Angle and angular momentum: Uncertainty relations, simultaneous measurement, and phase-space representation

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Reaching ultimate performance of quantum technologies requires the use of detection at quantum limits and access to all resources of the underlying physical system. We establish a full quantum analogy between the pair of angular momentum and exponential angular variable, and the structure of canonically conjugate position and momentum. This includes the notion of optimal simultaneous measurement of the angular momentum and angular variable, the identification of Einstein-Podolsky-Rosen-like variables and states, and, finally, a phase-space representation of quantum states. Our construction is based on close interconnection of the three concepts and may serve as a template for the treatment of other observables. This theory also provides a test bed for implementation of quantum technologies combining discrete and continuous quantum variables.

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I. INTRODUCTION

Quantum limitations establish challenging problems for contemporary science, and rapid progress in metrology and communications—two important pillars of our technological world—bring us closer to this largely unexplored ultimate regime. Though quantum effects are fundamentally distinct from our classical intuition, they are manifested in variables which have a classical interpretation. Conservation laws and the concept of complementary variables offer the opportunity to be safely guided through this unfamiliar world of intertwined quantum effects. Thus we see quantum limits more as a sophisticated network of the interconnected rules and subtle conditions rather than strict and impenetrable barriers.

Canonical pairs of variables like energy and time, position and momentum, and angular momentum and angle provide the textbook examples. For instance, the Schrödinger equation connecting the Hamiltonian with time evolution is a starting point of quantum mechanics, whereas detection of energy of an electromagnetic field at the level of single photons opened the era of quantum optics. Though these concepts are well understood, time is not an operator but a parameter controlling the interaction, so care must be employed in understanding the energy-time uncertainty relation. The celebrated pair of position and momentum is the most famous example of noncommuting variables and the starting point of quantum information science. The Heisenberg uncertainty principle, Einstein-Podolsky-Rosen (EPR) states [1] and their detection, coherent states and phase space representation formulated by Roy Glauber [2], the Arthurs-Kelly concept of approximate simultaneous detection [3] (see also [4]), as well as teleportation with continuous variables [5] are the important milestones on the long way toward harnessing quantum effects.

The angular momentum and angular variable have been treated similarly to energy and time rather than fully quantum (quadraturelike) variables forming a phase space providing a complete description. The purpose of this paper is to formulate full quantum description for this conjugated pair. Unlike approaches based on the angle operator [6,7], our formalism is built on periodic functions of angle operator [8,9] and shows the prominent role of the corresponding minimum uncertainty states (MUSs) in four tasks: the formulation of saturable uncertainty relations, the simultaneous detection of noncommuting variables, the construction of EPR-like variables and states, and, finally, the phase-space representation of quantum states.

Our paper is motivated by possible applications to metrology but more generally by overarching questions about optimal measurements limited by the uncertainty relations. The group E(2), the natural structure for angle and angular momentum, is an interesting test bed for the extension of techniques developed in the context of Heisenberg algebra. We mention for completeness some expressions valid for the general case of quasiperiodic representations [9,10] but leave the consequences of quasiperiodicity and its potential applications (as discussed, for instance, in Ref. [11]) for later work. As there is an extensive body of work related to optical angular momentum as a tool for quantum information processing [12–14], the theory developed here provides theoretical framework for a full quantum description based on the concept of complementary variables as a possible new platform fully implemented on the E(2) symmetry. Astonishing experimental progress with sources based on structured light with imprinted optical angular momentum [15,16] is a promise for the realization of such protocols and may trigger new experimental techniques oriented to state engineering and detection at quantum limits.

II. UNIVERSAL UNCERTAINTY RELATIONS

Noncommutativity is an essential differentiating concept between quantum and classical physics. We analyze

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in detail the concept for the paradigmatic pair of angular momentum $L = -i\partial_\phi$ and unitary exponential operator $E = e^{-i\phi}$, satisfying the commutation rule of Euclidean algebra $\mathfrak{e}(2)$: $[E, L] = E$. Rephrased in terms of Hermitian operators as $[S_\alpha, L] = iC_\alpha$, where $C_\alpha = (e^{-i\alpha}E^\dagger + e^{i\alpha}E)/2$ and $S_\alpha = (e^{-i\alpha}E^\dagger - e^{i\alpha}E)/2i$, the rule implies the uncertainty relations:

$$\langle(\Delta L)^2\rangle\langle(\Delta S_\alpha)^2\rangle \geqslant \tfrac{1}{4}|\langle C_\alpha\rangle|^2. \tag{1}$$

The corresponding MUSs (in the L representation) [9,17,18]

$$|n+\delta, \alpha\rangle = \frac{1}{\sqrt{I_0(2\kappa)}}\sum_{l\in\mathbb{Z}} e^{i(n-l)\alpha} I_{n-l}(\kappa)|l+\delta\rangle, \tag{2}$$

with $L|l+\delta\rangle = (l+\delta)|l+\delta\rangle$, yield the von Mises distribution for the angle ϕ: $|\langle\phi|n+\delta,\alpha\rangle|^2 = \exp[2\kappa\cos(\phi-\alpha)]/2\pi I_0(2\kappa)$. As a result the states $|n+\delta,\alpha\rangle$ will be referred to as von Mises states.

Here $n+\delta$, where $n \in \mathbb{Z}$ and $\delta \in [0, 1)$, is the angular momentum mean, α is an angle, $\kappa \geqslant 0$ represents the spread of angular variable, and $I_n(z)$ is the modified Bessel function [19] (see Appendix A for its definition and other properties). Note that we allow for angular momenta with generally fractional eigenvalues $l+\delta$, whence the angular momentum eigenstates $\{|l+\delta\rangle\}_{l\in\mathbb{Z}}$ possess quasiperiodic wave functions $\langle\phi|l+\delta\rangle = \exp[i(l+\delta)\phi]/\sqrt{2\pi}$ [9].

For fixed α, the von Mises states $|n+\delta,\beta\rangle$ with $\beta \neq \alpha + k\pi$, $k \in \mathbb{Z}$, do not saturate the uncertainty relations (1). However, by setting $\alpha = -\arg\langle E\rangle$ and $\Delta S = S_{\alpha=-\arg\langle E\rangle}$, we get the parameter-free uncertainty relations

$$\langle(\Delta L)^2\rangle\,\omega^2 \geqslant \frac{1}{4}, \quad \omega^2 = \frac{\langle(\Delta S)^2\rangle}{|\langle E\rangle|^2}, \tag{3}$$

which is saturated by *all* von Mises states. Importantly, the measure of the angular uncertainty ω^2 is complementary to angular momentum in the sense that

$$\langle(\Delta L)^2\rangle = \frac{\kappa}{2}\frac{I_1(2\kappa)}{I_0(2\kappa)}, \quad \omega^2 = \frac{1}{2\kappa}\frac{I_0(2\kappa)}{I_1(2\kappa)}, \tag{4}$$

where $\langle E^l\rangle = \exp(-il\alpha)I_l(2\kappa)/I_0(2\kappa)$ derived in Appendix B has been used. Though the uncertainty relations (3) appear superficially related to some previously published variants, there is a substantial difference in definition of the angular measure: contrary to the previous results [8,17], our state-dependent choice $\alpha = -\arg\langle E\rangle$ guarantees the saturability for all von Mises states. The uncertainty relations (3) thus represent the first important result of this paper.

The spread parameter κ has a similar meaning to squeezing but here for the angular momentum and the angular variable. Since the phase space of the pair angle and angular momentum has cylindrical topology [9], one can represent von Mises states by ellipses on the cylinder (see Fig. 1) similarly to the representation of squeezed states of a harmonic oscillator by ellipses in the plane. Moreover, the MUS of Eq. (2) form an overcomplete basis resolving the identity as [9]

$$\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}|n+\delta,\alpha\rangle\langle n+\delta,\alpha| = \mathbb{1}, \tag{5}$$

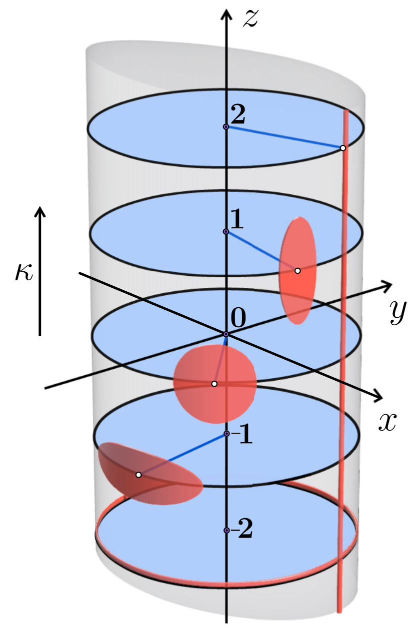

FIG. 1. Phase-space-representation of von Mises states $|n, \alpha\rangle$, $n \in \mathbb{Z}$, Eq. (2). The phase space consists of parallel equidistant rings (black rings), which are orthogonal to z axis and their centers possess the zth coordinate n. The von Mises state $|n, \alpha\rangle$ is represented by a noise ellipsis (red ellipsis) centered around a point on the circle with zth coordinate n and polar angle α (positive angle between blue line segment and positive x axis). The shape of the ellipsis depends on the value of the spread parameter κ, which is chosen to grow from the bottom to the top. Accordingly, the uncertainties $\langle(\Delta L)^2\rangle$ (ω^2), Eq. (4), grow (decrease) from the bottom to the top. The red ring represents von Mises state with $n = -2$ and $\kappa = 0$, which is an angular momentum eigenstate, so the other phase-space rings are images of the respective angular momentum eigenstates. The red vertical line represents the von Mises state with $n = 2$ in the limit of $\kappa \to \infty$. The red circle represents the von Mises state with $n = 0$ and symmetrical uncertainties $\langle(\Delta L)^2\rangle = \omega^2 = 1/2$ for $\kappa \doteq 1.292$.

and can be used as a generalized measurement for the discrete spectrum $n+\delta$ of angular momentum and the continuous values α of the angle. In the following, we set $\delta = 0$.

III. OPTIMAL SIMULTANEOUS MEASUREMENT

The deep analogy with x and p is obvious from the operator formalism behind the measurement on a signal (s) and ancilla (a) fields. Let us define the total sum angular momentum operator and the exponential angular difference operator:

$$\mathcal{L} = L_s + L_a, \quad \mathcal{E} = E_s E_a^\dagger. \tag{6}$$

Since $[\mathcal{L}, \mathcal{E}] = 0$, the operator $\mathcal{L}$ and any function of $\mathcal{E}$ and $\mathcal{E}^\dagger$ can be measured simultaneously and may serve as meter variables, in analogy with the pair of the EPR operators. We observe interestingly that if one assumes the unitary operator E is the exponential of some Hermitian angle operator, one would seemingly recover the same structure as the EPR pair for quadrature operators. However, such a conclusion cannot be justified here due to the issues of periodicity.

We now move to finding optimal simultaneous measurement of the noncommuting canonical pair L_s and S_s. The derivation is done in several steps implementing the measurement via joint measurement of the commuting bipartite observables $\mathcal{L}$ and $\mathcal{S} = (\mathcal{E}^\dagger - \mathcal{E})/2i$. We seek the measurement minimizing the uncertainty product $\langle(\Delta\mathcal{L})^2\rangle\langle(\Delta\mathcal{S})^2\rangle$ with $\Delta\mathcal{S} = \mathcal{S}_{\beta=\arg\langle E_a\rangle-\arg\langle E_s\rangle}$, where $\mathcal{S}_\beta = (e^{-i\beta}\mathcal{E}^\dagger - e^{i\beta}\mathcal{E})/2i$, with respect to the product state $|\varphi\rangle_s|\chi\rangle_a$. To minimize the unwanted influence of the ancilla on the measured signal quantities, we set up the following unbiased conditions $\langle L_a\rangle = 0, \arg\langle E_a\rangle = 0, \arg\langle E_a^2\rangle = 0,$ where the last two conditions guarantee preservation of the angular dependence of moments $\langle\mathcal{S}^l\rangle$, $l = 1, 2$, on the signal state. Under those conditions, the separate measurable uncertainties are simply given as

$$\begin{aligned}\langle(\Delta\mathcal{L})^2\rangle &= \langle(\Delta L_s)^2\rangle + \langle(\Delta L_a)^2\rangle,\\ \langle(\Delta\mathcal{S})^2\rangle &= |\langle E_a^2\rangle|\langle(\Delta S_s)^2\rangle + \langle(\Delta S_a)^2\rangle.\end{aligned} \tag{7}$$

The uncertainty for the total angular momentum includes additional noise from the ancillary field, as expected. Similarly, an analogous additive term is also present in the performance measure for the angular variable. There is also an additional multiplicative factor $|\langle E_a^2\rangle|$ related to the fact that the reference phase of ancillary field itself is uncertain. Cross-multiplying further the uncertainties in (7) and using the inequality of arithmetic and geometric means together with the uncertainty relations $\langle(\Delta L_{s,a})^2\rangle\langle(\Delta S_{s,a})^2\rangle \geqslant |\langle E_{s,a}\rangle|^2/4$, we finally get the following inequality:

$$\langle(\Delta\mathcal{L})^2\rangle\langle(\Delta\mathcal{S})^2\rangle \geqslant \tfrac{1}{4}\left(|\langle E_a\rangle| + |\langle E_s\rangle|\sqrt{|\langle E_a^2\rangle|}\right)^2. \tag{8}$$

This inequality is the second main result of this paper. It imposes a nontrivial fundamental constraint on accuracy with which the noncommuting observables L_s and S_s can be measured jointly. The right-hand side of the inequality represents the achievable lower bound for the simultaneous measurement. Indeed, the inequality is saturated by the MUS for both the system and ancilla fields satisfying, in addition, the condition $\langle(\Delta L_s)^2\rangle\langle(\Delta S_a)^2\rangle = |\langle E_a^2\rangle|\langle(\Delta L_a)^2\rangle\langle(\Delta S_s)^2\rangle$. Consequently, the lower bound is saturated by von Mises states $|\varphi\rangle_s = |n, \alpha, \kappa_s\rangle_s$ and $|\chi\rangle_a = |0, 0, \kappa_a\rangle_a$, with *different* spread parameters κ_s and κ_a connected by the condition

$$\kappa_s = \sqrt{|\langle E_a^2\rangle|}\kappa_a = \sqrt{\frac{I_2(2\kappa_a)}{I_0(2\kappa_a)}}\kappa_a. \tag{9}$$

Further details can be found in Appendix C.

In Fig. 2, we plot the optimally measurable uncertainty product (8) in comparison with the uncertainty relations (3) which give the constant lower bound of 1/4. Note that the bound for (8) is approximately four times larger as expected on the basis of the Arthurs-Kelly uncertainty relations [3], but only in the regime where the measurement resolves the angular variable well. This result is compared with the variance product mean $\langle(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2\rangle$ derived based on the She-Heffner formalism [20] in Appendix E. The analysis of the latter moment normalized with respect to $|\langle E_s\rangle|^2|\langle E_a\rangle|^2$ is surprising: it is even below the minimum value of uncorrelated product due to anticorrelations (see dashed blue line in Fig. 2). In other words, quantum mechanics allows us to specify the

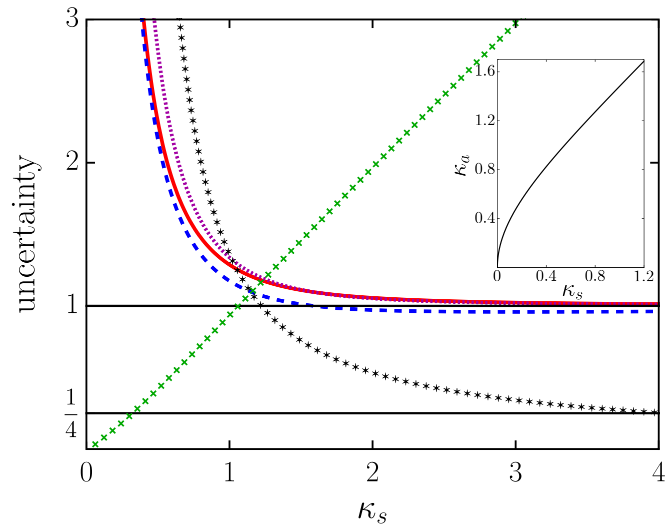

FIG. 2. Uncertainties and uncertainty products for angular momentum and angular variable for optimal states and measurements versus the signal-state spread parameter κ_s. Uncertainties $\langle(\Delta\mathcal{L})^2\rangle$ (green crosses) and $\Omega^2 = \langle(\Delta\mathcal{S})^2\rangle/|\langle E_s\rangle|^2|\langle E_a\rangle|^2$ (black stars), and uncertainty product $\langle(\Delta\mathcal{L})^2\rangle\Omega^2$ (solid red line) for optimal simultaneous measurement with von Mises signal and ancilla states with *different* spread parameters satisfying the optimal matching condition (9) whose inverse is depicted in the inset. The same uncertainty product for suboptimal simultaneous measurement with von Mises signal and ancilla states with *the same* spread parameters $\kappa_s = \kappa_a$ (dotted magenta line). The product mean $\langle(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2\rangle/|\langle E_s\rangle|^2|\langle E_a\rangle|^2$ for von Mises signal and ancilla states satisfying optimal matching condition (9) (dashed blue line). The uncertainty product for optimal simultaneous measurement approaches, asymptotically its lower bound of 1, which is four times larger than the lower bound of 1/4 for uncertainty relations (3). The product mean always lies below the uncertainty product for optimal measurement and it may even lie below 1. Equality $\langle(\Delta\mathcal{L})^2\rangle = \Omega^2 \doteq 1.099$ is achieved for $\kappa_s \doteq 1.146$ and $\kappa_a \doteq 1.632$.

states (and the measurement), where each canonically conjugated variable reaches its minimum in the uncertainty product, but the correlated product is even below that. This indicates stronger correlations linked to the fourth-order moments. Note that such an effect, though mild in our system, is not possible in the case of x and p operators.

The joint measurement is realized by a projection onto orthogonal common eigenvectors of operators $\mathcal{L}$ and $\mathcal{E}$ corresponding to eigenvalues N, Φ, respectively. The common eigenstates are given as

$$|N, \Phi\rangle_{sa} = \frac{1}{\sqrt{2\pi}}\sum_{l\in\mathbb{Z}} e^{-il\Phi}|l + N\rangle_s|-l\rangle_a, \tag{10}$$

resembling the EPR states for the position and momentum operators [1]: when the ancilla of the state is projected onto the von Mises ancilla state $|0, 0\rangle_a$ with the spread parameter κ, the signal collapses into the von Mises system state $|N, \Phi\rangle_s$ with the same κ as the ancilla. Below we further develop the analogy with EPR states by showing that the projective measurement onto the EPR-like states (10) plays the role analogous to a Bell measurement for position and momentum [5]. Generalization of the states of Eq. (10) to signal and ancilla with generally different fractional angular momenta is discussed in Appendix D. The full analogy between the structure of the EPR pair and states for quadrature operators and the angular momentum and angular variable represents the third main result of this paper.

IV. PHASE-SPACE REPRESENTATION

Existing attempts to construct a phase-space representation of angular momentum and angular variable focused exclusively on the Wigner function [21] using group-theoretical methods [22,23] or employing analogies with the harmonic oscillator [24,25]. Building on the latter ideas and our previous results, we develop a complete family of phase-space distributions exhibiting behaviors and connections very much like the quasiprobability distributions of the standard harmonic oscillator. Here, we only sketch the derivations, whereas the details can be found in Appendix F.

Our approach is based on identities linking the Fourier transformation of the projectors onto the EPR-like states (10) and von Mises states (2) with the ordering of the operators E and L:

$$2\pi(\mathcal{F}|n,\alpha\rangle_{sa}\langle n,\alpha|)(l,\phi) = D_s(l,\phi)D_a(-l,\phi), \tag{11}$$

$$(\mathcal{F}|n,\alpha\rangle_s\langle n,\alpha|)(l,\phi) = o(l,\phi)D_s(l,\phi), \tag{12}$$

where

$$(\mathcal{F}A)(l,\phi) = \sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}e^{i(l\alpha-\phi n)}A(n,\alpha)$$

is the Fourier transformation,

$$D(l,\phi) = e^{-il\frac{\phi}{2}}E^{-l}e^{-iL\phi} \tag{13}$$

is the displacement operator [25], and

$$o(l,\phi) = e^{il\frac{\phi}{2}}\langle l,\phi|0,0\rangle = \frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0(2\kappa)}. \tag{14}$$

The relation (11) follows immediately from the orthogonal expansion of the operator $\mathcal{E}^{-l}e^{-i\mathcal{L}\phi}$ in terms of the states (10), whereas the relation (12) is obtained by averaging Eq. (11) over the ancilla von Mises state $|0,0\rangle_a$ with spread parameter κ. Based on the equality (12), we can now construct the phase-space distributions for the angular momentum and angular variable, in a manner analogous to the quasiprobability Q function [26], Wigner function [21], and P function [2,27] of the harmonic oscillator. In particular, the averaging of the formula (12) over the rescaled density operator $\rho/(2\pi)$ immediately yields the relation between the characteristic function $C_Q(l,\phi) = (\mathcal{F}Q)(l,\phi)$ of the Q function,

$$Q(n,\alpha) = \frac{\langle n,\alpha|\rho|n,\alpha\rangle}{2\pi}, \tag{15}$$

and the Wigner characteristic function defined as $C_W(l,\phi) = \mathrm{Tr}[\rho D(l,\phi)]/2\pi$:

$$C_Q(l,\phi) = o(l,\phi)C_W(l,\phi). \tag{16}$$

The analogies with the phase-space distributions of the harmonic oscillator can be taken further by defining the diagonal representation of a density matrix ρ as a P distribution, analogous to the Glauber-Sudarshan quasiprobability distribution [2,27]. Recall first that the displacement operator (13) satisfies the following completeness property [25]:

$$\mathrm{Tr}[D^\dagger(l,\phi)D(l',\phi')] = 2\pi\delta_{ll'}\delta_{2\pi}(\phi-\phi'), \tag{17}$$

where $\delta_{2\pi}(\phi)$ is the 2π-periodic delta function. Thus, one can express any density matrix as

$$\rho = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}d\phi C_W(l,\phi)D^\dagger(l,\phi). \tag{18}$$

Insertion of $[o(l,\phi)]^{-1}o(l,\phi) = 1$ into the integrand and application of the unitarity of the Fourier transformation brings us straightforwardly to the P representation of any density matrix,

$$\rho = \sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}d\alpha P(n,\alpha)|n,\alpha\rangle\langle n,\alpha|, \tag{19}$$

where we introduced the analogy of the P function as the Fourier transformation $P(n,\alpha) = (\mathcal{F}C_P)(n,\alpha)$ of the corresponding characteristic function $C_P(l,\phi)$ defined by

$$C_W(l,\phi) = o(l,\phi)C_P(l,\phi). \tag{20}$$

From Eqs. (16) and (20), it is apparent that the Bessel overlap (14) plays for the pair of angular momentum and angular variable exactly the same role of a universal smoothing factor as the Gaussian overlap $\langle\alpha|0\rangle = \exp(-|\alpha|^2/2)$ of the vacuum state $|0\rangle$ and the coherent state $|\alpha\rangle$ of a harmonic oscillator. The connection between the respective phase-space distributions is given by the convolution with the kernel comprised by the Fourier transformation of the overlap (14). This phase-space structure and associated quasiprobability distributions related to operator ordering constitute the final major result of our paper.

V. APPLICATIONS AND IMPLEMENTATIONS

There are several theoretical and experimental attempts to use angular momentum and angle in a manner analogous to quadrature operators for the purpose of quantum information processing [28,29]. The role of the angular variable in quantum information processing adopting a modular variable approach was recognized in Ref. [30], and Bell inequalities for entangled angular variables were formulated in Ref. [31]. All those findings are indications of close analogies between quadrature operators and the pair of angular momentum and angle variable. The theory developed here may shed light on this topic by revealing the full potential of the phase-space description of complementary variables, periodic performance measures for description of angular variable, and complete analog for Bell variables and detection. All these are benefits of simultaneous measurements of $\mathcal{L}$ and $\mathcal{S}$ with optimal ancillary states, which is fully analogous to quadrature heterodyne detection. This allows us to translate protocols based on optical quadratures and heterodyne detection into the realm of the L and S variables. For instance, the coherent state cryptography protocol with heterodyne detection [32], which does not require switching of measurement bases, becomes the analogous no-switching protocol with von Mises states.

A. Quantum communication with von Mises states

The developed theory finds another application in quantum teleportation [5,33]. Specifically, the proposed generalized measurement plays the role of the Bell measurement for L and S, which can be used for quantum teleportation of von Mises states. Assume a sender Alice and a receiver Bob share an entangled state $|0,0\rangle_{AB}$ of the form given in Eq. (10) with $N=0$, $\Phi=0$, of two subsystems A and B. In addition, Alice has at her disposal an unknown von Mises state $|n,\alpha\rangle_{\text{in}}$ of an input system "in" and she wants to teleport this state to Bob. At the outset, Alice carries out measurement of the basis vectors given in (10) on her part of the joint state $|n,\alpha\rangle_{\text{in}}|0,0\rangle_{AB}$. If she obtains the measurement outcome (M,Ψ), Bob's subsystem collapses (up to an unimportant phase factor) into the state $D_B^{-1}(M,\Psi)|n,\alpha\rangle_B$. Alice then sends the outcomes via classical channel to Bob, who applies correcting displacement $D_B(M,\Psi)$ on his subsystem, thereby recreating the original state $|n,\alpha\rangle_B$. A generalization of such a protocol allowing teleportation of von Mises states between systems with *different* fractional angular momenta is provided in Appendix D. Realization of the proposed protocol would extend teleportation of finite superpositions of angular momentum eigenstates [34] to the genuine continuous-variable regime when states spanning entire unbounded state space are teleported.

B. Optical beams

It is a challenging task to implement von Mises states as optical beams by advanced techniques adopting twisted photons similar to Refs. [29,35]—either as nondiffracting Bessel or Laguerre-Gauss modes. Such states would truly play the role of squeezedlike states carrying information about both complementary observables of angular momentum and angular variable. The fascinating progress in compact generation of optical angular momentum states [16] together with optimal usage of information distributed into continuous and discrete variables represent a step toward unique communication schemes on a robust platform of optical beams.

C. Phase and intensity as conjugated variables

Although the quantum phase problem has a long history with many pitfalls [6], the canonical commutation relation for $\mathfrak{e}(2)$ can be modified to the case of phase and intensity of the signal field. Considerations inspired by the analysis of the phase of complex amplitudes allow us to formulate the following two-mode representation: $L=a_s^\dagger a_s - a_a^\dagger a_a$ and $E=\sqrt{(a_s+a_a^\dagger)/(a_s^\dagger+a_a)}$. The phase of the signal field enters through the phase of the complex amplitude $Y=a_s+a_a^\dagger$, $[Y,Y^\dagger]=0$. However, L and E are represented here by noncommuting operators and simultaneous detection would require the strategies discussed above.

VI. CONCLUSION

We developed a fully quantum description of the canonical pair of angular momentum and angular variable obeying commutation rules associated with group E(2). A central role is played by the von Mises MUSs, allowing the performance of optimal measurement as well as the provision of a phase-space representation of states. Since the optimality is linked to saturable uncertainty relations, our theory has important metrological consequences and may trigger experimental techniques oriented to state engineering and detection at quantum limits, fully employing the E(2) symmetry.

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APPENDIX A: MODIFIED BESSEL FUNCTION

Here we review useful formulas to help with some explicit calculations involving Bessel functions. The modified Bessel function of integer order n is defined by the integral formula [19]:

$$I_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{z\cos\phi+in\phi}. \tag{A1}$$

From the definition, one can easily see that $I_n(z)$ is real for real z and satisfies

$$I_n(z)=I_{-n}(z),\quad I_n(-z)=(-1)^n I_n(z),\quad I_n(0)=\delta_{n0}. \tag{A2}$$

In addition, the modified Bessel functions fulfill the recurrence relations [19]

$$I_{n-1}(z)-I_{n+1}(z)=\frac{2n}{z}I_n(z) \tag{A3}$$

and

$$I_{n-1}(z)+I_{n+1}(z)=2\frac{d}{dz}I_n(z). \tag{A4}$$

Our calculations with modified Bessel functions are greatly simplified by the addition theorem [19]

$$\sum_{m\in\mathbb{Z}}(-1)^m I_{r+m}(Z)I_m(z)e^{im\phi}=e^{ir\psi}I_r(\omega), \tag{A5}$$

where $r\in\mathbb{Z}$ and

$$\begin{gathered}\omega=\sqrt{Z^2+z^2-2Zz\cos\phi},\\ Z-z\cos\phi=\omega\cos\psi,\quad z\sin\phi=\omega\sin\psi.\end{gathered} \tag{A6}$$

In particular, the addition formula yields

$$\sum_{m\in\mathbb{Z}} I_m(\kappa)I_{m+r}(\kappa)e^{im\phi}=e^{-ir\frac{\phi}{2}}I_r\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right], \tag{A7}$$

with the special case

$$\sum_{m\in\mathbb{Z}} I_m^2(\kappa)=I_0(2\kappa). \tag{A8}$$

The modified Bessel functions can also be obtained from the following generating function [36]:

$$\sum_{m\in\mathbb{Z}} I_m(z)e^{im\phi}=e^{z\cos\phi}. \tag{A9}$$

APPENDIX B: PROPERTIES OF VON MISES STATES

In this Appendix, we summarize some useful properties of the von Mises states, Eq. (2) of the main text,

$$|n+\delta,\alpha\rangle=\frac{1}{\sqrt{I_0(2\kappa)}}\sum_{l\in\mathbb{Z}}e^{i(n-l)\alpha}I_{n-l}(\kappa)|l+\delta\rangle,\quad \text{(B1)}$$

where $\delta\in[0,1)$ and $\kappa\geqslant 0$.

Recall first that von Mises states (B1) are defined as the states saturating the uncertainty relations (1) of the main text:

$$\langle(\Delta L)^2\rangle\langle(\Delta S_\alpha)^2\rangle\geqslant\tfrac{1}{4}|\langle C_\alpha\rangle|^2.\quad \text{(B2)}$$

In the ϕ representation, von Mises states read

$$\psi_{n+\delta,\alpha}(\phi)=\frac{1}{\sqrt{2\pi I_0(2\kappa)}}e^{i(n+\delta)\phi+\kappa\cos(\phi-\alpha)},\quad \text{(B3)}$$

where the generating function (A9) has been used. The states can be seen as a special type of states introduced previously in Ref. [9], given in ϕ representation by

$$\tilde{\psi}^{\sigma}_{n+\delta,\alpha}(\phi)=\frac{1}{\sqrt{2\pi I_0(2s)}}e^{i[(n+\delta)(\phi-\alpha)+\sigma\sin(\phi-\alpha)]},\quad \text{(B4)}$$

where $\sigma=\gamma-is$. The states of (B4) can be shown to saturate the uncertainty relations

$$\langle(\Delta L)^2\rangle\langle(\Delta C_\alpha)^2\rangle\geqslant\tfrac{1}{4}(|\langle S_\alpha\rangle|^2+|\langle\{\Delta L,\Delta C_\alpha\}\rangle|^2),\quad \text{(B5)}$$

and their relationship to our states (B3) is given by

$$\tilde{\psi}^{-i\kappa}_{n+\delta,\alpha-\frac{\pi}{2}}(\phi)=e^{-i(n+\delta)(\alpha-\frac{\pi}{2})}\psi_{n+\delta,\alpha}(\phi).\quad \text{(B6)}$$

In what follows, it is advantageous to use the states (B3) as they represent the standard form of von Mises states in the ϕ representation with $\gamma=0$ guaranteeing vanishing of the anticommutator mean: $\langle\{\Delta L,\Delta S_\alpha\}\rangle=0$. This form is simpler for calculations yet it captures all essential features of MUSs for angular momentum and angular variable.

We start with the overlap $\langle n'+\delta,\alpha'|n+\delta,\alpha\rangle$ of two von Mises states with the same fractional parts δ. By inserting the resolution of identity $\int_{-\pi}^{\pi}d\phi|\phi\rangle\langle\phi|=\mathbb{1}$ into the overlap, we obtain

$$\begin{aligned}\langle n'+\delta,\alpha'|n+\delta,\alpha\rangle&=\int_{-\pi}^{\pi}d\phi\,\psi^*_{n'+\delta,\alpha'}(\phi)\psi_{n+\delta,\alpha}(\phi)\stackrel{1}{=}\frac{1}{I_0(2\kappa)}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{i(n-n')\phi+2\kappa\cos\left(\frac{\alpha-\alpha'}{2}\right)\cos\left[\phi-\left(\frac{\alpha+\alpha'}{2}\right)\right]}\\&\stackrel{2}{=}e^{i(n-n')\left(\frac{\alpha+\alpha'}{2}\right)}\frac{I_{n-n'}\left[2\kappa\cos\left(\frac{\alpha-\alpha'}{2}\right)\right]}{I_0(2\kappa)},\end{aligned}\quad \text{(B7)}$$

where to get equality 1, Eq. (B3) and the identity $\cos(\phi-\alpha)+\cos(\phi-\alpha')=2\cos[\phi-(\frac{\alpha+\alpha'}{2})]\cos(\frac{\alpha-\alpha'}{2})$ were used, whereas in equality 2 we used the definition (A1). Alternatively, the overlap formula can be derived using the definition (B1) and the addition theorem (A7) as

$$\begin{aligned}\langle n'+\delta,\alpha'|n+\delta,\alpha\rangle&=\frac{e^{i(n\alpha-n'\alpha')}}{I_0(2\kappa)}\sum_{l\in\mathbb{Z}}e^{il(\alpha'-\alpha)}I_{n-l}(\kappa)I_{n'-l}(\kappa)\\&=e^{i(n-n')\left(\frac{\alpha+\alpha'}{2}\right)}\frac{I_{n-n'}\left[2\kappa\cos\left(\frac{\alpha-\alpha'}{2}\right)\right]}{I_0(2\kappa)}.\end{aligned}\quad \text{(B8)}$$

Interestingly, since $I_n(0)=\delta_{n0}$ according to the last of Eq. (A2), von Mises states with $\alpha'=\alpha+(2k+1)\pi$, $k\in\mathbb{Z}$, and $n\neq n'$ are orthogonal. Thus, contrary to the usual intuition, the overcomplete von Mises-state basis contains not only nonorthogonal but also orthogonal states. A more generic overlap formula for states (B4) with generally different fractional parts can be found in Ref. [9].

The overlap formula (B7) together with the addition theorem (A7) allow us to calculate arbitrary moments of von Mises states. To show this, let us first note how the operators $\exp(-iL\phi)$ and E^{-l} act on von Mises states (B1),

$$\begin{aligned}e^{-iL\phi}|n+\delta,\alpha\rangle&=e^{-i(n+\delta)\phi}|n+\delta,\alpha+\phi\rangle,\\E^{-l}|n+\delta,\alpha\rangle&=|n+l+\delta,\alpha\rangle,\end{aligned}\quad \text{(B9)}$$

where in derivation of the second equality the relation $E^\dagger|n+\delta\rangle=|n+1+\delta\rangle$ has been used. Let us now adopt a conventional definition of the moment-generating function of a quantum state ρ as a mean $G(l,\phi)=\mathrm{Tr}[\rho\tilde{D}(l,\phi)]$ of the operator:

$$\tilde{D}(l,\phi)=E^{-l}e^{-iL\phi}.\quad \text{(B10)}$$

Making use of Eqs. (B9) and the overlap formula (B7), one can show easily that the moment-generating function for the von Mises state $|n+\delta,\alpha\rangle$ is given by

$$G(l,\phi)=e^{il\alpha}e^{-i\left(n+\delta-\frac{l}{2}\right)\phi}\frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0(2\kappa)}.\quad \text{(B11)}$$

From here, one can then get straightforwardly all moments as derivatives:

$$\langle E^{-l}L^N\rangle=i^N\frac{d^N}{d\phi^N}G(l,\phi)\bigg|_{\phi=0}.\quad \text{(B12)}$$

For $N=0$, we can combine Eqs. (B11) and (B12) to get

$$\langle E^{-l}\rangle=G(l,\phi)|_{\phi=0}=e^{il\alpha}\frac{I_l(2\kappa)}{I_0(2\kappa)}.\quad \text{(B13)}$$

Moving to $N>0$, let us now express the N-th derivative on the right-hand side of equation (B12) as $i^{N-1}(d^{N-1}/d\phi^{N-1})i(d/d\phi)G(l,\phi)$, calculate the first derivative $i(d/d\phi)G(l,\phi)$ with the help of generating function (B11), and use the recurrence relation (A4) to express the resulting formula for the first derivative in terms of $G(l,\phi)$ and $G(l\pm 1,\phi)$. This yields the N-th derivative of the generating function as a linear combination of $(N-1)$st derivatives of the generating functions $G(l,\phi)$ and $G(l\pm 1,\phi)$, which in turn leads, when combined with the formula (B12), to the following recurrence relation for the von Mises states:

$$\langle E^{-l}L^N\rangle = \frac{\kappa}{4}\{e^{i\alpha}\langle E^{-(l-1)}[L^{N-1}-(L-\mathbb{1})^{N-1}]\rangle - e^{-i\alpha}\langle E^{-(l+1)}[L^{N-1}-(L+\mathbb{1})^{N-1}]\rangle\} + \left(n+\delta-\frac{l}{2}\right)\langle E^{-l}L^{N-1}\rangle. \tag{B14}$$

Hence, we can rederive moments of the angular momentum [9]

$$\langle L\rangle = n+\delta, \quad \langle L^2\rangle = (n+\delta)^2 + \frac{\kappa}{2}\frac{I_1(2\kappa)}{I_0(2\kappa)} \tag{B15}$$

and

$$\langle(\Delta L)^2\rangle = \frac{\kappa}{2}\frac{I_1(2\kappa)}{I_0(2\kappa)}, \tag{B16}$$

or derive new moments, e.g.,

$$\langle(E)^{\pm 2}\Delta L\rangle = \pm e^{\mp i2\alpha}\frac{I_2(2\kappa)}{I_0(2\kappa)} \tag{B17}$$

and

$$\langle(E)^{\pm 2}(\Delta L)^2\rangle = \frac{e^{\mp i2\alpha}}{2I_0(2\kappa)}[I_2(2\kappa)+\kappa I_1(2\kappa)], \tag{B18}$$

where $(E)^{\pm 2}$ stands for the (± 2)nd power of E. Later, we use the latter joint moments to calculate the joint moment appearing in an alternative approach to simultaneous detection of incompatible observables put forward by She and Heffner [20].

Before doing this, let us briefly comment on another interesting property of von Mises states, which stems from the recurrence relation (B14), namely, as $\langle L\rangle = n+\delta$ for von Mises states, the joint moment $\langle E^{-l}L^N\rangle$ can be expressed via the mean $\langle L\rangle$ and joint moments involving at most $(N-1)$st power of the angular momentum operator. Repeated application of the recurrence relation (B14) on the moments on the right-hand side thus allows us to express any joint moment $\langle E^{-l}L^N\rangle$ only in terms of powers of the mean value $\langle L\rangle$ and the moments of powers of the operator E. This can be viewed as an analogy of a similar property of Gaussian quantum states [37]. These states are fully determined by the first-order and second-order moments of the quadrature operators and thus any higher-order moment can be expressed only in terms of the first two moments.

APPENDIX C: SIMULTANEOUS DETECTION OF ANGULAR MOMENTUM AND ANGULAR VARIABLE

In this Appendix, we derive a fundamental lower bound for a product of variances of the outcomes of simultaneous measurements of the noncommuting observables L_s and S_s. This can be done most easily using a joint measurement of commuting observables $\mathcal{L} = L_s + L_a$ and $\mathcal{S} = (\mathcal{E}^\dagger - \mathcal{E})/2i$ of the signal s and ancilla a, where $\mathcal{E} = E_s E_a^\dagger$. We seek the measurement which would reach the minimum of the uncertainty product $\langle(\Delta\mathcal{L})^2\rangle\langle(\Delta\mathcal{S})^2\rangle$ over the product state $|\varphi\rangle_s|\chi\rangle_a$ with $\Delta\mathcal{S} = \mathcal{S}_{\beta=\arg\langle E_a\rangle-\arg\langle E_s\rangle}$ and $\mathcal{S}_\beta = (e^{-i\beta}\mathcal{E}^\dagger - e^{i\beta}\mathcal{E})/2i$, acting on both the ancilla and system spaces. We require the measurement to preserve the signal angular momentum mean, i.e., $\langle\mathcal{L}\rangle = \langle L_s\rangle$, as well as the angular dependence of relevant moments of angular variable, $\langle\mathcal{S}^l\rangle$, $l=1,2$, on the signal state. This restricts the ancilla state as

$$\langle L_a\rangle = 0, \quad \arg\langle E_a\rangle = 0, \quad \arg\langle E_a^2\rangle = 0. \tag{C1}$$

Making use of the latter two conditions, one can cast the uncertainties of angular variables in the form

$$\langle(\Delta\mathcal{S})^2\rangle = \tfrac{1}{2}(1-e_s e_a\cos\psi_s), \quad \langle(\Delta S_a)^2\rangle = \tfrac{1}{2}(1-e_a),$$
$$\langle(\Delta S_s)^2\rangle = \tfrac{1}{2}(1-e_s\cos\psi_s), \tag{C2}$$

where $e_{s,a} = |\langle E_{s,a}^2\rangle|$ and $\psi_s = 2\arg\langle E_s\rangle - \arg\langle E_s^2\rangle$. Consequently, the measurable uncertainties are simply given as

$$\langle(\Delta\mathcal{L})^2\rangle = \langle(\Delta L_s)^2\rangle + \langle(\Delta L_a)^2\rangle,$$
$$\langle(\Delta\mathcal{S})^2\rangle = \langle(\Delta S_a)^2\rangle + e_a\langle(\Delta S_s)^2\rangle. \tag{C3}$$

Hence, the uncertainty product is lower bounded as

$$\begin{aligned}\langle(\Delta\mathcal{L})^2\rangle\langle(\Delta\mathcal{S})^2\rangle &= [\langle(\Delta L_s)^2\rangle + \langle(\Delta L_a)^2\rangle][\langle(\Delta S_a)^2\rangle + e_a\langle(\Delta S_s)^2\rangle]\\ &= \langle(\Delta L_a)^2\rangle\langle(\Delta S_a)^2\rangle + e_a\langle(\Delta L_s)^2\rangle\langle(\Delta S_s)^2\rangle + \langle(\Delta L_s)^2\rangle\langle(\Delta S_a)^2\rangle + e_a\langle(\Delta L_a)^2\rangle\langle(\Delta S_s)^2\rangle\\ &\overset{1}{\geqslant} \left[\sqrt{\langle(\Delta L_a)^2\rangle\langle(\Delta S_a)^2\rangle} + \sqrt{e_a\langle(\Delta L_s)^2\rangle\langle(\Delta S_s)^2\rangle}\right]^2\\ &\overset{2}{\geqslant} \tfrac{1}{4}\left(|\langle E_a\rangle| + |\langle E_s\rangle|\sqrt{|\langle E_a^2\rangle|}\right)^2.\end{aligned} \tag{C4}$$

Here, inequality 1 follows from

$$\left[\sqrt{\langle(\Delta L_s)^2\rangle\langle(\Delta S_a)^2\rangle} - \sqrt{e_a\langle(\Delta L_a)^2\rangle\langle(\Delta S_s)^2\rangle}\right]^2 \geqslant 0 \tag{C5}$$

and it is saturated if

$$\langle(\Delta L_s)^2\rangle\langle(\Delta S_a)^2\rangle = e_a\langle(\Delta L_a)^2\rangle\langle(\Delta S_s)^2\rangle. \tag{C6}$$

The inequality 2 is a consequence of the uncertainty relations

$$\langle(\Delta L_{s,a})^2\rangle\langle(\Delta S_{s,a})^2\rangle \geqslant \tfrac{1}{4}|\langle E_{s,a}\rangle|^2, \tag{C7}$$

and it is saturated by the von Mises MUS of both the signal and the ancilla. For the condition of Eq. (C6) to hold, we will see that the parameters κ_a and κ_s for these states must be related. Recall that for a general von Mises state $|n+\delta,\alpha\rangle$, one has $\langle L\rangle = n+\delta$ and $\langle E^l\rangle = \exp(-il\alpha)I_l(2\kappa)/I_0(2\kappa)$, Eqs. (B15) and (B13), and the unbiasedness conditions (C1) imply the optimal ancilla state to be the von Mises vacuum state $|0,0\rangle_a$. If the condition $\langle L_a\rangle = 0$ is relaxed, the optimal ancilla state reads $|\delta_a,0\rangle_a$, where $\delta_a \in [0,1)$. Similarly, the optimal signal state is also a von Mises state $|n+\delta_s,\alpha\rangle_s$. What is more, substituting the variances for signal and ancilla von Mises states,

$$\langle(\Delta L_j)^2\rangle = \frac{\kappa_j}{2}\frac{I_1(2\kappa_j)}{I_0(2\kappa_j)}, \quad \langle(\Delta S_j)^2\rangle = \frac{1}{2\kappa_j}\frac{I_1(2\kappa_j)}{I_0(2\kappa_j)}, \quad \text{(C8)}$$

$j=s,a$, into the condition of Eq. (C6) one finds that the signal and ancilla spread parameters κ_s and κ_a of optimal states must fulfill the nontrivial condition:

$$\kappa_s = \sqrt{\frac{I_2(2\kappa_a)}{I_0(2\kappa_a)}}\kappa_a. \quad \text{(C9)}$$

Thus, in accordance with our intuition, it is optimal to carry out a von Mises measurement on von Mises states, but contrary to our intuition, the spread parameter of the ancilla κ_a and of the measured state κ_s differ.

APPENDIX D: QUANTUM TELEPORTATION OF VON MISES STATES

This section deals with the unconditional teleportation of von Mises states. Let us consider two quantum systems A and B with angular momenta L_A and L_B, and angular variables E_A and E_B, respectively. The simultaneous measurement of the total orbital angular momentum $\mathcal{L}=L_A+L_B$ and of the sine of the angular difference $\mathcal{S}=(\mathcal{E}^\dagger-\mathcal{E})/2i=(E_A^\dagger E_B-E_AE_B^\dagger)/2i$ plays in the optimal simultaneous measurement of L_A and S_A the same role as the EPR operators x_A-x_B and p_A+p_B in the optimal simultaneous measurement of x_A and p_A. Since the latter measurement is nothing but the Bell measurement for continuous-variable systems [5], one expects that the former measurement will realize the Bell measurement for orbital angular momentum and angular variable. In the following, we confirm this by showing that the measurement can be used for perfect quantum teleportation [33] of unknown von Mises states.

Assume the two systems A and B under consideration carry generally different angular momenta characterized by fractional parts δ_A and δ_B, respectively. Consider further the vectors

$$\begin{aligned}&|N+\Delta_{AB},\Phi\rangle_{AB}\\ &\quad=\frac{1}{\sqrt{2\pi}}\sum_{l\in\mathbb{Z}}e^{-il\Phi}|l+\delta_A+N-I_{AB}\rangle_A|-l+\delta_B\rangle_B,\end{aligned} \quad \text{(D1)}$$

where $I_{jk}=[\delta_j+\delta_k]\in\{0,1\}$ and $\Delta_{jk}=(\delta_j+\delta_k)\bmod 1$, $\Delta_{jk}\in[0,1)$, $j,k=A,B$ are the integer part and the fractional part of $\delta_j+\delta_k$, respectively. The normalization factor $1/\sqrt{2\pi}$ ensures that the states are normalized as

$$\langle M+\Delta_{AB},\Psi|N+\Delta_{AB},\Phi\rangle=\delta_{MN}\delta_{2\pi}(\Psi-\Phi), \quad \text{(D2)}$$

where

$$\delta_{2\pi}(\phi)=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}e^{in\phi}=\sum_{n\in\mathbb{Z}}\delta(\phi-2n\pi) \quad \text{(D3)}$$

is the 2π-periodic delta function (or Dirac comb). From relations $E|n+\delta\rangle=|n-1+\delta\rangle$ and $E^\dagger|n+\delta\rangle=|n+1+\delta\rangle$, it further follows straightforwardly that

$$\begin{aligned}\mathcal{L}|N+\Delta_{AB},\Phi\rangle_{AB}&=(N+\Delta_{AB})|N+\Delta_{AB},\Phi\rangle_{AB},\\ \mathcal{E}|N+\Delta_{AB},\Phi\rangle_{AB}&=e^{-i\Phi}|N+\Delta_{AB},\Phi\rangle_{AB},\\ \mathcal{E}^\dagger|N+\Delta_{AB},\Phi\rangle_{AB}&=e^{i\Phi}|N+\Delta_{AB},\Phi\rangle_{AB},\end{aligned} \quad \text{(D4)}$$

where $\mathcal{E}=E_AE_B^\dagger$, and thus the vectors of Eq. (D1) are common eigenvectors of $\mathcal{L}$ and $\mathcal{S}$ corresponding to eigenvalues $(N+\Delta_{AB})$ and $\sin\Phi$, respectively.

Adopting the line of argument of Ref. [38] we can now design the following teleportation protocol. The goal of the protocol is to transmit faithfully an unknown von Mises state $|n+\delta_{\rm in},\alpha\rangle_{\rm in}$ of an input system "in" characterized by the fractional part of angular momentum $\delta_{\rm in}$, from a sender Alice to a receiver Bob. For this purpose, the participants can use the shared EPR-like state of Eq. (D1),

$$|\Delta_{AB},0\rangle_{AB}=\frac{1}{\sqrt{2\pi}}\sum_{l\in\mathbb{Z}}|l+\delta_A-I_{AB}\rangle_A|-l+\delta_B\rangle_B, \quad \text{(D5)}$$

corresponding to eigenvalue Δ_{AB} of $\mathcal{L}$ and zero eigenvalue of $\mathcal{S}$. First, Alice performs measurement of EPR-like states (D1) on subsystem "in" and her part A of the shared state. Provided that the outcomes of her measurement are (M,Ψ), the global state $|n+\delta_{\rm in},\alpha\rangle_{\rm in}|\Delta_{AB},0\rangle_{AB}$ collapses to the (unnormalized) state,

$$\begin{aligned}&{}_{{\rm in}A}\langle M+\Delta_{{\rm in}A},\Psi|n+\delta_{\rm in},\alpha\rangle_{\rm in}|\Delta_{AB},0\rangle_{AB}\\ &=\frac{e^{i(I_{AB}-\delta_B)\Psi}}{2\pi}e^{-i(M+I_{AB}-I_{{\rm in}A})\Psi}E_B^{M+I_{AB}-I_{{\rm in}A}}e^{iL_B\Psi}|n+\delta_B,\alpha\rangle_B\\ &=\frac{e^{i(I_{AB}+I_{{\rm in}A}-M-2\delta_B)\frac{\Psi}{2}}}{2\pi}D_B^{-1}(M+I_{AB}-I_{{\rm in}A},\Psi)|n+\delta_B,\alpha\rangle_B,\end{aligned} \quad \text{(D6)}$$

where

$$D(l,\phi)=e^{-il\frac{\phi}{2}}E^{-l}e^{-iL\phi} \quad \text{(D7)}$$

is the displacement operator [25] and where in the second equality we used the relation

$$E^le^{iL\phi}=e^{il\phi}e^{iL\phi}E^l=e^{il\frac{\phi}{2}}D^{-1}(l,\phi). \quad \text{(D8)}$$

Alice subsequently sends the outcomes of her measurement to Bob via a classical channel and he applies to his part of the shared state the correcting operation $D_B(M+I_{AB}-I_{{\rm in}A},\Psi)$. Up to an irrelevant phase factor and generally different fractional part δ_B from $\delta_{\rm in}$, Bob recreates a perfect replica $|n+\delta_B,\alpha\rangle_B$ of the original von Mises state on his system and thus he completes the teleportation.

The result above strengthens the attractiveness of a laboratory implementation of the von Mises measurement. First, the measurement would allow teleportation of von Mises states thereby extending teleportation of finite superpositions of angular momentum eigenstates [34] to the continuous-variable regime in which infinite superpositions of angular momentum eigenstates, which span entire infinite-dimensional Hilbert state space, are teleported. In addition, the presented protocol allows us, at least in principle, to teleport quantum states between systems with generally different fractional angular momenta. It can be expected that the utility of von Mises measurement will also further carry over to all other translations of quantum information protocols to angular momentum and angle, which utilize Bell measurement, such as entanglement swapping [39] or quantum cryptography without measurement switching [32].

Note, finally, that here we demonstrated perfect teleportation of von Mises states using the non-normalizable EPR-like state of Eq. (D1). Analysis of the realistic protocol with physical approximation of the state (D5), such as, for instance, the entangled state $\sum_{l\in\mathbb{Z}} c_{l,-l}|l\rangle_A|-l\rangle_B$ generated in the process of spontaneous parametric down-conversion [40], is outside the immediate scope of the present paper.

APPENDIX E: SHE-HEFFNER APPROACH TO SIMULTANEOUS MEASUREMENT

This Appendix contains analysis of the simultaneous detection of the angular momentum and angular variable based on the statistical perspective introduced in the seminal paper of She and Heffner [20]. Following their argumentation, simultaneous detection can be cast as a two-stage process: state preparation specified by the moments and repeated detection conditioned by the same constraints as in the state preparation step.

The EPR-like states of Eq. (D1) allow us to bridge the Arthurs-Kelly and She-Heffner approaches. Note first that the states (D1) satisfy the completeness condition

$$\sum_{N\in\mathbb{Z}}\int_{-\pi}^{\pi} d\Phi|N+\Delta_{sa},\Phi\rangle_{sa}\langle N+\Delta_{sa},\Phi| = \mathbb{1}_{sa}, \tag{E1}$$

where we have done the following identification: $s\equiv A$ and $a\equiv B$. With the help of the resolution of identity and the eigenvalue equations (D4), we can express the product of squares of operators $\Delta\mathcal{L}$ and $\Delta\mathcal{S}$ as

$$(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2 = \sum_{N\in\mathbb{Z}}\int_{-\pi}^{\pi} d\Phi(N+\Delta_{sa}-\langle\mathcal{L}\rangle)^2\sin^2(\Phi-\beta)|N+\Delta_{sa},\Phi\rangle_{sa}\langle N+\Delta_{sa},\Phi|, \tag{E2}$$

where once again $\beta=\arg\langle E_a\rangle-\arg\langle E_s\rangle$. Let us now calculate the partial average of the latter operator over the optimal ancilla state $|\delta_a,0\rangle_a$. Taking into account that for this ancilla $\langle L_a\rangle=\delta_a$, $\arg\langle E_a\rangle=0$ and ${}_a\langle\delta_a,0|N+\Delta_{sa},\Phi\rangle_{sa}=|N-I_{sa}+\delta_s,\Phi\rangle_s/\sqrt{2\pi}$, we get after some algebra the following signal operator, diagonal in the von Mises states $|N+\delta_s,\Phi\rangle_s$:

$$\langle(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2\rangle_a = \sum_{N\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\Phi}{2\pi}(N+\delta_s-\langle L_s\rangle)^2\sin^2(\Phi+\arg\langle E_s\rangle)|N+\delta_s,\Phi\rangle_s\langle N+\delta_s,\Phi|, \tag{E3}$$

where $\langle X_{sa}\rangle_a={}_a\langle\delta_a,0|X_{sa}|\delta_a,0\rangle_a$. Further, by averaging the latter operator over the signal state ρ_s, we get the analog of the She-Heffner integral [20] for the angular momentum and angular variable:

$$\langle(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2\rangle = \sum_{N\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\Phi}{2\pi}(N+\delta_s-\langle L_s\rangle)^2\sin^2(\Phi+\arg\langle E_s\rangle)_s\langle N+\delta_s,\Phi|\rho_s|N+\delta_s,\Phi\rangle_s. \tag{E4}$$

Making use of the expressions for the moments given in Eq. (B13) and Eqs. (B16)–(B18), we can finally calculate the She-Heffner moment for von Mises states with spread parameters κ_s and κ_a in the form

$$\langle(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2\rangle = \frac{1}{4I_0(2\kappa_s)I_0(2\kappa_a)}\left[\left(\frac{\kappa_s}{\kappa_a}+\frac{\kappa_a}{\kappa_s}\right)I_1(2\kappa_s)I_1(2\kappa_a)+2I_2(2\kappa_s)I_2(2\kappa_a)\right]. \tag{E5}$$

In Fig. 2 of the main text, we plot the properly normalized moment $\langle(\Delta\mathcal{L})^2(\Delta\mathcal{S})^2\rangle/|\langle E_s\rangle|^2|\langle E_a\rangle|^2$ versus the spread parameter κ_s and κ_a satisfying condition (C9). The figure reveals that the correlated uncertainties represented by the latter moment lie below the uncorrelated ones (C4) for the same von Mises states:

$$\langle(\Delta\mathcal{L})^2\rangle\langle(\Delta\mathcal{S})^2\rangle = \frac{1}{4}\left(|\langle E_a\rangle|+|\langle E_s\rangle|\sqrt{|\langle E_a^2\rangle|}\right)^2 = \frac{1}{4}\left[\frac{I_1(2\kappa_a)}{I_0(2\kappa_a)}+\sqrt{\frac{I_2(2\kappa_a)}{I_0(2\kappa_a)}}\frac{I_1(2\kappa_s)}{I_0(2\kappa_s)}\right]^2. \tag{E6}$$

APPENDIX F: PHASE-SPACE REPRESENTATION

In this Appendix, we show that von Mises states allow the development of a phase-space representation for angular momentum and angular variable, which closely resembles the phase-space representation for quadrature operators based on standard coherent states. For the sake of simplicity, we restrict our attention to integer angular momentum, the generalization to the fractional angular momenta being deferred for further research. The key mathematical tool used for the development of the phase-space methods is the Fourier transformation [24],

$$(\mathcal{F}A)(l,\phi) = \sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\alpha}{2\pi}e^{i(l\alpha-\phi n)}A(n,\alpha), \tag{F1}$$

of an operator (or function) $A(n,\alpha)$. Making use of the filtration property of the 2π-periodic delta function (D3) on the interval of the length 2π, one can show easily that the Fourier transformation (F1) fulfils the following analog of the Parseval formula,

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi} & d\phi(\mathcal{F}A)(l,\phi)(\mathcal{F}B)^\dagger(l,\phi)\\ &= \sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi} d\alpha A(n,\alpha)B^\dagger(n,\alpha),\end{aligned} \tag{F2}$$

where the symbol † stands for the Hermitian conjugate. Analogously, one can show that the Fourier transformation of a product is a convolution of the Fourier transformations of the factors

$$[\mathcal{F}(AB)](n,\alpha) = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}(\mathcal{F}A)(n-l,\alpha-\phi)(\mathcal{F}B)(l,\phi) = (\mathcal{F}A)*(\mathcal{F}B)(n,\alpha). \tag{F3}$$

Finally, for 2π-periodic $A(n,\alpha)$, the Fourier transformation (F1) is also its own inverse.

The phase-space representation relies on the identity linking the ordering of operators $\mathcal{L}$ and $\mathcal{E}$ with the Fourier transformation of the projectors onto common eigenstates of the operators, Eq. (D1), with $\Delta_{sa}=0$,

$$2\pi(\mathcal{F}|n,\alpha\rangle_{sa}\langle n,\alpha|)(l,\phi) = \mathcal{E}^{-l}e^{-i\mathcal{L}\phi} = D_s(l,\phi)D_a(-l,\phi), \tag{F4}$$

where $D_j(l,\phi)$ is the displacement operator of the subsystem $j=s,a$. The latter relation follows directly from the application of the operator $\mathcal{E}^{-l}e^{-i\mathcal{L}\phi}$ to the resolution of identity for states (D1), Eq. (E1) with $\Delta_{sa}=0$. Further, by averaging both sides of Eq. (F4) over the von Mises vacuum state $|0,0\rangle_a$ of the ancillary system a with spread parameter κ, we obtain

$$(\mathcal{F}|n,\alpha\rangle_s\langle n,\alpha|)(l,\phi) = o(l,\phi)D_s(l,\phi), \tag{F5}$$

where

$$o(l,\phi) = e^{il\frac{\phi}{2}}\langle l,\phi|0,0\rangle = \frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0(2\kappa)}. \tag{F6}$$

Here, to get the left-hand side we used ${}_a\langle 0,0|n,\alpha\rangle_{sa} = |n,\alpha\rangle_s/\sqrt{2\pi}$, and to calculate the mean ${}_a\langle 0,0|D_a(-l,\phi)|0,0\rangle_a$ on the right-hand side we used Eq. (B11). The Fourier transformation of the projector onto von Mises state (F5) plays a central role in our approach to development of the phase-space methods for the angular momentum and angular variable. An interesting feature of formula (F5) is the c-number function $o(l,\phi)$, Eq. (F6), in front of the displacement operator $D_s(l,\phi)$. Below we show, among other things, that for the angular momentum and angular variable, the overlap (F6) plays exactly the same role as plays overlap $\langle\alpha|0\rangle = \exp(-|\alpha|^2/2)$ of the vacuum state $|0\rangle$ and the coherent state $|\alpha\rangle$ of a harmonic oscillator.

The relation (F5) allows us to arrive in an elegant way to analogies of the (Husimi) Q function [26], Wigner function [21], and Glauber-Sudarshan P function [2,27] of the standard harmonic oscillator, namely, let us average the relation (with the index s dropped for simplicity) over the rescaled density operator $\rho/(2\pi)$, i.e.,

$$\mathrm{Tr}\left[\frac{\rho}{2\pi}(\mathcal{F}|n,\alpha\rangle\langle n,\alpha|)(l,\phi)\right] = \left[\mathcal{F}\frac{\langle n,\alpha|\rho|n,\alpha\rangle}{2\pi}\right](l,\phi) = o(l,\phi)\frac{1}{2\pi}\mathrm{Tr}[\rho D(l,\phi)]. \tag{F7}$$

In analogy with the phase-space distributions of a harmonic oscillator, we now introduce the Q function of a density matrix ρ by a prescription

$$Q(n,\alpha) = \frac{\langle n,\alpha|\rho|n,\alpha\rangle}{2\pi}, \tag{F8}$$

which is normalized as $\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}d\alpha Q(n,\alpha)=1$. Likewise, we define the Wigner characteristic function as the average of the displacement operator:

$$C_W(l,\phi) = \frac{1}{2\pi}\mathrm{Tr}[\rho D(l,\phi)]. \tag{F9}$$

As the Fourier transformation of the Q function is just its characteristic function, $(\mathcal{F}Q)(l,\phi) = C_Q(l,\phi)$, we get from the formula (F7) the relationship

$$C_Q(l,\phi) = o(l,\phi)C_W(l,\phi). \tag{F10}$$

Surprisingly, the analogy with the quadrature phase space can be developed even further. Recall first that the displacement operator (D7) exhibits the following completeness property [25]:

$$\mathrm{Tr}[D^\dagger(l,\phi)D(l',\phi')] = 2\pi\delta_{ll'}\delta_{2\pi}(\phi-\phi'). \tag{F11}$$

The property (F11) enables us to decompose any density matrix ρ as

$$\rho = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}d\phi C_W(l,\phi)D^\dagger(l,\phi). \tag{F12}$$

Consider now the Hermitian conjugate of the equality (F5) (with the index s again dropped)

$$(\mathcal{F}|n,\alpha\rangle\langle n,\alpha|)^\dagger(l,\phi) = o(l,\phi)D^\dagger(l,\phi). \tag{F13}$$

By multiplying both sides with $C_P(l,\phi) = [o(l,\phi)]^{-1}C_W(l,\phi)$ and performing summation over l and integration over ϕ, we get

$$\sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}d\phi C_P(l,\phi)(\mathcal{F}|n,\alpha\rangle\langle n,\alpha|)^\dagger(l,\phi) = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}d\phi C_W(l,\phi)D^\dagger(l,\phi) = \rho, \tag{F14}$$

where the rightmost equality is a consequence of Eq. (F12). If we now apply to the left-hand side the formula (F2), we obtain

$$\rho = \sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}d\alpha P(n,\alpha)|n,\alpha\rangle\langle n,\alpha|, \tag{F15}$$

where we defined the P function by the formula

$$P(n,\alpha) = (\mathcal{F}C_P)(n,\alpha). \tag{F16}$$

Equation (F15) reveals that any density matrix can be expressed in diagonal form in von Mises states. This is a direct analogy of the celebrated Glauber-Sudarshan representation [2,27] for the harmonic oscillator.

Summarizing the results, the characteristic functions of different phase-space distributions are related as

$$C_Q(l,\phi) = o(l,\phi)C_W(l,\phi) = o^2(l,\phi)C_P(l,\phi). \tag{F17}$$

The overlap $o(l,\phi)$, Eq. (F6), plays for the pair of angular momentum and angular variable the same role of a universal smoothing factor as plays the overlap $\langle\alpha|0\rangle = \exp(-|\alpha|^2/2)$ for the canonically conjugate quadrature operators, where [41]

$$C_Q(\alpha) = e^{-\frac{|\alpha|^2}{2}}C_W(\alpha) = e^{-|\alpha|^2}C_P(\alpha). \tag{F18}$$

Application of the Fourier transformation to Eq. (F17) and utilization of the formula (F3) finally yield the following relationship between the adjacent phase-space distributions:

$$\begin{aligned} Q(n,\alpha) &= [(\mathcal{F}o) * W](n,\alpha), \\ W(n,\alpha) &= [(\mathcal{F}o) * P](n,\alpha). \end{aligned} \tag{F19}$$

We see that the Fourier transformation of the overlap (F6) plays the role of a kernel of the convolution relating different phase-space distributions. As the P function of the von Mises state $|n,\alpha\rangle$ takes the form

$$P^{|n,\alpha\rangle}(m,\beta) = \delta_{nm}\delta_{2\pi}(\alpha-\beta), \tag{F20}$$

one finds from the second equality of (F19) the kernel to be

$$(\mathcal{F}o)(n,\alpha) = 2\pi W^{|0,0\rangle}(n,\alpha), \tag{F21}$$

where $W^{|0,0\rangle}(n,\alpha)$ is the Wigner function of the von Mises state $|0,0\rangle$. The Wigner function is given by a sum of two terms both involving the third Jacobi theta function [25] and we can combine it with formulas (F19) and (F21) to calculate phase-space distributions for other basic states of the investigated system. This program as well as further development of the phase-space methods introduced here are beyond the scope of the present paper and will be addressed elsewhere.

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