Process-optimized phase-covariant quantum cloning

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After the appearance of the no-cloning theorem, approximate quantum cloning machines (QCMs) has become a well-studied subject in quantum information theory. Among several measures to quantify the performance of a QCM, single-qudit fidelity and global fidelity have been most widely used. In this paper we compute the optimal global fidelity for phase-covariant cloning machines via semidefinite programming optimization, thereby completing a remaining gap in the previous results on QCMs. We also consider optimal simulations of the cloning and transpose cloning map, both by a direct optimization and by a composition of component-wise optimal QCMs. For the cloning map the composition method is suboptimal whereas for the transpose cloning map the method is asymptotically optimal.

I. INTRODUCTION

The famous no-cloning theorem tells us that there exists no quantum operation that copies an arbitrary pure state [1–3]. Despite this impossibility, the notion of approximate quantum cloning machines was introduced in 1996 [4], and the topic has received extensive development over the past twenty-five years. The inability to universally copy states is a fundamental signature of "quantumness" that has significant applications. In particular, the no-cloning theorem is essential for ensuring security in cryptographic tasks like quantum key distribution (QKD) [5] and quantum money [6,7].

A naturally arising question is whether it is possible to clone a state that is promised to be drawn from a smaller subset of pure states. For example, a family of states called *phase-covariant* states (also called *equatorial* states) is of particular interests as certain subsets of this family are used in quantum cryptography [5], phase estimation [8], and quantum clocks [9,10]. In fact, the problem of approximate phase-covariant cloning is closely connected to that of phase estimation both in terms of their mathematical structures and also their physical implementations [11]. The cloning of phase-covariant states has enjoyed a rich history of development [11–14]. Our objective here is to shed some new light on the problem by utilizing the technique of semidefinite programming to derive a closed-form expression for an optimal phase-covariant qudit cloner, which has not been explicitly documented in previous literature.

An ideal $N \to M$ quantum cloner is a nonlinear map

$$\rho^{\otimes N} \mapsto \rho^{\otimes M}$$

for any density matrix ρ of some d-dimensional system S, and a quantum cloning machine (QCM) approximates this operation via a physically realizable quantum channel. That is, an $N \to M$ QCM is a completely positive trace-preserving (CPTP) map $\mathcal{E}$ satisfying

$$\mathcal{E}(\rho^{\otimes N}) = \sigma^{\mathsf{S}_1 \cdots \mathsf{S}_M} \approx \rho^{\otimes M}.$$

Different criteria are used to categorize QCMs. For instance, the QCM is called symmetric if $\sigma^{\mathsf{S}_i} := \mathrm{Tr}_{\overline{\mathsf{S}}_i} \sigma$ is the same reduced state for all S_i, where $\mathrm{Tr}_{\overline{\mathsf{S}}_i}$ indicates a partial trace over all subsystems but S_i. An economic QCM is one that restricts to maps using no ancillary system other than the S_i output systems [11]. Finally, universality of a QCM means that it clones all the input states equally well, and a QCM is optimal if the cloning process is simulated optimally with respect to a given figure of merit [15].

The most commonly used figure of merit for cloning is the average single-qudit fidelity, which measures the distance between the input state ρ and the reduced state of each output copy σ^{S_i}. Another possible measure evaluates the average global fidelity between the QCM output state $\sigma^{\mathsf{S}_1 \cdots \mathsf{S}_M}$ and the ideal output state $\rho^{\otimes M}$. In Ref. [16], this global fidelity was called *process fidelity*, and the authors used the semidefinite programming (SDP) techniques introduced in Ref. [17] to optimize the process fidelity for cloning one qubit ($d = 2$). The goal of this paper is to extend this result by computing the optimal process fidelity for cloning qudits ($d > 2$). In particular, we consider the cloning of maximally coherent pure states having the form

$$|\theta\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} e^{i\theta_k} |k\rangle, \tag{1}$$

where $\boldsymbol{\theta}$ denotes a d tuple of angles $(\theta_1, \ldots, \theta_d)$, $\theta_i \in [0, 2\pi)$. For $d = 2$, this represents the family of equatorial states on the Bloch sphere. Up to a unitary transformation, the four BB84 states $\{|0\rangle, |1\rangle, |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}$ belong to this family [5].

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TABLE I. Optimal single-qudit fidelity and process fidelity of symmetric UQCM and phase-covariant QCM.

Dimension	Type	Optimal single-qubit fidelity	Optimal process fidelity
$1 \rightarrow 2$ qubits	Universal	$\frac{5}{6}$ [4,23–25]	$\frac{2}{3}$ [4]
$1 \rightarrow 2$ qudits	Universal	$\frac{d+3}{(2d+2)}$ [26]	$2/(d+1)$ [27]
$1 \rightarrow 2$ qubits	Phase covariant	$\frac{1}{2} + \frac{1}{\sqrt{8}}$ [18,28–30]	0.75 [16]
$1 \rightarrow 2$ qutrits	Phase covariant	$\frac{5+\sqrt{17}}{12}$ [12,13,20,31]	5/9 (this paper)
$1 \rightarrow 2$ qudits	Phase covariant	$\frac{1}{d} + \frac{d-2+\sqrt{d^2+4d-4}}{4d}$ [14]	$(2d-1)/d^2$ (this paper)

A (nonuniversal) QCM whose domain is restricted to the phase-encoded input states $|\boldsymbol{\theta}\rangle$ is called a phase-covariant quantum cloner. Bruß *et al.* first studied the cloning of BB84 states [18], and they showed that the optimal map that clones BB84 states also optimally clones any arbitrary phase-covariant states on the Bloch sphere. They obtained an optimal single-qubit fidelity of $\frac{1}{2} + \frac{1}{\sqrt{8}}$, which is greater than that of the universal cloner ($\frac{5}{6}$). Subsequent work has been conducted on the phase-covariant cloning of $1 \rightarrow M$ qubits [19], $1 \rightarrow 2$ qutrits [12,13,20], $1 \rightarrow M$ qudits [11,14], $N \rightarrow M$ qubits [21], $N \rightarrow M$ qudits for $M = kd + N$ [11], optimizing over the single qudit fidelity (see Ref. [22] for more).

In this paper, we present an explicit formula for an optimal $1 \rightarrow 2$ phase-covariant QCM, using the process fidelity as the figure of measure. This completes the picture of $1 \rightarrow 2$ optimal QCMs, as summarized in Table I. The organization of this paper as follows: In Sec. II we specify notation and definitions related to the process fidelity. In Sec. III we present our main results on optimal process fidelity of phase-covariant cloning. We introduce transpose cloning in Sec. IV and provide the optimal single-qudit and process fidelities. This allows us to consider in Sec. IV E the modular construction of larger QCMs by composing smaller ones. Finally, some concluding remarks are provided in Sec. V.

II. NOTATIONS AND DEFINITIONS

Let $B(\mathcal{H})$ denote the set of bounded operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{D} \subset B(\mathcal{H})$ be the set of density operators, i.e., positive trace-one elements, $\mathcal{D}_{\text{pure}} \subset \mathcal{D}$ the set of pure states, i.e., rank-one density operators, and $\mathcal{D}_\theta \subset \mathcal{D}_{\text{pure}}$ the set of states $|\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}|$ having the form of (1). The fidelity [32] of any two $\rho, \sigma \in \mathcal{D}$ is given by

$$F(\rho, \sigma) := (\text{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2.$$

Suppose $\mathcal{E}_{\text{ideal}} : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ is an "ideal" map that is not necessarily a quantum channel (not even necessarily linear), i.e, completely positive and trace-preserving (CPTP). We are interested in approximating $\mathcal{E}_{\text{ideal}}$ by a physically realizable quantum channel $\mathcal{E}$, or possibly approximating just the action of $\mathcal{E}_{\text{ideal}}$ on some restricted set of inputs S. There are various approaches to quantifying how well $\mathcal{E}$ approximates $\mathcal{E}_{\text{ideal}}$, and here we consider the process fidelity.

Definition 1. Let $S \subset \mathcal{D}$ be the set of states that are generated by the action of a compact group G, and let μ be the Haar measure induced by G. The *process fidelity* between an

arbitrary ideal map $\mathcal{E}_{\text{ideal}}$ and a CPTP map $\mathcal{E}$ is defined as

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|S) := \int_S F(\mathcal{E}_{\text{ideal}}(\sigma), \mathcal{E}(\sigma))d\mu(\sigma).$$

If $\mathcal{E}_{\text{ideal}}$ maps pure state to pure states and $S \subset \mathcal{D}_{\text{pure}}$, then the process fidelity is equivalent to

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|S) = \int_S \text{Tr}[\mathcal{E}_{\text{ideal}}(\sigma)\mathcal{E}(\sigma)]d\mu(\sigma). \quad (2)$$

A key feature of this expression is that the integrand is linear in $\mathcal{E}(\sigma)$. This will allow us to exploit symmetry properties of the Haar measure below. On the other hand, note that $\mathcal{E}_{\text{ideal}}$ need not be linear, such as with the cloning map $\rho \mapsto \rho \otimes \rho$.

Definition 2. Let $\text{CPTP}(\mathcal{H} \rightarrow \mathcal{K})$ be the set of all quantum channels mapping $B(\mathcal{H})$ to $B(\mathcal{K})$. The *optimal process fidelity* of a map $\mathcal{E}_{\text{ideal}} : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ is the maximum process fidelity over all possible quantum channels, i.e.,

$$F^*_{\text{proc}}(\mathcal{E}_{\text{ideal}}|S) := \max_{\mathcal{E} \in \text{CPTP}(\mathcal{H} \rightarrow \mathcal{K})} F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|S).$$

Given an orthonormal basis $\{|i\rangle\}_{i=1}^d$ for $\mathcal{H}$, the Choi-Jamiołkowski isomorphism [33,34] establishes an equivalence between every channel $\mathcal{E} \in \text{CPTP}(\mathcal{H} \rightarrow \mathcal{K})$ and an operator $J(\mathcal{E}) \in B(\mathcal{H}) \otimes B(\mathcal{K})$,

$$\mathcal{E} \leftrightarrow J(\mathcal{E}) := \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|). \quad (3)$$

The operator $J(\mathcal{E})$ is called the Choi matrix of $\mathcal{E}$ and the action of $\mathcal{E}$ can be directly expressed in terms of $J(\mathcal{E})$ as

$$\mathcal{E}(\rho) = \text{Tr}_{\mathcal{H}}(J(\mathcal{E})(\rho^T \otimes \mathbb{1}_{\mathcal{K}})),$$

where ρ^T is the transpose of ρ in some fixed basis. When $S \subset \mathcal{D}_{\text{pure}}$ and $\mathcal{E}_{\text{ideal}} : \mathcal{D}_{\text{pure}} \rightarrow \mathcal{D}_{\text{pure}}$, the process fidelity can be expressed in terms of the Choi matrix as

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|S) = \int_S \text{Tr}[J(\mathcal{E})\rho^T \otimes \mathcal{E}_{\text{ideal}}(\rho)]d\mu(\rho). \quad (4)$$

To compute $F^*_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|S)$, the problem then reduces to finding the Choi matrix $J(\mathcal{E})$ that maximizes Eq. (4).

III. PROCESS-OPTIMIZED PHASE-COVARIANT QUANTUM CLONING

1. Characterization of an average Choi map

The $1 \rightarrow 2$ ideal phase-covariant cloner $\mathcal{E}_{\text{ideal}}$ is given by (5)

$$\mathcal{E}_{\text{ideal}}(|\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}|) = |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}|^{\otimes 2} \quad (5)$$

for an arbitrary $|\theta\rangle\langle\theta| \in \mathcal{D}_\theta$. The goal is to find an optimal quantum channel $\mathcal{E}$ satisfying

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|\mathcal{D}_\theta) = F_{\text{proc}}^*(\mathcal{E}_{\text{ideal}}|\mathcal{D}_\theta).$$

It is useful to write $|\theta\rangle$ as $U(\theta)|\varphi_d^+\rangle$, where $U(\theta) = \sum_{k=1}^d e^{i\theta_k}|k\rangle\langle k|$ and $|\varphi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^d |k\rangle$. By the definition expressed in (4), the process fidelity for $\mathcal{E}$ can be written as

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|\mathcal{D}_\theta) = \text{Tr}[|\varphi_d^+\rangle\langle\varphi_d^+|^{\otimes 3}\mathcal{T}(J(\mathcal{E}))], \qquad (6)$$

where $\mathcal{T}(\cdot)$ denotes the "twirling" operation

$$\mathcal{T}(X) := \int U(\theta) \otimes U(-\theta)^{\otimes 2}(X)U(-\theta) \otimes U(\theta)^{\otimes 2}d\mu(\theta).$$

We exploit the symmetric properties of $\mathcal{T}(J(\mathcal{E}))$ to formulate the constraints of a semidefinite program to obtain maximum process fidelity. This is a standard trick [35], but we include it as lemmas for the sake of self-containment.

Let S_d be the permutation group of d elements. Given an orthonormal basis $\{|k\rangle\}_k$, define

$$U_\pi = \sum_{k=1}^d |\pi(k)\rangle\langle k|, \quad \pi \in S_d.$$

In other words, U_π permutes the basis vectors given a permutation $\pi \in S_d$. Next, let V_σ be the operator permuting three subsystems, where $\sigma \in S_3$. For example,

$$V_{(23)}|\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle = |\varphi_1\rangle \otimes |\varphi_3\rangle \otimes |\varphi_2\rangle.$$

Lemma 1. Suppose $\mathcal{E}$ is a quantum channel with Choi matrix $J(\mathcal{E})$. If we define the average Choi map for the channel $\mathcal{E}$ as

$$\widetilde{J}(\mathcal{E}) := \frac{1}{2|*|S_d} \sum_{\substack{\pi \in S_d \\ \sigma \in \{\text{id},(23)\}}} V_\sigma U_\pi^{\otimes 3}\mathcal{T}(J(\mathcal{E}))U_\pi^{\dagger\otimes 3}V_\sigma^\dagger, \qquad (7)$$

then we have

(1) $\widetilde{J}(\mathcal{E})$ is invariant under conjugation by $U(-\theta) \otimes U(\theta)^{\otimes 2}$ for any θ;

(2) $\widetilde{J}(\mathcal{E})$ is invariant under conjugation by $U_\pi^{\otimes 3}$ for all $\pi \in S_d$;

(3) $\widetilde{J}(\mathcal{E})$ is invariant under conjugation by $V_{(23)}$;

(4) $\widetilde{J}(\mathcal{E})$ is positive and $\text{Tr}_{23}(\widetilde{J}(\mathcal{E})) = \mathbb{1}$.

Note that property (iv) of Lemma 1 assures that $\widetilde{J}(\mathcal{E})$ is a valid Choi matrix of a quantum channel.

Lemma 2. Let $\mathcal{E}_{\text{ideal}}$ be the ideal phase-covariant cloning map and $\mathcal{E}$ an arbitrary quantum channel. Then

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|\mathcal{D}_\theta) = \text{Tr}[\varphi_d^{+\otimes 3}\widetilde{J}(\mathcal{E})]. \qquad (8)$$

Proof. Let $\pi' \in S_d$. We can observe that

$$\mathcal{T}(X) = U_{\pi'}^{\dagger\otimes 3}\mathcal{T}(U_{\pi'}^{\otimes 3}XU_{\pi'}^{\dagger\otimes 3})U_{\pi'}^{\otimes 3}.$$

Using (ii) of the Lemma 1, it follows that

$$\begin{aligned}
F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|\mathcal{D}_\theta) &= \text{Tr}[\varphi_d^{+\otimes 3}\mathcal{T}(J(\mathcal{E}))] \\
&= \text{Tr}[\varphi_d^{+\otimes 3}U_{\pi'}^{\dagger\otimes 3}\mathcal{T}(U_{\pi'}^{\otimes 3}J(\mathcal{E})U_{\pi'}^{\dagger\otimes 3})U_{\pi'}^{\otimes 3}] \\
&= \text{Tr}\left[\varphi_d^{+\otimes 3}\frac{1}{|*|S_d}\sum_{\pi \in S_d}U_\pi^{\otimes 3}\mathcal{T}(J(\mathcal{E}))U_\pi^{\dagger\otimes 3}\right] \\
&= \text{Tr}\left[\varphi_d^{+\otimes 3}\frac{1}{2|*|S_d}\sum_{\substack{\pi \in S_d \\ \sigma \in \{\text{id},(23)\}}}V_\sigma U_\pi^{\otimes 3}\mathcal{T}(J(\mathcal{E}))U_\pi^{\dagger\otimes 3}V_\sigma^\dagger\right] \\
&= \text{Tr}[\varphi_d^{+\otimes 3}\widetilde{J}(\mathcal{E})],
\end{aligned}$$

where we have used the facts that φ_d^+ is invariant under π' and $\varphi_d^{+\otimes 3}$ is invariant under $V_{(23)}$. ∎

In summary, we have

$$F_{\text{proc}}^*(\mathcal{E}_{\text{ideal}}|\mathcal{D}_\theta) = \max \text{Tr}[\varphi_d^{+\otimes 3}X], \qquad (9)$$

in which the maximization is taken over all operators $X \in B(\mathbb{C}^d)^{\otimes 3}$ satisfying properties (i)–(iv) in Lemma 1.

2. Optimization via semidefinite programming

Using the invariant properties of the average Choi matrix for quantum channel $\mathcal{E}$ in Lemma 2, we will construct a semidefinite program (SDP) to obtain an optimal $\mathcal{E}$ that maximizes the process fidelity. A particularly nice reference for applying semidefinite programming to quantum information problems is [36], and we apply the basic results here.

First, we characterize a $d^3 \times d^3$ Hermitian operator $X \neq 0$ that satisfies (i)–(iv) in Lemma 1. If we write

$$X = \sum_{i,j,k,l,m,n} x_{ijklmn}|ijk\rangle\langle lmn|,$$

then the invariance under $U(-\theta) \otimes U(\theta)^{\otimes 2}$ gives

$$x_{ijklmn}(1 - e^{i(-\theta_i+\theta_j+\theta_k+\theta_l-\theta_m-\theta_n)}) = 0. \qquad (10)$$

This can be satisfied if and only if $x_{ijklmn} = 0$ or $-\theta_i + \theta_j + \theta_k + \theta_l - \theta_m - \theta_n$ is identically zero. If we next add the invariance under under $V_{(23)}$ and $U_\pi^{\otimes 3}$, then X has the form

$\sum_{i=1}^{9} x_i X_i$, where $x_i \in \mathbb{C}$ and

$$X_1 = \sum_i |iii\rangle\langle iii|, \quad X_2 = \sum_{i \neq k} |iik\rangle\langle iik| + |iki\rangle\langle iki|,$$

$$X_3 = \sum_{i \neq k} |kii\rangle\langle kii|,$$

$$X_4 = \sum_{i \neq k} |kik\rangle\langle iii| + |iii\rangle\langle kik| + |kki\rangle\langle iii| + |iii\rangle\langle kki|,$$

$$X_5 = \sum_{i \neq k} |iik\rangle\langle iki| + |iki\rangle\langle iik|, \quad X_6 = \sum_{i \neq k \neq \ell} |ik\ell\rangle\langle ik\ell|,$$

$$X_7 = \sum_{i \neq k \neq \ell} |kk\ell\rangle\langle i\ell i| + |\ell k\ell\rangle\langle iik|,$$

$$X_8 = \sum_{i \neq k \neq \ell} |kk\ell\rangle\langle ii\ell| + |\ell k\ell\rangle\langle iki|, \quad X_9 = \sum_{i \neq k \neq \ell} |ik\ell\rangle\langle i\ell k|.$$

If we denote $\mathbf{x} = (x_i) \in \mathbb{C}^9$, then

$$\mathrm{Tr}\big[\varphi_d^{+\otimes 3} X\big] = \frac{1}{d^2} x_1 + \frac{(d-1)}{d^2}(2x_2 + x_3 + 4x_4 + 2x_5)$$
$$+ \frac{(d-1)(d-2)}{d^2}(x_6 + 2x_7 + 2x_8 + x_9).$$

The condition $\mathrm{Tr}_{23} X = \mathbb{1}$ corresponds to the equality

$$A(\mathbf{x}) := x_1 + (d-1)(2x_2 + x_3) + (d-1)(d-2)x_6 = 1.$$

Finally, the positivity constraint $X \geqslant 0$ allows us to express (9) in a simplified SDP. Let $\mathbf{a} = (a_i)$ where a_i is the coefficient of x_i's in $A(\mathbf{x})$, and $\mathbf{c} = (c_i)$ be the coefficient of x_i in $F_{\mathcal{E}}(\mathbf{x})$. Define $F_0 = 0_{d^3 \times d^3} \oplus [1]$, $F_i = X_i \oplus [-a_i]$. Then we have the primal form of the SDP:

$$\begin{aligned} \text{minimize} \quad & -\mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & F_0 + \sum_i x_i F_i \geqslant 0, \quad \mathbf{a}^T \mathbf{x} = 1. \end{aligned} \quad (11)$$

By Eq. (9), this SDP yields the value of $F_{\mathrm{proc}}^*(\mathcal{E}_{\mathrm{ideal}}|\mathcal{D}_\theta)$.

Now we present our main results.

Theorem 1 Let $\mathcal{E}_{\mathrm{ideal}}$ be the ideal $1 \to 2$ phase-covariant cloner. Then, the optimal process fidelity of $\mathcal{E}_{\mathrm{ideal}}$ is

$$F_{\mathrm{proc}}^*(\mathcal{E}_{\mathrm{ideal}}|\mathcal{D}_\theta) = \frac{2d-1}{d^2},$$

where d is the dimension of the input system.

Proof. Define

$$k_d = \frac{1}{2d-1},$$

$$\mathbf{x} = (k_d, k_d, 0, k_d, k_d, 0, k_d, k_d, 0),$$

where d is the dimension. It is straightforward to show that $X = \sum_i x_i X_i$ satisfies the positivity and the trace condition. So this $\mathbf{x}$ is a primal feasible solution and it yields $F_{\mathcal{E}}(\mathbf{x}) = \frac{2d-1}{d^2}$. Consider the dual form

$$\begin{aligned} \text{maximize} \quad & -\mathrm{Tr}F_0 Z \\ \text{subject to} \quad & \mathrm{Tr}F_i Z = -c_i, Z \geqslant 0. \end{aligned}$$

We construct Z such that $\mathrm{Tr}F_0 Z = (2d-1)/d^2$ and show that this Z is dual feasible. By strong duality, this implies that

$(2d-1)/d^2$ is indeed the optimal solution. Let $Z = \hat{Z} \oplus z$, where $\hat{Z} = \sum_i b_i X_i$, $b_i \in \mathbb{R}$ and $z = \frac{2d-1}{d^2}$. Then

$$\mathrm{Tr}F_0 Z = \frac{2d-1}{d^2},$$

and the constraints of the dual form become

$$\mathrm{Tr}[X_1 \hat{Z}] = z - \frac{1}{d^2},$$

$$\mathrm{Tr}[X_2 \hat{Z}] = 2\mathrm{Tr}[X_3 \hat{Z}] = 2(d-1)z - \frac{2(d-1)}{d^2},$$

$$\mathrm{Tr}[X_4 \hat{Z}] = 2\mathrm{Tr}[X_5 \hat{Z}] = -\frac{4(d-1)}{d^2},$$

$$\mathrm{Tr}[X_6 \hat{Z}] = (d-1)(d-2)z - \frac{(d-1)(d-2)}{d^2},$$

$$\mathrm{Tr}[X_7 \hat{Z}] = \mathrm{Tr}[X_8 \hat{Z}] = 2\mathrm{Tr}[X_9 \hat{Z}] = -\frac{2(d-1)(d-2)}{d^2}.$$

Solving these gives $b_1 = b_2 = b_3 = b_6 = \frac{2(d-1)}{d^3}$, $b_4 = b_5 = b_7 = b_8 = b_9 = -\frac{1}{d^3}$. We can rewrite $\hat{Z}$ as a linear combination of projections

$$\hat{Z} = \frac{1}{d^3}\Bigg[(2d - 2 - \sqrt{2})X_1 + \left(2d - \frac{9}{2} - \frac{3}{2\sqrt{2}}X_2\right)$$
$$+ (2d-2)X_3 + (2d-3)X_6 + 2\sqrt{2}P_A$$
$$+ 2\left(\frac{2}{\sqrt{2}} + 1\right)P_B + 2P_C + 2P_D + 2P_E \Bigg],$$

where we define

$$P_A = -\frac{1}{2\sqrt{2}}X_4 + \frac{1}{4}(X_1 + X_2) + \frac{1}{4}(X_1 + X_5),$$

$$P_B = \frac{1}{2}(X_2 - X_5), \quad P_C = \frac{1}{2}(X_2 - X_7),$$

$$P_D = \frac{1}{2}(X_2 - X_8), \quad P_E = \frac{1}{2}(X_6 - X_9).$$

When $d \geqslant 3$ all the coefficients are positive, and when $d = 2$ it can be easily checked that $\hat{Z} \geqslant 0$. This shows that Z is also positive, and therefore $\frac{2d-1}{d^2}$ is indeed the optimal solution. ∎

Note that one can calculate the single-qudit fidelity using a map $\mathcal{E}$ that is optimal for the process fidelity. Doing so leads to a single-qudit fidelity of $(d+1)/(2d-1)$, which is strictly smaller than the optimal value [14]. This means that the map yielding optimal process fidelity does not give optimal single-qudit fidelity unlike the universal cloners [27].

IV. TRANSPOSITION CLONING

A. Optimal phase-covariant transposition

The transposition map is another famous non-CP map. The ideal operation of transposition map is given by

$$\mathcal{E}_{\mathrm{ideal}} : |\psi\rangle\langle\psi| \mapsto |\psi\rangle\langle\psi|^T \ \forall \rho \in \mathcal{D}_{\mathrm{pure}}, \quad (12)$$

It has been shown that the optimal fidelity for approximate transposition channel of arbitrary pure states is $\frac{2}{d+1}$ [37]. What happens if we restrict the input states to phase-covariant states? In this case, Eq. (12) takes the form

$$\mathcal{E}_{\mathrm{ideal}} : |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \mapsto |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}|^T = |-\boldsymbol{\theta}\rangle\langle-\boldsymbol{\theta}| \ \forall \rho \in \mathcal{D}_\theta, \quad (13)$$

where $-\boldsymbol{\theta} = (-\theta_1, \ldots, -\theta_d)$. To compute the optimal process fidelity in this restricted case, we can use the same SDP analysis. The calculation yields to $F_{\text{proc}} = \frac{2}{d}$, and the Choi operator of an optimal map is given as

$$\widetilde{J}(\mathcal{E}) = \frac{1}{d-1} \sum_{i \neq j} (|ij\rangle\langle ij| + |ij\rangle\langle ji|).$$

Hence we see that restricting to maximally coherent states $|\boldsymbol{\theta}\rangle$ only yields an improved process fidelity of $\frac{2}{d}$ versus $\frac{2}{d+1}$ in the universal case. These have the same asymptotic scaling, yet in the qubit case perfect transposition can be achieved only for equatorial states since the map reduces to a π rotation on the Bloch sphere.

B. Process-optimized universal transposition cloning map

We can also define the ideal transposition cloning operation as the map that outputs two copies of the transpose of the input state, i.e.,

$$\mathcal{E}_{\text{ideal}} : \rho \mapsto \rho^T \otimes \rho^T \ \forall \ \rho \in \mathcal{D}_{\text{pure}}.$$

Here we compute the process fidelity of the $1 \to 2$ transposition cloning map for an arbitrary pure state as its input. Let μ be the induced Haar measure for the pure states $\mathcal{D}_{\text{pure}}$ and σ be the Haar measure on the set of unitary operators $\mathcal{U}(\mathcal{H})$. Define the average Choi operator as

$$\widetilde{J}(\mathcal{E}) := \int_{\mathcal{U}(\mathcal{H})} U^{\dagger \otimes 3} J(\mathcal{E}) U^{\otimes 3} d\sigma(U). \tag{14}$$

Notice that by the Haar invariance we get

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|\mathcal{D}_{\text{pure}})$$
$$= \int_{\mathcal{D}_{\text{pure}}} \text{Tr}[J(\mathcal{E})(\rho^T)^{\otimes 3}] d\mu(\rho)$$
$$= \text{Tr}\left[\int_{\mathcal{U}(\mathcal{H})} J(\mathcal{E})(U|0\rangle\langle 0|U^\dagger)^{\otimes 3} d\sigma(U) \right]$$
$$= \text{Tr}\left[\int_{\mathcal{U}(\mathcal{H})} U^{\dagger \otimes 3} J(\mathcal{E}) U^{\otimes 3} d\sigma(U)|0\rangle\langle 0|^{\otimes 3} \right]$$
$$= \text{Tr}[\widetilde{J}(\mathcal{E})|0\rangle\langle 0|^{\otimes 3}].$$

To find the optimal process fidelity, we exploit the fact that $\widetilde{J}(\mathcal{E})$ is $U^{\otimes 3}$-invariant. We then obtain the following proposition as a direct consequence of the results from work by Eggeling and Werner [38]:

Proposition 1. The optimal process fidelity of the $1 \to 2$ universal transpose cloning map is $6/(d^2 + 3d + 2)$.

Proof. Eggeling and Werner have shown that any quantum state ρ satisfying $U^{\otimes 3}\rho U^{\dagger \otimes 3} = \rho$ can be uniquely expressed as

$$\widetilde{J}(\mathcal{E}) = \sum_{i \in \{+,-,0,1,2,3\}} c_i R_i, \quad c_i \in \mathbb{R},$$

where

$$R_+ = \frac{1}{6}(\mathbb{1} + V_{(12)} + V_{(23)} + V_{(31)} + V_{(123)} + V_{(132)}),$$

$$R_- = \frac{1}{6}(\mathbb{1} - V_{(12)} - V_{(23)} - V_{(31)} + V_{(123)} + V_{(132)}),$$

$$R_0 = \frac{1}{3}(2 \cdot \mathbb{1} - V_{(123)} - V_{(321)}),$$

$$R_1 = \frac{1}{3}(2 \cdot \mathbb{1} - V_{(31)} - V_{(12)}),$$

$$R_2 = \frac{1}{\sqrt{3}}(V_{(12)} - V_{(31)}), \quad R_3 = \frac{i}{\sqrt{3}}(V_{(123)} - V_{(321)}).$$

Furthermore, it satisfies the conditions (i) $c_+, c_-, c_0 \geqslant 0$, (ii) $c_1^2 + c_2^2 + c_3^2 \leqslant c_0^2$, and (iii) $\text{Tr}[\rho R_+] + \text{Tr}[\rho R_-] + \text{Tr}[\rho R_-] = 1$. Let $\widetilde{J}(\mathcal{E})$ be the average Choi operator defined as (14). Then $\widetilde{J}(\mathcal{E})$ is invariant under $U^{\otimes 3}$ action, and the preceding statement holds except replacing (iii) with

$$\text{Tr}_{23}[R_+\widetilde{J}(\mathcal{E}) + R_-\widetilde{J}(\mathcal{E}) + R_0\widetilde{J}(\mathcal{E})] = \mathbb{1}.$$

Suppose $\widetilde{J}(\mathcal{E}) = \sum c_k R_k$. Then the objective function to maximize becomes

$$\text{Tr}[\widetilde{J}(\mathcal{E})|0\rangle\langle 0|^{\otimes 3}] = c_+,$$

while the constraints $\widetilde{J}(\mathcal{E}) \geqslant 0$ is equivalent to $c_+, c_-, c_0 \geqslant 0$ and $c_1^2 + c_2^2 + c_3^2 \leqslant c_0^2$ and the trace condition $\text{Tr}\widetilde{J}(\mathcal{E}) = d$ is equivalent to

$$\frac{d^2 + 3d + 2}{6}c_+ + \frac{d^2 - 3d + 2}{6}c_- + \frac{2(d^2-1)}{3}c_0 = 1.$$

Solving this linear programming gives

$$c_- = c_0 = c_1 = c_2 = c_3 = 0, \quad c_+ = \frac{6}{d^2 + 3d + 2}. \qquad \blacksquare$$

Eggeling and Werner use Schur-Weyl duality to characterize $U^{\otimes 3}$-invariant positive matrices, and details of the proof are in Refs. [39] and [38].

C. Process-optimized phase-covariant transposition cloning map

We next compare the previous result with the same type of cloner except with the domain further restricted to $\mathcal{D}_\theta$. The ideal map for phase-covariant transposition cloner is given by

$$\mathcal{E}_{\text{ideal}} : |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \mapsto |-\boldsymbol{\theta}\rangle\langle-\boldsymbol{\theta}|^{\otimes 2} \ \forall \ |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \in \mathcal{D}_\theta. \tag{15}$$

Different from the cloning case, the average Choi operator is defined as

$$\widetilde{J}(\mathcal{E}) := \frac{1}{|*|S_d|*|S_3} \sum_{\pi \in S_d} \sum_{\sigma \in S_3} V_\sigma U_\pi^{\otimes 3} \mathcal{T}(J(\mathcal{E})) U_\pi^{\dagger \otimes 3} V_\sigma^\dagger,$$

where the twirling operation is given by

$$\mathcal{T}(X) := \int U(-\boldsymbol{\theta})^{\otimes 3}(X) U(\boldsymbol{\theta})^{\otimes 3} d\mu(\boldsymbol{\theta}).$$

The positivity and trace condition of $\widetilde{J}$ remains the same and its invariance under the permutations of the basis vector also remains the same. The main difference is that $\widetilde{J}$ is invariant under $U(\boldsymbol{\theta})^{\otimes 3}$ instead of $U(-\boldsymbol{\theta}) \otimes U(\boldsymbol{\theta})^{\otimes 2}$, and it is also invariant under permuting any subsystems instead of just 2 and 3. Despite the difference, how we solve the optimal process fidelity is analogous to that of the cloning case. Here we just state the result as a theorem and include the proof in the Appendix for interested readers.
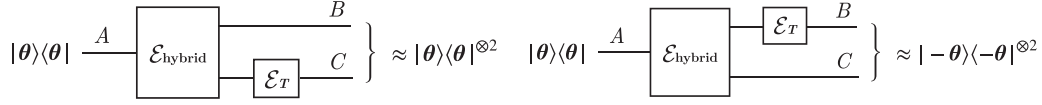
FIG. 1. Modular cloning channel and modular transpose cloning channel.

Theorem 2. Let $\mathcal{E}_{\text{ideal}}$ be given by Eq. (15). Then, the optimal process fidelity of $\mathcal{E}_{\text{ideal}}$ is

$$F^*_{\text{proc}}(\mathcal{E}_{\text{ideal}}|\mathcal{D}_\theta) = \begin{cases} 3/4, & d = 2 \\ 6/d^2, & d \geqslant 3. \end{cases}$$

D. Phase-covariant hybrid transposition cloning

Now consider a hybrid transposition cloning map

$$\mathcal{E}_{\text{ideal}} : |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \mapsto |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \otimes |-\boldsymbol{\theta}\rangle\langle-\boldsymbol{\theta}| \tag{16}$$

for any $|\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \in \mathcal{D}_\theta$. It is easy to see that the symmetries of the process fidelity between the optimal map and the above ideal map are identical to that of the cloning case, except we permute the system 1 and 3. In other words, we can reuse the constraints (i)–(iii) in Lemma 1 to construct the average Choi map for the hybrid cloning and make a modification for (iv) to be $\text{Tr}_{12}(\widetilde{J}(\mathcal{E}_{\text{ideal}})) = \mathbb{1}$, instead of taking the partial trace over systems 2 and 3. In fact, $U(-\boldsymbol{\theta}) \otimes U(\boldsymbol{\theta})^{\otimes 2}$ invariance guarantees the partial trace over any two system of $\widetilde{J}(\mathcal{E})$ to be a scalar multiple of $\mathbb{1}$. Hence, we can conclude that the process fidelity of the hybrid map is $(2d-1)/d^2$. Furthermore, if we let $\mathcal{E}_{\text{hybrid}}$ denote an optimal approximation of the hybrid transposition cloner (phase covariant) and $\mathcal{E}_{\text{cloner}}$ an optimal approximation of the $1 \to 2$ cloner (phase covariant), then

$$\widetilde{J}(\mathcal{E}_{\text{hybrid}}) = V_{(13)}\widetilde{J}(\mathcal{E}_{\text{clone}})V_{(13)}.$$

It is interesting that the two maps

$$|\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \mapsto |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \otimes |-\boldsymbol{\theta}\rangle\langle-\boldsymbol{\theta}|,$$
$$|\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \mapsto |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}| \otimes |\boldsymbol{\theta}\rangle\langle\boldsymbol{\theta}|$$

can be approximated with the same process fidelity. One might not expect this since the second can be obtained from the first by applying a transpose on the first system, and as computed above, the transposition itself has a process fidelity of $2/d$. Hence a composition of maps yields a highly nonoptimal approximation of the $1 \to 2$ phase-covariant cloner. We make this comparison more explicitly in the next section.

E. Modular building of quantum cloning machine

In this section we consider a modular approach of simulating some nonphysical process by breaking the latter down into smaller parts and then simulating each of those. We have just observed that a $1 \to 2$ phase-covariant cloner can be obtained by combining the hybrid cloner [Eq. (16)] with the transpose map [Eq. (13)]. The construction is depicted in Fig. 1, and the question we consider here is how well the combined optimal maps for the individual parts, $\mathcal{E}_{\text{hybrid}}$ and $\mathcal{E}_T$, respectively, compare with the optimal map computed in Theorem 1. The composition of $\mathcal{E}_{\text{hybrid}} : A \to B\widetilde{C}$ and $\mathcal{E}_{T_C} : \widetilde{C} \to C$ can be expressed in terms of their Choi

matrices as

$$(\mathcal{E}_{T_C} \circ \mathcal{E}_{\text{hybrid}})(\rho) = \text{Tr}_{A\widetilde{C}}[(J(\mathcal{E}_{\text{hybrid}})((\rho^T)^A \otimes \mathbb{1}^{B\widetilde{C}}))^{T_{\widetilde{C}}} \\ \times \otimes \mathbb{1}^C (J(\mathcal{E}_T)^{C\widetilde{C}} \otimes \mathbb{1}^{AB})],$$

where A, B, C denote the quantum systems indicated in Fig. 1 and the subscript $T_{\widetilde{C}}$ denotes the partial transpose of system $\widetilde{C}$.

Similarly, the transposition cloning of Eq. (15) can be obtained by combining the hybrid cloner $\mathcal{E}_{\text{hybrid}} : A \to \widetilde{B}C$ [Eq. (16)] with the transpose map $\mathcal{E}_{T_B} : \widetilde{B} \to B$ [Eq. (13)]. The construction is depicted in Fig. 1, and we compare with the optimal map computed in Theorem 2. The composition of maps has the form

$$(\mathcal{E}_{T_B} \circ \mathcal{E}_{\text{hybrid}})(\rho) = \text{Tr}_{A\widetilde{B}}[(J(\mathcal{E}_{\text{hybrid}})((\rho^T)^A \otimes \mathbb{1}^{\widetilde{B}C}))^{T_{\widetilde{B}}} \\ \otimes \mathbb{1}^B (J(\mathcal{E}_T)^{B\widetilde{B}} \otimes \mathbb{1}^{AC})],$$

where A, B, C denote the quantum systems indicated in Fig. 1 and the subscript $T_{\widetilde{B}}$ denotes the partial transpose of system $\widetilde{B}$.

The process fidelity of both maps are $\frac{3d-4}{d(d-1)(2d-1)}$, which is factor of d smaller than the optimal cloner in Theorem 1. However, it is comparable to that of optimal transposition cloning channel with process fidelity of $6/d^2$. Hence in this case, the modular approach is asymptotically optimal (i.e., as $d \to \infty$) in the process fidelity.

V. CONCLUSION

We have shown that the optimal process fidelity of the $1 \to 2$ phase-covariant cloner is $\frac{2d-1}{d^2}$. The obtained channel yields its single-qudit fidelity of $(d+1)/(2d-1)$, which is less than the previously known result by Ref. [14]. Conversely, the cloner by Ref. [14] also does not yield the optimal process fidelity. This means that the optimal phase-covariant cloners do not coincide with two different fidelity measures, as it is the case for the universal cloner.

We also defined a transpose cloning map, and showed that the optimal process fidelity for transpose cloning of arbitrary pure states is $6/(d^2 + 3d + 2)$ using the results from Ref. [38]. In comparison, the optimal fidelity for transpose cloning of phase-covariant states is $6/d^2$.

Future work in this direction would include finding an explicit formula for $1 \to M$ phase-covariant cloners. This will require a representation-theoretic approach involving representations of S_M with additional positivity and phase-covariant structure. In general, the $1 \to M$ phase-covariant cloner is interesting for applications in phase estimation. We expect that the modular approach presented in Fig. 1 may also be helpful in simulating such a map and other multisystem maps like it.

APPENDIX: SOLVING THE PROCESS FIDELITY FOR $1 \rightarrow 2$ PHASE-COVARIANT TRANSPOSE CLONER

Let $\mathcal{E}_{\text{ideal}} : |-\theta\rangle\langle-\theta| \mapsto |\theta\rangle\langle\theta|^{\otimes 2}$ be the $1 \rightarrow 2$ ideal phase-covariant transpose cloning map.

Lemma 3. Let $\mathcal{E}$ be a quantum channel. Define the average Choi operator $\widetilde{J}$ for the channel $\mathcal{E}$ as

$$\widetilde{J}(\mathcal{E}) := \frac{1}{|*|S_d| * |S_3} \sum_{\substack{\pi \in S_d \\ \sigma \in S_3}} V_\sigma U_\pi^{\otimes 3} \mathcal{T}(J(\mathcal{E})) U_\pi^{\dagger \otimes 3} V_\sigma^\dagger,$$

where the twirling operation $\mathcal{T}$ is defined as

$$\mathcal{T}(X) := \int U(\boldsymbol{\theta})^{\otimes 3}(X) U(-\boldsymbol{\theta})^{\otimes 3} d\mu(\boldsymbol{\theta}).$$

then we have

(1) $\widetilde{J}(\mathcal{E})$ is invariant under conjugation by $U(\boldsymbol{\theta})^{\otimes 3}$ for any $\boldsymbol{\theta}$;

(2) $\widetilde{J}(\mathcal{E})$ is invariant under conjugation by $U_\pi^{\otimes 3}$ for all $\pi \in S_d$;

(3) $\widetilde{J}(\mathcal{E})$ is invariant under permuting subsystems;

(4) $\widetilde{J}(\mathcal{E})$ is positive and $\text{Tr}_2(\widetilde{J}(\mathcal{E})) = \mathbb{1}$.

Proof. Property (i) directly follows from

$$U(\boldsymbol{\theta})^{\otimes 3} \mathcal{T}(J(\mathcal{E})) U(-\boldsymbol{\theta})^{\otimes 3} = \mathcal{T}(J(\mathcal{E})).$$

Same as the cloning case, (ii) and (iii) also directly follow from the definition of $\widetilde{J}$ and (iv) follows from the fact that twirling operation, permutation of basis or the system do not change the positivity of $J(\mathcal{E})$. Trace is also invariant under $\mathcal{T}$ and under conjugation by $U_\pi^{\otimes 3}$ or V_σ. ∎

With the above characterization of $\widetilde{J}$, we can establish the following in an analogous manner to the cloning case.

Lemma 4. Suppose $\widetilde{J}$ is defined as (7) for a quantum channel $\mathcal{E}$. Then we have

$$F_{\text{proc}}(\mathcal{E}_{\text{ideal}}, \mathcal{E}|\mathcal{D}_\theta) = \text{Tr}\big[\varphi_d^{+\otimes 3} \widetilde{J}(\mathcal{E})\big].$$

We omit the proof since it is more or less identical to Lemma 2 for the cloning case.

Similar to the cloning case, we characterize a $d^3 \times d^3$ Hermitian operator $X \neq 0$ that satisfies (i)–(iv) in Lemma 3. As a result, we can express X as $\sum_{i=1}^6 x_i X_i$, where $x_i \in \mathbb{C}$ and

$$X_1 = \sum_i |iii\rangle\langle iii|, \quad X_2 = \sum_{i \neq k} |iik\rangle\langle iik| + |iki\rangle\langle iki| + |kii\rangle\langle kii|,$$

$$X_3 = \sum_{i \neq k} |iik\rangle\langle kii| + |kii\rangle\langle iik| + |kii\rangle\langle iki| + |iik\rangle\langle iki| + |iki\rangle\langle iik|,$$

$$X_4 = \sum_{i \neq k \neq \ell} |ik\ell\rangle\langle ik\ell|, \quad X_5 = \sum_{i \neq k \neq \ell} |ik\ell\rangle\langle k\ell i| + |ik\ell\rangle\langle \ell ik|,$$

$$X_6 = \sum_{i \neq k \neq \ell} |ik\ell\rangle\langle ki\ell| + |ik\ell\rangle\langle \ell ki| + |ik\ell\rangle\langle i\ell k|.$$

If we denote $\mathbf{x} = (x_i) \in \mathbb{C}^6$, then

$$\text{Tr}\big[\varphi_d^{+\otimes 3} X\big] = \frac{1}{d^2} x_1 + \frac{(d-1)}{d^2}(3x_2 + 6x_3) + \frac{(d-1)(d-2)}{d^2}(x_4 + 2x_5 + 3x_6),$$

and the trace condition

$$A(\mathbf{x}) = x_1 + 3(d-1)x_2 + (d-1)(d-2)x_4 = 1.$$

We construct the SDP the identical way as (11) to prove the optimality of the process fidelity. Now we present the proof of the Theorem 2.

Proof. When the dimension $d = 2$, $\mathbf{x} = (0, 1/3, 1/3, 0, 0, 0)$ yields a primal feasible solution of $3/4$, and $\hat{Z} \oplus [3/4]$ is dual feasible, where

$$\hat{Z} = \tfrac{1}{4}(X_1 + X_2 + X_4) - \tfrac{1}{8}(X_3 + X_5 + X_6).$$

Let $d \geqslant 3$ and define

$$k_d = \frac{1}{(d-1)(d-2)}, \quad \mathbf{x} = (0, 0, 0, k_d, k_d, k_d).$$

It can be easily be checked that the resulting operator satisfies the positivity and trace condition. Analogous to the cloning case, we prove the optimality of $F_{\mathcal{E}}$ by finding a dual feasible $Z = \hat{Z} \oplus [z]$, where $z = 6/d^2$ and $\hat{Z} = \sum_i b_i X_i$, $b_i \in \mathbb{R}$. The constraints $\text{Tr} F_i Z = c_i$ yields

$$\text{Tr}[X_1 \hat{Z}] - z = -\frac{1}{d^2},$$

$$\text{Tr}[X_2 \hat{Z}] - 3(d-1)z = -\frac{3(d-1)}{d^2},$$

$$\text{Tr}[X_3 \hat{Z}] = -\frac{6(d-1)}{d^2},$$

$$\text{Tr}[X_4\hat{Z}] - (d-1)(d-2)z = -\frac{(d-1)(d-2)}{d^2},$$

$$\text{Tr}[X_5\hat{Z}] = -\frac{2(d-1)(d-2)}{d^2},$$

$$\text{Tr}[X_6\hat{Z}] = -\frac{3(d-1)(d-2)}{d^2},$$

and we get

$$b_1 = b_2 = b_4 = \frac{5}{d^3}, \quad b_3 = b_5 = b_6 = -\frac{1}{d^3}.$$

Next, we rewrite $\hat{Z}$ as the positive linear combination of projections to show positivity. Define the projections

$$P_A = \frac{2}{3}X_2 - \frac{1}{3}X_3,$$

$$P_B = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)X_4 - \frac{1}{4\sqrt{2}}X_5 - \frac{1}{4\sqrt{2}}X_6.$$

Then we can write $\hat{Z}$ as

$$\hat{Z} = \frac{5}{d^3}(X_1 + X_2 + X_4) - \frac{1}{d^3}(X_3 + X_5 + X_6)$$

$$= \frac{1}{d^3}[3P_1 + 4\sqrt{2}P_B + 5X_1 + 3X_2 + (3 - 2\sqrt{2})X_4],$$

and hence $\hat{Z} \geqslant 0$ and also $Z \geqslant 0$. Therefore, $\frac{6}{d^2}$ is indeed the optimal solution. ∎

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