Effects of rotation on a magnetic quadrupole moment system around a cylindrical cavity

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We analyze the interaction of the magnetic quadrupole moment of a neutral particle with magnetic and electric fields around a cylindrical cavity in a rotating reference frame. In contrast to the energy levels yielded by a Coulomb-type potential that stems from the interaction of the magnetic quadrupole moment with a nonuniform magnetic field, we show that the effects of rotation break the degeneracy of the energy levels. Moreover, we show that the effects of rotation can gives rise to an Aharonov-Bohm-type effect. Further, we discuss the appearance of persistent currents and the revival time.

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I. INTRODUCTION

Effects of rotation on Dirac fields [1], scalar fields [2], and the Landau quantization [3] have shown a great interesting in field theory. From the perspective of the classical physics, Landau and Lifshitz [4] showed that the line element of the Minkowski space time becomes singular at large distances for a system in a uniformly rotating frame. In quantum physics, in turn, the most well-known effect associated with rotation is the phase shift that appears in interferometry experiments [5–9]. It is called the Sagnac effect [5–7]. At present, this kind of phase shift is also known as geometric quantum phase, where we can include the Mashhoon effect [10] and the Aharonov-Carmi geometric phase [11] as geometric quantum phases that stem from the effects of rotation. Besides the appearance of geometric quantum phases, effects of rotation can contribute to the energy levels of a quantum system. As shown in Refs. [12–14], the energy levels can acquire a coupling between the angular momentum and the angular velocity of the rotating frame. From the perspective of observing effects of rotation in quantum system we cite the studies of quantum Hall effect [15], spintronics [16–18], quantum rings [19–21], Bose-Einstein condensation [22], and neutral particles systems [23–27].

In this work, we search for effects of rotation on the interaction of the magnetic quadrupole moment of a neutral particle with magnetic and electric fields around a cylindrical cavity. Besides the main interest in the magnetic quadrupole moment in systems with molecules [28–31] and atoms [32,33], the interest in the magnetic quadrupole moment of atoms and molecules also extends to studies in $\mathcal{PT}$ symmetry [34,35], $\mathcal{CP}$ symmetry [36], time-reversal symmetry in molecules [37], and chiral anomaly [38]. Noncommutative quantum mechanics has been dealt with in the magnetic quadrupole system in Refs. [39,40]. Geometric quantum phases have been studied in Refs. [41,42]. Recently, we have shown that bound states can be achieved for an attractive inverse-square-type potential and Coulomb-type potentials in the magnetic quadrupole moment system [42–44]. Thereby, by searching for effects of rotation in the magnetic quadrupole moment system around a cylindrical cavity, in this work we show that the degeneracy of the energy levels of a Coulomb-type potential can be broken. We also show that the effects of rotation can gives rise to an Aharonov-Bohm-type effect [45,46]. Further, we extend our discussion to the appearance of persistent currents [21,47] and quantum revivals [48–51].

The structure of this paper is as follows. In Sec. II, we analyze the effects of rotation on a Coulomb-type potential around a cylindrical cavity. In search of bound states in the rotating reference frame, we show that the effects of rotation break the degeneracy of energy levels in contrast to the case of absence of rotation [44]. Furthermore, we show that the effects of rotation can give rise to an Aharonov-Bohm-type effect [45,46]. In Sec. III, we discuss the appearance of persistent currents [21,47]. In Sec. IV, we discuss the quantum revivals [48–51]. In Sec. V, we present our conclusions.

II. QUANTUM DESCRIPTION IN THE ROTATING REFERENCE FRAME

According to Refs. [21,52–55], we can deal with a nonrelativistic quantum system in a rotating reference frame by writing the time-independent Schrödinger equation in the form:

$$\mathcal{E}\psi = \hat{\mathbb{H}}_0\,\psi - \vec{\omega}\cdot\hat{\mathbb{L}}\,\psi. \tag{1}$$

In Eq. (1), the quantum operator $\hat{\mathbb{H}}_0$ corresponds to the Hamiltonian operator of the particle system in the absence of rotation, $\vec{\omega}$ is the angular frequency of the rotating reference frame, and the operator $\hat{\mathbb{L}}$ corresponds to the angular momentum operator.

We will study a particle system that describes the interaction of the magnetic quadrupole moment of a (moving) neutral particle with magnetic and electric fields in a rotating reference frame. In short, the Hamiltonian operator of this magnetic quadrupole moment system in the absence of

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rotation is given by (with $\hbar=1$ and $c=1$) [23,24,42]

$$\hat{\mathbb{H}}_0=\frac{1}{2m}[\hat{p}-\vec{M}\times\vec{E}]^2-\vec{M}\cdot\vec{B}, \tag{2}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields (in the laboratory frame), respectively. In addition, the vector $\vec{M}$ is defined in such a way that its components are determined by $M_i=\sum_j M_{ij}\,\partial_j$, where M_{ij} is a symmetric and traceless tensor. Moreover, $\hat{p}=-i\vec{\nabla}$ is the momentum operator and m is the mass of the neutral particle.

In recent years, one of us has studied quantum effects associated with rotation on the interaction of the magnetic quadrupole moment with a nonuniform electric field produced by a nonuniform distribution of electric charges inside a nonconducting cylinder [23,24]. In the present work, we focus on the effects of rotation and the Aharonov-Bohm-type effect, which can appear when the magnetic quadrupole moment interacts with magnetic and electric fields around a cylindrical cavity. Henceforth, we assume that the magnetic quadrupole moment of the neutral particle has the components [42]:

$$M_{rz}=M_{zr}=M;$$
$$M_{rr}=M_{\varphi\varphi}=M;$$
$$M_{zz}=-2M, \tag{3}$$

where M is a constant ($M>0$).

Next, let us consider the magnetic field produced by the current density $\vec{J}=-\frac{B_0}{r}\,\hat{\varphi}$ ($B_0>0$ is a constant and $\hat{\varphi}$ is a unit vector in the azimuthal direction) inside a long conducting cylinder, which possesses an inner radius r_0. This magnetic field is in the z direction and its expression is given by $\vec{B}_1=B_0\,\ln\frac{r}{r_0}\,\hat{z}$ [44]. From Eq. (2), the interaction of the magnetic quadrupole moment (3) with this magnetic field yields the effective scalar potential: $V_{\rm eff}(r)=-\vec{M}\cdot\vec{B}=-\frac{M\,B_0}{r}$.

In addition, when we consider the presence of the time-dependent magnetic field $\vec{B}_2=\frac{E_0\,t}{r}\,\hat{\varphi}$ ($E_0>0$ is a constant), then, it produces the induced electric field: $\vec{E}=E_0\,\ln\frac{r}{r_0}\,\hat{z}$. Thereby, when the magnetic quadrupole moment (3) interacts with the induced electric field, this interaction gives rise to the appearance of a geometric quantum phase [41,42]:

$$\phi_1=\oint\vec{A}_{\rm eff}\cdot d\vec{r}=-2\pi\,M\,E_0, \tag{4}$$

where the effective vector potential is defined as $\vec{A}_{\rm eff}=\vec{M}\times\vec{E}$.[1] It is worth noting that the magnetic quadrupole moment (3) does not interact with the time-dependent magnetic field $\vec{B}_2$. Therefore, there is no contribution to the effective scalar potential $V_{\rm eff}(r)$ that stems from the magnetic field $\vec{B}_2$ [42].

Returning to Eq. (1), hence, we have that the angular momentum operator can be written in terms of the effective vector potential $\vec{A}_{\rm eff}=\vec{M}\times\vec{E}$. Its expression is given by $\hat{\mathbb{L}}=\vec{r}\times(\hat{p}-\vec{A}_{\rm eff})$. From Eq. (4), we have $\vec{A}_{\rm eff}=\frac{\phi_1}{2\pi\,r}\,\hat{\varphi}$ [42].

[1]Note that $\vec{A}_{\rm eff}=\vec{M}\times\vec{E}=(\hat{r}\times\hat{z})\,M_{rr}(\partial_r E_z)=(-\hat{\varphi})\,\frac{M\,E_0}{r}$, where we have used Eq. (3) and $E_z=E_0\,\ln\frac{r}{r_0}$ in order to have a non-null $\vec{M}\times\vec{E}$. Thereby, with $d\vec{r}=r\,d\varphi\,\hat{\varphi}$, we have in Eq. (4): $\phi_1=\oint\vec{A}_{\rm eff}\cdot d\vec{r}=-\int_0^{2\pi}\frac{M\,E_0}{r}\times r\,d\varphi=-2\pi\,M\,E_0$.

Furthermore, for the two-dimensional system, we have $\vec{r}=r\,\hat{r}$ ($\hat{r}$ is a unit vector in the radial direction).

From now on, we consider a rotating reference frame with a constant angular velocity: $\vec{\omega}=\omega\,\hat{z}$. Then, in the region $r>r_0$, the Schrödinger equation (1) becomes (with $\hbar=1$ and $c=1$)

$$\mathcal{E}\psi=-\frac{1}{2m}\nabla^2\psi-\frac{i}{m}\frac{\phi_1}{2\pi\,r^2}\frac{\partial\psi}{\partial\varphi}+\frac{1}{2m}\left(\frac{\phi_1}{2\pi\,r}\right)^2\psi$$
$$-\frac{M\,B_0}{r}\,\psi+\omega\left[i\,\frac{\partial}{\partial\varphi}-\frac{\phi_1}{2\pi}\right]\psi, \tag{5}$$

where the operator ∇^2 is the Laplacian (in cylindrical coordinates). Let us write $\psi(r,\varphi,z)=e^{ikz}\,e^{i\ell\varphi}u(r)$, where k is a constant, $\ell=0,\pm1,\pm2,\ldots$ and $u(r)$ is an unknown function. After substituting $\psi(r,\varphi,z)=e^{ikz}\,e^{i\ell\varphi}u(r)$ into Eq. (5), we obtain the radial equation:

$$u''+\frac{1}{r}u'-\frac{\left(\ell+\frac{\phi_1}{2\pi}\right)^2}{r^2}u+\frac{2m\,MB_0}{r}\,u$$
$$+\left[2m\mathcal{E}-k^2+2m\omega\left(\ell+\frac{\phi_1}{2\pi}\right)\right]u=0. \tag{6}$$

By searching for bound states solutions to the Schrödinger equation (5), we assume that $\mathcal{E}<0$ from now on. We also take $k=0$ and define the parameter $\tau=\sqrt{-2m\mathcal{E}-2m\omega(\ell+\frac{\phi_1}{2\pi})}$. In this way, the radial equation (6) becomes:

$$u''+\frac{1}{r}u'-\frac{\left(\ell+\frac{\phi_1}{2\pi}\right)^2}{r^2}u+\frac{2m\,MB_0}{r}\,u-\tau^2\,u=0. \tag{7}$$

Next, we define the parameter $y=2\tau\,r$, and thus, Eq. (7) becomes

$$u''+\frac{1}{y}\,u'-\frac{\left(\ell+\frac{\phi_1}{2\pi}\right)^2}{y^2}\,u+\frac{\delta}{y}\,u-\frac{1}{4}\,u=0, \tag{8}$$

where $\delta=\frac{m\,MB_0}{\tau}$.

Let us take a solution to Eq. (8) in which $u(y)\to0$ when $y\to\infty$. This solution can be written as

$$u(y)=e^{-\frac{y}{2}}\,y^{|\ell+\frac{\phi_1}{2\pi}|}\,U\left(\left|\ell+\frac{\phi_1}{2\pi}\right|+\frac{1}{2}-\delta,\;2\left|\ell+\frac{\phi_1}{2\pi}\right|+1;\;y\right), \tag{9}$$

where $U(a,b;y)=U(|\ell+\frac{\phi_1}{2\pi}|+\frac{1}{2}-\delta,\;2|\ell+\frac{\phi_1}{2\pi}|+1;\;y)$ is the confluent hypergeometric function regular at $y\to\infty$ [56].

Then, let us impose that the wave function vanishes at $r=r_0$ (where r_0 is the inner radius of the long conducting cylinder), i.e., there is an infinite wall at $r=r_0$ [44,57,58]. With $y_0=2\tau\,r_0$, we have that $u(y_0)=0$. With the purpose of obtaining the eigenvalues of energy explicitly, we consider the case where $U(a,b;y_0)\propto\cos(\sqrt{2b\,y_0-4a\,y_0}-\frac{b\pi}{2}+a\pi+\frac{\pi}{4})$. According to Ref. [56], this particular case of the confluent hypergeometric function $U(a,b;y)$ is given for y_0 and b fixed, and for large a. Thereby, from $u(y_0)=0$, we obtain

$$\mathcal{E}_{n,\ell}=-\frac{m\,\pi^2(M\,B_0)^2}{2\left[\sqrt{8m\,M\,B_0\,r_0}-n\pi-\frac{\pi}{4}\right]^2}-\omega\left(\ell+\frac{\phi_1}{2\pi}\right), \tag{10}$$

where $n = 0, 1, 2, 3, \ldots$ is the radial quantum number and $\ell = 0, \pm1, \pm2, \pm3, \ldots$ is the angular momentum quantum number.

Hence, Eq. (10) shows the energy levels in the rotating reference frame, which stem from the interaction of a magnetic quadrupole moment of the neutral particle (3) with the magnetic field $\vec{B}_1 = B_0 \ln\frac{r}{r_0}\,\hat{z}$ and the induced electric field $\vec{E} = E_0 \ln\frac{r}{r_0}\,\hat{z}$ around the cylindrical cavity (which corresponds to the region $r > r_0$). The discrete spectrum of energy (10) is influenced by the rotation, where its contribution is given by the last term of the right-hand side of Eq. (10). This contribution corresponds to an analog of the Page-Werner *et al.* term [12–14], i.e., it yields the coupling between the angular velocity of the rotating frame ω and the effective angular momentum quantum number $\ell_{\text{eff}} = \ell + \phi_1/2\pi$. Besides, the energy levels (10) are also influenced by the geometric quantum phase ϕ_1, which gives rise to an Aharonov-Bohm-type effect for bound states [46].

Note that with $\omega \to 0$ we recover the energy levels in absence of rotation [44]. As shown in Ref. [44], in the absence of rotation, the interaction of a magnetic quadrupole moment of the neutral particle (3) with the magnetic field $\vec{B}_1 = B_0 \ln\frac{r}{r_0}\,\hat{z}$ around the cylindrical cavity yields energy levels which are infinitely degenerated (with respect to the quantum number ℓ or ℓ_{eff}). On the other hand, in the present work, the presence of the analog of the Page-Werner *et al.* term [12–14] in the energy levels (10) breaks the degeneracy of the energy levels obtained in Ref. [44]. In addition, the Aharonov-Bohm-type effect [46] occurs only in the presence of rotation [57].

Another aspect to observe is the upper limit of the radial quantum number. Due to the fact that $\tau = \sqrt{-2m\mathcal{E} - 2m\omega(\ell + \frac{\phi_1}{2\pi})} > 0$, then, the radial quantum number takes values inside the range: $0 \leqslant n \leqslant n_{\max}$. This upper limit is determined by

$$n_{\max} < \frac{\sqrt{8mMB_0 r_0}}{\pi} - \frac{1}{4}. \tag{11}$$

Without the upper limit (11), we would have $\tau < 0$.

Finally, the bound states associated with the energy levels (10) can be achieved for large values of r in agreement with Refs. [41,42,44]. This occurs due to the fact that the electric current density $\vec{J}$ can disturb the system, but it vanishes for large values of r.

III. PERSISTENT CURRENTS

From Eq. (10), we can observe that $\mathcal{E}_{n,\ell}(\phi_1 \pm 2\pi) = \mathcal{E}_{n,\ell\pm1}(\phi_1)$, which means that the energy eigenvalues are a periodic function of the geometric quantum phase ϕ_1. The corresponding periodicity is $\phi_0 = \pm 2\pi$. According to Refs. [21,47], persistent currents can appear in the quantum system due to the dependence of the energy levels on the geometric quantum phase ϕ_1. The persistent currents (at temperature $T = 0$) can be obtained through the Byers-Yang relation [21,47]:

$$\mathcal{I} = -\sum_{n,\ell}\frac{\partial\mathcal{E}_{n,\ell}}{\partial\phi_1} = \frac{\omega}{2\pi}. \tag{12}$$

The non-null persistent current (12) shows us that the persistent currents can appear in this system only in the presence of rotation. It depends only on the angular velocity of the rotating reference frame. This occurs because the analog of the Page-Werner *et al.* term [12–14] brings the dependence of the energy levels (10) on the geometric quantum phase ϕ_1. With $\omega \to 0$, in turn, no persistent current exists.

IV. QUANTUM REVIVALS

Quantum revivals are obtained when the wave function recovers its initial shape at a time called the revival time [48–51]. In this work, we have dealt with a bidimensional system, which is characterized by having two quantum numbers $\{\nu_1 = n,\ \nu_2 = \ell\}$. In this case, the eigenvalues of energy can be expanded about central values ν_1' and ν_2' of these quantum numbers through the Taylor series as [48,49,59–61]

$$\begin{aligned}\mathcal{E}_{\nu_1,\nu_2} \approx{}& \mathcal{E}_{\nu_1',\nu_2'} + \left(\frac{\partial\mathcal{E}}{\partial\nu_1}\right)_{\nu_1'\nu_2'}(\nu_1-\nu_1') + \left(\frac{\partial\mathcal{E}}{\partial\nu_2}\right)_{\nu_1'\nu_2'}(\nu_2-\nu_2')\\ &+ \frac{1}{2}\left(\frac{\partial^2\mathcal{E}}{\partial\nu_1^2}\right)_{\nu_1'\nu_2'}(\nu_1-\nu_1')^2 + \frac{1}{2}\left(\frac{\partial^2\mathcal{E}}{\partial\nu_2^2}\right)_{\nu_1'\nu_2'}(\nu_2-\nu_2')^2\\ &+ \left(\frac{\partial^2\mathcal{E}}{\partial\nu_1\partial\nu_2}\right)_{\nu_1'\nu_2'}(\nu_1-\nu_1')(\nu_2-\nu_2') + \cdots\end{aligned} \tag{13}$$

The revival times, hence, are defined as follows [48,49]:

$$\begin{aligned}\tau^{(1)} &= \frac{4\pi\hbar}{\left|\left(\frac{\partial^2\mathcal{E}}{\partial\nu_1^2}\right)_{\nu_1'\nu_2'}\right|};\quad \tau^{(2)} = \frac{4\pi\hbar}{\left|\left(\frac{\partial^2\mathcal{E}}{\partial\nu_2^2}\right)_{\nu_1'\nu_2'}\right|}\\ \tau^{(12)} &= \frac{2\pi\hbar}{\left|\left(\frac{\partial^2\mathcal{E}}{\partial\nu_1\partial\nu_2}\right)_{\nu_1'\nu_2'}\right|}.\end{aligned} \tag{14}$$

The revival time $\tau^{(12)}$ is called the cross-revival time. From the energy levels (10), the revival times (14) are defined in terms of the quantum numbers $\{n, \ell\}$. Then, with respect to the radial quantum number n, the corresponding revival time is

$$\tau^{(1)} = \frac{4\pi}{\left|\frac{\partial^2\mathcal{E}_{n,\ell}}{\partial n^2}\right|} = \frac{4\left[\sqrt{8mMB_0 r_0} - n\pi - \frac{\pi}{4}\right]^4}{3m\pi^3(MB_0)^2}. \tag{15}$$

The revival time related to the angular momentum quantum number is

$$\tau^{(2)} = \frac{4\pi}{\left|\frac{\partial^2\mathcal{E}_{n,\ell}}{\partial\ell^2}\right|} = 0. \tag{16}$$

Finally, the cross-revival time [49] is

$$\tau^{(12)} = \frac{2\pi}{\left|\left(\frac{\partial^2\mathcal{E}_{n,\ell}}{\partial n\partial\ell}\right)\right|} = 0. \tag{17}$$

Hence, there is just one non-null revival time in the present system, which is not influenced by the effects of rotation. As a consequence, it is not influenced by the geometric quantum phase ϕ_1.

V. CONCLUSIONS

Hence, the main aspect of the bound states achieved around the cylindrical cavity in the rotating reference frame is the break of the degeneracy of the energy levels. In contrast to the energy levels in the absence of rotation obtained in Ref. [44], the presence of the analog of the Page-Werner *et al.* term [12–14] in the energy levels (10) is responsible for breaking the degeneracy of the energy levels. In addition, the analog of the Page-Werner *et al.* term [12–14] is responsible for yielding the dependence of the energy levels on the geometric quantum phase, and thus, by yielding the Aharonov-Bohm-type effect for bound states [46].

Furthermore, due to the dependence of the energy levels on the geometric quantum phase, we have seen that a persistent current arises in this system. However, the persistent current can appear only in the rotating reference frame. With $\omega \to 0$, hence, there is no persistent current.

Finally, we have seen that there is only one non-null revival time related to the radial quantum number n. Besides, this revival time is not influenced by the effects of rotation and, as a consequence, there is no influence of the geometric quantum phase on it.

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