Quantum regression theorem for multi-time correlators: A detailed analysis in the Heisenberg picture

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The quantum regression theorem is one of the central results in open quantum systems and is extensively used for computing multi-point correlation functions. Traditionally it is derived for two-time correlators in the Markovian limit employing the Schrödinger picture. In this paper we make use of the Heisenberg picture to derive the quantum regression theorems for multi-time correlation functions, which in the special limit reduce to the well-known two-time regression theorem. For the multi-time correlation function we find that the regression theorem takes the same form as it takes for the two-time correlation function with a mild restriction that one of the times should be greater than all other time variables. Interestingly, the Heisenberg picture also allows us to derive an analog of regression theorem for out-of-time-ordered correlators. We further extend our study for the case of non-Markovian dynamics and report the modifications to the standard quantum regression theorem. We illustrate all of the above results using the paradigmatic dissipative spin-boson model.

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I. INTRODUCTION

Correlation functions are important dynamical quantities which are often related to experimentally measurable quantities. In the context of an open quantum system [1–5] with the system of interest following a Markovian dynamics, the quantum regression theorem (QRT) [1,2,5,6] turns out to be one of the most useful and practical tools to compute the correlation functions [2]. The QRT states that the knowledge of time evolution of a single-point function is sufficient to determine the time evolution of two-point or multi-point correlation functions. More explicitly, the validity of QRT requires that there exists a complete set of system operators $A_\mu, \mu = 1, 2, \dots$ such that

$$\frac{d}{dt}\langle A_\mu(t)\rangle = \sum_\lambda M_{\mu\lambda}\langle A_\lambda(t)\rangle. \tag{1}$$

Then the QRT for the two- and three-point function reads as

$$\begin{aligned}\frac{d}{d\tau}\langle O(t)A_\mu(t+\tau)\rangle &= \sum_\lambda M_{\mu\lambda}\langle O(t)A_\lambda(t+\tau)\rangle,\\ \frac{d}{d\tau}\langle O_1(t)A_\mu(t+\tau)O_2(t)\rangle &= \sum_\lambda M_{\mu\lambda}\langle O_1(t)A_\lambda(t+\tau)O_2(t)\rangle,\\ \frac{d}{d\tau}\langle A_\mu(t+\tau)O_1(t)O_3(t)\rangle &= \sum_\lambda M_{\mu\lambda}\langle A_\lambda(t+\tau)O_1(t)O_3(t)\rangle.\end{aligned} \tag{2}$$

Interestingly, it is easy to generalize the QRT in Eq. (2) for arbitrary N-point correlation functions of the form

$$\langle A_1(t)A_2(t+\tau)A_3(t)\cdots A_n(t)\rangle, \tag{3}$$

where the position of the operator with the argument $t+\tau$ can be arbitrary. Note that the above QRTs are given for multi-point correlation functions which are dependent on two times t and τ. Recently there has been a lot of research activity to understand systems that follow non-Markovian dynamics [7–12], and an attempt has been made to show violation of the regression theorem for such systems [6,8,13–18]. In spite of its great importance and interest, there has been a lack of systematic derivation of the regression-type theorem for cases beyond the two-time correlation function and its extension for non-Markovian systems. In general, the regression theorem may not hold for general time configurations such as

$$\langle A(t_1)B(t_2)C(t_3)\rangle, \tag{4}$$

or more generally,

$$\langle A_1(t_1)A_2(t_2)\cdots A_n(t_n)\rangle. \tag{5}$$

Here we would like to understand if there exist QRT type relations for such a class of correlation functions, including the out-of-time-ordered correlators (OTOCs),[1] which are a special class of correlation functions [19].

In this paper we derive the regression theorem for multi-time correlators in the Markovian limit using the Heisenberg picture [20]. We further extend our analysis for systems following non-Markovian dynamics. The paper is organized as follows: In Sec. II we first start with deriving the QRT for

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[1]OTOC is defined as $\langle O_1(t_1)O_2(t_2)O_3(t_1)O_4(t_2)\rangle$. Even though it's a two-time correlator, its form is significantly different from Eq. (3).

two-point functions and then extend our analysis to three, four, and further generalize to N-point functions with general time arrangements. We also point out for what special time arrangements the QRT may not hold. In Sec. III we derive a regressionlike expression for the OTOC. In Sec. IV we illustrate all these above findings for a paradigmatic dissipative spin-boson model. In Sec. V we give a systematic derivation of the equation similar to the QRT for non-Markovian systems. We skip most of the lengthy derivations to the Appendix to keep the discussion in the main text transparent.

II. QRT USING THE HEISENBERG PICTURE

In this section we derive QRT for multi-time correlation functions. We make extensive use of the Heisenberg picture formulation for open quantum systems, recently described in [20], and for completeness we also review this formalism in Appendix A and further discuss the Markovian limit in Appendix A 2. Our first aim here is to derive the well-known forms of QRTs for two-point and special three-point functions, as discussed in the Introduction, using the Heisenberg picture. We then aim for generalizing the QRT for more generic N-point correlation functions defined with multi-time arguments. We also discuss the limitations of the QRT in the Markovian limit and also generalize our study to non-Markovian systems.

Let us start by writing the Hamiltonian of the total system $H = H_S + H_R + \lambda H_{SR}$, where H_S is the Hamiltonian of the system of interest, H_R represents the Hamiltonian for the reservoir (bath), and H_{SR} is the coupling Hamiltonian between the system and the reservoir. We keep the parameter λ to keep track of the order of the perturbation with respect to the system-bath interaction. We also make the standard choice for the initial condition of the total density operator at $t = 0$ by considering a product initial state between the system and the reservoir and write $\rho_{SR}(t = 0) = \rho_S \otimes \rho_R$. The interaction between the system and the reservoir is turned on at $t = 0^+$. Since in the Heisenberg picture the operators evolve in time, it is important to define the reduced system operators. The one-point reduced operator is defined as $O_S(t) = \mathrm{Tr}_R[O(t)\rho_R]$, where the operator $O(t)$ evolves unitarily with respect to the full Hamiltonian H. The expectation value of the operator O at time t then can be written as

$$\langle O(t)\rangle = \mathrm{Tr}_S[\mathrm{Tr}_R[O(t)\rho_R]\rho_S] = \mathrm{Tr}_S[O_S(t)\rho_S]. \tag{6}$$

In a similar manner, one can define the arbitrary N-point reduced operator as

$$[O_1(t_1)O_2(t_2)....O_N(t_N)]_S = \mathrm{Tr}_R[O_1(t_1)O_2(t_2)....O_N(t_N)\rho_R]. \tag{7}$$

This definition of reduced operator has a property that $[O_1(t_1)O_2(t_2)\cdots O_N(t_N)]_S \neq O_{1S}(t_1)O_{2S}(t_2)\cdots O_N(t_N)$ as a result of finite system-bath coupling.

To derive the QRT in the Heisenberg picture, we write an equation analogous to Eq. (1) in the Heisenberg picture by assuming that there exists a complete set of reduced system operators $A_{\mu S}(t)$ that satisfies[2] the following relation:

$$\frac{d}{dt}A_{\mu S}(t) = \sum_{\lambda} M_{\mu\lambda}A_{\lambda S}(t). \tag{8}$$

The above equation implies that the operators form a closed set between themselves.

A. QRT for two-point correlation functions

To keep our discussion simple, we first focus on deriving the QRT for two-point correlation functions. Following the definition in (7) for two-point reduced operators, one can write [20] (please see Appendix A for details of the derivation)

$$[O_1(t_1)O_2(t_2)]_S = O_{1S}(t_1)O_{2S}(t_2) + I[O_{1S}(t_1), O_{2S}(t_2)], \tag{9}$$

where $O_{1S}(t_1), O_{2S}(t_2)$ are reduced one-point operators and $I[O_{1S}(t_1), O_{2S}(t_2)]$ is called the irreducible part capturing the information about coupled system-bath dynamics.[3]

One can explicitly calculate the quantity I up to the second order of the system-bath coupling (λ^2) in both Markovian and non-Markovian limits. Given the above expression, the two-point correlation functions can be easily computed by performing an additional trace over the initial system density operator, i.e.,

$$\langle O_1(t_1)O_2(t_2)\rangle = \mathrm{Tr}_S[(O_1(t_1)O_2(t_2))_S\rho_S(0)]. \tag{10}$$

To derive the QRT, we set $O_2 = A_\mu$, $O_1 = O$, and consider $t_2 > t_1$. Now taking derivative with respect to t_2 in Eq. (9), the first term of the right-hand side immediately gives a QRT-like expression,

$$\frac{d}{dt_2}[O_S(t_1)A_{\mu S}(t_2)] = \sum_{\lambda} M_{\mu\lambda}[O_S(t_1)A_{\lambda S}(t_2)], \tag{11}$$

thanks to Eq. (8). Interestingly, one can now show that for $t_2 > t_1$ and in the Markovian limit, the irreducible part $I[O_S(t_1), A_{\mu S}(t_2)]$ also satisfies a regressionlike expression, given as [please see Eq. (B1)]

$$\frac{d}{dt_2}I[O_S(t_1), A_{\mu S}(t_2)] = \sum_{\lambda} M_{\mu\lambda}I[O_S(t_1), A_{\lambda S}(t_2)]. \tag{12}$$

As a result of the above two equations, we receive the QRT for arbitrary two-point system operators as

$$\frac{d}{dt_2}[O(t_1)A_\mu(t_2)]_S = \sum_{\lambda} M_{\mu\lambda}[O(t_1)A_\lambda(t_2)]_S, \tag{13}$$

from which we trivially receive the standard QRT in terms of correlation functions,

$$\frac{d}{dt_2}\langle O(t_1)A_\mu(t_2)\rangle = \sum_{\lambda} M_{\mu\lambda}\langle O(t_1)A_\lambda(t_2)\rangle, \tag{14}$$

which matches with (2). Furthermore, in the expression of $I[O_S(t_1), A_{\mu S}(t_2)]$, it can be shown that [see Eqs. (A16) and

[2]In one of the examples discussed later, we explicitly construct operator A_μ.

[3]See Eq. (A16) and Eqs. (A25)–(A27) in Appendix A 2 for details.

(A25)–(A27)] if we swap the position of $O_S(t_1)$ and $A_{\mu S}(t_2)$, Eq. (12) still is respected, i.e.,

$$\frac{d}{dt_2}I[A_{\mu S}(t_2), O_S(t_1)] = \sum_\lambda M_{\mu\lambda}I[A_{\lambda S}(t_2), O_S(t_1)], \quad (15)$$

and as a result we receive another form of QRT as

$$\frac{d}{dt_2}\langle A_\mu(t_2)O(t_1)\rangle = \sum_\lambda M_{\mu\lambda}\langle A_\lambda(t_2)O(t_1)\rangle. \quad (16)$$

It is important to note that if we consider the other time sequence, i.e., if $t_2 < t_1$, and take the derivative with respect to t_2, what we receive does not obey the standard QRT. This is what one also receives by working in the Schrödinger picture. We next discuss the generalization of our analysis for higher-point and multi-time correlators.

B. QRT for three-point, multi-time correlation functions

Following similar steps as before, we can define the three-point multi-time reduced operators, which up to the second order of the system-bath coupling (λ^2 order) is given as [20]

$$\begin{aligned}[O_1(t_1)O_2(t_2)O_3(t_3)]_S = {}& O_{1S}(t_1)O_{2S}(t_2)O_{3S}(t_3) + W_{1,2,3}\{O_{1S}(t_1)I[O_{2S}(t_2), O_{3S}(t_3)]\}\\ &+ W_{1,2,3}\{I[O_{1S}(t_1), O_{2S}(t_2)]O_{3S}(t_3)\} + W_{1,2,3}\{I[O_{1S}(t_1), O_{3S}(t_3)]O_{2S}(t_2)\},\end{aligned} \quad (17)$$

where the operator $W_{1,2,3}$ ensures that the operator product is ordered such that O_{1S} comes before O_{2S}, and O_{2S} comes before O_{3S} (please see Appendix A 1 b for more details). The three-point multi-time correlation functions can be computed as

$$\langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle = \mathrm{Tr}_S\,[[O_1(t_1)O_2(t_2)O_3(t_3)]_S\rho_S(0)]. \quad (18)$$

Now once again, to derive the QRT we first set $O_3 = A_\mu$ and assume $t_i < t_3$ with $i = 1, 2$. Taking the derivative of Eq. (17) with respect to t_3, the first term of the right-hand side gives

$$\frac{d}{dt_3}[O_{1S}(t_1)O_{2S}(t_2)A_{\mu S}(t_3)] = \sum_\lambda M_{\mu\lambda}[O_{1S}(t_1)O_{2S}(t_2)A_{\lambda S}(t_3)], \quad (19)$$

where we have used Eq. (8). One can then show that [please see Eq. (B8) of Appendix B 2] for the time sequence $t_i < t_3$ with $i = 1, 2$ and invoking the Markovian limit, the second term of the right-hand side of Eq. (17) satisfies the equation

$$\frac{d}{dt_3}W_{1,2,3}\{O_{1S}(t_1)I[O_{2S}(t_2), A_{\mu S}(t_3)]\} = \sum_\lambda M_{\mu\lambda}W_{1,2,3}\{O_{1S}(t_1)I[O_{2S}(t_2), A_{\lambda S}(t_3)]\}. \quad (20)$$

Interestingly, the third and the fourth term also follow identical equations such as the above. Finally, summing up all these contributions, we receive a multi-time QRT-like form involving the reduced system operators:

$$\frac{d}{dt_3}[O_1(t_1)O_2(t_2)A_\mu(t_3)]_S = \sum_\lambda M_{\mu\lambda}[O_1(t_1)O_2(t_2)A_\lambda(t_3)]_S, \quad (21)$$

from which we trivially receive the QRT for the three-point correlation functions as

$$\frac{d}{dt_3}\langle O_1(t_1)O_2(t_2)A_\mu(t_3)\rangle = \sum_\lambda M_{\mu\lambda}\langle O_1(t_1)O_2(t_2)A_\lambda(t_3)\rangle. \quad (22)$$

Interestingly, if we swap the position of $O_2(t_2)$ and $A_\mu(t_3)$, one can obtain a similar QRT. Equation (22) is the regression theorem for three-point correlation functions that involve three different times t_1, t_2, and t_3. The above QRT in (22) reduces to the following standard result [Eq. (2)] if we swap the position of $O_2(t_2)$ and $A_\mu(t_3)$ and set $t_1 = t_2$,

$$\frac{d}{dt_3}\langle O_1(t_1)A_\mu(t_3)O_2(t_1)\rangle = \sum_\lambda M_{\mu\lambda}\langle O_1(t_1)A_\lambda(t_3)O_2(t_1)\rangle. \quad (23)$$

Note that the QRT in our case holds irrespective of the position of $A_\mu(t_3)$. However, in Eq. (22), if we take derivative with respect to t_1 or t_2 instead of t_3 (maximum time), interestingly, we do not receive a QRT-like relation.

C. QRT for four-point and general N-point, multi-time correlation functions

Following almost similar steps as before, one can work out the regression theorem for multi-time four-point correlation functions, and in fact, it is possible to generalize this analysis for N-point functions as well. Here we present the central results (see Appendix B 3 for more details). For four-point, multi-time correlation functions we receive a QRT in the Markovian limit,

$$\frac{d}{dt_4}\langle O_1(t_1)O_2(t_2)O_3(t_3)A_\mu(t_4)\rangle = \sum_\lambda M_{\mu\lambda}\langle O_1(t_1)O_2(t_2)O_3(t_3)A_\lambda(t_4)\rangle. \quad (24)$$

Let us emphasize that (24) holds as long as $t_4 > t_i$ with $i = 1, 2, 3$. Also note that (24) holds irrespective of the position of $A_\mu(t_4)$, as was observed for the QRT for three-point functions. This entire analysis can be generalized to N-point multi-time correlation functions as

$$\frac{d}{dt_N}\langle O_1(t_1)O_2(t_2)......O_{N-1}(t_{N-1})A_\mu(t_N)\rangle = \sum_\lambda M_{\mu\lambda}\langle O_1(t_1)O_2(t_2)......O_{N-1}(t_{N-1})A_\lambda(t_N)\rangle, \quad (25)$$

where $t_i < t_N$ and $i = 1, 2, ...N-1$. Once again, the operator $A_\mu(t_N)$ can take any place, and if we take derivative with respect to t_k instead of the highest time t_N, then we do not receive the regression-type formula.

III. QRT FOR OUT-OF-TIME-ORDERED CORRELATORS

As an application of the developed formalism, we now extend our analysis to compute the OTOC, which is an excellent measure of quantum chaos, many-body localization, information scrambling, etc. [21–28]. OTOC has received significant attention in recent times, with its applicability ranging from quantum information theory to condensed matter physics to quantum gravity. Very recently, OTOC has found its application in the context of open quantum systems, as the coupling of the system with a dissipative or dephasing bath naturally leads to information scrambling [22,29]. Motivated by this, in this section we derive the QRT-like formula for OTOC correlators in the Markovian limit. Details of the derivation are provided in Appendix C. Here we present the main result. Let us first define the four-point reduced operator of the form

$$[O_1(t_1)A_\mu(t_2)O_3(t_1)A_\nu(t_2)]_S, \quad (26)$$

where the system reduced operator $A_{\mu S}(t_2)$ satisfies Eq. (8). One can then receive the following regressionlike formula for the OTOC,

$$\begin{aligned}\frac{d}{dt_2}\langle O_1(t_1)A_\mu(t_2)O_3(t_1)A_\nu(t_2)\rangle &= \sum_\lambda M_{\mu\lambda}\langle O_1(t_1)A_\lambda(t_2)O_3(t_1)A_\nu(t_2)\rangle \\ &+ \sum_{\lambda'} M_{\nu\lambda'}\langle O_1(t_1)A_\mu(t_2)O_3(t_1)A_{\lambda'}(t_2)\rangle \\ &+ \langle W_{1,2,3,4}\{O_{1S}(t_1)F[A_{\mu S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\}\rangle, \quad (27)\end{aligned}$$

where we assume that $t_2 > t_1$. Interestingly, the expression in Eq. (27) is almost identical to the regression theorem, except for the last term. As before, $W_{1,2,3,4}$ ensures the time ordering of the operators. The explicit form of the operator $F[A_{\mu S}(t_2), A_{\nu S}(t_2)]$ is given in Appendix C [Eq. (C5)]. In Ref. [19], a regressionlike formula for OTOC in the Markovian limit was recently derived using a different approach. Equations (25) and (27) are the central results of our paper.

IV. EXAMPLE: DISSIPATIVE SPIN-BOSON MODEL

In this section we illustrate the above-derived results for the paradigmatic dissipative spin-boson model by calculating various correlation functions in the Heisenberg picture. In particular, we verify the QRTs for two-, three-, and four-point correlators. In order to perform these calculations, we first derive the master equation for the reduced one-point operator and then proceed to calculate multi-time, multi-point correlation functions. The details of the derivations are provided in Appendix D.

The Hamiltonian for a dissipative spin-1/2 system, coupled to a bath consisting of an infinite collection of harmonic oscillators with different normal-mode frequencies, can be written as

$$\begin{aligned}H &= H_S + H_R + H_{SR} \\ &= \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \alpha_k(b_k^\dagger\sigma_- + b_k\sigma_+), \quad (28)\end{aligned}$$

where ω_0 is the frequency of the qubit. The bath is characterized by the eigenmode frequency ω_k referring to the kth oscillator, with $b_k(b_k^\dagger)$ the corresponding annihilation (creation) operator. The last term in the above Hamiltonian represents the standard dissipative coupling term between the spin-1/2 system and the harmonic bath, with α_k being the coupling strength between the kth mode and the qubit. For simplicity, we always work in the zero-temperature limit ($T = 0$). We first derive the master equation correct up to the second order of the system-bath coupling and further make a secular approximation [1]. One can show that the reduced one-point operator obeys the following equation at zero temperature ($T = 0$):

$$\frac{d}{dt}O_S(t) = i\frac{\omega_0'}{2}[\sigma_z, O_S(t)] + \frac{\gamma}{2}[2\sigma_+O_S(t)\sigma_- - \sigma_+\sigma_-O_S(t) - O_S(t)\sigma_+\sigma_-], \quad (29)$$

where $\omega_0' = \omega_0 + \Delta$ is the renormalized frequency of the qubit due to the coupling with the environment with

$$\Delta = \mathrm{P}\int_0^\infty \frac{g(\omega')|\alpha(\omega')|^2 d\omega'}{\omega_0 - \omega'}, \quad (30)$$

where P refers to the principal value of the integral. Here $g(\omega)$ represents the density of states of the bath oscillators, and $\gamma = 2\pi g(\omega_0)|\alpha(\omega_0)|^2$ represents the decay rate.

Using the above master equation for the operator, it is easy to show that

$$\frac{d}{dt}\begin{bmatrix}\sigma_{xS}\\ \sigma_{yS}\\ \sigma_{zS}\end{bmatrix} = \begin{bmatrix}-\frac{\gamma}{2} & -\omega_0' & 0\\ \omega_0' & -\frac{\gamma}{2} & 0\\ 0 & 0 & -\gamma\end{bmatrix}\begin{bmatrix}\sigma_{xS}\\ \sigma_{yS}\\ \sigma_{zS}\end{bmatrix} - \gamma\begin{bmatrix}0_{2\times2}\\ 0_{2\times2}\\ I_{2\times2}\end{bmatrix}, \quad (31)$$

where $0_{2\times2}$ and $I_{2\times2}$ are the 2×2 null matrix and the identity matrix, respectively. It is therefore easy to see that for this model there exist two different sets of closed operators that follow Eq. (8). These are given as

$$A_\mu = \{\sigma_z, I_{2\times 2}\}, \quad M_{\mu\lambda} = \begin{bmatrix} -\gamma & -\gamma \\ 0 & 0 \end{bmatrix}$$
$$A_\nu = \{\sigma_x, \sigma_y\}, \quad M'_{\nu\lambda'} = \begin{bmatrix} -\frac{\gamma}{2} & -\omega_0' \\ \omega_0' & -\frac{\gamma}{2} \end{bmatrix}. \tag{32}$$

With this in hand, we are now ready to asses the validity of QRT results that we derived in the previous section.

A. Two-point correlation function

Let us first verify the regression theorem for a two-point reduced operator (13) for this model. The spin-boson Hamiltonian in Eq. (28) can alternatively be expressed in the following form:

$$H = \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \alpha_k \left(\frac{b_k + b_k^\dagger}{2}\sigma_x + i\frac{b_k^\dagger - b_k}{2}(-\sigma_y) \right). \tag{33}$$

By comparing the last term of Eq. (33) with $H_{SR} = \sum_i S^i \otimes R^i$ we can identify the following system operators:

$$S^1 = \sigma_x \quad \text{and} \quad S^2 = -\sigma_y. \tag{34}$$

Now, in the interaction picture with respect to H_S, the system operators are written as $\tilde{S}_i(t) = \sum_\omega S_\omega^i e^{i\omega t}$, and one can easily show that

$$S^1_{\omega_0} = \sigma_-, \quad S^1_{-\omega_0} = \sigma_+ \quad \text{and}$$
$$S^2_{\omega_0} = -i\,\sigma_-, \quad S^2_{-\omega_0} = i\,\sigma_+. \tag{35}$$

Note that in the above expression ω takes two possible values $\pm\omega_0$. We are interested here to validate the regression theorem for correlation functions of the type $(\sigma_x(t_1)\sigma_x(t_2))$ and $(\sigma_x(t_1)\sigma_y(t_2))$ for $t_2 > t_1$. We therefore now proceed and calculate the corresponding irreducible components following Eqs. (A25) and (A26). By putting $O_{1S} = O_{2S} = \sigma_{xS}$ in Eq. (A25) and taking the secular approximation (please see Appendix A 3), we receive

$$I_1[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = -t_1 \sum_\omega \sum_{i,j} \sigma_{xS}(t_1) S^i_\omega \sigma_{xS}(t_2) S^j_{-\omega} \beta_1^{ij}(-\omega). \tag{36}$$

Using the above equation, we can easily show that the $I_1[\sigma_{xS}(t_1), \sigma_{xS}(t_2)]$ term is zero. However, if we put $O_{1S} = O_{2S} = \sigma_{xS}$ in Eq. (A26) and take the secular approximation, we receive

$$I_2[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = \gamma t_1 [\sigma_+ \sigma_{xS}(t_1)\sigma_{xS}(t_2)\sigma_-] = \gamma t_1 e^{i\omega_0'(t_2-t_1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{37}$$

$$I_3[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = \gamma t_1 [\sigma_{xS}(t_1)\sigma_+ \sigma_- \sigma_{xS}(t_2)] = \gamma t_1 e^{i\omega_0'(t_2-t_1)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{38}$$

Similarly, we calculate $I_4[\sigma_{xS}(t_1), \sigma_{xS}(t_2)]$ but it vanishes, i.e.,

$$I_4[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = I_1[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{39}$$

Using Eq. (A16) we get

$$I[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = \gamma t_1 e^{i\omega_0'(t_2-t_1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{40}$$

By putting $O_{1S} = \sigma_{xS}$, $O_{2S} = \sigma_{yS}$ in Eqs. (A25), (A26) and performing the identical steps, we receive

$$I[\sigma_{xS}(t_1), \sigma_{yS}(t_2)] = -i\gamma t_1 e^{i\omega_0'(t_2-t_1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{41}$$

With the full irreducible components in hand for the two different correlators, we now use Eq. (9) to compute the following two-point reduced operators, given as

$$(\sigma_x(t_1)\sigma_x(t_2))_S = \begin{bmatrix} \left[1 - \frac{\gamma}{2}(t_2 - t_1)\right] e^{-i\omega_0'(t_2-t_1)} + 2i\gamma t_1 \sin\omega_0'(t_2 - t_1) & 0 \\ 0 & \left[1 - \frac{\gamma}{2}(t_2 - t_1)\right] e^{i\omega_0'(t_2-t_1)} \end{bmatrix}, \tag{42}$$

$$(\sigma_x(t_1)\sigma_y(t_2))_S = \begin{bmatrix} i\left[1 - \frac{\gamma}{2}(t_2 - t_1)\right] e^{-i\omega_0'(t_2-t_1)} - 2i\gamma t_1 \cos\omega_0'(t_2 - t_1) & 0 \\ 0 & -i\left[1 - \frac{\gamma}{2}(t_2 - t_1)\right] e^{i\omega_0'(t_2-t_1)} \end{bmatrix}. \tag{43}$$

To verify the QRT, let us first set $O_1 = \sigma_x$, $A_\mu = \{\sigma_x, \sigma_y\}$ in Eq. (13). Now, using Eqs. (42) and (43), we explicitly calculate the left-hand side and right-hand side of Eq. (13) by taking the derivative with respect to the maximum time t_2. We find that they are equal, which verifies Eq. (13). As an immediate consequence, we conclude that the QRT follows for two-point correlation functions Eq. (14) for the arbitrary initial density matrix for the system.

B. Three-point, multi-time correlation function

We next move to verify the QRT for the three-point reduced operator as given in Eq. (21). We first set $O_1 = O_2 = \sigma_x$, $A_\mu = \{\sigma_x, \sigma_y\}$ in Eq. (21) and assume $t_1 < t_2 < t_3$. Using Eq. (17), we calculate the following three-point reduced operators (please see the details of the calculation in Appendix D),

$$(\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3))_S = \begin{bmatrix} 0 & \left[1-\frac{1}{2}\gamma(t_1+t_3-t_2)\right]e^{i\omega_0'(t_1+t_3-t_2)} \\ \left[1-\frac{1}{2}\gamma(-t_1+t_2+t_3)\right]e^{-i\omega_0'(t_1+t_3-t_2)} \\ +\gamma(t_2-t_1)e^{i\omega_0'(-t_1-t_2+t_3)} & 0 \end{bmatrix} \tag{44}$$

and

$$(\sigma_x(t_1)\sigma_x(t_2)\sigma_y(t_3))_S = \begin{bmatrix} 0 & -i\left[1-\frac{1}{2}\gamma(t_1+t_3-t_2)\right]e^{i\omega_0'(t_1+t_3-t_2)} \\ i\left[1-\frac{1}{2}\gamma(-t_1+t_2+t_3)\right]e^{-i\omega_0'(t_1+t_3-t_2)} \\ -i\gamma(t_2-t_1)e^{i\omega_0'(-t_1-t_2+t_3)} & 0 \end{bmatrix}. \tag{45}$$

Now using Eqs. (44) and (45), we explicitly calculate the left-hand side and right-hand side of Eq. (21). We find that they are equal, which verifies Eq. (21). Note that we choose a particular order of time, but we can show Eq. (21) holds as long as $t_i < t_3$ with $i = 1, 2$. There are no constraints on the order of (t_1, t_2).

C. Verification of OTOC

We next provide one example to asses the validity of our expression for OTOC. For the spin-boson model, it is easy to compute the following four-point reduced operator and one receives

$$\frac{d}{dt_2}[\sigma_x(t_1)\sigma_z(t_2)\sigma_x(t_1)\sigma_z(t_2)]_S = 2\gamma\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{46}$$

It is easy to check that the corresponding right-hand side of the OTOC also gives the same result. In a similar way, OTOC can be checked for

$$\frac{d}{dt_2}[\sigma_z(t_1)\sigma_z(t_2)\sigma_z(t_1)\sigma_z(t_2)]_S = -8\gamma\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{47}$$

V. GENERALIZATION OF QRT FOR THE NON-MARKOVIAN CASE

The results for QRT presented in the previous sections can be extended for the non-Markovian case. For simplicity, we here focus on systems with bosonic bath and linear system-bath interaction, but one can generalize this study for a more generic type of system-bath interaction as well. We derive a Lindblad-type equation up to order λ^2 for this setup which takes into account the non-Markovian evolution. We then derive the correction to the QRT for the non-Markovian case by focusing only on the two-point correlation functions [14]. Extension to higher-point multi-time correlators can be similarly obtained, even for the non-Markovian case.

A. Lindblad-type non-Markovian equation for one-point reduced operator

Let the Hamiltonian of the composite system be

$$H = H_S + H_R + H_{SR} = H_S + \sum_k \omega_k b_k^\dagger b_k + \sum_k \alpha_k (L b_k^\dagger + L^\dagger b_k). \tag{48}$$

Here, the system is coupled with the bath through a generic system operator L. Let us assume that the initial density operator of the total system can be written as $\rho_{SR}(0) = \rho_S \otimes \rho_R$. Now, following the master equation in the Heisenberg picture [please see Eq. (A10)], one can show that the reduced density operator for the system obeys the following non-Markovian master equation at zero temperature ($T = 0$) and is correct up to the second order of system-bath coupling,

$$\frac{d}{dt}O_S(t) = i[H_S, O_S(t)] + \int_0^t d\tau\, \alpha(\tau)[L^\dagger(0), O_S(t)]\tilde{L}(-\tau) + \int_0^t d\tau\, \alpha^*(\tau)\tilde{L}^\dagger(-\tau)[O_S(t), L(0)], \tag{49}$$

where $\tilde{L}(t) = U_0(t) L U_0^\dagger(t)$ is the coupled system operator in the interaction picture with $U_0(t)$ representing the free evolution due to the Hamiltonian H_S. $\alpha(\tau)$ denotes the bath correlation function, which is given as

$$\alpha(\tau) = \sum_k |\alpha_k|^2 \,\mathrm{Tr}_R[\tilde{b}_k(0)\tilde{b}_k^\dagger(-\tau)\rho_R] = \sum_k |\alpha_k|^2 e^{-i\omega_k \tau}. \tag{50}$$

B. Extension of QRT to non-Markovian case

Having obtained the non-Markovian master equation in the Heisenberg picture, we now extend the QRT for the non-Markovian dynamics. To achieve that let us first assume that there exists a complete set of system operators $A_{\mu S}(t)$ such that

$$\frac{d}{dt}A_{\mu S}(t) = \sum_\lambda M_{\mu\lambda}(t) A_{\lambda S}(t). \tag{51}$$

Note the crucial explicit time dependence in $M_{\mu\lambda}(t)$ for the non-Markovian case, which is typically time independent for the case of Markovian dynamics. Using the above results, it is easy to generalize QRT for the non-Markovian case, and it is given by (see Appendix E for derivation)

$$\begin{aligned}\frac{d}{dt_2}[O(t_1)A_\mu(t_2)]_S = \sum_\lambda M_{\mu\lambda}(t_2)[O(t_1)A_\lambda(t_2)]_S\\ &- \int_0^{t_1} d\tau_1\,\alpha(t_2-\tau_1)\,O_S(t_1)\tilde{L}^\dagger(-\tau_1)A_{\mu S}(t_2)\tilde{L}(-t_2)\\ &- \int_0^{t_1} d\tau_1\,\alpha(t_2-\tau_1)\,\tilde{L}^\dagger(-\tau_1)O_S(t_1)\tilde{L}(-t_2)A_{\mu S}(t_2)\\ &+ \int_0^{t_1} d\tau_1\,\alpha(t_2-\tau_1)\,\tilde{L}^\dagger(-\tau_1)O_S(t_1)A_{\mu S}(t_2)\tilde{L}(-t_2)\\ &+ \int_0^{t_1} d\tau_1\,\alpha(t_2-\tau_1)\,O_S(t_1)\tilde{L}^\dagger(-\tau_1)\tilde{L}(-t_2)A_{\mu S}(t_2),\end{aligned} \tag{52}$$

where we have assumed $t_2 > t_1$ and that the operator $A_{\mu S}$ follows Eq. (51). The above equation is the extension of QRT to the non-Markovian dynamics. In the Markovian limit Eq. (52) correctly reproduces the QRT, as given in Eq. (11). To illustrate, in the Markovian limit, the system timescale is much larger than the bath characteristic timescale τ_B, that is $t_2 - \tau_1 \gg \tau_B$, and the bath correlation function $\alpha(t_2 - \tau_1)$ vanishes beyond τ_B in (52). So in the Markovian limit, all the integrals in Eq. (52) vanish, thus reproducing the standard QRT (11) for the two-point function.

C. Example: Dissipative spin-boson model

To illustrate the above result, we once again focus on the dissipative spin-boson model as defined in Eq. (28). Following the non-Markovian master equation in Eq. (49), we receive

$$\frac{d}{dt}O_S(t) = i\frac{\omega_0}{2}[\sigma_z, O_S(t)] + \lambda^2\int_0^t d\tau\alpha(\tau)[\sigma_+, O_S(t)]\sigma_- e^{i\omega_0\tau} + \lambda^2\int_0^t d\tau\alpha^*(\tau)\sigma_+[O_S(t), \sigma_-]e^{-i\omega_0\tau}. \tag{53}$$

It is easy to check that in the Markovian limit the above equation reduces to Eq. (29). We now consider $O_S = \sigma_{xS}$ in Eq. (53), which gives us the following solution:

$$\sigma_{xS}(t) = \begin{bmatrix} 0 & \left(1-\lambda^2\int_0^t d\tau\gamma(\tau)\right)e^{i\int_0^t d\tau\omega_0'(\tau)} \\ \left(1-\lambda^2\int_0^t d\tau\gamma(\tau)\right)e^{-i\int_0^t d\tau\omega_0'(\tau)} & 0 \end{bmatrix}, \tag{54}$$

where

$$\begin{aligned}\omega_0'(t) &= \omega_0 + \lambda^2\mathrm{Im}[R(t)],\\ \gamma(t) &= \mathrm{Re}[R(t)],\\ R(t) &= \int_0^t d\tau\alpha(\tau)e^{i\omega_0\tau}.\end{aligned} \tag{55}$$

Using Eq. (E2), we then calculate all four irreducible terms I_i, and using them we find the following two-point reduced operator:

$$(\sigma_x(t_1)\sigma_x(t_2))_S = \begin{bmatrix} M(t_1,t_2) & 0 \\ 0 & M^*(t_1,t_2)\end{bmatrix} + \eta(t_1,t_2)N(t_1,t_2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{56}$$

where

$$\begin{aligned}M(t_1,t_2) &= \left(1-\lambda^2\int_0^{t_1} dt_1'\gamma(t_1') - \lambda^2\int_0^{t_2} dt_2'\gamma(t_2')\right)N(t_1,t_2),\\ N(t_1,t_2) &= e^{i\int_0^{t_1} dt_1'\omega_0'(t_1')}\,e^{-i\int_0^{t_2} dt_2'\omega_0'(t_2')},\\ \eta(t_1,t_2) &= \lambda^2\int_0^{t_2} d\tau_2\int_0^{t_1} d\tau_1\alpha(\tau_2-\tau_1)e^{i\omega_0(\tau_2-\tau_1)}.\end{aligned} \tag{57}$$

If we now take the derivative of Eq. (56) with respect to t_2, we do not receive the standard QRT derived in Eq. (13); rather, we receive the extended QRT derived in Eq. (52). In the Markovian limit, the bath correlation function $\alpha(\tau)$ decays very fast with time, and Eq. (56) reduces to Eq. (42). Also in the Markovian limit, the extended QRT reduces to the standard QRT.

VI. DISCUSSION

While defining the multi-time correlation function, it is natural to work in the Heisenberg picture. Even though the multi-time correlation function is of great physical importance, the Heisenberg picture has not received much attention in the context of quantum open systems. In this paper we make use of the recently developed Heisenberg picture technique [20] to calculate correlation functions in the Markovian limit and derive the quantum regression theorem. In particular, we generalize the regression theorem for multi-time correlation functions with general time arguments. What we observe is that the form of the regression theorem remains the same for two or multi-time correlation functions as long as the following mild restriction on the time arrangements is met, $t_i < t_N$ with $i = 1, 2, ...N - 1$. We also extend our approach to compute out-of-time-ordered correlators in the Markovian limit and find that regression theorem gets a modification from the known two-time regression theorem. We further extend our study to the non-Markovian dynamics. However, in this case the QRT receives a complicated correction term with the two-point correlation function requiring information about the four-point function. As a possible future direction, an interesting topic would be to use the Heisenberg picture [20] and go beyond the standard second-order perturbation scheme. For example, one can consider exactly solvable systems such as the spin-boson dephasing model or the well-known Caldeira-Leggett model and investigate the possibility to sum up the perturbation series exactly in the Heisenberg picture.

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APPENDIX A: REVIEW OF HEISENBERG PICTURE DYNAMICS

For completeness, in this Appendix we briefly review the Heisenberg picture results in [20]. We follow the same notation and convention as is used in [20]. Let us consider the Hamiltonian of the total system is given by $H = H_S + H_R + \lambda H_{SR}$, where we have introduced an extra parameter λ to keep track of the order of perturbation in terms of system-bath coupling. We assume that the interaction between system and bath is turned on at $t = 0$. Before turning on the interaction, the system and bath were decoupled and their total density matrix can be written as $\rho_{SR} = \rho_S \otimes \rho_R$. In the Heisenberg picture, the density matrix is time independent. The expectation value of any operator can be written as

$$\langle O(t)\rangle = \mathrm{Tr}_S[\mathrm{Tr}_R[O(t)\rho_R]\rho_S] = \mathrm{Tr}_S[O_S(t)\rho_S], \tag{A1}$$

where the reduced one-point operator is defined as $O_S(t) = \mathrm{Tr}_R[O(t)\rho_R]$. Similarly, we can define the N-point reduced operator as

$$[O_1(t_1)O_2(t_2)....O_N(t_N)]_S = \mathrm{Tr}_R[O_1(t_1)O_2(t_2)....O_N(t_N)\rho_R]. \tag{A2}$$

This definition has a property that $[O_1(t_1)O_2(t_2)]_S \neq O_{1S}(t_1)O_{2S}(t_2)$. However, we can express the reduced N-point operator in terms of one-point reduced operators using what are called image operators [20]. The image operator of any operator $O(t)$ is defined as

$$O_{\alpha\beta}(t) = T_\alpha^\dagger O(t) T_\beta, \tag{A3}$$

with $T_\alpha = \sum_i |i\alpha\rangle\langle i|$, where $\{|i\rangle\}$ is an orthonormal basis of H_S and $\{|\alpha\rangle\}$ is an orthonormal basis of H_R. It is easy to see that

$$\sum_\alpha T_\alpha T_\alpha^\dagger = I, \tag{A4}$$

where I is the identity operator. One can show that the N-point image operators can be written in terms of one-point image operators as

$$[O_1(t_1)O_2(t_2)....O_N(t_N)]_{\alpha\beta} = \sum_{\gamma_1,\dots,\gamma_{N-1}} O_{1\alpha\gamma_1}(t_1)O_{2\gamma_1\gamma_2}(t_2)....O_{N\gamma_{N-1}\beta}(t_N). \tag{A5}$$

The N-point reduced operators defined in (A2) can also be expressed in terms of one-point image operators by inserting (A4) and using (A5) as follows:[4]

$$[O_1(t_1)O_2(t_2)....O_N(t_N)]_S = \sum_{\alpha,\beta,\gamma_1,\dots,\gamma_{N-1}} O_{1\alpha\gamma_1}(t_1)O_{2\gamma_1\gamma_2}(t_2)....O_{N\gamma_{N-1}\beta}(t_N)\rho_{R\beta\alpha}. \tag{A6}$$

We next consider a general form of the interaction Hamiltonian between system and bath and write

$$H_{SR} = \sum_i S^i \otimes R^i, \tag{A7}$$

where S^i is a Hermitian operator acting on the system's Hilbert space, and R^i is a Hermitian operator in the bath's Hilbert space. The corresponding image operators of H_{SR} are [using Eq. (A3)]

$$H_{SR\alpha\gamma} = \sum_i S^i R^i_{\alpha\gamma}, \tag{A8}$$

where $R^i_{\alpha\gamma}$ is the α, γth element of R^i ($|\alpha\rangle$, $|\gamma\rangle$ are the eigenstates of bath Hamiltonian H_R). We define the interaction picture operators $\tilde{H}_{SR\alpha\gamma}(t)$ as

$$\tilde{H}_{SR\alpha\gamma}(t) = \sum_i \tilde{S}^i(t)\tilde{R}^i{}_{\alpha\gamma}(t), \tag{A9}$$

where $\tilde{S}^i(t) = U_0(t)S^iU_0^\dagger(t) = \sum_\omega S^i_\omega e^{i\omega t}$, and $\tilde{R}^i{}_{\alpha\gamma}(t) = R^i_{\alpha\gamma}e^{-i(E_\alpha - E_\gamma)t}$; here $U_0(t) = e^{-iH_S t}$, and E_α, E_γ are the

[4]We can explicitly express the one-point image operator in terms of the one-point reduced operator [20].

eigenvalues of bath Hamiltonian H_R. Then the exact equation (written to all orders in λ) that satisfies the reduced one-point system operator $O_S(t)$ is [20]

$$\frac{d}{dt}O_S(t) = i[H_S, O_S(t)] + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{n_1,n_2,..n_k=1}^{\infty}(-1)^k\lambda^{n+n_1+\cdots+n_k}D_tP_s^nP_s^{n_1}\dots P_s^{n_k}O_S(t), \tag{A10}$$

where the superoperator $D_tP_s^n$ is defined as

$$D_tP_s^nA(t) = \sum_{r=0}^{n}\sum_{\alpha,\beta,\gamma} i^{n-2r}U_0^\dagger(t)\frac{d}{dt}\big[U_0(t)K_{\gamma\alpha}^{(n-r)\dagger}(t)U_0^\dagger(t)\big]U_0(t)A(t)K_{\gamma\beta}^r(t)\rho_{B\beta\alpha}$$
$$+\sum_{r=0}^{n}\sum_{\alpha,\beta,\gamma} i^{n-2r}K_{\gamma\alpha}^{(n-r)\dagger}(t)A(t)U_0^\dagger(t)\frac{d}{dt}\big[U_0(t)K_{\gamma\beta}^r(t)U_0^\dagger(t)\big]U_0(t)\rho_{B\beta\alpha}, \tag{A11}$$

where

$$K_{\alpha\beta}^r(t) = e^{i(E_\alpha-E_\beta)t}U_0^\dagger(t)\tilde{K}_{\alpha\beta}^r(t)U_0(t),$$
$$\tilde{K}_{\alpha\beta}^n(t) = \sum_{\gamma_1,\dots,\gamma_{n-1}}\int_0^t dt_1\dots\int_0^{t_{n-1}} dt_n\tilde{H}_{SR\alpha\gamma_1}(t_1)\dots\tilde{H}_{SR\gamma_{n-1}\beta}(t_n) \tag{A12}$$

with

$$\tilde{K}_{\alpha\beta}^0(t) = \delta_{\alpha\beta}. \tag{A13}$$

1. Correlation function in the Heisenberg picture

a. Expression for two-point reduced operators

We now compute the two-point reduced operator in the Heisenberg picture, which can be written as [20]

$$[O_1(t_1)O_2(t_2)]_S = O_{1S}(t_1)O_{2S}(t_2) + I[O_{1S}(t_1), O_{2S}(t_2)], \tag{A14}$$

where $I[O_{1S}(t_1), O_{2S}(t_2)]$ is the irreducible part.[5] This irreducible part can be expressed up to λ^2 order, following Eq. (A6), as

$$I[O_{1S}(t_1), O_{2S}(t_2)]$$
$$= \sum_{n_0^1,l_0^1,n_0^2,l_0^2}\sum_{\alpha,\beta,\gamma,\gamma_0,\gamma_0'}\big((-i\lambda)^{n_0^1}K_{\gamma_0\alpha}^{n_0^1}(t_1)\big)^\dagger O_{1S}(t_1)\big((-i\lambda)^{l_0^1}K_{\gamma_0\gamma}^{l_0^1}(t_1)\big)\big((-i\lambda)^{n_0^2}K_{\gamma_0'\gamma}^{n_0^2}(t_2)\big)^\dagger O_{2S}(t_2)\big((-i\lambda)^{l_0^2}K_{\gamma_0'\beta}^{l_0^2}(t_2)\big)\rho_{R\beta\alpha}, \tag{A15}$$

such that $n_0^1 + l_0^1 = 1$ and $n_0^2 + l_0^2 = 1$, so these are the following four possible combinations:

(1) $n_0^1 = 0, l_0^1 = 1$ and $n_0^2 = 0, l_0^2 = 1$
(2) $n_0^1 = 1, l_0^1 = 0$ and $n_0^2 = 0, l_0^2 = 1$
(3) $n_0^1 = 0, l_0^1 = 1$ and $n_0^2 = 1, l_0^2 = 0$
(4) $n_0^1 = 1, l_0^1 = 0$ and $n_0^2 = 1, l_0^2 = 0$.

We can then write I as the sum over these four combinations i.e.,

$$I = I_1 + I_2 + I_3 + I_4. \tag{A16}$$

One can write down the expressions for each I_i. For example, using (A12), (A13),(A15), we obtain

$$I_1 = -\lambda^2\sum_{\omega,\omega'}\sum_{i,j} O_{1S}(t_1)S_\omega^iO_{2S}(t_2)S_{\omega'}^j\int_0^{t_1} d\tau_1e^{-i\omega\tau_1}\int_0^{t_2} d\tau_2e^{-i\omega'\tau_2}\mathrm{Tr}_R[\tilde{R}^i(-\tau_1)\tilde{R}^j(-\tau_2)\rho_R], \tag{A17}$$

where $\tau_1 = t_1 - t_1'$ and $\tau_2 = t_2 - t_2'$.

Similarly, we can show that the other contributions give, s

$$I_2 = -\lambda^2\sum_{\omega,\omega'}\sum_{i,j} S_\omega^{i\dagger}O_{1S}(t_1)O_{2S}(t_2)S_{\omega'}^j\int_0^{t_1} d\tau_1e^{i\omega\tau_1}\int_0^{t_2} d\tau_2e^{-i\omega'\tau_2}\,\mathrm{Tr}_R[\tilde{R}^i(-\tau_1)\tilde{R}^j(-\tau_2)\rho_R],$$

$$I_3 = -\lambda^2\sum_{\omega,\omega'}\sum_{i,j} O_{1S}(t_1)S_\omega^iO_{2S}(t_2)S_{\omega'}^{j\dagger}\int_0^{t_1} d\tau_1\,e^{-i\omega\tau_1}\int_0^{t_2} d\tau_2e^{i\omega'\tau_2}\,\mathrm{Tr}_R[\tilde{R}^i(-\tau_1)\tilde{R}^j(-\tau_2)\rho_R],$$

[5]It can't be expressed simply as the multiplication of two one-point reduced operators, but it's a function of one-point reduced operator and it starts from λ^2 order.

$$I_4 = -\lambda^2 \sum_{\omega,\omega'}\sum_{i,j} S_\omega^{i\dagger} O_{1S}(t_1) S_{\omega'}^{j\dagger} O_{2S}(t_2) \int_0^{t_1} d\tau_1 e^{i\omega\tau_1} \int_0^{t_2} d\tau_2\, e^{i\omega'\tau_2}\, \mathrm{Tr}_R[\tilde{R}^i(-\tau_1)\tilde{R}^j(-\tau_2)\rho_R]. \tag{A18}$$

Note that all these expressions are correct up to order λ^2 and are valid for arbitrary dynamics.

b. Expression for three-point reduced operator

Similarly, we can work out expressions for a three-point reduced operator up to λ^2 order. We receive [20]

$$[O_1(t_1)O_2(t_2)O_3(t_3)]_S = O_{1S}(t_1)O_{2S}(t_2)O_{3S}(t_3) + W_{1,2,3}\{I[O_{1S}(t_1), O_{2S}(t_2)]O_{3S}(t_3)\} + W_{1,2,3}\{O_{1S}(t_1)I[O_{2S}(t_2), O_{3S}(t_3)]\} + W_{1,2,3}\{I[O_{1S}(t_1), O_{3S}(t_3)]O_{2S}(t_2)\}. \tag{A19}$$

The operator $W_{1,2,3}$ makes sure that the operator product is ordered such that O_{1S} comes before O_{2S}, and O_{2S} comes before O_{3S}. Let us illustrate this by one example. Considering the last term of the above equation (A19) we get

$$W_{1,2,3}\{I[O_{1S}(t_1), O_{3S}(t_3)]O_{2S}(t_2)\} = \lambda^2 \sum_{n_0^1, l_0^1, n_0^2, l_0^2}\ \sum_{\alpha,\beta,\gamma,\gamma_0,\gamma_0'} \left(K_{\gamma_0\alpha}^{n_0^1}(t_1)\right)^\dagger O_{1S}(t_1) K_{\gamma_0\gamma}^{l_0^1}(t_1) O_{2S}(t_2) \times i^{n_0^1+n_0^2-l_0^1-l_0^2}\left(K_{\gamma_0'\gamma}^{n_0^2}(t_3)\right)^\dagger O_{3S}(t_3) K_{\gamma_0'\beta}^{l_0^2}(t_3)\rho_{R\beta\alpha}, \tag{A20}$$

such that $n_0^1 + l_0^1 = 1$ and $n_0^2 + l_0^2 = 1$.

c. Expression for four-point reduced operators

We can now calculate the four-point function as well. The four-point reduced operator (up to λ^2 order) is [20]

$$\begin{aligned}[O_1(t_1)O_2(t_2)O_3(t_3)O_4(t_4)]_S = {} & O_{1S}(t_1)O_{2S}(t_2)O_{3S}(t_3)O_{4S}(t_4) + W_{1,2,3,4}\{I[O_{1S}(t_1), O_{2S}(t_2)]O_{3S}(t_3)O_{4S}(t_4)\} \\ & + W_{1,2,3,4}\{O_{1S}(t_1)I[O_{2S}(t_2), O_{3S}(t_3)]O_{4S}(t_4)\} + W_{1,2,3,4}\{I[O_{1S}(t_1), O_{3S}(t_3)]O_{2S}(t_2)O_{4S}(t_4)\} \\ & + W_{1,2,3,4}\{I[O_{4S}(t_4), O_{3S}(t_3)]O_{2S}(t_2)O_{1S}(t_1)\} + W_{1,2,3,4}\{I[O_{1S}(t_1), O_{4S}(t_4)]O_{3S}(t_3)O_{2S}(t_2)\} \\ & + W_{1,2,3,4}\{O_{1S}(t_1)I[O_{2S}(t_2), O_{4S}(t_4)]O_{3S}(t_3)\},\end{aligned} \tag{A21}$$

where functions W and I have the same property as before. For our purposes we need to calculate the explicit form of the function I. Below we give an explicit example.

2. Results in the Markovian limit

In the Markovian limit [20] the bath correlation function, i.e., $\mathrm{Tr}_R[\tilde{R}(t)\tilde{R}(t-\tau)\rho_R]$, is a rapidly decaying function of τ. Using this property we can show that the general equation (A10) reduces to the well-known master equation for the reduced operator [20],

$$\frac{d}{dt}O_S(t) = iH_S O_S(t) + (i\lambda)^2 \sum_{\omega,\omega'}\sum_{i,j} J^{ij}(\omega)\left[S_\omega^{i^\dagger} S_{\omega'}^j O_S(t) - S_\omega^{i^\dagger} O_S(t) S_{\omega'}^j\right] + \text{H.c.}, \tag{A22}$$

where $J^{ij}(\omega)$ is the Fourier transformation of the bath correlation functions, and S_ω^i is the Fourier decomposition of $\tilde{S}^i(t)$ and is given as

$$J^{ij}(\omega) = \int_0^\infty d\tau e^{-i\omega\tau}\, \mathrm{Tr}_R[\tilde{R}^i(0)\tilde{R}^j(-\tau)\rho_R], \quad \tilde{S}^i(t) = \sum_\omega S_\omega^i e^{i\omega t}. \tag{A23}$$

One can also simplify the expressions for the I in the Markovian limit. Let us first analyze Eq. (A17) in this limit. As mentioned before, since in the Markovian limit the bath correlation function $\mathrm{Tr}_R[\tilde{R}(-\tau_1)\tilde{R}(-\tau_2)\rho_R]$ is a rapidly decaying function of $\tau_2 - \tau_1$ only, this implies we can ignore the bath correlation after a characteristic timescale τ_B, determined by bath dynamics. Using this fact, it becomes easy to analyze Eq. (A17).

Let us now make an important choice and assume that t_2 is the maximum time, i.e., $t_2 > t_1$. One can easily perform the calculation in the other limit as well. We now proceed and first compute the τ_2 integration. Significant contribution to the integral will come within the range $|\tau_2 - \tau_1| \leqslant \tau_B$, which gives $\tau_1 - \tau_B \leqslant \tau_2 \leqslant \tau_B + \tau_1$. Using this fact, we can write Eq. (A17) as

$$\begin{aligned} I_1 &= -\lambda^2 \sum_{\omega,\omega'}\sum_{i,j} O_{1S}(t_1) S_\omega^i O_{2S}(t_2) S_{\omega'}^j \int_0^{t_1} d\tau_1 e^{-i\omega\tau_1} \int_{\tau_1-\tau_B}^{\tau_1+\tau_B} d\tau_2 e^{-i\omega'\tau_2}\mathrm{Tr}_R[\tilde{R}^i(-\tau_1)\tilde{R}^j(-\tau_2)\rho_R] \\ &= -\lambda^2 \sum_{\omega,\omega'}\sum_{i,j} O_{1S}(t_1) S_\omega^i O_{2S}(t_2) S_{\omega'}^j \int_0^{t_1} d\tau_1 e^{-i\omega\tau_1} \int_{-\tau_B}^{\tau_B} d\tau e^{-i\omega'(\tau_1-\tau)}\mathrm{Tr}_R[\tilde{R}^i(0)\tilde{R}^j(\tau)\rho_R], \end{aligned} \tag{A24}$$

where in the last line we have used the variable $\tau = \tau_1 - \tau_2$ to rewrite the integral. In the Markovian limit, the correlation function decays very fast beyond τ_B, which implies that the integration limit in the last line of (A24) can be extended to infinity. This gives

$$I_1 = -\lambda^2 \sum_{\omega,\omega'} \sum_{i,j} O_{1S}(t_1) S^i_\omega O_{2S}(t_2) S^j_{\omega'} \int_0^{t_1} d\tau_1 e^{-i(\omega+\omega')\tau_1} \int_{-\infty}^{\infty} d\tau e^{i\omega'\tau}\, \mathrm{Tr}_R[\tilde{R}^i(0)\tilde{R}^j(\tau)\rho_R],$$
$$= -\sum_{\omega,\omega'} \sum_{i,j} O_{1S}(t_1) S^i_\omega O_{2S}(t_2) S^j_{\omega'} \alpha_1(\omega,\omega',t_1) \beta_1^{ij}(\omega'), \tag{A25}$$

where in the last line we have defined α_1 and β_1 for simplicity, and their explicit expression is given below (A27). By following the identical steps, we can find I_2, I_3, and I_4,

$$I_2 = \sum_{\omega,\omega'} \sum_{i,j} S^{i\dagger}_\omega O_{1S}(t_1) O_{2S}(t_2) S^j_{\omega'}\ \alpha_2(\omega,\omega',t_1)\ \beta_2^{ij}(\omega'),$$
$$I_3 = \sum_{\omega,\omega'} \sum_{i,j} O_{1S}(t_1) S^i_\omega S^{j\dagger}_{\omega'} O_{2S}(t_2)\ \alpha_3(\omega,\omega',t_1)\ \beta_3^{ij}(\omega'),$$
$$I_4 = -\sum_{\omega,\omega'} \sum_{i,j} S^{i\dagger}_\omega O_{1S}(t_1) S^{j\dagger}_{\omega'} O_{2S}(t_2)\ \alpha_4(\omega,\omega',t_1)\ \beta_4^{ij}(\omega'), \tag{A26}$$

where

$$\alpha_1(\omega,\omega',t_1) = \lambda^2 \int_0^{t_1} d\tau_1 e^{-i(\omega+\omega')\tau_1} \quad \beta_1^{ij}(\omega') = \int_{-\infty}^{\infty} d\tau\, e^{i\omega'\tau}\, \mathrm{Tr}_R[\tilde{R}^i(0)\tilde{R}^j(\tau)\rho_R]$$
$$\alpha_2(\omega,\omega',t_1) = \lambda^2 \int_0^{t_1} d\tau_1 e^{i(\omega-\omega')\tau_1} \quad \beta_3^{ij}(\omega') = \int_{-\infty}^{\infty} d\tau\, e^{-i\omega'\tau}\, \mathrm{Tr}_R[\tilde{R}^i(0)\tilde{R}^j(\tau)\rho_R]$$
$$\alpha_3(\omega,\omega',t_1) = \lambda^2 \int_0^{t_1} d\tau_1 e^{-i(\omega-\omega')\tau_1} \quad \beta_2^{ij}(\omega') = \beta_1^{ij}(\omega')$$
$$\alpha_4(\omega,\omega',t_1) = \lambda^2 \int_0^{t_1} d\tau_1 e^{i(\omega+\omega')\tau_1} \quad \beta_4^{ij}(\omega') = \beta_3^{ij}(\omega'). \tag{A27}$$

Using Eqs. (A25), (A26) we get the explicit form of the irreducible part I [Eq. (A14)] in the Markovian limit.

3. Secular approximation

We can further simplify the expression of α_i's, as defined in Eq. (A27), using the secular approximation. Let us first substitute $s_1 = \lambda^2\tau_1$ and $\sigma = \lambda^2 t_1$ in the expression of α_1 defined in (A27),

$$\alpha_1(\omega,\omega',t_1) = \lambda^2 \int_0^{t_1} d\tau_1 e^{-i(\omega+\omega')\tau_1} = \int_0^{\sigma} ds_1 e^{-i\frac{(\omega+\omega')}{\lambda^2}s_1}. \tag{A28}$$

Now the Riemann-Lebesgue lemma states that if $f(t)$ is an integrable function in $[a,b]$, then

$$\lim_{x\to\infty} \int_a^b dt e^{ixt} f(t) = 0. \tag{A29}$$

In the weak-coupling limit, i.e., in the limit $\lambda \to 0$ (keeping s_1 and σ finite), if we compare Eq. (A28) with the Riemann-Lebesgue lemma (A29), we can conclude that α_1 is nonzero only when $(\omega+\omega') = 0$, i.e., $\alpha_1(\omega,\omega',t_1) = \lambda^2 t_1 \delta_{\omega,-\omega'}$. Similarly, we can show that $\alpha_4(\omega,\omega',t_1) = \alpha_1(\omega,\omega',t_1) = \lambda^2 t_1 \delta_{\omega,-\omega'}$ and $\alpha_2(\omega,\omega',t_1) = \alpha_3(\omega,\omega',t_1) = \lambda^2 t_1 \delta_{\omega,\omega'}$. We use this Markov-secular approximation to obtain our results for the dissipative spin-boson model, as presented in the main text.

Note that using the same Markovian-secular approximation, the quantum master equation for the one-point reduced operator Eq. (A22) reduces to

$$\frac{d}{dt} O_S(t) = iH_S O_S(t) + (i\lambda)^2 \sum_{\omega} \sum_{i,j} J^{ij}(\omega)\big[S^{i^\dagger}_\omega S^j_\omega O_S(t) - S^{i^\dagger}_\omega O_S(t) S^j_\omega\big] + \text{H.c.} \tag{A30}$$

As mentioned earlier, this master equation is used to derive Eq. (29) for the dissipative spin-boson model in the main text.

APPENDIX B: SOME FURTHER DETAILS ON QRT IN THE MARKOVIAN LIMIT

In this Appendix we discuss some further details on QRT in the Markovian limit and in particular, some useful properties of the irreducible function I as also discussed in the main text. One of the interesting properties of the function I in (A25), (A26) is to note that its dependence on the maximum time is quite simple. This in turn helps us to obtain the quantum regression theorem.

1. Irreducible part for two-point function

Let us first set $O_2 = A_\mu$ and assume $t_1 < t_2$ in Eq. (A14). Taking the derivative of Eq. (A14) with respect to t_2, the first term of the right-hand side of Eq. (A14) trivially gives the regression-type form [using Eq. (8)]. Equations (A16), (A25)–(A27) gives us the explicit form of the second term, i.e., the irreducible part of Eq. (A14). Notice that the t_2 dependency in the irreducible part $I[O_S(t_1), A_{\mu S}(t_2)]$ comes from $A_{\mu S}(t_2)$ only. So if we take the derivative of I with respect to t_2, it simply gives

$$\frac{d}{dt_2} I[O_S(t_1), A_{\mu S}(t_2)] = \sum_\lambda M_{\mu\lambda} I[O_S(t_1), A_{\lambda S}(t_2)]. \tag{B1}$$

In addition, in the expression of $I[O_S(t_1), A_{\mu S}(t_2)]$ [see Eqs. (A25)–(A27)], it is easy to notice that if we swap the position of $O_S(t_1)$ and $A_{\mu S}(t_2)$, Eq. (B1) still holds, i.e.,

$$\frac{d}{dt_2} I[A_{\mu S}(t_2), O_S(t_1)] = \sum_\lambda M_{\mu\lambda} I[A_{\lambda S}(t_2), O_S(t_1)]. \tag{B2}$$

Interestingly, if we consider $t_2 < t_1$ and take derivative of Eq. (A14) with respect to t_2, then we will not get the regression-type form. The simple reason behind this in the Heisenberg picture is that in the expression of I, t_2 dependency comes from both $A_{\mu S}$ and α_i's defined in (A27). Now taking derivative with respect to t_2 will give rise to complicated terms,[6] i.e.,

$$\frac{d}{dt_2} \langle O(t_1) A_\mu(t_2)\rangle \neq \sum_\lambda M_{\mu\lambda} \langle O(t_1) A_\lambda(t_2)\rangle. \tag{B3}$$

2. Irreducible part for three-point function

Now let us first set $O_3 = A_\mu$ and assume $t_i < t_3$ with $i = 1, 2$ in Eq. (A19). Taking the derivative of Eq. (A19) with respect to t_3, the first term of the right-hand side gives [using Eq. (8)]

$$\frac{d}{dt_3} [O_{1S}(t_1) O_{2S}(t_2) A_{\mu S}(t_3)] = \sum_\lambda M_{\mu\lambda} [O_{1S}(t_1) O_{2S}(t_2) A_{\lambda S}(t_3)]. \tag{B4}$$

Now the third term of the right-hand side of Eq. (A19) is, using Eq. (A16),

$$W_{1,2,3}\{O_{1S}(t_1) I[O_{2S}(t_2), A_{\mu S}(t_3)]\} = W_{1,2,3}\{O_{1S}(t_1)(I_1 + I_2 + I_3 + I_4)\}. \tag{B5}$$

The first term of the right-hand side of Eq. (B5) in the Markovian limit is given by, using Eq. (A26),

$$W_{1,2,3}\{O_{1S}(t_1) I_1[O_{2S}(t_2), A_{\mu S}(t_3)]\} = -\lambda^2 \sum_{\omega,\omega'} \sum_{i,j} O_{1S}(t_1) O_{2S}(t_2) S^i_\omega A_{\mu S}(t_3) S^j_{\omega'}\, \alpha_1(\omega, \omega', t_2)\, \beta_1^{ij}(\omega'). \tag{B6}$$

If we differentiate Eq. (B6) with respect to t_3, we get [using Eq. (8)]

$$\frac{d}{dt_3} W_{1,2,3}\{O_{1S}(t_1) I_1[O_{2S}(t_2), A_{\mu S}(t_3)]\} = \sum_\lambda M_{\mu\lambda} W_{1,2,3}\{O_{1S}(t_1) I_1[O_{2S}(t_2), A_{\lambda S}(t_3)]\}, \tag{B7}$$

since in the expression of $W_{1,2,3}\{O_{1S}(t_1) I_1[O_{2S}(t_2), A_{\mu S}(t_3)]\}$, t_3 dependency comes from $A_{\mu S}(t_3)$ only. Similarly, we can show that all the other terms of Eq. (B5) follows the identical equation to (B7). This finally gives

$$\frac{d}{dt_3} W_{1,2,3}\{O_{1S}(t_1) I[O_{2S}(t_2), A_{\mu S}(t_3)]\} = \sum_\lambda M_{\mu\lambda} W_{1,2,3}\{O_{1S}(t_1) I[O_{2S}(t_2), A_{\lambda S}(t_3)]\}. \tag{B8}$$

3. Irreducible part for four-point function

Set $O_4 = A_\mu$, $t_i < t_4$ with $i = 1, 2, 3$ in Eq. (A21), and if we take the derivative of Eq. (A21) with respect to t_4 then the first term of the right-hand side will simply give [using Eq. (8)]

$$\frac{d}{dt_4} [O_{1S}(t_1) O_{2S}(t_2) O_{3S}(t_3) A_{\mu S}(t_4)] = \sum_\lambda M_{\mu\lambda} [O_{1S}(t_1) O_{2S}(t_2) O_{3S}(t_3) A_{\lambda S}(t_4)]. \tag{B9}$$

We can straightforwardly conclude that the second term will obey the following equation [using Eq. (8)]:

$$\frac{d}{dt_4} W_{1,2,3,4}\{I[O_{1S}(t_1), O_{2S}(t_2)] O_{3S}(t_3) A_{\mu S}(t_4)\} = \sum_\lambda M_{\mu\lambda} W_{1,2,3,4}\{I[O_{1S}(t_1), O_{2S}(t_2)] O_{3S}(t_3) A_{\lambda S}(t_4)\}. \tag{B10}$$

[6]We shall see that this observation is true for the general multi-time correlation function. In the Schrödinger picture as well, we have seen that the regression theorem holds only for derivatives with respect to the highest time.

By giving the exactly similar argument we can show that the third and fourth term of the right-hand side of Eq. (A21) will follow identical to the above equation. Now using Eq. (B1) we can show that the fifth term of the right-hand side of the same equation will give

$$\frac{d}{dt_4}W_{1,2,3,4}\{I[A_{\mu S}(t_4), O_{3S}(t_3)]O_{2S}(t_2)O_{1S}(t_1)\} = \sum_{\lambda} M_{\mu\lambda}W_{1,2,3,4}\{I[A_{\lambda S}(t_4), O_{3S}(t_3)]O_{2S}(t_2)O_{1S}(t_1)\}, \tag{B11}$$

since in the expression of $I[A_{\mu S}(t_4), O_{3S}(t_3)]$, the t_4 dependency comes from $A_{\mu S}(t_4)$ only. By giving a similar argument, we can show that the last two terms of Eq. (A21) will also follow the identical equation. Then, finally, we arrive at the following equation [using Eq. (A21)]:

$$\frac{d}{dt_4}[O_1(t_1)O_2(t_2)O_3(t_3)A_\mu(t_4)]_S = \sum_{\lambda} M_{\mu\lambda}[O_1(t_1)O_2(t_2)O_3(t_3)A_\lambda(t_4)]_S. \tag{B12}$$

APPENDIX C: OUT-OF-TIME-ORDERED CORRELATORS (OTOCS)

In this section we calculate the out-of-time-order correlator (OTOC). More specifically, we want to derive a regression-type theorem for OTOC [19]. Now set $t_1 = t_3$, $t_2 = t_4$, $t_2 > t_1$, $O_2(t_2) = A_\mu(t_2)$, and $O_4(t_2) = A_\nu(t_2)$ in Eq. (A21), and then we get

$$\begin{aligned}[O_1(t_1)A_\mu(t_2)O_3(t_1)A_\nu(t_2)]_S = {} & O_{1S}(t_1)A_{\mu S}(t_2)O_{3S}(t_1)A_{\nu S}(t_2) + W_{1,2,3,4}\{I[O_{1S}(t_1), A_{\mu S}(t_2)]O_{3S}(t_1)A_{\nu S}(t_2)\}\\ &+ W_{1,2,3,4}\{O_{1S}(t_1)I[A_{\mu S}(t_2), O_{3S}(t_1)]A_{\nu S}(t_2)\} + W_{1,2,3,4}\{I[O_{1S}(t_1), O_{3S}(t_1)]A_{\mu S}(t_2)A_{\nu S}(t_2)\}\\ &+ W_{1,2,3,4}\{I[A_{\nu S}(t_2), O_{3S}(t_1)]A_{\mu S}(t_2)O_{1S}(t_1)\} + W_{1,2,3,4}\{I[O_{1S}(t_1), A_{\nu S}(t_2)]O_{3S}(t_1)A_{\mu S}(t_2)\}\\ &+ W_{1,2,3,4}\{O_{1S}(t_1)I[A_{\mu S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\}.\end{aligned} \tag{C1}$$

Differentiate the above equation by t_2, and then the first term of the right-hand side gives

$$\frac{d}{dt_2}[O_{1S}(t_1)A_{\mu S}(t_2)O_{3S}(t_1)A_{\nu S}(t_2)] = \sum_{\lambda} M_{\mu\lambda}[O_{1S}(t_1)A_{\lambda S}(t_2)O_{3S}(t_1)A_{\nu S}(t_2)] + \sum_{\lambda'} M_{\nu\lambda'}[O_{1S}(t_1)A_{\mu S}(t_2)O_{3S}(t_1)A_{\lambda' S}(t_2)].$$

Similarly, the second term of the right-hand side gives

$$\begin{aligned}\frac{d}{dt_2}W_{1,2,3,4}\{I[O_{1S}(t_1), A_{\mu S}(t_2)]O_{3S}(t_1)A_{\nu S}(t_2)\} = {} & \sum_{\lambda} M_{\mu\lambda}W_{1,2,3,4}\{I[O_{1S}(t_1), A_{\lambda S}(t_2)]O_{3S}(t_1)A_{\nu S}(t_2)\}\\ &+ \sum_{\lambda'} M_{\nu\lambda'}W_{1,2,3,4}\{I[O_{1S}(t_1), A_{\mu S}(t_2)]O_{3S}(t_1)A_{\lambda' S}(t_2)\}.\end{aligned} \tag{C2}$$

All the other terms in (C1) also follow the identical equation except for the last term. The last term gives

$$\frac{d}{dt_2}W_{1,2,3,4}\{O_{1S}(t_1)I[A_{\mu S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\} = W_{1,2,3,4}\left\{O_{1S}(t_1)\frac{d}{dt_2}(I[A_{\mu S}(t_2), A_{\nu S}(t_2)])O_{3S}(t_1)\right\}. \tag{C3}$$

Using the expression of I,

$$\frac{d}{dt_2}I[A_{\mu S}(t_2), A_{\nu S}(t_2)] = \sum_{\lambda} M_{\mu\lambda}I[A_{\lambda S}(t_2), A_{\nu S}(t_2)] + \sum_{\lambda'} M_{\nu\lambda'}I[A_{\mu S}(t_2), A_{\lambda' S}(t_2)] + F[A_{\mu S}(t_2), A_{\nu S}(t_2)], \tag{C4}$$

where

$$\begin{aligned}F[A_{\mu S}(t_2), A_{\nu S}(t_2)] = {} & -\lambda^2 \sum_{\omega,\omega'}\sum_{i,j} S_\omega^{i\dagger}A_{\mu S}(t_2)S_{\omega'}^{j\dagger}A_{\nu S}(t_2)\, e^{i(\omega+\omega')t_2}\ \beta_4^{ij}(\omega')\\ &+ \lambda^2 \sum_{\omega,\omega'}\sum_{i,j} S_\omega^{i\dagger}A_{\mu S}(t_2)A_{\nu S}(t_2)S_{\omega'}^{j}\, e^{i(\omega-\omega')t_2}\ \beta_2^{ij}(\omega') + \lambda^2 \sum_{\omega,\omega'}\sum_{i,j} A_{\mu S}(t_2)S_\omega^{i}S_{\omega'}^{j\dagger}A_{\nu S}(t_2)\, e^{-i(\omega-\omega')t_2}\ \beta_3^{ij}(\omega')\\ &- \lambda^2 \sum_{\omega,\omega'}\sum_{i,j} A_{\mu S}(t_2)S_\omega^{i}A_{\nu S}(t_2)S_{\omega'}^{j}\, e^{-i(\omega+\omega')t_2}\ \beta_1^{ij}(\omega'),\end{aligned} \tag{C5}$$

and Eq. (C3) becomes

$$\begin{aligned}\frac{d}{dt_2}W_{1,2,3,4}\{O_{1S}(t_1)I[A_{\mu S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\} = {} & \sum_{\lambda} M_{\mu\lambda}W_{1,2,3,4}\{O_{1S}(t_1)I[A_{\lambda S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\}\\ &+ \sum_{\lambda'} M_{\nu\lambda'}W_{1,2,3,4}\{O_{1S}(t_1)I[A_{\mu S}(t_2), A_{\lambda' S}(t_2)]O_{3S}(t_1)\}\\ &+ W_{1,2,3,4}\{O_{1S}(t_1)F[A_{\mu S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\}.\end{aligned} \tag{C6}$$

Finally, adding all the different contributions we receive

$$\frac{d}{dt_2}[O_1(t_1)A_\mu(t_2)O_3(t_1)A_\nu(t_2)]_S = \sum_\lambda M_{\mu\lambda}[O_1(t_1)A_\lambda(t_2)O_3(t_1)A_\nu(t_2)]_S + \sum_{\lambda'} M_{\nu\lambda'}[O_1(t_1)A_\mu(t_2)O_3(t_1)A_{\lambda'}(t_2)]_S + W_{1,2,3,4}\{O_{1S}(t_1)F[A_{\mu S}(t_2), A_{\nu S}(t_2)]O_{3S}(t_1)\}. \quad (C7)$$

We observe that the regression theorem for OTOC takes a different form than discussed in the previous section.

APPENDIX D: EXAMPLE—DISSIPATIVE SPIN HALF SYSTEM

In this Appendix we provide the details of the calculation for the dissipative spin-1/2 system as discussed in the main text.

1. Correlation function in the Heisenberg picture

Here we give out expressions for three- and four-point reduced operators using the Heisenberg picture.

a. Three-point functions

To get the three-point reduced operators we have to calculate all the W terms of Eq. (A19). Once we receive the expression for the irreducible component I, it is easy to compute W. Here we give out expressions for various W's, which are necessary to compute the three-point correlation $\langle\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\rangle$. In this case to compute the W's we need $I[\sigma_{xS}(t), \sigma_{xS}(t')]$. This is already computed in the main text (see Sec. IV A) and is given as

$$I[\sigma_{xS}(t_1), \sigma_{xS}(t_2)] = \gamma t_1 e^{i\omega_0'(t_2-t_1)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \quad (D1)$$

We then obtain the various W's as

$$W_{1,2,3}\{I[\sigma_{xS}(t_1), \sigma_{xS}(t_2)]\sigma_{xS}(t_3)\} = \gamma t_1\begin{bmatrix}0 & e^{i\omega_0'(-t_1+t_2+t_3)}\\ e^{i\omega_0'(-t_1+t_2-t_3)} & 0\end{bmatrix}, \quad (D2)$$

$$W_{1,2,3}\{I[\sigma_{xS}(t_1), \sigma_{xS}(t_3)]\sigma_{xS}(t_2)\} = -\gamma t_1\begin{bmatrix}0 & e^{i\omega_0'(-t_1+t_2+t_3)}\\ e^{i\omega_0'(-t_1-t_2+t_3)} & 0\end{bmatrix}, \quad (D3)$$

$$W_{1,2,3}\{\sigma_{xS}(t_1)I[\sigma_{xS}(t_2), \sigma_{xS}(t_3)]\} = \gamma t_2\begin{bmatrix}0 & e^{i\omega_0'(t_1-t_2+t_3)}\\ e^{i\omega_0'(-t_1-t_2+t_3)} & 0\end{bmatrix}. \quad (D4)$$

As a result, the three-point reduced operator is given as

$$\sigma_{xS}(t_1)\sigma_{xS}(t_2)\sigma_{xS}(t_3) = \left[1 - \frac{\gamma}{2}(t_1+t_2+t_3)\right]\begin{bmatrix}0 & e^{i\omega_0'(t_1-t_2+t_3)}\\ e^{i\omega_0'(-t_1+t_2-t_3)} & 0\end{bmatrix}. \quad (D5)$$

b. Four-point correlation function

Now we want to verify the regression theorem for four-point function (24) in this example. To do that let us first set $O_1 = O_2 = O_3 = \sigma_x, A_\mu = \{\sigma_x, \sigma_y\}$ in Eq. (24) and assume $t_1 < t_2 < t_3 < t_4$. We then compute the following reduced operators, which turns out to be *diagonal*. More explicitly, the matrix elements are given as

$$[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_x(t_4)]_S|_{11} = \left[1 - \tfrac{1}{2}\gamma(t_1+t_3-t_2-t_4)\right]e^{i\omega_0'(t_1+t_3-t_2-t_4)} + \gamma\, t_1\, e^{i\omega_0'(-t_1+t_2-t_3+t_4)} + \gamma\,(t_3-t_2)e^{i\omega_0'(t_1-t_2-t_3+t_4)}$$

$$[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_x(t_4)]_S|_{22} = \left[1 - \tfrac{1}{2}\gamma(t_1+t_2+t_3+t_4)\right]e^{i\omega_0'(-t_1+t_2-t_3+t_4)} + \gamma\, t_1\, e^{i\omega_0'(-t_1+t_2-t_3+t_4)} + \gamma\, t_3\, e^{i\omega_0'(-t_1+t_2-t_3+t_4)},$$

and

$$[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_x(t_4)]_S|_{12} = [\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_x(t_4)]_S|_{21} = 0. \quad (D6)$$

In a similar way we receive

$$[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_y(t_4)]_S|_{11} = i\left[1 - \tfrac{1}{2}\gamma(t_1+t_3-t_2-t_4)\right]e^{i\omega_0'(t_1+t_3-t_2-t_4)} - i\,\gamma\, t_1\, e^{i\omega_0'(-t_1+t_2-t_3+t_4)} - i\,\gamma\,(t_3-t_2)\, e^{i\omega_0'(t_1-t_2-t_3+t_4)}$$

$$[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_y(t_4)]_S|_{22} = -i\left[1 - \tfrac{1}{2}\gamma(t_1+t_2+t_3+t_4)\right]e^{i\omega_0'(-t_1+t_2-t_3+t_4)} - i\,\gamma\, t_1\, e^{i\omega_0'(-t_1+t_2-t_3+t_4)} - i\,\gamma\, t_3\, e^{i\omega_0'(-t_1+t_2-t_3+t_4)},$$

$$[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_y(t_4)]_S|_{12} = [\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_y(t_4)]_S|_{21} = 0. \quad (D7)$$

Now, taking the derivative of Eq. (D6) with respect to t_4, we receive the QRT as

$$\frac{d}{dt_4}[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_x(t_4)]_S = -\frac{1}{2}\gamma\,[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_x(t_4)]_S - \omega_0'[\sigma_x(t_1)\sigma_x(t_2)\sigma_x(t_3)\sigma_y(t_4)]_S. \tag{D8}$$

This immediately verifies Eq. (24). Note that we choose here a particular order of time, but note that one can show the validity of the QRT as long as $t_i < t_4$ with $i = 1, 2, 3$. There are no constraints on the order of (t_1, t_2, t_3).

APPENDIX E: EXTENSION OF QRT FOR TWO-POINT REDUCED OPERATORS FOR NON-MARKOVIAN DYNAMICS

In this Appendix we provide details about the extension of the QRT for two-point reduced operators for systems following non-Markovian dynamics. The main results for this Appendix are provided in Sec. V of the main text.

Recall that the two-point reduced operator can be written as

$$[O_1(t_1)O_2(t_2)]_S = O_{1S}(t_1)O_{2S}(t_2) + I[O_{1S}(t_1), O_{2S}(t_2)], \tag{E1}$$

where the complete irreducible function I consists of four components, $I = I_1 + I_2 + I_3 + I_4$. We once again assume t_2 as the maximum time, i.e., $t_2 > t_1$, and then using Eqs. (A17) and (A18), we obtain the irreducible components for the non-Markovian dynamics correct up to order λ^2. We receive

$$\begin{aligned}
I_1[O_{1S}(t_1), O_{2S}(t_2)] &= -\lambda^2\int_0^{t_1}d\tau_1\int_0^{t_2}d\tau_2[\alpha(\tau_2-\tau_1)O_{1S}(t_1)\tilde{L}^\dagger(-\tau_1)O_{2S}(t_2)\tilde{L}(-\tau_2)]I_2[O_{1S}(t_1), O_{2S}(t_2)]\\
I_2[O_{1S}(t_1), O_{2S}(t_2)] &= \lambda^2\int_0^{t_1}d\tau_1\int_0^{t_2}d\tau_2[\alpha(\tau_2-\tau_1)\tilde{L}^\dagger(-\tau_1)O_{1S}(t_1)O_{2S}(t_2)\tilde{L}(-\tau_2)]I_3[O_{1S}(t_1), O_{2S}(t_2)]\\
I_3[O_{1S}(t_1), O_{2S}(t_2)] &= \lambda^2\int_0^{t_1}d\tau_1\int_0^{t_2}d\tau_2\,[\alpha(\tau_2-\tau_1)O_{1S}(t_1)\tilde{L}^\dagger(-\tau_1)\tilde{L}(-\tau_2)O_{2S}(t_2)]I_4[O_{1S}(t_1), O_{2S}(t_2)]\\
I_4[O_{1S}(t_1), O_{2S}(t_2)] &= -\lambda^2\int_0^{t_1}d\tau_1\int_0^{t_2}d\tau_2\,[\alpha(\tau_2-\tau_1)\tilde{L}^\dagger(-\tau_1)O_{1S}(t_1)\tilde{L}(-\tau_2)O_{2S}(t_2)],
\end{aligned} \tag{E2}$$

where recall that $\tilde{L}(t) = U_0(t)LU_0^\dagger(t)$ is the system operator in the interaction picture that is coupled with the bath, and $\alpha(\tau)$ is the bath correlation function. Then by differentiating Eq. (E2) with respect to the maximum time t_2 we receive

$$\begin{aligned}
\frac{d}{dt_2}I_1 &= -\lambda^2\bigg[\int_0^{t_1}\int_0^{t_2}d\tau_1 d\tau_2\,\alpha(\tau_2-\tau_1)(O_{1S}(t_1)\tilde{L}^\dagger(-\tau_1)[iH_S, O_{2S}(t_2)]\tilde{L}(-\tau_2))\\
&\quad-\int_0^{t_1}d\tau_1\,\alpha(t_2-\tau_1)O_{1S}(t_1)\tilde{L}^\dagger(-\tau_1)O_{2S}(t_2)\tilde{L}(-t_2)\bigg]\\
\frac{d}{dt_2}I_2 &= \lambda^2\bigg[\int_0^{t_1}\int_0^{t_2}d\tau_1 d\tau_2\,\alpha(\tau_2-\tau_1)(\tilde{L}^\dagger(-\tau_1)O_{1S}(t_1)[iH_S, O_{2S}(t_2)]\tilde{L}(-\tau_2))\\
&\quad+\int_0^{t_1}d\tau_1\,\alpha(t_2-\tau_1)\tilde{L}^\dagger(-\tau_1)O_{1S}(t_1)O_{2S}(t_2)\tilde{L}(-t_2)\bigg]\\
\frac{d}{dt_2}I_3 &= \lambda^2\bigg[\int_0^{t_1}\int_0^{t_2}d\tau_1 d\tau_2\,\alpha(\tau_2-\tau_1)(O_{1S}(t_1)\tilde{L}^\dagger(-\tau_1)\tilde{L}(-\tau_2)[iH_S, O_{2S}(t_2)])\\
&\quad+\int_0^{t_1}d\tau_1\,\alpha(t_2-\tau_1)O_{1S}(t_1)\tilde{L}^\dagger(-\tau_1)\tilde{L}(-t_2)O_{2S}(t_2)\bigg]\\
\frac{d}{dt_2}I_4 &= -\lambda^2\bigg[\int_0^{t_1}\int_0^{t_2}d\tau_1 d\tau_2\,\alpha(\tau_2-\tau_1)(\tilde{L}^\dagger(-\tau_1)O_{1S}(t_1)\tilde{L}(-\tau_2)[iH_S, O_{2S}(t_2)])\\
&\quad-\int_0^{t_1}d\tau_1\,\alpha(t_2-\tau_1)\tilde{L}^\dagger(-\tau_1)O_{1S}(t_1)\tilde{L}(-t_2)O_{2S}(t_2)\bigg].
\end{aligned} \tag{E3}$$

Now, as done for the Markovian dynamics, in this case also we assume that there exists a closed set of system operators such that

$$\frac{d}{dt}A_{\mu S}(t) = \sum_\lambda M_{\mu\lambda}(t)A_{\lambda S}(t). \tag{E4}$$

Once again, note the crucial time dependence in $M_{\mu\lambda}(t)$ for the non-Markovian dynamics. By setting the operators $O_1 = O$ and $O_2 = A_\mu$ and by using Eqs. (E3) and (E4), we receive for the first irreducible component I_1,

$$\frac{d}{dt_2} I_1[O_S(t_1), A_{\mu S}(t_2)] = \sum_\lambda M_{\mu\lambda}(t_2) I_1[O_S(t_1), A_{\lambda S}(t_2)] - \lambda^2 \int_0^{t_1} d\tau_1\, \alpha(t_2 - \tau_1) O_S(t_1) \tilde{L}^\dagger(-\tau_1) A_{\mu S}(t_2) \tilde{L}(-t_2). \tag{E5}$$

Following the similar steps we receive similar equations for I_2, I_3, and I_4, which finally provides us with the central equation presented in the main text Eq. (52). We therefore receive an extension of the QRT-like expression for the non-Markovian dynamics, where the additional correction term depends crucially on the correlation timescale of the bath correlation functions. As argued in the main text, in the Markovian limit one can show that the additional correction term vanishes and one recovers the standard QRT.

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