$$\|\mathrm{Im}(|v'\rangle)\|\leqslant\||v_1\rangle\|+\||v_2\rangle\|\in O(\sqrt{N\log N}).$$

Because $|v\rangle=\frac{1}{2}|-\rangle^{\perp}-\frac{\iota}{2}|v'\rangle$, this shows in particular that $\|\mathrm{Re}(|v\rangle)\|\in O(\sqrt{N\log N})$ and $\|\mathrm{Im}(|v\rangle)\|\in O(1)$. We combine this with the bounds on ρ to obtain $\|\mathrm{Re}(|\psi\rangle)\|\in O(\frac{1}{\sqrt{\log N}})$ as stated. ■

Lemmas 10 and 11 show that the real part of $|\zeta\rangle$ tends to $\frac{1}{2}(|\pi_z\rangle+|\circlearrowleft\rangle)$ as N increases. This implies that $|\pi_z\rangle+|\circlearrowleft\rangle$ lies asymptotically in the slowest rotational subspace of WF_1. We make this precise in the following lemma.

Lemma 12. Let Π_{φ_1} denote the projection onto the slowest rotational subspace of WF_1, which is spanned by the eigenvectors with eigenphases $\pm\varphi_1$. Then

$$\|\Pi_{\varphi_1}(|\pi_z\rangle+|\circlearrowleft\rangle)\|=\sqrt{2}-O\left(\frac{1}{\log N}\right). \quad (33)$$

Proof. Let $\mathrm{Re}(a)$ denote the real part of a. Observe that

$$|\zeta\rangle+|\overline{\zeta}\rangle=|\pi_z\rangle+|\circlearrowleft\rangle+2\mathrm{Re}(a)|-\rangle^{\perp}+2\mathrm{Re}(|\psi\rangle).$$

By Lemma 10, we have $|\mathrm{Re}(a)| \in O(\frac{1}{\sqrt{\log N}})$, and by Lemma 11 we have $\|\mathrm{Re}(|\psi\rangle)\| \in O(\frac{1}{\sqrt{\log N}})$. Therefore,

$$\||\zeta\rangle + |\bar{\zeta}\rangle\| = \sqrt{2} + O\left(\frac{1}{\log N}\right).$$

We can also see that

$$(\langle\zeta| + \langle\bar{\zeta}|)(|\pi_z\rangle + |\circlearrowleft\rangle) = 2.$$

Noting that Π_{φ_1} denotes the projection onto $\mathrm{span}\{|\zeta\rangle, |\bar{\zeta}\rangle\}$, this implies that

$$\|\Pi_{\varphi_1}(|\pi_z\rangle + |\circlearrowleft\rangle)\| \geqslant \sqrt{2} - O\left(\frac{1}{\log N}\right).$$

■

C. Proof of corollary

Applying the operator WF_1 to the initial state $|\pi_z\rangle$ yields an optimal algorithm for finding $|\circlearrowleft\rangle$. The proof of this corollary follows from a similar argument to the proof of Theorem 1. We apply our characterization of the slowest rotational subspace of WF_1, given by Lemmas 8 and 12, to show the action of WF_1 on $|\pi_z\rangle$ is asymptotically restricted to this single subspace. The result is a Grover-like algorithm that rotates $|\pi_z\rangle$ to our desired state $|\circlearrowleft\rangle$.

Corollary 6 *Fix $s = 1 - \frac{1}{N+1}$ and suppose $n_r = n_c$. Then there exists a constant $c > 0$ such that after $c\sqrt{N \log N}$ applications of WF_1 to $|\pi_z\rangle$, measuring the state will produce $|\circlearrowleft\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{\log N})$.*

Proof. Recall the decomposition of $|\pi_z\rangle$ in (22). By Lemma 7, we know that $|\mathsf{U}_0\rangle$ is a $(+1)$ eigenvector of WF_1. Letting Π_{φ_1} denote the projection onto the slowest rotational subspace of WF_1, we decompose $|\pi_z\rangle + |\circlearrowleft\rangle$ as

$$|\pi_z\rangle + |\circlearrowleft\rangle = \Pi_{\varphi_1}(|\pi_z\rangle + |\circlearrowleft\rangle) + |\perp\rangle.$$

for some vector $|\perp\rangle$. By Lemma 12, we know that $\||\perp\rangle\| \in O(\frac{1}{\log N})$. We also know from Lemma 8 that the slowest rotational subspace has eigenphase $\varphi_1 \in \Theta(\frac{1}{\sqrt{N \log N}})$. Therefore, there exists a constant c such that $c\sqrt{N \log N} = \lfloor \frac{\pi}{\varphi_1} \rfloor = k$. After k applications of WF_1 to $|\pi_z\rangle$, we get the state

$$\begin{aligned}(\mathsf{WF}_1)^k|\pi_z\rangle &= \tfrac{1}{2}(|\pi_z\rangle - |\circlearrowleft\rangle) - \tfrac{1}{2}\Pi_{\varphi_1}(|\pi_z\rangle + |\circlearrowleft\rangle) + |\rho\rangle \\ &= -|\circlearrowleft\rangle + \tfrac{1}{2}|\perp\rangle + |\rho\rangle.\end{aligned}$$

Here, $|\rho\rangle$ is some state that captures both the result of applying $(\mathsf{WF}_1)^k$ to $|\perp\rangle$ and the small error incurred by the rounding of $\frac{\pi}{\varphi_1}$, and has norm $\||\rho\rangle\| \in O(\frac{1}{\log N})$. Thus, measuring the state will produce $|\circlearrowleft\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{\log N})$ as stated. ■

VI. FINDING WITH A MEMORYLESS WALK

We now present the proof of our main theorem, which is stated as follows:

Theorem 1 (*Main result*) *Fix $s = 1 - \frac{1}{N+1}$ and suppose $n_r = n_c$.* Then there exists a constant $c > 0$ such that after $c\sqrt{N \log N}$ applications of U to $|\pi\rangle$, measuring the state will produce $|\circlearrowleft\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{\log N})$.

Our proof uses the decomposition of $|\pi_z\rangle$ given in (22). We show that the state $\frac{1}{2}(|\pi_z\rangle + |\circlearrowleft\rangle)$ lies asymptotically in the slowest rotational subspace of WF. Therefore, WF can be used to rotate $|\pi_z\rangle$ to a state close to $-|\circlearrowleft\rangle$. Applying the change of basis CZ yields the result as stated.

The proof is based on relating the slowest rotational subspaces of WF_1 and $\mathsf{WF} = (\mathsf{WF}_1)\mathsf{F}_2$. We continue to apply the results from Appendix A, this time to analyze the real operator WF_1 composed with the one-dimensional reflection F_2. We show the slowest rotational subspaces of WF_1 and WF have the same asymptotic bound on the rotational angle and that both asymptotically contain $\frac{1}{2}(|\pi_z\rangle + |\circlearrowleft\rangle)$. By using WF_1 as an intermediate operator, we are thus able to tightly characterize the slowest rotational subspace of a real operator W, composed with a two-dimensional rotation F.

A. Relationship with WF_1

We begin by relating the slowest rotational subspaces of WF and WF_1. Note that when $s = 1 - \frac{1}{N+1}$, the vector $|f_2\rangle$ has the form

$$|f_2\rangle = \frac{\sqrt{3N}\sqrt{N+4}}{2(N+1)}|+\rangle + \frac{2-N}{2(N+1)}|f_1\rangle. \tag{34}$$

Let the eigenphases of WF_1 different from $0, \pi$ be denoted by $\pm\varphi_k$ for $1 \leqslant k \leqslant m$, where $0 < |\varphi_1| \leqslant |\varphi_2| \leqslant \cdots \leqslant |\varphi_m| < \pi$. Let the associated eigenvectors be $|A_k^{\pm}\rangle$. Both $|f_2\rangle$ and WF_1 are real valued, so we can decompose $|f_2\rangle$ into the eigenbasis of WF_1 as

$$|f_2\rangle = g_0|A_0\rangle + \sum_{k=1}^{m} g_k(|A_k^+\rangle + |A_k^-\rangle) + g_{-1}|A_{-1}\rangle, \tag{35}$$

where $|A_0\rangle$ is a $(+1)$ eigenvector, $|A_{-1}\rangle$ is a (-1) eigenvector, and all g_i are non-negative real numbers.

By Lemma 5, $\Pi_{kl}|+\rangle \perp \Pi_{kl}|f_1\rangle$ for each invariant subspace W_{kl} of W (including W_{00}). Therefore, the eigenvectors $\Pi_{kl}|+\rangle$ of W are also eigenvectors of WF_1 with the same eigenphases θ_{kl}.

Lemma 13. The decomposition of $|f_2\rangle$ in (35) satisfies

$$g_0^2 = \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right), \tag{36}$$

$$g_1^2 \in O\left(\frac{1}{\log N}\right). \tag{37}$$

Proof. Recall that $0 < \varphi_1 < |\theta_{kl}|$ for all nonzero eigenphases θ_{kl} of W. Using the property that $\Pi_{kl}|+\rangle$ is an eigenvector of WF_1 with eigenvalue θ_{kl}, this implies that $\langle A_1^+|\Pi_{kl}|+\rangle = 0$ for all k, l, so $\langle A_1^+|+\rangle = 0$. Let $\Pi_{(+1)}$ denote the projection onto the $(+1)$ eigenspace of WF_1. Then we have $\|\Pi_{(+1)}|+\rangle\|^2 = \|\Pi_{00}|+\rangle\|^2 = \frac{2(N+2)}{3N}$ by Lemma 5.

We bound $\|\Pi_{(+1)}|f_1\rangle\|^2$ by observing that

$$\begin{aligned}\|\Pi_{(+1)}|f_1\rangle\|^2 &= \sum_{\theta_{kl}=\pi} \|\Pi_{kl}|f_1\rangle\|^2 \\ &= \frac{N}{N+4}\sum_{\theta_{kl}=\pi} \|\Pi_{kl}|-\rangle\|^2 \\ &= \frac{N}{N+4}\sum_{\theta_{kl}=\pi} \frac{2\dim(\mathsf{W}_{kl})}{N} \in O\left(\frac{1}{\sqrt{N}}\right),\end{aligned}$$

where the final bound follows from the property that there are $O(\sqrt{N})$ terms in the sum.

Thus, using (34) and the property that $\Pi_{(+1)}|+\rangle$ and $\Pi_{(+1)}|f_1\rangle$ are orthogonal,

$$\begin{aligned} g_0^2 &= \frac{3N(N+4)}{4(N+1)^2}\|\Pi_{(+1)}|+\rangle\|^2 + \frac{(2-N)^2}{4(N+1)^2}\|\Pi_{(+1)}|f_1\rangle\|^2 \\ &= \frac{3N(N+4)}{4(N+1)^2}\left(\frac{2(N+2)}{3N}\right) + O\left(\frac{1}{\sqrt{N}}\right) \\ &= \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

To bound g_1^2, consider the eigenvector decomposition given in (27) in the case where $\alpha = \varphi_1$. Then $|A_1^+\rangle$ is the normalized version of $|\zeta\rangle$. By definition, we have $\||\zeta\rangle\|^2 \geqslant \frac{1}{2}$, so

$$\begin{aligned} g_1^2 &= |\langle A_1^+|f_2\rangle|^2 \\ &= \cos^2(2\eta)|\langle A_1^+|f_1\rangle|^2 + \sin^2(2\eta)|\langle A_1^+|+\rangle|^2 \\ &= \cos^2(2\eta)|\langle A_1^+|f_1\rangle|^2 \\ &\leqslant |\langle\zeta|f_1\rangle|^2/\||\zeta\rangle\|^2 \\ &\leqslant 2|\langle\zeta|f_1\rangle|^2 \\ &= 2\left|a\sqrt{\frac{N}{N+4}}\left(1-\frac{4}{N}\right) - \frac{2}{\sqrt{N+4}}\right|^2 \in O\left(\frac{1}{\log N}\right), \end{aligned}$$

where the final bound follows from Lemma 10. ■

Lemma 13 shows that $|f_2\rangle$ has a large constant overlap with the $(+1)$ eigenspace of WF_1, and a vanishing overlap with the slowest rotational space. Intuitively, this suggests the reflection F_2 will have little effect on the slowest rotational subspace of the intermediate walk WF_1. As discussed in Sec. V, this subspace is where most of the action of the walk takes place. This observation is the basis for the proof of Lemma 15.

In the next lemma, we use the constraints from Lemma 16 to show that the smallest positive eigenphase of WF is asymptotically close to φ_1.

Lemma 14. Let β denote the smallest eigenphase of WF. Then

$$\beta = \varphi_1 - O\left(\frac{1}{\sqrt{N}(\log N)^{3/2}}\right). \tag{38}$$

Note that in particular, this implies $\beta \in \Theta(\frac{1}{\sqrt{N\log N}})$.

Proof. First, we derive an upper bound on β using the flip-flop theorem. By Lemma 5, $|+\rangle$ intersects every eigenspace of W. Each of the eigenvectors $\Pi_{kl}|+\rangle$ is also an eigenvector of WF_1, so in particular, this implies that $g_0 > 0$, $g_{-1} > 0$, and $g_k > 0$ for the k corresponding to the eigenphases θ_{kl}. Therefore, by Theorem 17, $0 < \beta < \varphi_1$. Applying Lemma 8, we obtain $\beta \in O(\frac{1}{\sqrt{N\log N}})$.

To obtain the upper bound on $\varphi_1 - \beta$, we apply Lemma 16, which states that β must satisfy

$$g_0^2\cot\left(\frac{\beta}{2}\right) + \sum_{k=1}^m g_k^2\left[\cot\left(\frac{\varphi_k+\beta}{2}\right) - \cot\left(\frac{\varphi_k-\beta}{2}\right)\right] \\ -g_{-1}^2\tan\left(\frac{\beta}{2}\right) = 0.$$

We apply trigonometric identities to rewrite this as

$$g_0^2\cot\left(\frac{\beta}{2}\right) - 2\cot\left(\frac{\beta}{2}\right)\sum_{k=1}^m g_k^2\left(\frac{\cot^2\left(\frac{\varphi_k}{2}\right)+1}{\cot^2\left(\frac{\beta}{2}\right)-\cot^2\left(\frac{\varphi_k}{2}\right)}\right) \\ -g_{-1}^2\tan\left(\frac{\beta}{2}\right) = 0.$$

Applying our upper bound on β, we know that $g_{-1}^2\tan^2(\frac{\beta}{2}) \in O(\frac{1}{N\log N})$, so we have

$$\begin{aligned} 2\sum_{k=1}^m g_k^2\frac{\cot^2\left(\frac{\varphi_k}{2}\right)+1}{\cot^2\left(\frac{\beta}{2}\right)-\cot^2\left(\frac{\varphi_k}{2}\right)} && (39)\\ = g_0^2 - O\left(\frac{1}{N\log N}\right) && (40)\\ = \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right), && (41) \end{aligned}$$

where the last equality follows by Lemma 13.

Because the smallest positive eigenphase of W has order $\Theta(\frac{1}{\sqrt{N}})$, Theorem 17 implies that $\varphi_k \in \Omega(\frac{1}{\sqrt{N}})$ for $k \geqslant 2$. Therefore, $\cot^2(\frac{\varphi_k}{2}) \in O(N)$ for $k \geqslant 2$, while $\cot^2(\frac{\beta}{2}) \in \Omega(N\log N)$. Also note that $\sum_k g_k^2 \leqslant 1$. This means that

$$\sum_{k=2}^m g_k^2\frac{\cot^2\left(\frac{\varphi_k}{2}\right)+1}{\cot^2\left(\frac{\beta}{2}\right)-\cot^2\left(\frac{\varphi_k}{2}\right)} \in O\left(\frac{1}{\log N}\right). \tag{42}$$

Combining (41) and (42), we get

$$2g_1^2\frac{\cot^2\left(\frac{\varphi_1}{2}\right)+1}{\cot^2\left(\frac{\beta}{2}\right)-\cot^2\left(\frac{\varphi_1}{2}\right)} = \frac{1}{2} - O\left(\frac{1}{\log N}\right).$$

We know that $\cot^2(\frac{\varphi_1}{2}) \in O(N\log N)$ by Lemma 8. We also have $g_1^2 \in O(\frac{1}{\log N})$ by Lemma 13. This implies that

$$\cot^2\left(\frac{\beta}{2}\right) - \cot^2\left(\frac{\varphi_1}{2}\right) \in O(N).$$

Because $\cot(\frac{\beta}{2}) + \cot(\frac{\varphi_1}{2}) \in \Omega(\sqrt{N\log N})$, it must be the case that

$$\cot\left(\frac{\beta}{2}\right) - \cot\left(\frac{\varphi_1}{2}\right) \in O\left(\frac{\sqrt{N}}{\sqrt{\log N}}\right).$$

Applying the Taylor expansion for cotangent, we get

$$\frac{1}{\beta} - \frac{1}{\varphi_1} = \frac{\varphi_1-\beta}{\varphi_1\beta} \in O\left(\frac{\sqrt{N}}{\sqrt{\log N}}\right).$$

We know that $\frac{1}{\varphi_1} \in \Theta(\sqrt{N\log N})$, so this implies that $\frac{1}{\beta} \in \Theta(\sqrt{N\log N})$. Thus,

$$\varphi_1 - \beta \in O\left(\frac{1}{\sqrt{N}(\log N)^{3/2}}\right).$$

■

Finally, we show that $|\pi_z\rangle + |\circlearrowleft\rangle$ lies asymptotically in the slowest rotational subspace of WF. To do so, we apply our bounds from Lemmas 13 and 14 to show that the slowest eigenvectors of WF are asymptotically close to the slowest eigenvectors of WF_1.

Lemma 15. Let Π_β denote the projection onto the slowest rotational subspace of WF, which is spanned by the eigenvectors with eigenphases $\pm\beta$. Then

$$\|\Pi_\beta(|\pi_z\rangle + |\circlearrowleft\rangle)\| = \sqrt{2} - O\left(\frac{1}{\log N}\right). \tag{43}$$

Proof. We use the decomposition of $|f_2\rangle$ in (35). By Lemma 16, the (unnormalized) eigenvector of WF associated with eigenphase β is $|e_\beta\rangle = |f_2\rangle + \imath|e_\beta^\perp\rangle$, where

$$\begin{aligned}|e_\beta^\perp\rangle = g_0 \cot\left(\frac{\beta}{2}\right)|A_0\rangle \\ + \sum_{k=1}^{m} g_k\left[\cot\left(\frac{\beta-\varphi_k}{2}\right)|A_k^+\rangle + \cot\left(\frac{\beta+\varphi_k}{2}\right)|A_k^-\rangle\right] \\ - g_{-1}\tan\left(\frac{\beta}{2}\right)|A_{-1}\rangle.\end{aligned} \tag{44}$$

Let $|B_1^+\rangle$ denote the normalization of $|e_\beta\rangle$, and let $|B_1^-\rangle$ denote the conjugate of $|B_1^+\rangle$. Then Π_β is a projection onto the span of $|B_1^+\rangle$ and $|B_1^-\rangle$.

We know from the proof of Lemma 12 that

$$|(\langle A_1^+| + \langle A_1^-|)(|\pi_z\rangle + |\circlearrowleft\rangle)| = 2 - O\left(\frac{1}{\log N}\right),$$

where $|A_1^+\rangle$, $|A_1^-\rangle$ are the normalizations of $|\zeta\rangle$ and $|\overline{\zeta}\rangle$, respectively. We prove Lemma 15 by showing that $|A_1^+\rangle$ and $|A_1^-\rangle$ have large overlap with $|B_1^+\rangle$ and $|B_1^-\rangle$, respectively.

We know from (44) that

$$\begin{aligned}\||e_\beta\rangle\|^2 &= \||e_\beta^\perp\rangle\|^2 + \||f_2\rangle\|^2 \\ &= g_0^2\cot^2\left(\frac{\beta}{2}\right) + \sum_{k=1}^{m} g_k^2\left[\cot^2\left(\frac{\beta-\varphi_k}{2}\right)\right. \\ &\quad \left. + \cot^2\left(\frac{\beta+\varphi_k}{2}\right)\right] + g_{-1}^2\tan^2\left(\frac{\beta}{2}\right) + 1.\end{aligned}$$

By Lemmas 13 and 14, we know that $g_0^2\cot^2(\frac{\beta}{2}) \in O(N\log N)$ and that $g_{-1}^2\tan^2(\frac{\beta}{2}) \in O(\frac{1}{N\log N})$. Recall from (42) that

$$\sum_{k=2}^{m} g_k^2\left[\cot^2\left(\frac{\beta-\varphi_k}{2}\right) + \cot^2\left(\frac{\beta+\varphi_k}{2}\right)\right] \in O\left(\frac{1}{\log N}\right).$$

Finally, we know from Lemma 14 that $\cot^2(\frac{\beta+\varphi_1}{2}) \in O(N\log N)$. Thus,

$$\||e_\beta\rangle\|^2 = g_1^2\cot^2\left(\frac{\beta-\varphi_1}{2}\right) + O(N\log N).$$

By Lemma 13, $g_1^2 \in O(\frac{1}{\log N})$, and by Lemma 14, $\cot^2(\frac{\beta-\varphi_1}{2}) \in \Omega(N(\log N)^3)$. Therefore,

$$\begin{aligned}|\langle A_1^+|B_1^+\rangle|^2 &= \frac{|\langle A_1^+|e_\beta\rangle|^2}{\||e_\beta\rangle\|^2} \\ &= \frac{|\langle A_1^+|f_2\rangle + \imath\langle A_1^+|e_\beta^\perp\rangle|^2}{\||e_\beta\rangle\|^2} \\ &= \frac{\left|g_1 + \imath g_1\cot\left(\frac{\beta-\varphi_1}{2}\right)\right|^2}{\||e_\beta\rangle\|^2} \\ &= 1 - O\left(\frac{1}{\log N}\right),\end{aligned}$$

so $|\langle A_1^+|B_1^+\rangle| = 1 - O(\frac{1}{\log N})$. Similarly, one can show that

$$\begin{aligned}|\langle A_1^-|B_1^+\rangle| &\in O\left(\frac{1}{\log^2 N}\right), \\ |\langle A_1^+|B_1^-\rangle| &\in O\left(\frac{1}{\log^2 N}\right), \\ |\langle A_1^-|B_1^-\rangle| &= 1 - O\left(\frac{1}{\log N}\right).\end{aligned}$$

Combining these results, we get

$$|(\langle B_1^+| + \langle B_1^-|)(|A_1^+\rangle + |A_1^-\rangle)| = 2 - O\left(\frac{1}{\log N}\right).$$

Therefore, letting $\Pi_A = (|A_1^+\rangle + |A_1^-\rangle)(\langle A_1^+| + \langle A_1^-|)$,

$$\begin{aligned}\|\Pi_\beta(|\pi_z\rangle + |\circlearrowleft\rangle)\| &\geqslant \frac{1}{\sqrt{2}}|(\langle B_1^+| + \langle B_1^-|)(|\pi_z\rangle + |\circlearrowleft\rangle)| \\ &\geqslant \frac{1}{2\sqrt{2}}|(\langle B_1^+| + \langle B_1^-|)\Pi_A(|\pi_z\rangle + |\circlearrowleft\rangle)| \\ &= \sqrt{2} - O\left(\frac{1}{\log N}\right).\end{aligned}$$

■

B. Proof of main result

Through Lemmas 14 and 15, we have an asymptotic description of the slowest rotational subspace of WF. The description shows that as N increases, the action of WF on $|\pi_z\rangle$ approaches a rotation in this slowest rotational subspace. This property is what allows us to map our initial state to the target state $|\circlearrowleft\rangle$ with probability approaching 1. Applying the relationship $\mathsf{WF} = (\mathsf{U})_z$, we thus obtain the proof of our main result.

Theorem 1 (Main result) Fix $s = 1\text{-}\frac{1}{N+1}$ *and suppose* $n_r = n_c$. Then there exists a constant $c > 0$ such that after $c\sqrt{N\,log\,N}$ applications of U to $|\pi\rangle$ measuring the state will produce $|\circlearrowleft\rangle$with probability $1-e(N)$, where $e(N) \in O(\frac{1}{\log})$.

Proof. First, observe that for any k,

$$\mathsf{U}^k|\pi\rangle = \mathsf{cz}(\mathsf{WF})^k\mathsf{cz}|\pi\rangle = \mathsf{cz}(\mathsf{WF})^k|\pi_z\rangle.$$

Thus, we prove that after $c\sqrt{N\log N}$ applications of WF to $|\pi_z\rangle$, measuring the state will produce $\mathsf{cz}|\circlearrowleft\rangle = |\circlearrowleft\rangle$ with the stated probability.

Recall from (22) that $|\pi_z\rangle$ can be decomposed as

$$|\pi_z\rangle = \tfrac{1}{2}|\mathsf{U}_0\rangle + \tfrac{1}{2}(|\pi_z\rangle + |\circlearrowleft\rangle).$$

By Lemma 7, we know that $|\mathsf{U}_0\rangle$ is a $(+1)$ eigenvector of WF. Letting Π_β denote the projection onto the slowest rotational subspace of WF, we decompose $|\pi_z\rangle + |\circlearrowleft\rangle$ as

$$|\pi_z\rangle + |\circlearrowleft\rangle = \Pi_\beta(|\pi_z\rangle + |\circlearrowleft\rangle) + |\perp\rangle$$

for some vector $|\perp\rangle$. By Lemma 15, we know that $\||\perp\rangle\| \in O(\frac{1}{\log N})$. We also know from Lemma 14 that applying WF to a vector in the slowest rotational subspace will result in a rotation of the vector by the angle $\beta \in \Theta(\frac{1}{\sqrt{N \log N}})$. Therefore, there exists a constant c such that $c\sqrt{N \log N} = \lfloor \frac{\pi}{\beta} \rfloor = k$. After k applications of WF to $|\pi_z\rangle$, we get the state

$$(\mathsf{WF})^k|\pi_z\rangle = \tfrac{1}{2}(|\pi_z\rangle - |\circlearrowleft\rangle) - \tfrac{1}{2}\Pi_\beta(|\pi_z\rangle + |\circlearrowleft\rangle) + |\rho\rangle$$
$$= -|\circlearrowleft\rangle + \tfrac{1}{2}|\perp\rangle + |\rho\rangle.$$

Here, $|\rho\rangle$ is some state that captures both the result of applying $(\mathsf{WF})^k$ to $|\perp\rangle$ and the small error incurred by the rounding of $\frac{\pi}{\beta}$, and has norm $\||\rho\rangle\| \in O(\frac{1}{\log N})$. Thus, measuring the state will produce $|\circlearrowleft\rangle$ with probability $1 - e(N)$, where $e(N) \in O(\frac{1}{\log N})$. ∎

VII. MULTIPLE MARKED VERTICES

Interpolated walks were initially defined by Krovi *et al.* [26], who added weighted self-loop edges to a quantum walk on the edge space of a graph. They show that if there is a unique marked vertex, a quantum interpolated walk can find it with constant success probability and a quadratic speedup over classical random walks. In the case of multiple marked vertices, it was recently shown that interpolated walks can still find marked vertices with constant success probability, and with almost the same speedup [27,28]. In this spirit, one might hope that interpolated memoryless walks could be used to find marked vertices with a quadratic speedup over classical random walks when there are multiple marked vertices.

Unfortunately, the straightforward generalization of our memoryless walk to the case of multiple marked vertices does not achieve this goal. Portugal *et al.* [10] show that for any staggered walk on a graph, there is some configuration of marked vertices for which it is unable to find. We show that this property holds for our construction as well. In particular, if an entire 2×2 tessellation square of vertices on the lattice is marked, then our initial state is a stationary state of the walk.

We consider the following extension of our memoryless walk to the case of multiple marked vertices. We introduce a unique self-loop state $|\circlearrowleft_{ij}\rangle$ and a self-loop weight s_{ij} for each marked vertex $|g_{ij}\rangle = |i, j\rangle$. We then define the interpolated states as

$$|\tilde{g}_{ij}\rangle = \sqrt{s_{ij}}|g_{ij}\rangle + \sqrt{1 - s_{ij}}|\circlearrowleft_{ij}\rangle,$$

and the input-dependent reflection as

$$\tilde{\mathsf{G}} = \mathsf{I} - 2\sum |\tilde{g}_{ij}\rangle\langle\tilde{g}_{ij}|. \tag{45}$$

Now, consider the case where the marked vertices are those in a 2×2 square. Without loss of generality, we can assume that these are the vertices $|0, 0\rangle$, $|0, 1\rangle$, $|1, 0\rangle$, and $|1, 1\rangle$, whose uniform superposition gives the tessellation state $|a_{00}\rangle$. If all s_{ij} are equal, one can compute that

$$\mathsf{A}\tilde{\mathsf{G}}|a_{00}\rangle = -\tilde{\mathsf{G}}|a_{00}\rangle = \tilde{\mathsf{G}}\mathsf{A}|a_{00}\rangle.$$

Furthermore, since these are the only marked vertices, $\mathsf{A}\tilde{\mathsf{G}}|i, j\rangle = \mathsf{A}|i, j\rangle = \tilde{\mathsf{G}}\mathsf{A}|i, j\rangle$ for any unmarked $|i, j\rangle$. Therefore,

$$\mathsf{B}\tilde{\mathsf{G}}\mathsf{A}\tilde{\mathsf{G}}|\pi\rangle = \mathsf{B}\mathsf{A}\tilde{\mathsf{G}}\tilde{\mathsf{G}}|\pi\rangle = \mathsf{B}\mathsf{A}|\pi\rangle = |\pi\rangle,$$

showing that our initial state is a stationary state of the walk. This shows that if an entire tessellation square is marked, our walk cannot be used to find any of those marked vertices.

By this observation, a main technical obstacle to applying our construction to find when there are multiple marked vertices is the case where an entire tessellation unit is marked. To overcome this, we propose a randomized approach. For each marked vertex, consider removing it from the marked set with some fixed probability, such as $\frac{1}{2}$. This would break the symmetries that cause $|\pi\rangle$ to be a stationary state of the walk with constant probability. This idea would also be simple to implement, and could be applied to memoryless walks on general graphs. Given a graph with an interpolated memoryless walk that can be used to find a unique marked vertex, it is interesting to consider whether this randomized approach could be used to obtain a memoryless walk that can find with constant probability when there are multiple marked vertices as well.

VIII. CONCLUSION

We give a quantum walk that uses minimal memory and $\Theta(\sqrt{N \log N})$ steps to find a unique marked vertex on a two-dimensional lattice. In doing so, we show how interpolated walks can be adapted to the memoryless setting. By adding a self-loop to the marked vertex, our walk boosts the probability of measuring the marked state from $O(\frac{1}{\log N})$ to $1 - O(\frac{1}{\log N})$, while preserving the simplicity of the tessellation-based structure.

We give a precise analysis of how the self-loop affects the walk dynamics by showing that our walk asymptotically reduces to a rotation in a single two-dimensional subspace. Applying this rotation evolves the initial state to the self-loop state, from which the marked state can be obtained by straightforward amplitude amplification.

As part of our proof, we give a precise description of the slowest rotational subspace of our memoryless walk operator. This is done using its decomposition into a real operator composed with a two-dimensional rotation. The techniques we use to analyze such an operator are general enough they have the potential to be used in the analysis of other walks as well. This includes developing and analyzing memory-optimal spatial search algorithms for other types of graph.

We also show that the immediate extension of our interpolated construction cannot be used in the case of multiple marked vertices, as there are configurations of marked vertices for which it fails to find. We then propose a suggestion for avoiding these configurations. It is interesting to consider whether our idea can be used to give an algorithm that can find with minimum memory, even when there are multiple marked vertices.

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APPENDIX A: COMPOSITION WITH A REFLECTION

We discuss techniques to characterize the spectra for the composition of a real operator with a one-dimensional reflection. Operators with this structure appear at multiple points in our work. In this section, we present two lemmas from the quantum walk literature that we apply in our analysis of both WF_1 and WF.

Consider an arbitrary real unitary operator T acting on a space $\mathcal{H}$, and let $|s\rangle \in \mathcal{H}$ be a state with real amplitudes. Define $\mathsf{S} = \mathsf{I} - |s\rangle\langle s|$ to be the reflection of state $|s\rangle$. The goal of this section is to describe the spectra of the operator TS.

Because T is real valued, its eigenvalues different from ± 1 come in complex conjugate pairs. We denote the eigenvalues as $e^{\pm\imath\phi_k}$ for $k = 1, 2, \ldots, m$, corresponding to the eigenvectors $|T_k^{\pm}\rangle$. We then decompose $|s\rangle$ into the eigenbasis of T as

$$|s\rangle = s_0|T_0\rangle + \sum_k s_k(|T_k^+\rangle + |T_k^-\rangle) + s_{-1}|T_{-1}\rangle. \tag{A1}$$

Here, $|T_0\rangle$ and $|T_{-1}\rangle$ are eigenvectors of T with eigenvalues $+1$ and -1, respectively. The coefficients s_0, s_{-1}, and all s_k are chosen to be non-negative real numbers by multiplying the eigenvectors with appropriate phases. This decomposition allows us to state the following lemma, originally given by Ref. [3]:

Lemma 16. Consider the (unnormalized) state $|e_\alpha\rangle = |s\rangle + \imath|e_\alpha{}^\perp\rangle$, where

$$|e_\alpha{}^\perp\rangle = s_0\cot\left(\frac{\alpha}{2}\right)|T_0\rangle + \sum_k s_k\left[\cot\left(\frac{\alpha-\phi_k}{2}\right)|T_k^+\rangle + \cot\left(\frac{\alpha+\phi_k}{2}\right)|T_k^-\rangle\right] - s_{-1}\tan\left(\frac{\alpha}{2}\right)|T_{-1}\rangle,$$

and $|e_\alpha{}^\perp\rangle$ is orthogonal to $|s\rangle$. If α is a solution of the equation

$$s_0^2\cot\left(\frac{\alpha}{2}\right) + \sum_k s_k^2\left[\cot\left(\frac{\alpha-\phi_k}{2}\right) + \cot\left(\frac{\alpha+\phi_k}{2}\right)\right] - s_{-1}^2\tan\left(\frac{\alpha}{2}\right) = 0,$$

then $|e_\alpha\rangle$ is an eigenvector of TS with eigenvalue $e^{\imath\alpha}$.

This lemma allows us to determine the eigenvectors and eigenvalues of TS by specifying a set of constraints they must satisfy. The lemma is applied in Ref. [3], and with slight variations in Refs. [5,25,30], to obtain bounds on the smallest eigenphase of a walk operator. We use the lemma for the same purpose, applying it to obtain a lower bound for the smallest eigenphase of WF_1 and WF in Lemmas 8 and 14. We also use a similar technique in our analysis of the eigenvector $|\zeta\rangle$ in Lemma 9, where we derive a set of constraints and use them to find properties of a and $|\psi\rangle$.

The next theorem we state describes the behavior of the eigenphases of TS in relation to those of T. The flip-flop theorem of Ref. [30] describes how the eigenphases of the operators interlace, with the exact pattern of interlacing depending on the eigenspaces of T that $|s\rangle$ intersects. We limit the theorem statement to the case we apply in this paper, where $|s\rangle$ intersects the $(+1)$ eigenspace, the (-1) eigenspace, and at least one other eigenspace of T.

Theorem 17 (Flip-flop theorem). Consider any real unitary T and let $|s\rangle$ be a state with real amplitudes in the same space. Denote the positive eigenphases of T different from $0, \pi$ by $0 < \phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_m < \pi$. If $s_0 \neq 0$, $s_{-1} \neq 0$ and $s_k \neq 0$ for some k, then TS has $m+1$ two-dimensional eigenspaces, and no $(+1)$ or (-1) eigenspaces which overlap $|s\rangle$. The positive eigenphases α_j of TS satisfy the inequality $0 < \alpha_0 < \phi_1 \leqslant \alpha_1 \leqslant \cdots \leqslant \phi_m \leqslant \alpha_m < \pi$.

We apply this theorem in Lemmas 8 and 14 to obtain an upper bound on the smallest positive eigenphases of WF_1 and WF, respectively. One of the contributions of our work is to show how Theorem 17 can be used in combination with Lemma 16 to tightly bound these eigenphases. We show that this approach can be used in the case of an operator composed with a reflection, and then by applying a second reflection, to an operator composed with a two-dimensional rotation.

APPENDIX B: DECOMPOSITION OF $|+\rangle$ AND $|-\rangle$

To analyze the behavior of WF_1 and WF, we specify how W acts on vectors in the nontrivial eigenspaces of F. Recall from the definitions in (8) and (9) that $|+\rangle$ and $|-\rangle$ are orthonormal vectors that have the same span as $|g\rangle$ and $|a_{00}\rangle$. Together with $|\circlearrowleft\rangle$, they span a space that includes the two-dimensional subspace on which F acts nontrivially. In this Appendix, we prove Lemma 5, which describes how $|+\rangle$ and $|-\rangle$ decompose into the invariant subspaces of W.

To simplify notation, define

$$s_{kl}^+ = \frac{1}{2}(r_{kl}^+ + r_{kl}^-) = \sqrt{1 + \frac{\sin\tilde{l}}{p_{kl}}},$$

$$s_{kl}^- = \frac{1}{2}(r_{kl}^+ - r_{kl}^-) = \epsilon_l\sqrt{1 - \frac{\sin\tilde{l}}{p_{kl}}},$$

and

$$d_{kl}^+ = \frac{1}{2}(c_{kl}^+ + c_{kl}^-) = \sqrt{1 + \frac{\sin\tilde{k}}{p_{kl}}},$$

$$d_{kl}^- = \frac{1}{2}(c_{kl}^+ - c_{kl}^-) = \epsilon_k\sqrt{1 - \frac{\sin\tilde{k}}{p_{kl}}}.$$

In the case where $k = l = 0$, we define $s_{00}^+ = d_{00}^+ = \sqrt{2}$ and $s_{00}^- = d_{00}^- = 0$. Note that $r_{kl}^\pm = s_{kl}^+ \pm s_{kl}^-$ and $c_{kl}^\pm = d_{kl}^+ \pm d_{kl}^-$.

Recall that both $|a_{00}\rangle$ and $|g\rangle$ lie in the span of the basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. We compute the projections of these basis states onto the components of the eigenvectors of W:

$$\langle 0|u_{kl}\rangle = \sqrt{2}\langle 0|r_{kl}\rangle\langle 0|\phi_r^k\rangle = \frac{1}{\sqrt{2n_r}}r_{kl}^-, \quad \langle 0|v_{kl}\rangle = \sqrt{2}\langle 0|c_{kl}\rangle\langle 0|\phi_c^l\rangle = \frac{1}{\sqrt{2n_c}}c_{kl}^-,$$

$$\langle 1|u_{kl}\rangle = \sqrt{2}\langle 1|r_{kl}\rangle\langle 1|\phi_r^k\rangle = \frac{1}{\sqrt{2n_r}}r_{kl}^+\omega_{n_r}^k, \quad \langle 1|v_{kl}\rangle = \sqrt{2}\langle 1|c_{kl}\rangle\langle 1|\phi_c^l\rangle = \frac{1}{\sqrt{2n_c}}c_{kl}^+\omega_{n_c}^l,$$

$$\langle 0|u_{kl}^1\rangle = \sqrt{2}\langle 0|r_{kl}^1\rangle\langle 0|\phi_r^k\rangle = -\frac{1}{\sqrt{2n_r}}r_{kl}^+, \quad \langle 0|v_{kl}^1\rangle = \sqrt{2}\langle 0|c_{kl}^1\rangle\langle 0|\phi_c^l\rangle = -\frac{1}{\sqrt{2n_c}}c_{kl}^+,$$

$$\langle 1|u_{kl}^1\rangle = \sqrt{2}\langle 1|r_{kl}^1\rangle\langle 1|\phi_r^k\rangle = \frac{1}{\sqrt{2n_r}}r_{kl}^-\omega_{n_r}^k, \quad \langle 1|v_{kl}^1\rangle = \sqrt{2}\langle 1|c_{kl}^1\rangle\langle 1|\phi_c^l\rangle = \frac{1}{\sqrt{2n_c}}c_{kl}^-\omega_{n_c}^l.$$

Now, using the property that

$$\frac{1}{2}\left(r_{kl}^+\omega_{n_r}^k + r_{kl}^-\right) = \omega_{n_r}^{k/2}\left[\cos\left(\frac{\tilde{k}}{2}\right)s_{kl}^+ + \imath\sin\left(\frac{\tilde{k}}{2}\right)s_{kl}^-\right],$$
$$\frac{1}{2}\left(r_{kl}^-\omega_{n_r}^k - r_{kl}^+\right) = \omega_{n_r}^{k/2}\left[-\cos\left(\frac{\tilde{k}}{2}\right)s_{kl}^- + \imath\sin\left(\frac{\tilde{k}}{2}\right)s_{kl}^+\right],$$

we compute

$$\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|u_{kl}\rangle = \frac{1}{\sqrt{n_r}}\omega_{n_r}^{k/2}\left[\cos\left(\frac{\tilde{k}}{2}\right)s_{kl}^+ + \imath\sin\left(\frac{\tilde{k}}{2}\right)s_{kl}^-\right],$$
$$\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|u_{kl}^1\rangle = \frac{1}{\sqrt{n_r}}\omega_{n_r}^{k/2}\left[-\cos\left(\frac{\tilde{k}}{2}\right)s_{kl}^- + \imath\sin\left(\frac{\tilde{k}}{2}\right)s_{kl}^+\right],$$
$$\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|v_{kl}\rangle = \frac{1}{\sqrt{n_c}}\omega_{n_c}^{l/2}\left[\cos\left(\frac{\tilde{l}}{2}\right)d_{kl}^+ + \imath\sin\left(\frac{\tilde{l}}{2}\right)d_{kl}^-\right],$$
$$\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|v_{kl}^1\rangle = \frac{1}{\sqrt{n_c}}\omega_{n_c}^{l/2}\left[-\cos\left(\frac{\tilde{l}}{2}\right)d_{kl}^- + \imath\sin\left(\frac{\tilde{l}}{2}\right)d_{kl}^+\right].$$

Excluding the case $k = l = 0$, the squared amplitudes of the projections are then

$$\|\langle 0|u_{kl}\rangle\|^2 = \frac{1}{n_r}\left(1 - \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right), \quad \left\|\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|u_{kl}\rangle\right\|^2 = \frac{1}{n_r}\left(1 + \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right),$$
$$\|\langle 0|u_{kl}^1\rangle\|^2 = \frac{1}{n_r}\left(1 + \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right), \quad \left\|\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|u_{kl}^1\rangle\right\|^2 = \frac{1}{n_r}\left(1 - \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right),$$
$$\|\langle 0|v_{kl}\rangle\|^2 = \frac{1}{n_c}\left(1 - \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right), \quad \left\|\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|v_{kl}^1\rangle\right\|^2 = \frac{1}{n_c}\left(1 - \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right).$$
$$\|\langle 0|v_{kl}^1\rangle\|^2 = \frac{1}{n_c}\left(1 + \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right), \quad \left\|\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|v_{kl}\rangle\right\|^2 = \frac{1}{n_c}\left(1 + \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right),$$

Fact 18. For any subspace W_{kl},

$$\langle g|\Pi_{kl}|a_{00}\rangle = \begin{cases}\frac{1}{2} & \text{if } k = l = 0\\ 0 & \text{otherwise}\end{cases}. \tag{B1}$$

Proof. Recall that $|g\rangle = |00\rangle$ and $|a_{00}\rangle = \frac{1}{2}(|0\rangle + |1\rangle)\otimes(|0\rangle + |1\rangle)$. Consider any k, l not both 0. Then using the expansions above, it can be derived that

$$\langle g|w_{k'l}^{11}\rangle\langle w_{k'l}^{11}|a_{00}\rangle = -\langle g|w_{kl}^{00}\rangle\langle w_{kl}^{00}|a_{00}\rangle$$

and

$$\langle g|w_{kl'}^{11}\rangle\langle w_{kl'}^{11}|a_{00}\rangle = -\langle g|w_{kl}^{00}\rangle\langle w_{kl}^{00}|a_{00}\rangle.$$

Using the property that $(k')' = k$, this further implies that $\langle g|w_{k'l'}^{00}\rangle\langle w_{k'l'}^{00}|a_{00}\rangle = \langle g|w_{kl}^{00}\rangle\langle w_{kl}^{00}|a_{00}\rangle$. By definition of the invariant subspaces W_{kl} in Sec. IV B, this shows that for any k, l not both 0, $\langle g|\Pi_{kl}|a_{00}\rangle = 0$. It follows that $\langle g|\Pi_{00}|a_{00}\rangle = \langle g|a_{00}\rangle = \frac{1}{2}$. ■

Proof. Observe that for any k, l not both zero,

$$\|\langle g|w_{kl}^{00}\rangle\|^2 + \|\langle g|w_{kl'}^{11}\rangle\|^2 + \|\langle g|w_{k'l}^{11}\rangle\|^2 + \|\langle g|w_{k'l'}^{00}\rangle\|^2$$
$$= \frac{1}{N}\left(1 - \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right)\left(1 - \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right)$$
$$+ \frac{1}{N}\left(1 - \frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right)\left(1 + \frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right)$$

$$+\frac{1}{N}\left(1+\frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right)\left(1-\frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right)$$
$$+\frac{1}{N}\left(1+\frac{\sin\tilde{k}\cos\tilde{l}}{p_{kl}}\right)\left(1+\frac{\cos\tilde{k}\sin\tilde{l}}{p_{kl}}\right)=\frac{4}{N}$$
$$=\|\langle a_{00}|w_{kl}^{00}\rangle\|^2+\|\langle a_{00}|w_{kl'}^{11}\rangle\|^2+\|\langle a_{00}|w_{k'l}^{11}\rangle\|^2$$
$$+\|\langle a_{00}|w_{k'l'}^{00}\rangle\|^2.$$

Therefore, for any subspace W_{kl} with $kl \neq 00$, we have $\langle g|\Pi_{kl}|g\rangle = \langle a_{00}|\Pi_{kl}|a_{00}\rangle = \frac{\dim(\mathsf{W}_{kl})}{N}$. Next, recall that W_{00} refers to the $(+1)$ eigenspace of W. We know that both $|g\rangle$ and $|a_{00}\rangle$ are normalized, so

$$\langle g|\Pi_{00}|g\rangle = \langle a_{00}|\Pi_{00}|a_{00}\rangle = 1-\sum_{kl\neq 00}\frac{\dim(\mathsf{W}_{kl})}{N}=\frac{N+4}{2N}.$$

Applying Fact 18, this implies that

$$\sqrt{3}\langle -|\Pi_{kl}|+\rangle$$
$$=\langle g|\Pi_{kl}|g\rangle+\langle g|\Pi_{kl}|a_{00}\rangle-\langle a_{00}|\Pi_{kl}|g\rangle-\langle a_{00}|\Pi_{kl}|a_{00}\rangle$$
$$=0,$$

for any subspace W_{kl}. This proves (16).

To prove (17), we compute

$$\langle +|\Pi_{kl}|+\rangle=\frac{1}{3}[\langle g|\Pi_{kl}|g\rangle-\langle g|\Pi_{kl}|a_{00}\rangle-\langle a_{00}|\Pi_{kl}|g\rangle$$
$$+\langle a_{00}|\Pi_{kl}|a_{00}\rangle]$$
$$=\frac{2\dim(\mathsf{W}_{kl})}{3N},$$

and similarly for (18).

Using the property that $|+\rangle$ and $|-\rangle$ are normalized, this implies that

$$\|\Pi_{00}|+\rangle\|^2=1-\sum_{kl\neq 00}\frac{2\dim(\mathsf{W}_{kl})}{3N}=\frac{2(N+2)}{3N},$$

which proves (19). We can similarly compute that $\|\Pi_{00}|-\rangle\|^2=\frac{4}{N}$, proving (20). ■

APPENDIX C: SUMS

In this section, we prove asymptotic bounds on a set of sums over the spectra of W. We assume a square lattice, with $n_r = n_c = \sqrt{N}$.

Fact 19. Suppose $0 < \alpha < \theta_{kl}$ for all k, l, and consider the sum

$$\sum_{kl\neq 00}\dim(\mathsf{W}_{kl})\cot\left(\frac{\theta_{kl}-\alpha}{2}\right).$$

If $\alpha \in \Theta(\frac{1}{\sqrt{N}})$, then the sum has order $\Omega(\sqrt{N}\log N)$.

If $\alpha \in o(\frac{1}{\sqrt{N}})$, then the sum has order $\Theta(\alpha N\log N)$.

Proof. Instead of taking the sum over the subspaces W_{kl}, which partition the domain of W, we convert to a sum over k and l. Recall that each pair $0 \leqslant k, l \leqslant \sqrt{N}/2-1$ corresponds to two eigenvectors of W: $|w_{kl}^{00}\rangle$ with eigenphase θ_{kl} and $|w_{kl}^{11}\rangle$ with eigenphase $-\theta_{kl}$. Using this property, we rewrite the sum as

$$\sum_{kl\neq 00}\dim(\mathsf{W}_{kl})\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)$$
$$=\sum_{k=0}^{\frac{\sqrt{N}}{2}-1}\sum_{\substack{l=0\\ \text{not both }0}}^{\frac{\sqrt{N}}{2}-1}\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)-\cot\left(\frac{\theta_{kl}+\alpha}{2}\right)$$
$$=2\sum_{k=0}^{\frac{\sqrt{N}}{2}-1}\sum_{\substack{l=0\\ \text{not both }0}}^{\frac{\sqrt{N}}{2}-1}\frac{\cot\left(\frac{\alpha}{2}\right)\left(\cot^2\left(\frac{\theta_{kl}}{2}\right)+1\right)}{\cot^2\left(\frac{\alpha}{2}\right)-\cot^2\left(\frac{\theta_{kl}}{2}\right)},\tag{C1}$$

where the final equality follows from angle sum identities.

Next, observe that by the definition of θ_{kl},

$$\cot^2\left(\frac{\theta_{kl}}{2}\right)+1=\frac{2}{1-\cos\theta_{kl}}=\frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}}.$$

We therefore consider the sum

$$\sum_{k=0}^{\frac{\sqrt{N}}{2}-1}\sum_{\substack{l=0\\ \text{not both }0}}^{\frac{\sqrt{N}}{2}-1}\frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}}.$$

For the terms where $l = 0$, we get

$$\sum_{k=1}^{\frac{\sqrt{N}}{2}-1}\frac{1}{1-\cos^2\tilde{k}}\in\Theta(N),$$

and similarly for $k = 0$. The remaining terms satisfy

$$\sum_{k=1}^{\frac{\sqrt{N}}{2}-1}\sum_{l=1}^{\frac{\sqrt{N}}{2}-1}\frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}}\in\Theta(N\log N).$$

Now, consider (C1) in the case where $\alpha \in \Theta(\frac{1}{\sqrt{N}})$. We have

$$2\sum_{k=0}^{\frac{\sqrt{N}}{2}-1}\sum_{\substack{l=0\\ \text{not both }0}}^{\frac{\sqrt{N}}{2}-1}\frac{\cot\left(\frac{\alpha}{2}\right)\left(\cot^2\left(\frac{\theta_{kl}}{2}\right)+1\right)}{\cot^2\left(\frac{\alpha}{2}\right)-\cot^2\left(\frac{\theta_{kl}}{2}\right)}$$
$$\geqslant 2\sum_{k=0}^{\frac{\sqrt{N}}{2}-1}\sum_{\substack{l=0\\ \text{not both }0}}^{\frac{\sqrt{N}}{2}-1}\frac{\cot\left(\frac{\alpha}{2}\right)\left(\cot^2\left(\frac{\theta_{kl}}{2}\right)+1\right)}{\cot^2\left(\frac{\alpha}{2}\right)}$$
$$=\frac{2}{\cot\left(\frac{\alpha}{2}\right)}\sum_{k=0}^{\frac{\sqrt{N}}{2}-1}\sum_{\substack{l=0\\ \text{not both }0}}^{\frac{\sqrt{N}}{2}-1}\frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}}.$$

Therefore,

$$\sum_{kl\neq 00}\dim(\mathsf{W}_{kl})\cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\in\Omega(\sqrt{N}\log N).$$

In the case where $\alpha \in o(\frac{1}{\sqrt{N}})$, the denominator in (C1) is dominated by the term $\cot^2(\frac{\alpha}{2})$. Therefore, the expression has the same asymptotic order as

$$\frac{1}{\cot\left(\frac{\alpha}{2}\right)} \sum_{k=0}^{\frac{\sqrt{N}}{2}-1} \sum_{\substack{l=0 \\ \text{not both } 0}}^{\frac{\sqrt{N}}{2}-1} \frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}} \in \Theta(\alpha N \log N),$$

proving the second clause. ∎

Fact 20. Suppose $0 < \alpha < \theta_{kl}$ for all k, l, and that $\alpha \in o(\frac{1}{\sqrt{N}})$. Then

$$\sum_{kl\neq 00} \dim(\mathsf{W}_{kl})\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right) - \cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2 \in \Theta(\alpha^2 N^2).$$

Proof.

$$\sum_{kl\neq 00} \dim(\mathsf{W}_{kl})\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right) - \cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2$$

$$= \sum_{k=0}^{\frac{\sqrt{N}}{2}-1} \sum_{\substack{l=0 \\ \text{not both } 0}}^{\frac{\sqrt{N}}{2}-1} \left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right) - \cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2.$$

By a similar derivation as in Fact (19), this sum has the same order as

$$\frac{1}{\cot^2\left(\frac{\alpha}{2}\right)} \sum_{k=0}^{\frac{\sqrt{N}}{2}-1} \sum_{\substack{l=0 \\ \text{not both } 0}}^{\frac{\sqrt{N}}{2}-1} \left(\frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}}\right)^2.$$

Using the Taylor expansion of cosine, for $l = 0$ we get

$$\sum_{k=0}^{\frac{\sqrt{N}}{2}-1} \left(\frac{1}{1-\cos^2\tilde{k}}\right)^2 \in \Theta(N^2),$$

and similarly for $k = 0$.

Finally,

$$\sum_{k=1}^{\frac{\sqrt{N}}{2}-1} \sum_{l=1}^{\frac{\sqrt{N}}{2}-1} \left(\frac{1}{1-\cos^2\tilde{k}\cos^2\tilde{l}}\right)^2 \in \Theta(N^2).$$

Therefore,

$$\sum_{kl\neq 00} \dim(\mathsf{W}_{kl})\left[\cot\left(\frac{\theta_{kl}+\alpha}{2}\right) - \cot\left(\frac{\theta_{kl}-\alpha}{2}\right)\right]^2 \in \Theta(\alpha^2 N^2).$$

∎

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