Real transmission and reflection zeros of periodic structures with a bound state in the continuum

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For lossless periodic structures with proper symmetry, the transmission and reflection spectra often have peaks and dips that are truly 100% and 0%, respectively. The full peaks and zero dips typically appear near resonant frequencies, and they are robust with respect to structural perturbations that preserve the required symmetry. However, current theories on the existence of full peaks and zero dips are incomplete and difficult to use. For periodic structures with a bound state in the continuum (BIC), we present a theory on the existence of real transmission and reflection zeros that correspond to the zero dips in the transmission and reflection spectra. Our theory is relatively simple, complete, and easy to use. Numerical examples are presented to validate the theory.

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I. INTRODUCTION

For periodic structures sandwiched between two homogeneous media, the transmission and reflection spectra often have interesting and useful features. The peaks and dips can be very sharp. A peak and a dip may appear close to each other and form an asymmetric line shape. The study on "anomalous" transmission and reflection has a long history [1,2]. It is widely accepted that the rapid change from a peak to a dip, the "anomaly" first observed by Wood [1], is in fact a particular case of Fano resonance [2–5]. The asymmetric line shape is formed from the interference between the resonant and nonresonant wave field components [5]. The resonant wave field component is caused by the excitation of an eigenmode of the periodic structure satisfying an outgoing radiation condition. The eigenmode is either a resonant mode with a complex frequency or a leaky mode with a complex propagation constant. The nonresonant field component exists in a direct passway. In case there is no direct passway, the spectra exhibit a Lorentzian line shape with only a single peak or dip [5].

For lossless periodic structures, the peaks and dips in the transmission and reflection spectra can actually be 100% and 0%, respectively. Popov *et al.* [4] first realized that structural symmetry is important to the appearance of full peaks and zero dips. As functions of frequency and wave number, the reflection and transmission coefficients vanish at their corresponding zeros (which are complex in general). For lossless periodic structures with a proper symmetry, the transmission/reflection zeros are either real or form complex conjugate pairs [4,6–10]. A real transmission zero corresponds to a zero dip in the transmission spectrum and a full peak in the reflection spectrum. Since a simple zero cannot be turned to a complex conjugate pair by a small perturbation, the zero dips and full peaks in the transmission/reflection spectra are robust with respect to small structural perturbations that preserve the required symmetry [10]. However, even for structures with the required symmetry, the appearance of a real transmission/reflection zero (near a resonance) is not guaranteed. To the best of our knowledge, there is no general theory on the existence of full peaks or zero dips.

Shipman and Tu [11] developed a theory on the existence of full peaks and zero dips for symmetric lossless periodic structures with a bound state in the continuum (BIC). A BIC is a guided mode that decays exponentially in the homogeneous media surrounding the periodic structure, and it exists in the radiation continuum, namely, there are propagating plane waves in the homogeneous media having the same frequency and wave number as the BIC [12–16]. Importantly, a BIC is a special point in a band of resonant modes. The theory of Shipman and Tu is applicable to resonant modes near a BIC. It is mathematically rigorous, but the technical conditions are specified on quantities that are difficult to calculate. They identified a generic condition under which a real transmission zero and a real reflection zero likely exist near the frequency of the BIC, and they also studied a nongeneric case.

It should be pointed that the temporal coupled-mode theory (TCMT) [5,17,18] can approximate full peaks and zero dips in transmission and reflection spectra. In a recent work [19], we showed that a direct approximation to the exact scattering matrix gives the same approximate transmission/reflection spectra as the TCMT. However, the approximate formulas have limitations in accuracy, are valid only under proper conditions, and do not provide a rigorous justification for the existence of real transmission and reflection zeros.

In this paper, we present a theory on the existence of real transmission and reflection zeros. Similar to the work of Shipman and Tu [11], our theory is applicable to lossless periodic structures with a BIC, but the symmetry requirement in our

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study is less restrictive. Moreover, our results are relatively simple and more complete, and the technical conditions are only specified on directly computable quantities. The rest of this paper is organized as follows. In Sec. II, we recall the definitions and basic properties for scattering matrices, resonant modes, and BICs. In Sec. III, we give a brief summary for the theories of Popov *et al* [4] and Shipman and Tu [11]. In Sec. IV, we present our theory with details on the assumptions, derivations, and results, and also discuss the difference with existing theories. To valid our theory, we present numerical examples in Sec. V. The paper is concluded with a brief discussion in Sec. VI.

II. BACKGROUND

We consider a lossless two-dimensional (2D) periodic structure that is invariant in z, periodic in y with period L, and sandwiched between two homogeneous media given for $x > D$ and $x < -D$, respectively. The dielectric function $\varepsilon(x, y)$ of the structure is real and periodic in y, and $\varepsilon(x, y) = \varepsilon_0 \geqslant 1$ for $|x| > D$. In the homogeneous media for $|x| > D$, we specify two time-harmonic E-polarized plane incident waves with a positive angular frequency ω and real wave vectors $(\pm\alpha, \beta)$ satisfying

$$-\frac{\pi}{L} < \beta \leqslant \frac{\pi}{L}, \tag{1}$$

$$|\beta| < \frac{\omega}{c}\sqrt{\varepsilon_0} < \frac{2\pi}{L} - |\beta|, \tag{2}$$

$$\alpha = \sqrt{(\omega/c)^2\varepsilon_0 - \beta^2}, \tag{3}$$

where c is the speed of light in vacuum and $\alpha > 0$. The z component of the total electric field, denoted as u, can be expanded as

$$u(x, y) = b_1^+ e^{i[\beta y + \alpha(x+D)]} + b_1^- e^{i[\beta y - \alpha(x+D)]} + \sum_{j\neq 0} b_{1j} e^{i\beta_j y + \tau_j(x+D)}, \quad x < -D, \tag{4}$$

$$u(x, y) = b_2^+ e^{i[\beta y - \alpha(x-D)]} + b_2^- e^{i[\beta y + \alpha(x-D)]} + \sum_{j\neq 0} b_{2j} e^{i\beta_j y - \tau_j(x-D)}, \quad x > D, \tag{5}$$

where b_1^+ and b_2^+ are the amplitudes of the given incident waves in the left and right homogeneous media, respectively, b_1^- and b_2^- are the amplitudes of the outgoing plane waves,

$$\beta_j = \beta + 2\pi j/L, \quad \tau_j = \sqrt{\beta_j^2 - (\omega/c)^2\varepsilon_0} \tag{6}$$

for $j \neq 0$, τ_j is positive, b_{1j} and b_{2j} are the amplitudes of the evanescent waves. The scattering matrix S satisfies

$$S\begin{bmatrix} b_1^+ \\ b_2^+ \end{bmatrix} = \begin{bmatrix} b_1^- \\ b_2^- \end{bmatrix} \tag{7}$$

for any b_1^+ and b_2^+. We write down the entries of S as

$$S = \begin{bmatrix} r & \tilde{t} \\ t & \tilde{r} \end{bmatrix}, \tag{8}$$

where r and t are the reflection and transmission coefficients for the left incident wave, $\tilde{r}$ and $\tilde{t}$ are those of the right incident wave.

Clearly, the scattering matrix S depends on both ω and β. The definition of S can be extended to complex ω by analytic continuation. For a real β, S satisfies

$$S^{-1}(\omega, \beta) = S^*(\overline{\omega}, \beta), \tag{9}$$

$$S^{\mathsf{T}}(\omega, \beta) = S(\omega, -\beta), \tag{10}$$

where $\overline{\omega}$ is the complex conjugate of ω, S^{T} is the transpose of S, and $S^*(\overline{\omega}, \beta)$ is the conjugate transpose of $S(\overline{\omega}, \beta)$ [4,10]. Equations (9) and (10) are related to energy conservation and reciprocity, respectively. Notice that if ω is real, S is unitary; and if $\beta \neq 0$, S is typically nonsymmetric. If the periodic structure has a proper symmetry, the scattering matrix can be further simplified [4]. Specifically, we have three cases.

(a) If $\epsilon(x, y) = \epsilon(-x, -y)$, then $r = \tilde{r}$;
(b) If $\epsilon(x, y) = \epsilon(x, -y)$, then $t = \tilde{t}$;
(c) If $\epsilon(x, y) = \epsilon(-x, y)$, then $t = \tilde{t}$ and $r = \tilde{r}$.

Without incident waves, the periodic structure can support Bloch eigenmodes given by

$$u(x, y) = e^{i\beta y}\phi(x, y) \tag{11}$$

where β is the Bloch wave number and ϕ is periodic in y with period L. For $x \to \pm\infty$, an eigenmode should either decay exponentially or radiate out power to infinity (i.e., satisfy the outgoing radiation condition). For a real β satisfying (1) and the real part of ω satisfying (2), expansions (4) and (5) are applicable to an eigenmode, provided that we set $b_1^+ = b_2^+ = 0$. Notice that b_1^- and b_2^- are the coefficients of outgoing waves. Since the structure is passive and nonabsorbing (i.e., ε is real and positive), the existence of outgoing waves is only possible when ω has a nonzero imaginary part, so that the eigenmode decays with time. Such an eigenmode with a real β and a complex ω, and satisfying the outgoing radiation condition, is a resonant mode (also called a resonant state or quasi-normal mode) [20,21]. The resonant modes form bands. Each band is given by a dispersion relation $\omega = \omega_\star(\beta)$, where $\omega_\star(\beta)$ is a complex-valued continuous function of a real variable β. Notice that α and τ_j given in (3) and (6) are complex for the resonant modes. Under special circumstances, both b_1^- and b_2^- are zero, then ω is real, the eigenmode decays exponentially as $x \to \pm\infty$, and it is a guided mode above the light line, also called a bound state in the continuum (BIC). Note that a BIC is a special point in a band of resonant modes. If we denote the wave number and frequency of a BIC by $\beta_\dagger$ and $\omega_\dagger$, respectively, then $\omega_\dagger = \omega_\star(\beta_\dagger)$.

Since the inverse of the scattering matrix S maps $[b_1^-, b_2^-]^{\mathsf{T}}$ to $[b_1^+, b_2^+]^{\mathsf{T}}$, and $b_1^+ = b_2^+ = 0$ for a resonant mode, we have

$$S^{-1}(\omega_\star, \beta)\begin{bmatrix} b_1^- \\ b_2^- \end{bmatrix} = S^*(\overline{\omega}_\star, \beta)\begin{bmatrix} b_1^- \\ b_2^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

This above implies that $\omega_\star$ is a pole of S, and $\overline{\omega}_\star$ is a zero of S. The entries of S can also vanish at special values of ω and β. We assume the left transmission coefficient $t = t(\omega, \beta)$ has a zero set given by a function $\omega = \omega_t^\circ(\beta)$. Namely, $t(\omega_t^\circ(\beta), \beta) = 0$ for any β. Similarly, the left reflection coefficient r has a zero set given by a function $\omega = \omega_r^\circ(\beta)$. For a fixed β, we call $\omega_t^\circ(\beta)$ a transmission zero and $\omega_r^\circ(\beta)$ a reflection zero. In general, for a real β, the transmission and reflection zeros are complex. We are concerned with the existence of real transmission and reflection zeros.

III. EXISTING THEORIES

The possible existence of real transmission and reflection zeros in symmetric periodic structures was first investigated by Popov *et al.* [4]. These authors identified the three different symmetries (listed in Sec. II) for which $t = \tilde{t}$, $r = \tilde{r}$, or both $t = \tilde{t}$ and $r = \tilde{r}$, respectively; and showed that if $t = \tilde{t}$, then either a transmission zero is real or there is a pair of complex conjugate transmission zeros. The case for $r = \tilde{r}$ is similar. This implies that a real transmission or reflection zero for a symmetric periodic structure is robust with respect to structural perturbations that preserve the symmetry. However, the possibility of complex conjugate pairs cannot be ruled out, and it remains unclear why real transmission and reflection zeros are widely observed in practice.

If for a real β, the periodic structure has a high-Q nondegenerate resonant mode with a complex frequency $\omega_\star = \omega_0 - i\gamma$ where $\gamma \ll \omega_0$, and there is no other resonant modes near $\omega_\star$, then the reflection and transmission coefficients can be written as

$$r(\omega) = \frac{a(\omega)}{\omega - \omega_\star}, \quad t(\omega) = \frac{b(\omega)}{\omega - \omega_\star}, \tag{13}$$

where a and b are analytic functions of ω near $\omega_\star$. Since t and r are bounded by 1 in magnitude, if ω is close to ω_0, $|a|/\omega_0$ and $|b|/\omega_0$ must also be small, but this does not imply that a or b (thus r or t) must have a single zero in a domain that is symmetric about the real axis and contains $\omega_\star$. Consequently, it is not possible to predict the existence of real transmission and reflection zeros by considering only the symmetry of the structure and the resonant modes.

Shipman and Tu [11] considered periodic structures with a reflection symmetry in x [case (c) of Sec. II], and showed that the existence of real transmission and reflection zeros is a generic phenomenon. Their theory is restricted to structures with a BIC and involves technical conditions are rather difficult to verify. More specifically, Shipman and Tu defined an operator A so that the diffraction problem of Sec. II becomes

$$Au = p, \tag{14}$$

where both A and p depend on ω and β, and p is related to the incident wave. They further considered the linear eigenvalue problem

$$Av = \lambda v, \tag{15}$$

where λ is an eigenvalue depending on both ω and β, and λ vanishes at the BIC point $(\omega_\dagger, \beta_\dagger)$. The technical conditions are specified using λ, λr and λt and their partial derivatives at $(\omega_\dagger, \beta_\dagger)$.

The theory of Shipman and Tu is difficult to use in practice, because A is abstractly defined without an explicit expression, and the eigenvalue problem Eq. (15) does not have a clear physical interpretation. Moreover, since Shipman and Tu only considered case (c), it is not clear whether their method can be extended to cases (a) and (b). In our view, a simpler and more intuitive theory on the existence of real transmission and reflection zeros is highly desirable.

IV. THEORY

In this section, we present a theory to clarify the conditions for the existence of real transmission and reflection zeros. We consider a lossless periodic structure as described in Sec. II, and assume there is a BIC with frequency $\omega_\dagger$ and Bloch wave number $\beta_\dagger$ satisfying conditions (1) and (2). The BIC is supposed to be a special point in a band of nondegenerate resonant modes with dispersion relation $\omega = \omega_\star(\beta) = \omega_0(\beta) - i\gamma(\beta)$, where ω_0 and $-\gamma$ are the real and imaginary parts of $\omega_\star$, and they satisfy $\gamma(\beta_\dagger) = 0$, $\omega_0(\beta_\dagger) = \omega_\star(\beta_\dagger) = \omega_\dagger$, and $\gamma(\beta) > 0$ for β near but not equal to $\beta_\dagger$.

Our theory depends on the analyticity of a few functions. First of all, $\omega_\star$ is an analytic function of real variable β [11]. Therefore $\omega - \omega_\star(\beta)$, as a function of complex variable ω and real variable β, is analytic in ω and β. It is known that the reflection and transmission coefficients, r and t, as functions of ω and β, are not continuous at $(\omega_\dagger, \beta_\dagger)$ [11]. However, we claim that the two functions a and b given by

$$a(\omega, \beta) = [\omega - \omega_\star(\beta)]r(\omega, \beta),$$
$$b(\omega, \beta) = [\omega - \omega_\star(\beta)]t(\omega, \beta),$$

are analytic in ω and β in a neighborhood of the BIC point $(\omega_\dagger, \beta_\dagger)$. This implies that the singularity of r and t at $(\omega_\dagger, \beta_\dagger)$ can be removed by multiplying $\omega - \omega_\star(\beta)$. Notice that $a(\omega_\dagger, \beta_\dagger) = b(\omega_\dagger, \beta_\dagger) = 0$. The above proposition is different from our statement about Eq. (13), where β is fixed, the dependence on β is suppressed, $\mathrm{Im}(\omega_\star) \neq 0$, and a and b are analytic in a single complex variable ω. The analyticity of a and b can be established following the proof for the analyticity of λt and λr by Shipman and Tu [11], where λ is the eigenvalue defined in Eq. (15).

Similarly, for the right reflection and transmission coefficients, the two functions $\tilde{a} = [\omega - \omega_\star(\beta)]\tilde{r}$ and $\tilde{b} = [\omega - \omega_\star(\beta)]\tilde{t}$ are analytic in ω and β near $(\omega_\dagger, \beta_\dagger)$. The scattering matrix can be written as

$$S(\omega, \beta) = \frac{1}{\omega - \omega_\star(\beta)} \begin{bmatrix} a(\omega, \beta) & \tilde{b}(\omega, \beta) \\ b(\omega, \beta) & \tilde{a}(\omega, \beta) \end{bmatrix} \tag{16}$$

for $\omega \neq \omega_\star(\beta)$, where a, b, $\tilde{a}$, and $\tilde{b}$ are analytic in ω and β, and vanish at $(\omega_\dagger, \beta_\dagger)$.

Next, we consider the function

$$f(\omega, \beta) = \frac{\omega - \omega_\star(\beta)}{\omega - \overline{\omega}_\star(\beta)} \det S(\omega, \beta). \tag{17}$$

For a fixed $\beta \neq \beta_\dagger$, $\omega_\star = \omega_\star(\beta)$ is complex with a nonzero imaginary part. Since the resonant mode is nondegenerate, $\omega_\star$ is a simple pole of $\det S$ and $\overline{\omega}_\star$ is a simple zero of $\det S$. Therefore, for the fixed β, f given in Eq. (17) is an analytic function of ω. For $\beta = \beta_\dagger$ and $\omega \neq \omega_\dagger$, we have $f(\omega, \beta) = \det S(\omega, \beta_\dagger)$. It is known that for a fixed β, the reflection and transmission coefficients are continuous in ω. Therefore we define $f(\omega_\dagger, \beta_\dagger)$ by

$$f(\omega_\dagger, \beta_\dagger) = \lim_{\omega \to \omega_\dagger} \det S(\omega, \beta_\dagger) = \det S(\omega_\dagger, \beta_\dagger). \tag{18}$$

Numerical results suggest that $f(\omega, \beta)$ is an analytic function of ω and β for ω near $\omega_\dagger$ and β near $\beta_\dagger$. Unfortunately, a formal mathematical proof is currently not available.

Using Eq. (9), we can easily show that

$$\overline{f}(\overline{\omega}, \beta) f(\omega, \beta) = 1, \tag{19}$$

where $\overline{f}(\overline{\omega}, \beta)$ is the complex conjugate of $f(\overline{\omega}, \beta)$. In particular, if ω is real, then $|f(\omega, \beta)| = 1$. Since $|f(\omega_\dagger, \beta_\dagger)| = 1$, we can choose a small neighborhood (denoted as Ω) of $(\omega_\dagger, \beta_\dagger)$, and choose a proper branch cut to define a complex square root function, so that $g(\omega, \beta) = \sqrt{f(\omega, \beta)}$ is nonzero and analytic (in ω and β) on Ω. With this function g, we can rewrite the scattering matrix as

$$S(\omega, \beta) = \frac{g(\omega, \beta)}{\omega - \omega_\star(\beta)} \begin{bmatrix} R(\omega, \beta) & \tilde{T}(\omega, \beta) \\ T(\omega, \beta) & \tilde{R}(\omega, \beta) \end{bmatrix}, \tag{20}$$

where $(R, T, \tilde{R}, \tilde{T}) = (a, b, \tilde{a}, \tilde{b})/g$. Clearly, R, T, $\tilde{R}$ and $\tilde{T}$ are analytic functions of ω and β on Ω, and they vanish at $(\omega_\dagger, \beta_\dagger)$. These functions are introduced, because they have useful properties. Using Eq. (9), we can show that

$$\tilde{T}(\omega, \beta) = -\overline{T}(\overline{\omega}, \beta), \tag{21}$$

$$\tilde{R}(\omega, \beta) = \overline{R}(\overline{\omega}, \beta). \tag{22}$$

In addition, it is also easy to verify that

$$\frac{\partial R}{\partial \omega}(\omega_\dagger, \beta_\dagger) = \frac{r_\dagger}{g_\dagger}, \quad \frac{\partial T}{\partial \omega}(\omega_\dagger, \beta_\dagger) = \frac{t_\dagger}{g_\dagger}, \tag{23}$$

where $r_\dagger = r(\omega_\dagger, \beta_\dagger)$, $t_\dagger = t(\omega_\dagger, \beta_\dagger)$ and $g_\dagger = g(\omega_\dagger, \beta_\dagger)$.

For case (a) of Sec. II, we have $r = \tilde{r}$, and thus

$$R(\omega, \beta) = \overline{R}(\overline{\omega}, \beta). \tag{24}$$

This implies that $R(\omega, \beta)$ is a real analytic function of two real variables ω and β near $(\omega_\dagger, \beta_\dagger)$. Since $R(\omega_\dagger, \beta_\dagger) = 0$, we can analyze the zero set of R using the partial derivatives of R at $(\omega_\dagger, \beta_\dagger)$.

(1) $\partial_\omega R \neq 0$ at $(\omega_\dagger, \beta_\dagger)$. According to Eq. (23), this condition is equivalent to $r_\dagger \neq 0$. In the ω-R plane, $R = R(\omega, \beta_\dagger)$ is a curve passing through zero at $\omega_\dagger$ with a nonzero slope. When β is slightly changed, the curve is slightly shifted. The new curve, given by $R = R(\omega, \beta)$, still passes through zero near $\omega_\dagger$. More precisely, according to the implicit function theorem, when $\partial_\omega R$ is nonzero at $(\omega_\dagger, \beta_\dagger)$, $R(\omega, \beta) = 0$ can be uniquely solved near $\beta_\dagger$, and the solution is a function $\omega = \omega_r^\circ(\beta)$, such that $\omega_r^\circ(\beta_\dagger) = \omega_\dagger$ and $R(\omega_r^\circ(\beta), \beta) = 0$ for β near $\beta_\dagger$. Therefore, if $r_\dagger \neq 0$ and β is close to $\beta_\dagger$, there is a real reflection zero $\omega_r^\circ(\beta)$ near $\omega_\dagger$. This is the generic case and $r_\dagger \neq 0$ is the generic condition.

(2) At $(\omega_\dagger, \beta_\dagger)$, $\partial_\omega R = 0$, $\partial_\omega^2 R \neq 0$, $\partial_\beta^2 R \neq 0$, $\partial_\omega^2 R$, and $\partial_\beta^2 R$ have the same sign. This is a nongeneric case. The condition $\partial_\omega R = 0$ implies $r_\dagger = 0$, and thus $\partial_\beta R = 0$ at $(\omega_\dagger, \beta_\dagger)$. This is the case for which $(\omega_\dagger, \beta_\dagger)$ is a local extremum of R. The function $R(\omega, \beta)$ is nonzero for real (ω, β) near but not equal to $(\omega_\dagger, \beta_\dagger)$. There is no real reflection zero for β near but not equal to $\beta_\dagger$.

(3) At $(\omega_\dagger, \beta_\dagger)$, $\partial_\omega R = 0$, $\partial_\omega^2 R \neq 0$, $\partial_\beta^2 R \neq 0$, $\partial_\omega^2 R$, and $\partial_\beta^2 R$ have the opposite sign. This is another nongeneric case. In the ω-R plane, $R = R(\omega, \beta_\dagger)$ obtains a local minimum (or maximum) at $\omega_\dagger$ and is exactly zero at $\omega_\dagger$. For β near $\beta_\dagger$, the minimum (or maximum) of the curve $R = R(\omega, \beta)$ is negative (or positive), and the curve passes zero at two values of ω near $\omega_\dagger$. On the surface given by $R = R(\omega, \beta)$, $(\omega_\dagger, \beta_\dagger)$ is a saddle point. For β near but not equal to $\beta_\dagger$, there are real two reflection zeros near $\omega_\dagger$.

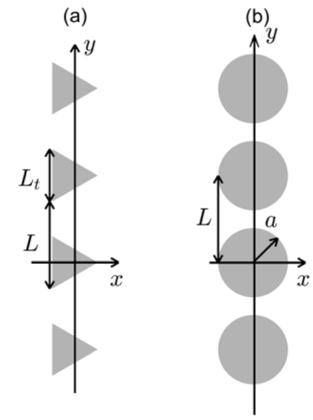

FIG. 1. Two periodic arrays of cylinders with period L in the y direction: (a) triangular cylinders with side length L_t; (b) circular cylinders with radius a.

There are other nongeneric cases. For example, when $\partial_\omega R = 0$, $\partial_\omega^2 R \neq 0$, we can consider the case $\partial_\beta^2 R = 0$ and $\partial_\beta^3 R \neq 0$. This leads to one real reflection zero for $\beta < \beta_\dagger$ (or $\beta > \beta_\dagger$) and no real reflection zero for $\beta > \beta_\dagger$ (or $\beta < \beta_\dagger$). We can also consider new cases assuming $\partial_\omega R = \partial_\omega^2 R = 0$ and $\partial_\omega^3 R \neq 0$. However, these additional nongeneric cases are too special and difficult to find in practice.

For case (b), we have $t = \tilde{t}$, thus

$$T(\omega, \beta) = -\overline{T}(\overline{\omega}, \beta). \tag{25}$$

Clearly, if ω is real, T is pure imaginary. Following the approach for analyzing R, we also identify the generic case and two main nongeneric cases as follows.

(1) $\partial_\omega T \neq 0$ at $(\omega_\dagger, \beta_\dagger)$, equivalent to $t_\dagger \neq 0$. This is the generic case and the generic condition is $t_\dagger \neq 0$. For β near $\beta_\dagger$, there is one transmission zero $\omega_t^\circ(\beta)$ near $\omega_\dagger$.

(2) At $(\omega_\dagger, \beta_\dagger)$, $\partial_\omega T = 0$, $\partial_\omega^2 T \neq 0$, $\partial_\beta^2 T \neq 0$, and $(\partial_\omega^2 T)(\partial_\beta^2 T) < 0$. This is a nongeneric case. $\mathrm{Im}(T)$ has a local extremum at $(\omega_\dagger, \beta_\dagger)$. There is no real transmission zero for β near $\beta_\dagger$ and $\beta \neq \beta_\dagger$.

(3) At $(\omega_\dagger, \beta_\dagger)$, $\partial_\omega T = 0$, $\partial_\omega^2 T \neq 0$, $\partial_\beta^2 T \neq 0$, and $(\partial_\omega^2 T)(\partial_\beta^2 T) > 0$. This is another nongeneric case. On the surface given by $\mathrm{Im}(T)$ as a function of real variables ω and β, $(\omega_\dagger, \beta_\dagger)$ is a saddle point. For any β near $\beta_\dagger$ and $\beta \neq \beta_\dagger$, there are two real transmission zeros.

For case (c), we have both $t = \tilde{t}$ and $r = \tilde{r}$. Therefore the above results on transmission and reflection zeros for cases (a) and (b), respectively, are valid for case (c).

V. NUMERICAL EXAMPLES

To validate our theory, we present several numerical examples involving periodic arrays of dielectric cylinders. In Fig. 1, we depict two periodic arrays with triangular and circular cylinders, respectively. The arrays are periodic in y with period L. The triangular cylinders have one surface parallel to the y axis, and their cross sections are equilateral triangles with side length L_t. The radius of the circular cylinders is a.

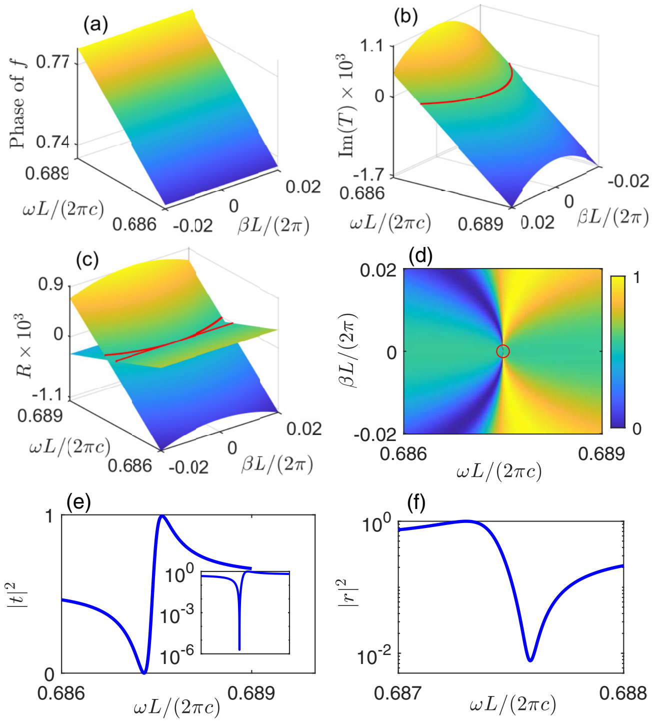

FIG. 2. Transmission and reflection near a BIC in a periodic array of triangular cylinders with $L_t = 0.45L$: (a) the phase of f in unit of π, (b) $\mathrm{Im}(T)$, (c) $\mathrm{Re}(R)$ and $\mathrm{Im}(R)$, (d) $|t|^2$, (e) $|t|^2$ for $\beta = 0.01(2\pi/L)$ (with a logarithmic plot in the inset). (f): $|r|^2$ for $\beta = 0.01(2\pi/L)$ (in a logarithmic scale). The solid red curves in (b) and (c) correspond to $T = 0$, $\mathrm{Re}(R) = 0$ and $\mathrm{Im}(R) = 0$, respectively. The circle in (d) corresponds the BIC.

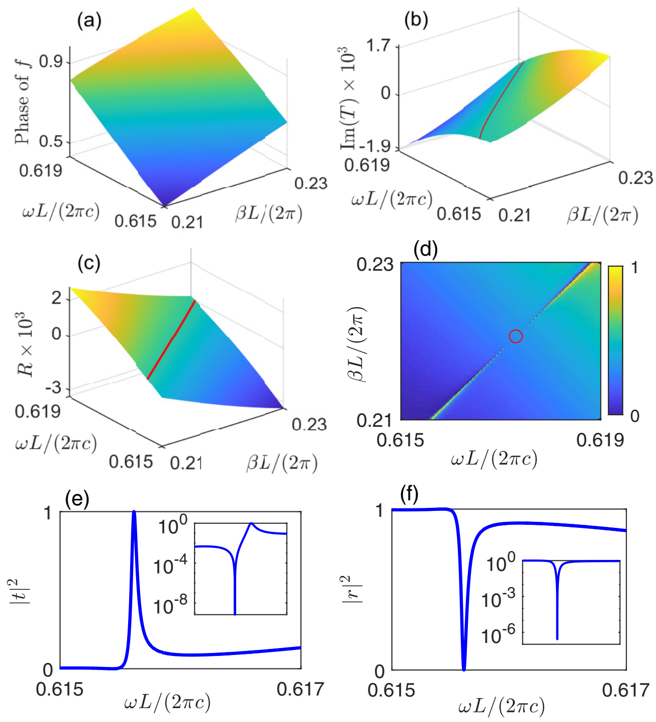

FIG. 3. Transmission and reflection near a BIC in a periodic array of circular cylinders with radius $a = 0.3L$: (a) the phase of f in unit of π, (b) $\mathrm{Im}(T)$, (c) R, (d) $|t|^2$ with the BIC marked by "∘," (e) $|t|^2$ for $\beta = 0.21(2\pi/L)$, and (f) $|r|^2$ for $\beta = 0.21(2\pi/L)$. The solid red curves in (b) and (c) correspond to $T = 0$ and $R = 0$, respectively. Insets in (e) and (f) are logarithmic plots.

All cylinders have the same dielectric constant $\varepsilon_1 = 10$ and are surrounded by air (with dielectric constant $\varepsilon_0 = 1$).

Our theory is for periodic structures with a BIC. It is well known that many different BICs may exist in a periodic array of dielectric cylinders [22–30]. Our first example is for the array of triangular cylinders shown in Fig. 1(a). For $L_t = 0.45L$, we found a BIC with frequency $\omega_\dagger = 0.6875(2\pi c/L)$ and wave number $\beta_\dagger = 0$. Since $\beta_\dagger$ is zero and the electric field is an odd function of y, this BIC is an anti-symmetric standing wave. The transmission coefficient at $(\omega_\dagger, \beta_\dagger)$ satisfies $|t_\dagger| = 0.7530$. Since the periodic array has a reflection symmetry in y and $t_\dagger \neq 0$, there should be a real transmission zero ω_t° for β near $\beta_\dagger = 0$. In Figs. 2(a)–2(d), we show the phase f, T, R and $|t|^2$ for (ω, β) near $(\omega_\dagger, \beta_\dagger)$. Our theory relies on the analyticity of function f in ω and β. Since $|f(\omega, \beta)| = 1$ for real (ω, β), we show the phase of f (in unit of π) in Fig. 2(a). The periodic array has a symmetry corresponding to case (b) of Sec. II, thus T is pure imaginary and R is complex for real (ω, β). In Figs. 2(b) and 2(c), we show $\mathrm{Im}(T)$, $\mathrm{Re}(R)$ and $\mathrm{Im}(R)$, and highlight their zero sets by the solid red curves. Notice that the two curves for R touch tangentially at $(\omega_\dagger, \beta_\dagger)$. In Fig. 2(d), we show transmittance $|t|^2$ as a function of ω and β, where the BIC point $(\omega_\dagger, \beta_\dagger)$ is marked by a small circle. To show peaks and dips more clearly, we plot transmission and reflection spectra ($|t|^2$ and $|r|^2$ as functions of ω) for $\beta = 0.01(2\pi/L)$ in Figs. 2(e) and 2(f), respectively. There is indeed a zero dip (corresponding to the real transmission zero) in the transmission spectrum, but the dip in the reflection spectrum is not zero, and the peak in the transmission spectrum is not 100%.

The second example is for a periodic array of circular cylinders with $a = 0.3L$. In this array, there is a propagating BIC with $\beta_\dagger = 0.2206(2\pi/L)$ and $\omega_\dagger = 0.6173(2\pi c/L)$. The transmission coefficient at the BIC point satisfies $|t_\dagger| = 0.5568$. The symmetry of the structure corresponds to case (c) of Sec. II. Since $t_\dagger \neq 0$ and $r_\dagger \neq 0$, both transmission and reflection coefficients have one real zero for β close to $\beta_\dagger$. In Figs. 3(a)–3(d), we show the phase of f, $\mathrm{Im}(T)$, R, and $|t|^2$ as functions of real ω and β. The real transmission and reflection zeros form curves in the ω-β plane and shown as the solid red curves in Figs. 3(b) and 3(c), respectively. The transmittance $|t|^2$ is shown in Fig. 3(d) for (ω, β) near $(\omega_\dagger, \beta_\dagger)$. To show the peaks and dips more clearly, we plot transmission and reflection spectra for $\beta = 0.21(2\pi/L)$ in Figs. 3(e) and 3(f), respectively. The zero dips in the spectra correspond to the real transmission and reflection zeros.

The first two examples cover only generic cases with $t_\dagger \neq 0$ and $r_\dagger \neq 0$. The third example is designed to illustrate a nongeneric case. The structure is still a periodic array of circular cylinders. Many BICs in the array exist continuously with respect to radius a. For $a = 0.2074L$, there is a BIC (an antisymmetric standing wave) with $\beta_\dagger = 0$, $\omega_\dagger = 0.5835(2\pi c/L)$, and $r_\dagger = 0$. In Fig. 4, we show the phase of f, imaginary part of T, R, and $|t|^2$ as functions of ω and β, and transmission and reflection spectra for $\beta = 0.01(2\pi/L)$.

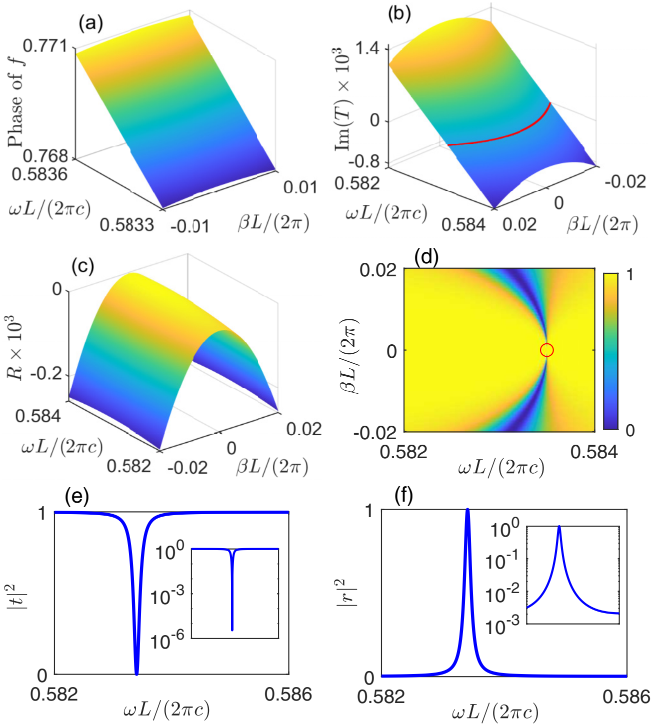

FIG. 4. Transmission and reflection near a BIC in a periodic array of circular cylinders with radius $a = 0.2074L$: (a) the phase of f in unit of π, (b) $\mathrm{Im}(T)$, (c) R, (d) $|t|^2$ with the BIC marked by "$\circ$," (e) transmission spectrum, and (f) reflection spectrum for $\beta = 0.01(2\pi/L)$. The solid red curve in (b) corresponds to $T = 0$. The sets in (e) and (f) are logarithmic plots.

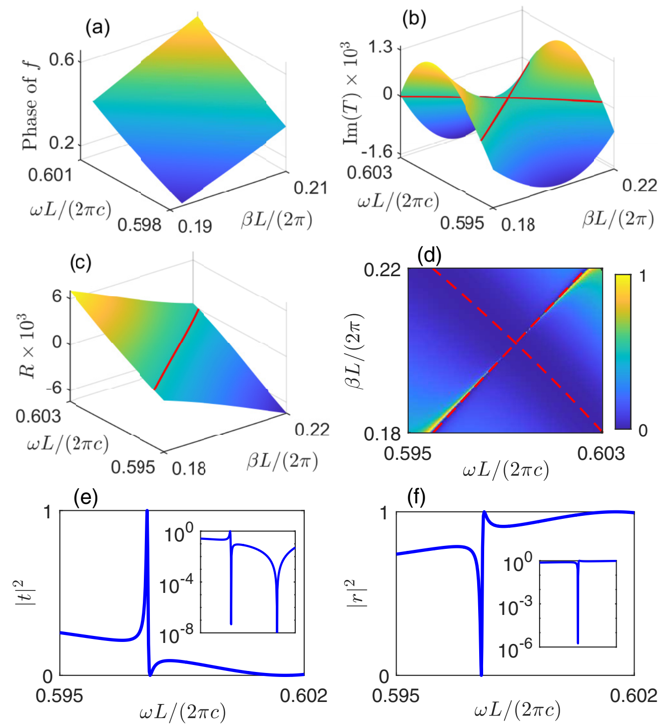

FIG. 5. Transmission and reflection near a BIC in a periodic array of circular cylinders with radius $a = 0.3087L$: (a) the phase of f in unit of π, (b) $\mathrm{Im}(T)$, (c) R, (d) $|t|^2$, (e) $|t|^2$ for $\beta = 0.19(2\pi/L)$, and (f) $|r|^2$ for $\beta = 0.19(2\pi/L)$. The solid red curves in (b) and (c), and the dashed red curves in (d), are the real zero sets of T (or t) and R. The insets in (e) and (f) are logarithmic plots.

Since $|t_\dagger| = 1$, there is a real transmission zero for each β near $\beta_\dagger = 0$, and it corresponds to the red curve in Fig. 4(b). On the other hand, $(\omega_\dagger, \beta_\dagger)$ is a local maximum point of R as shown in Fig. 4(c), and there are no real reflection zeros for β near $\beta_\dagger$. For $\beta = 0.01(2\pi/L)$, the transmission spectrum, shown in Fig. 4(e), has an inverted Lorentzian line shape and a zero dip. The reflection spectrum, shown in Fig. 4(f), has a full peak and no dip in the frequency range.

The fourth example is also designed to exhibit a nongeneric case. It is well known that a periodic array of circular cylinders can support propagating BICs (with a nonzero Bloch wave number) that depend on the radius a continuously. For radius $a = 0.3087L$, we found a propagating BIC with $\beta_\dagger = 0.2018(2\pi/L)$ and $\omega_\dagger = 0.5994(2\pi c/L)$, and $t_\dagger = 0$. In Figs. 5(a)–5(d), we show the phase of f, $\mathrm{Im}(T)$, R, and $|t|^2$, respectively, as functions of real (ω, β). It is clear that $(\omega_\dagger, \beta_\dagger)$ is saddle point of $\mathrm{Im}(T)$. For each β near $\beta_\dagger$, the transmission coefficient has two real zeros. As shown in Fig. 5(b), the zero set of T form two intersecting curves in the ω-β plane. The real transmission zeros are also shown in Fig. 5(d) as the red dashed curves. In Figs. 5(e) and 5(f), we show the transmission and reflection spectra, respectively, for $\beta = 0.19(2\pi/L)$. As shown clearly in the logarithmic plot (in the inset), there are two zero dips in the transmission spectrum. Since $|r_\dagger| = 1$, there is one real reflection zero for each β near $\beta_\dagger$. The zero set of R is shown as the red curve in Fig. 5(c). The reflection spectrum of Fig. 5(f) shows clearly a single zero dip.

The two examples above illustrate the nongeneric cases where $(\omega_\dagger, \beta_\dagger)$ is an extremum point of R and a saddle point of $\mathrm{Im}(T)$, respectively. We have also found numerical examples (still for a periodic array of circular cylinders) where $(\omega_\dagger, \beta_\dagger)$ is an extremum point of $\mathrm{Im}(T)$ or a saddle point of R. Since the results are quite similar, we skip these examples.

VI. CONCLUSION

For lossless periodic structures with a proper symmetry, the transmission and reflection spectra often have peaks and dips that are truly 100% and zero, respectively. When the transmission and reflection coefficients are considered as functions of the frequency ω, they vanish at their corresponding zeros, but the zeros are complex in general. A zero dip in the transmission/reflection spectrum corresponds to a real zero of the transmission/reflection coefficient. Existing theories on real transmission/reflection zeros have limitations and may be difficult to use [4,11]. In this paper, a relatively simple theory on the existence (and nonexistence) of real transmission/reflection zeros is developed. The key step is to scale the transmission and reflection coefficients, t and r, to T and R, such that for structures with a proper symmetry, T is a pure imaginary analytic function and R is a real analytic function of real ω and β. We identified the generic case and two nongeneric cases, for which the number of real transmission/reflection zeros is 1, 0, or 2, respectively. The nongeneric cases appear when the reflection or transmission coefficient at $(\omega_\dagger, \beta_\dagger)$ (the BIC frequency and wave number) vanishes, and when $(\omega_\dagger, \beta_\dagger)$ is either an extremum point or a saddle point of R or $\mathrm{Im}(T)$. Our theory is validated by numerical examples involving periodic arrays of dielectric cylinders.

It should be pointed out that our theory relies on the analyticity of function f given in Eq. (17), but a rigorous mathematical proof is not available. In addition, the theory is only applicable to periodic structures with a BIC. The transmission and reflection spectra are those for incident waves with a wave number β near $\beta_\dagger$. For resonant scattering problems without wave number β, for example, the scattering problem for a local defect in a closed waveguide, the transmission and reflection spectra can be approximated using the transmission and reflection coefficients at the real resonant frequency [19]. However, it is not clear whether this approximate theory can be used to establish the existence of real transmission/reflection zeros rigorously.

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