


# Proposed Fermi-surface reservoir engineering and application to realizing unconventional Fermi superfluids in a driven-dissipative nonequilibrium Fermi gas

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We develop a theory to describe the dynamics of a driven-dissipative many-body Fermi system to pursue our proposal to realize exotic quantum states based on reservoir engineering. Our idea is to design the shape of a Fermi surface so as to have multiple Fermi edges by properly attaching multiple reservoirs with different chemical potentials to a fermionic system. These emerged edges give rise to additional scattering channels that can destabilize the system into unconventional states, which is exemplified in this work by considering a driven-dissipative attractively interacting Fermi gas. By formulating a quantum kinetic equation using the Nambu-Keldysh Green's function technique, we explore nonequilibrium steady states in this system and assess their stability. We find that, in addition to the Bardeen—Cooper—Schrieffer-type isotropic pairing state, a Fulde-Ferrell-type anisotropic superfluid state being accompanied by Cooper pairs with nonzero center-of-mass momentum exists as a stable solution, even in the absence of a magnetic Zeeman field. Our result implies a great potential of realizing quantum matter beyond the equilibrium paradigm by engineering the shape and topology of Fermi surfaces in both electronic and atomic systems.

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## I. INTRODUCTION

The last two decades witnessed great progress in understanding and controlling many-body systems out of equilibrium [1–9]. When the system is driven out of equilibrium, restrictions such as the fluctuation-dissipation theorem are generally lifted. The lack of these constraints gives additional “free hands” for the system to exhibit exotic states that are otherwise prohibited in equilibrium. Floquet time crystals [10–15], light-induced superconducting-like states [16–18], long-range orders in two dimensions [19,20], as well as nonreciprocal phase transitions [21–25] are a few such examples. Among these, the strategy of dissipatively controlling many-body states by carefully designing the coupling between reservoirs and a system, which is often referred to as “reservoir engineering,” is recognized as a promising route to obtain the desired state [26–32]. For example, by an appropriate design of reservoir-system coupling, it was shown to be possible to implement a nontrivial topological state [26], universal quantum computing [27], as well as nonreciprocal coupling [30]. Although most of these consider Markovian reservoirs, it was pointed out that a non-Markovian reservoir is also useful as a dissipative stabilizer of strongly correlated states, such as the Mott insulator [31] and the fractional quantum Hall state [32].

In this paper, we apply non-Markovian reservoir engineering to a many-body Fermi system. In particular, we propose

to design the shape of a Fermi surface by attaching multiple reservoirs with different chemical potentials  $\mu_\alpha$  to the main system to explore exotic quantum many-body states that have not been realized, discussed, or known in condensed matter physics. In the simplest model shown in Fig. 1(a), for example, the two reservoirs supply the fermion to the system up to their respective chemical potentials, giving rise to a nonequilibrium steady state (NESS) with a two-step structure in the Fermi momentum distribution  $n_{p,\sigma=\uparrow,\downarrow}$ . [See the pop-up in Fig. 1(a).] As we discuss in the main text (see also Appendix C), we expect our proposal can be well achieved experimentally in ultracold Fermi gas systems, as well as electron systems, by using current state-of-the-art techniques.

If each edge imprinted on  $n_{p,\sigma}$  works like a Fermi surface it means that one can produce multiple Fermi surfaces from one Fermi sphere. Then, in the model case in Fig. 1(a), an  $s$ -wave attractive interaction  $-U$  is expected to produce four types of Cooper pairs (A) to (D), shown in Fig. 1(b). Among them, while (C) and (D) are essentially the same as the ordinary Bardeen—Cooper—Schrieffer (BCS) pairing, (A) and (B) are unconventional pairings with nonzero center-of-mass momentum. (A) and (B) are analogous to the unconventional Fulde-Ferrell (FF) state [see Fig. 1(c)] discussed in superconductivity under an external magnetic field [33–37], spin-polarized Fermi gases [38–43], and color superconductivity in quantum chromodynamics [44]. We recall that the FF state is usually realized in the spin-imbalanced case, where FF Cooper pairs are formed between  $\uparrow$ -spin fermions around the larger Fermi surface in Fig. 1(c) and  $\downarrow$ -spin ones around the smaller Fermi surface, as symbolically written as

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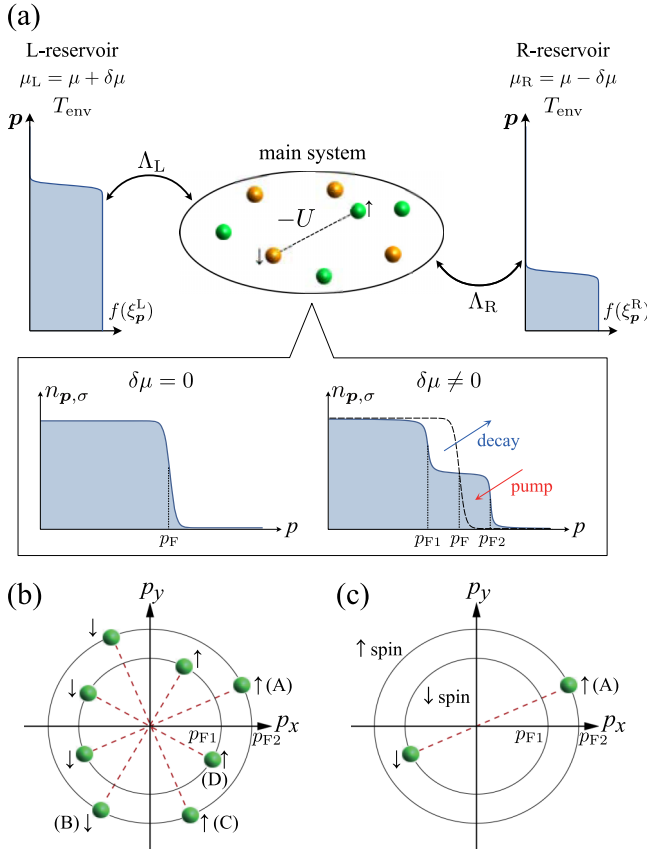


FIG. 1. (a) Model nonequilibrium driven-dissipative two-component Fermi gas with an  $s$ -wave pairing interaction  $-U (< 0)$ . The main system is coupled with two reservoirs ( $\alpha = L, R$ ) with different chemical potentials  $\mu_L = \mu + \delta\mu$  and  $\mu_R = \mu - \delta\mu$ . Both the reservoirs consist of free fermions in the thermal equilibrium state at the environment temperature  $T_{\text{env}}$ .  $f(\xi_p^\alpha)$  is the Fermi distribution function, where  $\xi_p^\alpha = \varepsilon_p - \omega_\alpha$  is the kinetic energy, measured from  $\omega_\alpha$ .  $\Lambda_\alpha$  describes tunneling between the main system and the  $\alpha$  reservoir. The pumping and decay of Fermi atoms by the two reservoirs bring about two edges at  $p_{F1} = \sqrt{2m\mu_R}$  and  $p_{F2} = \sqrt{2m\mu_L}$  in the Fermi momentum distribution  $n_{p,\sigma}$  in the main system (where  $\sigma = \uparrow, \downarrow$  describe two atomic hyperfine states). (b) Expected types (A)–(D) of Cooper pairs, when the two edges work like two Fermi surfaces. (c) Ordinary (thermal equilibrium) Fulde-Ferrell (FF) pairing state under an external magnetic field.

$|(\text{A})\rangle = |-\mathbf{p}_{F1}, \downarrow\rangle|\mathbf{p}_{F2}, \uparrow\rangle$ . In contrast, the model driven-dissipative Fermi gas in Fig. 1(a) is *not* accompanied by any spin imbalance, but each spin component has two “Fermi edges” at  $p_{F1}$  and  $p_{F2}$ . This leads to the pairing  $|(\text{B})\rangle = |\mathbf{p}_{F1}, \uparrow\rangle|-\mathbf{p}_{F2}, \downarrow\rangle$ , in addition to  $|(\text{A})\rangle$ . In a sense, the nonequilibrium FF-like (NFF) state may be viewed as a mixture of two FF states under external magnetic fields  $\mathbf{B}$  and  $-\mathbf{B}$ .

We note that possible routes to the FF state in the spin-balanced case were discussed in the literature, where the shift of the single-particle energy induced by the external current [45], a size effect [46], an interatomic interaction [47], and an artificial field [48–50] were proposed to realize this unconventional Fermi superfluid. However, these ideas are all in the thermal equilibrium case with the ordinary Fermi distribution

function, which is quite different from our idea in the nonequilibrium state.

In what follows, we confirm our scenario by dealing with the model driven-dissipative Fermi gas in Fig. 1(a). Our principal results are captured in Fig. 2. Figures 2(a) and 2(b) show the steady-state phase diagram of a driven-dissipative Fermi gas, with respect to half the chemical potential difference  $\delta\mu = [\mu_L - \mu_R]/2$  between the two reservoirs, the atomic damping rate  $\gamma$  coming from the system-reservoir couplings, and the environment temperature  $T_{\text{env}}$ . We clarify that all the states appearing in Fig. 2 are (meta)stable in the sense that the time evolution of a small *deviation* from each state always decays. (Detailed results of the stability analysis are presented in Sec. IV B.) The expected NFF state appears in region II, where the chemical potential difference  $\delta\mu/\mu$  is large enough to produce a clear two-step structure in  $n_{p,\sigma}$ , but the damping  $\gamma/\mu$  is not strong enough to smear out this structure. We note that the BCS-type superfluid (NBCS) is also stable in region II. This so-called bistability is a characteristic nonequilibrium phenomenon and was observed in various systems [51–53]. This is quite different from the thermal equilibrium case, where the ground state is uniquely identified as the state with the lowest free energy. In region III, the bistability of the NBCS and the normal state occurs.

In the bistability regions II and III, which state (among the multiple metastable states) is realized would depend on how the parameters are varied to reach these regions. When one varies  $\delta\mu$  adiabatically, we argue that the hysteresis shown in Fig. 2(c) appears: As  $\delta\mu$  increases from  $\delta\mu = 0$ , the NBCS would be maintained both in regions II and III. As one decreases  $\delta\mu$  from region IV, on the other hand, the phase transition from the normal state to NFF would occur at the boundary between regions II and III.

To close this section, let us briefly comment on the connection between our previous work [54,55] and the present study. In Refs. [54,55], we theoretically studied the properties of a nonequilibrium driven-dissipative Fermi gas in the *normal* phase and found that the chemical potential bias applied by two reservoirs gives rise to the anomalous enhancement of FF-type pairing fluctuations. In this paper, we extend these previous studies [54,55] to the superfluid phase and derive the quantum kinetic equation, to clarify the properties and stabilities of the unconventional Fermi superfluid state associated with this FF-type pairing.

This paper is organized as follows. In Sec. II, we extend the BCS theory to the nonequilibrium steady state by employing the Nambu-Keldysh Green’s function technique. We also give a concrete experimental setup to realized our proposal, both in an ultracold Fermi gas and electron systems. In Sec. III, using the same technique, we derive a quantum kinetic equation to evaluate the time evolution of the superfluid order parameter, under the initial condition that it slightly deviates from the mean-field value. We show our results in Sec. IV. We first show possible mean-field solutions for nonequilibrium superfluid steady states. We then assess their stability from the time evolution of the superfluid order parameter to draw the phase diagram in Figs. 2(a) and 2(b). Throughout this paper, we set  $\hbar = k_B = 1$ , and the system volume  $V$  is taken to be unity, for simplicity.

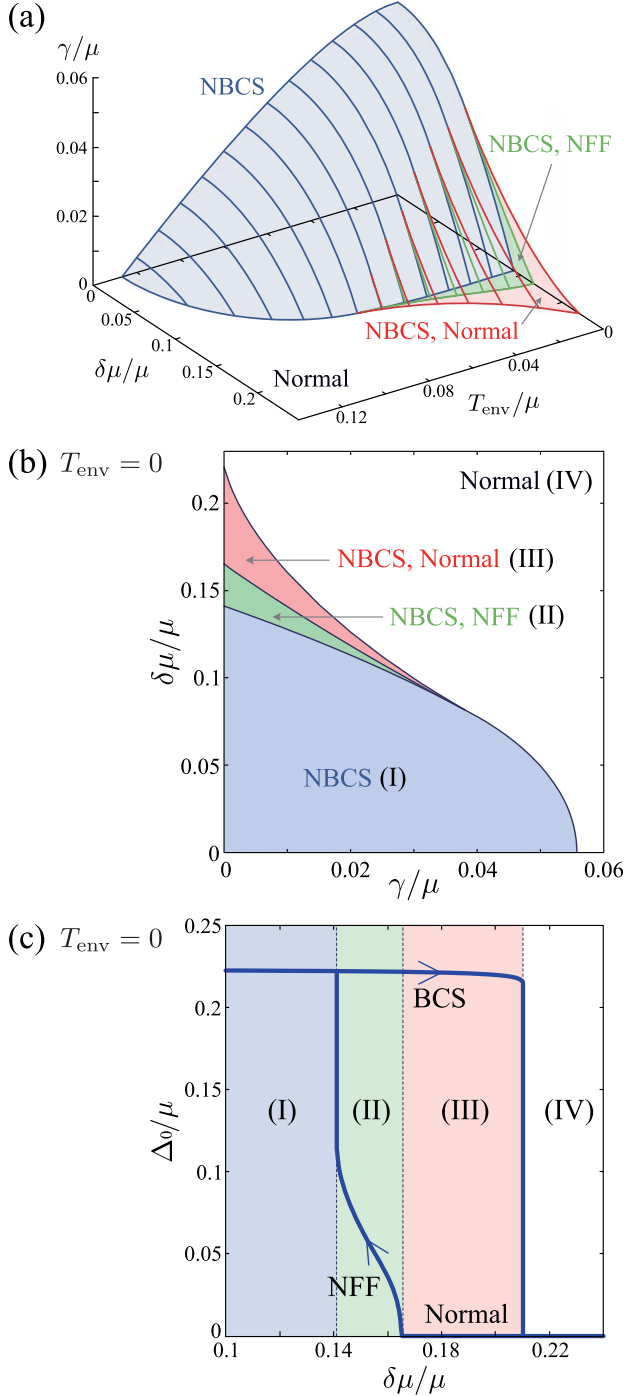


FIG. 2. Summary of our main results in this paper. (a) Steady-state phase diagram of a driven-dissipative two-component Fermi gas, with respect to half the chemical potential difference  $\delta\mu = [\mu_L - \mu_R]/2$  between the two reservoirs, the damping rate  $\gamma$  caused by system-reservoir couplings, and the environment temperature  $T_{\text{env}}$  (that are all scaled by the averaged chemical potential  $\mu = [\mu_L + \mu_R]/2$ ). This figure shows the weak-coupling case when  $(p_F a_s)^{-1} = -1$  (where  $p_F = \sqrt{2m\mu}$ ) under the vanishing current condition  $\mathbf{J}_{\text{net}} = 0$ . In the phase diagram, NBCS and NFF represent the nonequilibrium BCS ( $\mathbf{Q} = 0$ ) and FF-like ( $\mathbf{Q} \neq 0$ ) states, where  $\mathbf{Q}$  is the center-of-mass momentum of a Cooper pair. (b) The steady-state phase diagram at  $T_{\text{env}} = 0$ . (c) Hysteresis phenomenon in the regions II and III, shown in panel (b). In panel (c), we set  $T_{\text{env}} = 0$  and  $\gamma \rightarrow 0^+$ .

## II. BCS THEORY OF NONEQUILIBRIUM SUPERFLUID STEADY STATE

### A. Model driven-dissipative nonequilibrium Fermi gas

The model driven-dissipative two-component Fermi gas shown in Fig. 1(a) is described by the Hamiltonian

$$H = H_{\text{sys}} + H_{\text{env}} + H_{\text{mix}}, \quad (1)$$

where each term has the form, in the Nambu representation [56,57],

$$H_{\text{sys}} = - \int d\mathbf{r} \Psi^\dagger(\mathbf{r}) \frac{\nabla_{\mathbf{r}}^2}{2m} \tau_3 \Psi(\mathbf{r}) - U \int d\mathbf{r} \Psi^\dagger(\mathbf{r}) \tau_+ \Psi(\mathbf{r}) \Psi^\dagger(\mathbf{r}) \tau_- \Psi(\mathbf{r}), \quad (2)$$

$$H_{\text{env}} = \sum_{\alpha=L,R} \int d\mathbf{R} \Phi_\alpha^\dagger(\mathbf{R}) \left[ -\frac{\nabla_{\mathbf{R}}^2}{2m} - \omega_\alpha \right] \tau_3 \Phi_\alpha(\mathbf{R}), \quad (3)$$

$$H_{\text{mix}} = \sum_{\alpha=L,R} \sum_{i=1}^{N_i} [\Lambda_\alpha \Phi_\alpha^\dagger(\mathbf{R}_i^\alpha) e^{i\tau_3 \mu_\alpha t} \tau_3 \Psi(\mathbf{r}_i^\alpha) + \text{H.c.}]. \quad (4)$$

In these Hamiltonians,

$$\Psi(\mathbf{r}) = \begin{pmatrix} \psi_\uparrow(\mathbf{r}) \\ \psi_\downarrow(\mathbf{r}) \end{pmatrix}, \quad (5)$$

$$\Phi_\alpha(\mathbf{R}) = \begin{pmatrix} \phi_{\alpha,\uparrow}(\mathbf{R}) \\ \phi_{\alpha,\downarrow}(\mathbf{R}) \end{pmatrix}, \quad (6)$$

represent the two-component Nambu fields describing fermions in the main system and the  $\alpha = L, R$  reservoir, respectively [where  $\psi_\sigma(\mathbf{r})$  ( $\phi_{\alpha,\sigma}(\mathbf{R})$ ) is the annihilation operator of a fermion with pseudospin  $\sigma = \uparrow, \downarrow$  and particle mass  $m$  in the main system (reservoir)]. The corresponding Pauli matrices  $\tau_i$  ( $i = 1, 2, 3$ ), as well as  $\tau_\pm = [\tau_1 \pm i\tau_2]/2$ , act on the particle-hole space.

$H_{\text{sys}}$  in Eq. (2) describes the main system in Fig. 1(a), consisting of a two-component Fermi gas with an  $s$ -wave pairing interaction  $-U$  ( $< 0$ ). Because  $H_{\text{sys}}$  involves the ultraviolet divergence, as usual in cold Fermi gas physics, we remove this singularity by measuring the interaction strength in terms of the  $s$ -wave scattering length  $a_s$  [58]. It is related to the pairing interaction  $-U$  as

$$\frac{4\pi a_s}{m} = -\frac{U}{1 - U \sum_p \frac{1}{2\varepsilon_p}}, \quad (7)$$

where  $\varepsilon_p = \mathbf{p}^2/(2m)$  is the kinetic energy of a Fermi atom with an atomic  $m$ . In this paper, we only deal with the weak-coupling regime by setting  $(p_F a_s)^{-1} = -1$ . Here,  $p_F = \sqrt{2m\mu}$ , where  $\mu \equiv [\mu_R + \mu_L]/2$  ( $> 0$ ) is the averaged chemical potential between the two reservoirs [where  $\mu_\alpha$  is the chemical potential of the  $\alpha$  reservoir in Fig. 1(a)].

The two reservoirs ( $\alpha = L, R$ ) are described by the Hamiltonian  $H_{\text{env}}$  in Eq. (3). As schematically shown in Fig. 3,  $-\omega_{\alpha=L,R}$  in Eq. (3) gives the bottom of the energy band in the  $\alpha$  reservoir, when the energy is measured from the bottom ( $\varepsilon_{p=0} = 0$ ) of the energy band in the main system. We assume that both reservoirs are huge compared to the main system (which is satisfied by taking  $\omega_\alpha$  to be sufficiently large

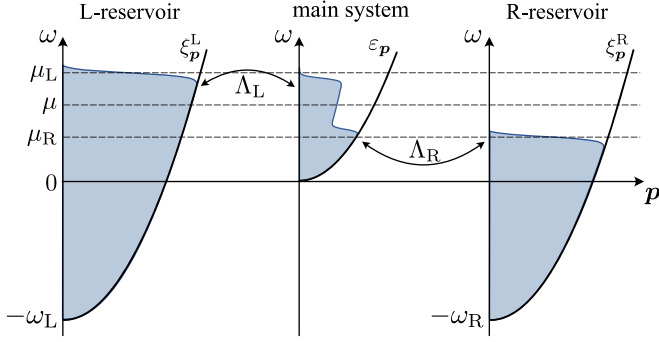


FIG. 3. Schematic energy band structure of our model. We measure the energy from the bottom ( $\varepsilon_{p=0} = 0$ ) of the energy band in the main system. We set  $\mu \ll \omega_\alpha$  so that the reservoirs are huge compared to the main system.

compared to  $\mu$ ) and are always in the thermal equilibrium state at the common environment temperature  $T_{\text{env}}$  [59]. The particle occupation in each reservoir obeys the ordinary Fermi distribution function at  $T_{\text{env}}$ ,

$$f(\omega) = \frac{1}{e^{\omega/T_{\text{env}}} + 1}. \quad (8)$$

The coupling between the main system and the reservoirs is described by  $H_{\text{mix}}$  in Eq. (4) with the coupling strength  $\Lambda_{\alpha=L,R}$ . For simplicity, we set  $\Lambda_L = \Lambda_R \equiv \Lambda$  in what follows. The particle tunneling is assumed to occur between randomly distributing spatial positions  $\mathbf{R}_i^\alpha$  in the  $\alpha$  reservoir and  $\mathbf{r}_i^\alpha$  in the main system ( $i = 1, \dots, N_i \gg 1$ ). As we will see later [see Eq. (24) and the paragraphs nearby], the random tunneling points introduced here assures the supply or decay of the particle from or to the reservoir to be uniform [60]. This mimics the experimental situations that we propose below, where the injection of particles are also approximately uniform (to be discussed later). This phenomenological modeling parameters  $\Lambda$  and  $N_i$  would only appear in the expression of the linewidth  $\gamma$  [see Eq. (28)], which we interpret as a fitting parameter that would be determined experimentally [63–65].

In Eq. (4), the factor  $\exp(i\tau_3 \mu_\alpha t)$  describes the situation that the energy band in the  $\alpha$  reservoir is filled up to  $\mu_\alpha$  (at  $T_{\text{env}} = 0$ ), as schematically shown in Fig. 3 [54,55]. By imposing the chemical-potential bias  $\mu_L = \mu + \delta\mu$  and  $\mu_R = \mu - \delta\mu$  ( $\delta\mu > 0$ ) between the two reservoirs, we realize the nonequilibrium steady state in the main system. In this paper, we fix the value of the average chemical potential  $\mu$ , and tune the nonequilibrium situation of the main system by adjusting the tunneling matrix element  $\Lambda$ , as well as the chemical-potential bias  $\delta\mu$  [66].

At the end of this subsection, we comment on this model and its feasibility in experiments. We expect the proposed setup can experimentally be realized, both in ultracold Fermi

systems and electron systems. In the first systems, several groups succeeded in splitting a cloud of a trapped Fermi gas into two [67–72]. This system is known to be well described by two systems coupled via the tunneling between them [73–76], similarly to our Hamiltonian [Eq. (1)]. Thus, we strongly believe that our system composed of *three* systems (the main system and the left and right reservoirs) can be realized by extending this two-terminal setup into three, and applying a magnetic field to tune the strength of a pairing interaction associated with a Feshbach resonance only to the main system by using the techniques developed in Refs. [77–82]. In this setup, the tunneling process between the system and reservoirs would be overwhelmingly complicated: When a particle is injected from the reservoir to the system, they relax, decohere, and diffuse via inelastic collision processes. However, since this process seems to occur very quickly in the two-terminal configuration in cold atomic systems, the pumping can be regarded as being effectively uniform (which justifies our modeling with random tunneling points). This is supported by the verification of the Landauer formula [70] and the ac and dc Josephson effects [83], which uses the above assumption of fast dissipation over space. In Appendix C, we further argue that it is possible to achieve the parameter regime necessary to realize the unconventional states (i.e.,  $\gamma/\mu < 0.01$ ) within the current experimental techniques.

In electron systems, a metal (that turns into a superconductor at low temperature) under a strong voltage bias ( $eV \gg k_B T$ ) is also a promising experimental setup to realize our proposed nonequilibrium FF-type superconducting state. In fact, a nonequilibrium quasiparticle distribution with the two-step structure analogous to the one depicted in Fig. 1(a) was observed in voltage-biased mesoscopic wires [84–86], as well as in carbon nanotubes [87]. The nonequilibrium quasiparticle distribution in a voltage-biased mesoscopic superconducting wire was also investigated theoretically based on the quasiclassical theory, which showed that a nonequilibrium distribution with the two-step structure is realized near the center of the wire, when the wire is sufficiently long compared to the superconducting coherence length [88,89]. Thus, we expect that the exotic superconducting state induced by the nonequilibrium quasiparticle distribution can also be observed in voltage-biased mesoscopic superconducting wires connected to normal-metal electrodes [89–91] or thin superconducting films sandwiched between electrodes [92–94]. In Appendix C, we provide further arguments on the feasibility of the realization of our proposal.

## B. Nonequilibrium Nambu-Keldysh Green's function

To deal with the nonequilibrium superfluid state in the main system, we conveniently employ the  $4 \times 4$  matrix Nambu-Keldysh Green's function [95,96], given by

$$\hat{\mathcal{G}}(1, 2) = \begin{pmatrix} \mathcal{G}^R(1, 2) & \mathcal{G}^K(1, 2) \\ 0 & \mathcal{G}^A(1, 2) \end{pmatrix} = \begin{pmatrix} -i\Theta(t_1 - t_2)\langle[\Psi(1) \diamond \Psi^\dagger(2)]_+\rangle & -i\langle[\Psi(1) \diamond \Psi^\dagger(2)]_-\rangle \\ 0 & i\Theta(t_2 - t_1)\langle[\Psi(1) \diamond \Psi^\dagger(2)]_+\rangle \end{pmatrix}, \quad (9)$$

where  $\Theta(t)$  is the step function, and the abbreviated notations  $1 \equiv (\mathbf{r}_1, t_1)$  and  $2 \equiv (\mathbf{r}_2, t_2)$  are used. In Eq. (9),  $\mathcal{G}^R$ ,  $\mathcal{G}^A$ , and  $\mathcal{G}^K$  are the  $2 \times 2$  matrix retarded, advanced, and Keldysh Green's functions in the two-component Nambu space, respectively. In Eq. (9),

$$[\Psi(1) \diamond \Psi^\dagger(2)]_\pm = \Psi(1) \diamond \Psi^\dagger(2) \pm \Psi^\dagger(2) \diamond \Psi(1), \quad (10)$$

where “ $\diamond$ ” denotes the operation,

$$\Psi(1) \diamond \Psi^\dagger(2) = \begin{pmatrix} \psi_\uparrow(1)\psi_\uparrow^\dagger(2) & \psi_\uparrow(1)\psi_\downarrow^\dagger(2) \\ \psi_\downarrow^\dagger(1)\psi_\uparrow^\dagger(2) & \psi_\downarrow^\dagger(1)\psi_\downarrow^\dagger(2) \end{pmatrix}, \quad (11)$$

$$\Psi^\dagger(2) \diamond \Psi(1) = \begin{pmatrix} \psi_\uparrow^\dagger(2)\psi_\uparrow(1) & \psi_\downarrow(2)\psi_\uparrow(1) \\ \psi_\uparrow^\dagger(2)\psi_\downarrow^\dagger(1) & \psi_\downarrow(2)\psi_\downarrow^\dagger(1) \end{pmatrix}. \quad (12)$$

For later convenience, we also introduce the  $4 \times 4$  matrix lesser Green's function  $\mathcal{G}^<(1, 2)$ , which is related to  $\mathcal{G}^{R,K,A}(1, 2)$  as

$$\begin{aligned} \mathcal{G}^<(1, 2) &= i(\Psi^\dagger(2) \diamond \Psi(1)) \\ &= \frac{1}{2}[\mathcal{G}^K(1, 2) - \mathcal{G}^R(1, 2) + \mathcal{G}^A(1, 2)]. \end{aligned} \quad (13)$$

We briefly note that the diagonal and off-diagonal components of  $\mathcal{G}^<$  are related to the particle density and the pair amplitude, respectively.

In the Nambu-Keldysh scheme, effects of the pairing interaction  $-U$  and the system-reservoir couplings  $\Lambda_{\alpha=L,R}$  can be summarized by the  $4 \times 4$  matrix self-energy correction

$$\hat{\Sigma}(1, 2) = \begin{pmatrix} \Sigma^R(1, 2) & \Sigma^K(1, 2) \\ 0 & \Sigma^A(1, 2) \end{pmatrix} = \hat{\Sigma}_{\text{int}}(1, 2) + \hat{\Sigma}_{\text{env}}(1, 2), \quad (14)$$

$$\begin{aligned} \hat{\Sigma}_{\text{int}}(1, 2) &= iU \sum_{s=\pm} \sum_{\alpha=1,2} (\tau_s \otimes \eta_\alpha^+) \text{Tr}_N \text{Tr}_K [(\tau_{-s} \otimes \eta_\alpha^-) \hat{\mathcal{G}}(1, 2)] \delta(1-2) \\ &= \frac{iU}{2} \sum_{s=\pm} \begin{pmatrix} \tau_s \text{Tr}_N [\tau_{-s} \mathcal{G}^K(1, 2)] & \tau_s \text{Tr}_N [\tau_{-s} \mathcal{G}^R(1, 2) + \tau_{-s} \mathcal{G}^A(1, 2)] \\ \tau_s \text{Tr}_N [\tau_{-s} \mathcal{G}^R(1, 2) + \tau_{-s} \mathcal{G}^A(1, 2)] & \tau_s \text{Tr}_N [\tau_{-s} \mathcal{G}^K(1, 2)] \end{pmatrix} \delta(1-2). \end{aligned} \quad (18)$$

Here,

$$\eta_\alpha^+ = \frac{1}{\sqrt{2}} \sigma_{2-\alpha}, \quad \eta_\alpha^- = \frac{1}{\sqrt{2}} \sigma_{\alpha-1} \quad (19)$$

are vertex matrices [54], where  $\sigma_{i=1,2,3}$  are the Pauli matrices acting on the Keldysh space.  $\text{Tr}_N$  and  $\text{Tr}_K$  stand for taking the trace over the Nambu and the Keldysh spaces, respectively.

We introduce the superfluid order parameter  $\Delta(\mathbf{r}_1, t_1)$ , which is related to the off-diagonal component of

which appears in the nonequilibrium Nambu-Keldysh Dyson equation [95,96]

$$\hat{\mathcal{G}}(1, 2) = \hat{\mathcal{G}}_0(1, 2) + [\hat{\mathcal{G}}_0 \circ \hat{\Sigma} \circ \hat{\mathcal{G}}](1, 2). \quad (15)$$

Here,

$$\begin{aligned} [A \circ B](1, 2) &= \int d\mathbf{r}_3 \int_{-\infty}^{\infty} dt_3 A(\mathbf{r}_1, t_1, \mathbf{r}_3, t_3) B(\mathbf{r}_3, t_3, \mathbf{r}_2, t_2) \\ &= \int d3 A(1, 3) B(3, 2), \end{aligned} \quad (16)$$

and

$$\begin{aligned} \hat{\mathcal{G}}_0(1, 2) &= \sum_{\mathbf{p}} \int \frac{d\omega}{2\pi} e^{i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - i\omega(t_1 - t_2)} \hat{\mathcal{G}}_0(\mathbf{p}, \omega) \\ &= \sum_{\mathbf{p}} \int \frac{d\omega}{2\pi} e^{i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2) - i\omega(t_1 - t_2)} \\ &\quad \times \begin{pmatrix} \frac{1}{\omega_+ - \varepsilon_{\mathbf{p}} \tau_3} & -2\pi i \delta(\omega - \varepsilon_{\mathbf{p}}) [1 - 2f_{\text{ini}}(\omega)] \\ 0 & \frac{1}{\omega_- - \varepsilon_{\mathbf{p}} \tau_3} \end{pmatrix} \end{aligned} \quad (17)$$

is the bare Green's function in the *initial* thermal equilibrium state at  $t = -\infty$ , where the system-reservoir couplings  $\Lambda_{\alpha=L,R}$ , as well as the pairing interaction  $-U$ , were absent. (Note that  $\hat{\mathcal{G}}_0(1, 2)$  only depends on the relative coordinate as  $\hat{\mathcal{G}}_0(1, 2) = \hat{\mathcal{G}}_0(1-2)$ .) In Eq. (17),  $\omega_\pm = \omega \pm i\delta$  (where  $\delta$  is an infinitesimally small positive number), and  $f_{\text{ini}}(\omega) = 1/[e^{\omega/T_{\text{ini}}} + 1]$  is the Fermi distribution function with  $T_{\text{ini}}$  being the initial temperature of the main system at  $t = -\infty$ . Although the Dyson equation (15) looks like it depends on the initial state through  $\hat{\mathcal{G}}_0(1, 2)$ , we will soon find that the dressed Green's function  $\hat{\mathcal{G}}(1, 2)$  in the final nonequilibrium steady state, which we are interested in, actually loses the initial memory [54,55,61–65,97,98].

In Eq. (14),  $\hat{\Sigma}_{\text{int}}$  and  $\hat{\Sigma}_{\text{env}}$  describe the effects of the pairing interaction  $-U$  and the system-reservoir couplings  $\Lambda_{\alpha=L,R}$  ( $= \Lambda$ ), respectively. In the mean-field BCS approximation [63–65], the first is diagrammatically drawn as Fig. 4(a), which gives

$\mathcal{G}^K$  and  $\mathcal{G}^<$  as

$$\begin{aligned} \Delta(\mathbf{r}_1, t_1) &\equiv U \langle \psi_\downarrow(1) \psi_\uparrow(1) \rangle = -\frac{iU}{2} \mathcal{G}^K(1, 1)_{12} \\ &= -iU \mathcal{G}^<(1, 1)_{12}. \end{aligned} \quad (20)$$

Then, Eq. (18) can be simply written as

$$\begin{aligned} \hat{\Sigma}_{\text{int}}(1, 2) &= \begin{pmatrix} -\Delta(1)\tau_+ - \Delta^*(1)\tau_- & 0 \\ 0 & -\Delta(1)\tau_+ - \Delta^*(1)\tau_- \end{pmatrix} \\ &\quad \times \delta(1-2). \end{aligned} \quad (21)$$

We note that the off-diagonal components of  $\hat{\Sigma}_{\text{int}}(1, 2)$  identically vanish because  $\mathcal{G}^{\text{R(A)}}(1, 1)_{12} = \mathcal{G}^{\text{R(A)}}(1, 1)_{21} = 0$ .

For the self-energy correction  $\hat{\Sigma}_{\text{env}}$  in Eq. (14), we take into account the system-reservoir couplings within the second-order Born approximation, as diagrammatically shown in Fig. 4(b). An evaluation of this diagram gives

$$\hat{\Sigma}_{\text{env}}(\mathbf{p}, \mathbf{p}', t_1, t_2) = \int d\mathbf{r}_1 \int d\mathbf{r}_2 \hat{\Sigma}_{\text{env}}(1, 2) e^{-i(\mathbf{p}\cdot\mathbf{r}_1 + \mathbf{p}'\cdot\mathbf{r}_2)} = |\Lambda|^2 \sum_{\alpha=L,R} \sum_{i,j}^{N_t} \hat{D}_\alpha(\mathbf{R}_i^\alpha - \mathbf{R}_j^\alpha, t_1 - t_2) e^{-i\tau_3 \mu_\alpha (t_1 - t_2)} e^{-i(\mathbf{p}\cdot\mathbf{r}_1 + \mathbf{p}'\cdot\mathbf{r}_2)}, \quad (22)$$

where

$$\begin{aligned} \hat{D}_\alpha(\mathbf{R}_i^\alpha - \mathbf{R}_j^\alpha, t_1 - t_2) &= \sum_{\mathbf{q}} \int \frac{d\omega}{2\pi} e^{i\mathbf{q}\cdot(\mathbf{R}_i^\alpha - \mathbf{R}_j^\alpha) - i\omega(t_1 - t_2)} \hat{D}_0(\mathbf{q}, \omega) = \sum_{\mathbf{q}} \int \frac{d\omega}{2\pi} e^{i\mathbf{q}\cdot(\mathbf{R}_i^\alpha - \mathbf{R}_j^\alpha) - i\omega(t_1 - t_2)} \\ &\times \begin{pmatrix} \frac{1}{\omega_+ - \xi_{\mathbf{q}}^\alpha \tau_3} & -2\pi i \delta(\omega - \xi_{\mathbf{q}}^\alpha) \tanh\left(\frac{\omega}{2T_{\text{env}}}\right) \\ 0 & \frac{1}{\omega_- - \xi_{\mathbf{q}}^\alpha \tau_3} \end{pmatrix} \end{aligned} \quad (23)$$

is the noninteracting Nambu-Keldysh Green's function in the  $\alpha$  reservoir, with  $\xi_{\mathbf{q}}^\alpha = \varepsilon_{\mathbf{q}} - \omega_\alpha$  being the kinetic energy measured from the bottom energy  $-\omega_\alpha$  of the band in the  $\alpha$  reservoir. (Note that the reservoirs are assumed to be in thermal equilibrium.) When one takes the spatial averages over the randomly distributing tunneling positions  $\mathbf{R}_i^\alpha$  and  $\mathbf{r}_i^\alpha$  in Eq. (22), the resulting self-energy  $\langle \hat{\Sigma}_{\text{env}}(\mathbf{p}, \mathbf{p}', t_1, t_2) \rangle_{\text{av}}$  recovers its translational invariance as [54,63–65]

$$\langle \hat{\Sigma}_{\text{env}}(\mathbf{p}, \mathbf{p}', t_1, t_2) \rangle_{\text{av}} = \hat{\Sigma}_{\text{env}}(\mathbf{p}, t_1, t_2) \delta_{\mathbf{p}, \mathbf{p}'}, \quad (24)$$

where

$$\hat{\Sigma}_{\text{env}}(\mathbf{p}, t_1, t_2) = N_t |\Lambda|^2 \sum_{\mathbf{q}, \alpha=L,R} \hat{D}_\alpha(\mathbf{q}, t_1 - t_2) e^{-i\mu_\alpha (t_1 - t_2) \tau_3}. \quad (25)$$

Carrying out the Fourier transformation with respect to the relative time  $t_1 - t_2$ , we have

$$\hat{\Sigma}_{\text{env}}(\mathbf{p}, \omega) = N_t |\Lambda|^2 \sum_{\mathbf{q}, \alpha=L,R} \hat{D}_\alpha(\mathbf{q}, \omega - \mu_\alpha \tau_3). \quad (26)$$

For simplicity, we employ the so-called wide-band limit approximation [99], that is, we assume white reservoirs with the constant density of states  $\rho_\alpha(\omega) \equiv \rho$ . This approximation is justified when the reservoirs are so huge that the energy dependence of the reservoir density of states around the Fermi level can be ignored [54], which is just the situation we are considering ( $\mu \ll \omega_\alpha$ ). Then, replacing the  $\mathbf{q}$  summation in Eq. (26) by the  $\xi^\alpha$  integration, we obtain

$$\begin{aligned} \hat{\Sigma}_{\text{env}}(\mathbf{p}, \omega) &= \begin{pmatrix} -2i\gamma \tau_0 & -2i\gamma \left[ \tanh\left(\frac{\omega - \tau_3 \mu_L}{2T_{\text{env}}}\right) + \tanh\left(\frac{\omega - \tau_3 \mu_R}{2T_{\text{env}}}\right) \right] \tau_0 \\ 0 & 2i\gamma \tau_0 \end{pmatrix}. \end{aligned} \quad (27)$$

Here,

$$\gamma = \pi N_t \rho |\Lambda|^2 \quad (28)$$

is the quasiparticle damping rate and  $\tau_0$  is the  $2 \times 2$  unit matrix acting on the particle-hole Nambu space.

### C. Extension of BCS theory to nonequilibrium steady state

In this paper, we explore stable nonequilibrium superfluid steady states, having the following type of the order

parameter:

$$\Delta(\mathbf{r}, t) = \Delta_0 e^{i\mathbf{Q}\cdot\mathbf{r}} e^{-2i\mu t}. \quad (29)$$

Without loss of generality, we take  $\Delta_0 > 0$ . When  $\mathbf{Q} = 0$ , Eq. (29) describes the BCS-type uniform superfluid, which was discussed in exciton(-polariton) systems [23,61,63–65,97,98]. When  $\mathbf{Q} \neq 0$ , Eq. (29) has the same form as the order parameter in the Fulde-Ferrell (FF) superfluid state, discussed in superconductivity under an external magnetic field [33–37], as well as in a spin-polarized Fermi gas [38–43]. Although the Larkin-Ovchinnikov-type solution [34]  $\Delta(\mathbf{r}, t) = \Delta_0 \cos(\mathbf{Q} \cdot \mathbf{r}) e^{-2i\mu t}$  is also conceivable in our model, leaving this possibility as our future study, we only deal with the FF-type solution in this paper. Regarding this, we emphasize that the main system has *no* spin imbalance.

To treat the superfluid order parameter  $\Delta(\mathbf{r}, t)$  in Eq. (29), it is convenient to formally remove the time and spatial dependence from it, which is achieved by employing the following gauge transformation [95]:

$$\begin{aligned} \hat{\mathcal{G}}(1, 2) &= \begin{pmatrix} \tilde{\mathcal{G}}^{\text{R}}(1, 2) & \tilde{\mathcal{G}}^{\text{K}}(1, 2) \\ 0 & \tilde{\mathcal{G}}^{\text{A}}(1, 2) \end{pmatrix} \\ &\equiv e^{-i\chi(1)\tau_3 \otimes \sigma_0} \hat{\mathcal{G}}(1, 2) e^{i\chi(2)\tau_3 \otimes \sigma_0}, \end{aligned} \quad (30)$$

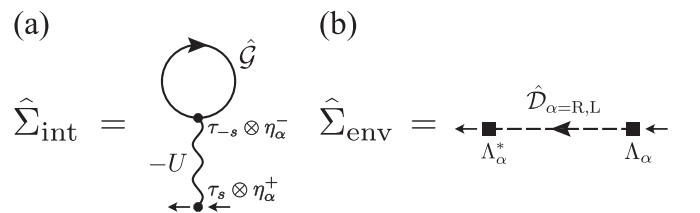


FIG. 4. Self-energy corrections. (a)  $\hat{\Sigma}_{\text{int}}$  describes effects of the pairing interaction  $-U$  in the mean-field BCS approximation. The solid line is the dressed Nambu-Keldysh Green's function  $\hat{\mathcal{G}}$  in the main system. The wavy line is the pairing interaction  $-U$ , which is accompanied by the vertices  $\tau_{\pm s} \otimes \eta_{\alpha}^{\pm}$  at both ends (where  $s = \pm$ ), acting on the Nambu  $\otimes$  Keldysh space. (b)  $\hat{\Sigma}_{\text{env}}$  describes effects of the system-reservoir couplings  $\Lambda_{\alpha=L,R}$  in the second-order Born approximation. The dashed line is the Green's function  $\hat{D}_{\alpha=L,R}$  in the  $\alpha$  reservoir. The solid square represents the tunneling matrix  $\Lambda_{\alpha=L,R}$  between the system and the  $\alpha$  reservoir.

$$\begin{aligned}\hat{\Sigma}(1, 2) &= \begin{pmatrix} \tilde{\Sigma}^R(1, 2) & \tilde{\Sigma}^K(1, 2) \\ 0 & \tilde{\Sigma}^A(1, 2) \end{pmatrix} \\ &\equiv e^{-i\chi(1)\tau_3 \otimes \sigma_0} \hat{\Sigma}(1, 2) e^{i\chi(2)\tau_3 \otimes \sigma_0},\end{aligned}\quad (31)$$

where

$$\chi(\mathbf{r}, t) = \frac{1}{2} \mathbf{Q} \cdot \mathbf{r} - \mu t, \quad (32)$$

and  $\sigma_0$  is the  $2 \times 2$  unit matrix acting on the Keldysh space. The Nambu-Keldysh Dyson equation (15) after this manipulation is given by

$$\hat{\hat{\Sigma}}(1, 2) = \hat{\hat{G}}_0(1, 2) + [\hat{\hat{G}}_0 \circ \hat{\hat{\Sigma}} \circ \hat{\hat{G}}](1, 2). \quad (33)$$

$$\begin{aligned}\hat{\hat{G}}_0(\mathbf{p}, \omega) &= \int d\mathbf{r} \int dt e^{i(\omega t - \mathbf{p} \cdot \mathbf{r})} \hat{G}_0(\mathbf{r}, t) = \hat{G}_0(\mathbf{p} + \frac{\mathbf{Q}}{2} \tau_3, \omega + \mu \tau_3) \\ &= \begin{pmatrix} \frac{1}{\omega + \xi_{\mathbf{p} + (\mathbf{Q}/2)\tau_3} \tau_3} & -2\pi i \delta(\omega - \xi_{\mathbf{p} + (\mathbf{Q}/2)\tau_3} \tau_3) [1 - 2f_{\text{ini}}(\omega + \mu \tau_3)] \\ 0 & \frac{1}{\omega - \xi_{\mathbf{p} + (\mathbf{Q}/2)\tau_3} \tau_3} \end{pmatrix},\end{aligned}\quad (36)$$

with  $\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} - \mu$ . The interaction component  $\hat{\Sigma}_{\text{int}}$  of the self-energy  $\hat{\Sigma} = \hat{\Sigma}_{\text{int}} + \hat{\Sigma}_{\text{env}}$  in Eq. (33) is obtained from Eq. (21) as

$$\hat{\Sigma}_{\text{int}}(1, 2) = \hat{\Sigma}_{\text{int}}(1 - 2) = \begin{pmatrix} -\Delta_0 \tau_1 & 0 \\ 0 & -\Delta_0 \tau_1 \end{pmatrix} \delta(1 - 2). \quad (37)$$

In the energy and momentum space, this self-energy has the form

$$\hat{\Sigma}_{\text{int}}(\mathbf{p}, \omega) = \begin{pmatrix} -\Delta_0 \tau_1 & 0 \\ 0 & -\Delta_0 \tau_1 \end{pmatrix}. \quad (38)$$

In the same manner, the self-energy  $\hat{\Sigma}_{\text{env}}(1, 2)$  coming from the system-reservoir couplings can also be obtained from Eq. (27). Thus, it has the form, in the energy and momentum space,

$$\begin{aligned}\hat{\Sigma}_{\text{env}}(\mathbf{p}, \omega) &= \hat{\Sigma}_{\text{env}}\left(\mathbf{p} + \frac{\mathbf{Q}}{2} \tau_3, \omega + \mu \tau_3\right) \\ &= \begin{pmatrix} -2i\gamma \tau_0 & -2i\gamma [\tanh(\frac{\omega - \delta\mu}{2T_{\text{env}}}) + \tanh(\frac{\omega + \delta\mu}{2T_{\text{env}}})] \tau_0 \\ 0 & 2i\gamma \tau_0 \end{pmatrix}.\end{aligned}\quad (39)$$

The self-energy  $\hat{\Sigma}(\mathbf{p}, \omega)$  involved in the Dyson equation (35) is then given by the sum of Eqs. (38) and (39).

Solving the Dyson equation (35), we obtain

$$\tilde{G}^R(\mathbf{p}, \omega) = \sum_{\eta=\pm} \frac{1}{\omega + 2i\gamma - E_{\mathbf{p}, \mathbf{Q}}^{0, \eta}} \Xi_{\mathbf{p}, \mathbf{Q}}^{0, \eta}, \quad (40)$$

$$\tilde{G}^A(\mathbf{p}, \omega) = \sum_{\eta=\pm} \frac{1}{\omega - 2i\gamma - E_{\mathbf{p}, \mathbf{Q}}^{0, \eta}} \Xi_{\mathbf{p}, \mathbf{Q}}^{0, \eta}, \quad (41)$$

$$\tilde{G}^K(\mathbf{p}, \omega) = \sum_{\eta=\pm} \frac{4i\gamma [1 - 2F(\omega)]}{[\omega - \eta E_{\mathbf{p}, \mathbf{Q}}^{0, \eta}]^2 + 4\gamma^2} \Xi_{\mathbf{p}, \mathbf{Q}}^{0, \eta}, \quad (42)$$

where

$$E_{\mathbf{p}, \mathbf{Q}}^{0, \pm} = \sqrt{(\xi_{\mathbf{p}, \mathbf{Q}}^{(s)})^2 + \Delta_0^2} \pm \xi_{\mathbf{p}, \mathbf{Q}}^{(a)} \equiv E_{\mathbf{p}, \mathbf{Q}}^0 \pm \xi_{\mathbf{p}, \mathbf{Q}}^{(a)}, \quad (43)$$

Here,

$$\hat{\hat{G}}_0(1, 2) = \hat{\hat{G}}_0(1 - 2) = e^{i[\mu(t_1 - t_2) - \frac{\mathbf{Q}}{2} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] \tau_3} \hat{G}_0(1 - 2) \quad (34)$$

is the gauge-transformed bare Green's function. In the energy and momentum space, Eq. (33) has the form

$$\begin{aligned}\hat{\hat{G}}(\mathbf{p}, \omega) &= \begin{pmatrix} \tilde{G}^R(\mathbf{p}, \omega) & \tilde{G}^K(\mathbf{p}, \omega) \\ 0 & \tilde{G}^A(\mathbf{p}, \omega) \end{pmatrix} \\ &= \hat{\hat{G}}_0(\mathbf{p}, \omega) + \hat{\hat{G}}_0(\mathbf{p}, \omega) \hat{\hat{\Sigma}}(\mathbf{p}, \omega) \hat{\hat{G}}_0(\mathbf{p}, \omega),\end{aligned}\quad (35)$$

where

$$\Xi_{\mathbf{p}, \mathbf{Q}}^{0, \pm} = \frac{1}{2} \left[ \tau_0 \pm \frac{\xi_{\mathbf{p}, \mathbf{Q}}^{(s)}}{E_{\mathbf{p}, \mathbf{Q}}^0} \tau_3 \mp \frac{\Delta_0}{E_{\mathbf{p}, \mathbf{Q}}^0} \tau_1 \right], \quad (44)$$

with  $\xi_{\mathbf{p}, \mathbf{Q}}^{(s)} = [\xi_{\mathbf{p} + \mathbf{Q}/2} + \xi_{-\mathbf{p} + \mathbf{Q}/2}]/2$ , and  $\xi_{\mathbf{p}, \mathbf{Q}}^{(a)} = [\xi_{\mathbf{p} + \mathbf{Q}/2} - \xi_{-\mathbf{p} + \mathbf{Q}/2}]/2$ . In Eq. (42),

$$F(\omega) = \frac{1}{2} [f(\omega + \delta\mu) + f(\omega - \delta\mu)] \quad (45)$$

works as the nonequilibrium distribution function in the main system [although  $T_{\text{env}}$  is used in  $f(\omega)$ , see Eq. (8)]. When we set  $\omega = \xi_{\mathbf{p}}$ , the resulting momentum distribution  $F(\xi_{\mathbf{p}})$  has two Fermi-surface-like edges at  $p_{F1} = \sqrt{2m\mu_R}$  and  $p_{F2} = \sqrt{2m\mu_L}$ , as is schematically drawn in Fig. 1(a).

The superfluid order parameter  $\Delta(\mathbf{r}, t)$  in Eq. (29) is self-consistently determined from Eq. (20). To evaluate this equation, we note that the gauge-transformed lesser Green's function  $\tilde{G}^<(\mathbf{p}, \omega)$  in the energy and momentum space is obtained from Eqs. (13) and (40) to (42) as

$$\begin{aligned}\tilde{G}^<(\mathbf{p}, \omega) &= \frac{1}{2} [\tilde{G}^K(\mathbf{p}, \omega) - \tilde{G}^R(\mathbf{p}, \omega) + \tilde{G}^A(\mathbf{p}, \omega)] \\ &= \sum_{\eta=\pm} \frac{4i\gamma F(\omega)}{[\omega - \eta E_{\mathbf{p}, \mathbf{Q}}^{0, \eta}]^2 + 4\gamma^2} \Xi_{\mathbf{p}, \mathbf{Q}}^{0, \eta}.\end{aligned}\quad (46)$$

Using Eq. (46) in evaluating Eq. (20), we obtain the gap equation

$$1 = U \sum_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \times \frac{4\gamma [\omega - \xi_{\mathbf{p}, \mathbf{Q}}^{(a)}] [1 - 2F(\omega)]}{[(\omega - E_{\mathbf{p}, \mathbf{Q}}^{0+})^2 + 4\gamma^2][(\omega + E_{\mathbf{p}, \mathbf{Q}}^{0-})^2 + 4\gamma^2]}. \quad (47)$$

To quickly see the relation to the ordinary BCS gap equation, we take the thermal equilibrium and uniform limit, by setting  $(\gamma, \delta\mu, \mathbf{Q}) \rightarrow (+0, 0, 0)$ . Then, Eq. (47) is reduced to

$$1 = U \sum_{\mathbf{p}} \frac{1}{2E_{\mathbf{p}}} \tanh\left(\frac{E_{\mathbf{p}}}{2T_{\text{env}}}\right), \quad (48)$$

where  $E_{\mathbf{p}} = E_{\mathbf{p}, \mathbf{Q}=0}^0 = \sqrt{\xi_{\mathbf{p}}^2 + \Delta_0^2}$ . Equation (48) is just the same form as the ordinary BCS gap equation [57], when one

interprets  $\mu$  and  $T_{\text{env}}$  as the Fermi chemical potential and the temperature in the main system, respectively. Thus, Eq. (47) may be viewed as a nonequilibrium extension of the BCS gap equation.

Because  $\Delta(\mathbf{r}, t)$  in Eq. (29) involves two parameters  $\Delta_0$  and  $\mathbf{Q}$ , we actually need one more equation to completely fix these parameters [100]. For this purpose, we impose the vanishing condition for the net current  $\mathbf{J}_{\text{net}}$  in the main system. In the Nambu-Keldysh formalism, it is given by

$$\mathbf{J}_{\text{net}} = \sum_{\sigma=\uparrow,\downarrow} \sum_{\mathbf{p}} [\mathbf{p} + \mathbf{Q}/2] n_{\mathbf{p},\sigma} = 0, \quad (49)$$

where the Fermi momentum distribution  $n_{\mathbf{p},\sigma}$  in the pseudospin  $\sigma$  component is related to the diagonal component of the lesser Green's function  $\tilde{\mathcal{G}}^<(\mathbf{p}, \omega)$  in Eq. (46) as

$$n_{\mathbf{p},\uparrow} = -i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\mathcal{G}}^<(\mathbf{p}, \omega)_{11}, \quad (50)$$

$$n_{\mathbf{p},\downarrow} = 1 - i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\mathcal{G}}^<(-\mathbf{p}, \omega)_{22}. \quad (51)$$

We solve the gap equation (47) under the condition in Eq. (49), to self-consistently determine  $(\Delta_0, \mathbf{Q})$  for a given set  $(\gamma, \delta\mu, T_{\text{env}})$  of the environment parameters.

We note that the net current  $\mathbf{J}_{\text{net}}$  is the current flowing *inside* the main system, but *not* to be confused with the current flowing from the reservoir to the main system (this last is always nonzero unless we consider the chemical equilibrium case  $\mu_L = \mu_R$ ). Although we assume here that the first is zero, strictly speaking, there is no *a priori* way of determining it since it ultimately depends on the choice of a boundary condition [101]. Hence, one may also consider more general cases with  $\mathbf{J}_{\text{net}} \neq 0$ . Even in such a case, however, the essential physics remains unchanged: The nonequilibrium FF-type superfluid state with a nonzero net current can be realized as a stable nonequilibrium steady state (unless the current exceeds the Landau velocity, which would destroy all superfluid states). Thus, we choose the simplest case of the vanishing-current boundary condition with Eq. (49) for the sake of brevity.

We also note that the normal state ( $\Delta_0 = \mathbf{Q} = 0$ ) always satisfies the gap equation (47) and the vanishing current condition in Eq. (49). However, as pointed out in our previous work [54,55], the normal state becomes unstable when the particle-particle scattering vertex  $\chi(\mathbf{Q}, \nu)$  develops a pole at  $\nu = 2\mu$ . Here,  $\mathbf{Q}$  and  $\nu$  are the center-of-mass momentum and the total energy of two particles participating in the Cooper channel, respectively. Within the random phase approximation in terms of  $-U$ , one has [54,55], in NESS,

$$\chi(\mathbf{Q}, \nu = 2\mu) = \frac{-U}{1 - \frac{U}{4\pi} \sum_{\eta,\zeta=\pm} \frac{1}{\xi_{\mathbf{p},\mathbf{Q}}^{\eta,\zeta}} \tan^{-1} \left( \frac{\xi_{\mathbf{p},\mathbf{Q}}^{\eta,\zeta}}{2\gamma} \right)}, \quad (52)$$

where  $\xi_{\mathbf{p},\mathbf{Q}}^{\eta,\zeta} = \xi_{\mathbf{p}+\eta\mathbf{Q}/2} + \zeta\delta\mu$ . We determine the region where the normal state is stable on the phase diagram from the condition that  $\chi(\mathbf{Q}, \nu = 2\mu) < 0$  for any  $\mathbf{Q}$ , that is,  $\chi(\mathbf{Q}, \nu)$  has no pole at  $\nu = 2\mu$ .

### III. QUANTUM KINETIC THEORY TO ASSESS THE STABILITY OF NONEQUILIBRIUM FERMION SUPERFLUIDS

#### A. Quantum kinetic equation

So far, we obtained the self-consistent equations that the steady-state solutions satisfy. To make sure that the obtained solutions are physical, we study the stability of such solutions. In this section, we explain how one can check such stability in a nonequilibrium setup.

The stability of the obtained solution can be examined by tracing the time evolution of the superfluid order parameter under the initial condition that it is slightly deviated from the mean-field value at  $t = 0$ . This is done here by deriving a quantum kinetic equation (QKE). We can then determine the stability of the steady-state solution by observing whether such deviation decays or grows over time.

The central quantity of interest is the Wigner-transformed Nambu lesser Green's function [95,96], given by

$$\tilde{\mathcal{G}}^<(\mathbf{p}, \omega, \mathbf{r}, t) = \int d\mathbf{r}_r \int dt_r e^{i(\omega t_r - \mathbf{p} \cdot \mathbf{r}_r)} \tilde{\mathcal{G}}^<(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2). \quad (53)$$

Here,  $\mathbf{r}_r = \mathbf{r}_1 - \mathbf{r}_2$  and  $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$  are, respectively, the relative coordinate and the center-of-mass coordinate. For time variables, we also introduced  $t_r = t_1 - t_2$  and  $t = (t_1 + t_2)/2$  in Eq. (53). It is useful to study such a quantity since the off-diagonal component of this quantity is directly related to the (gauge-transformed) superfluid order parameter  $\tilde{\Delta}(\mathbf{r}, t)$ . This can be readily seen by summing up  $\tilde{\mathcal{G}}^<(\mathbf{p}, \omega, \mathbf{r}, t)$  over momentum  $\mathbf{p}$  and frequency  $\omega$ :

$$\begin{aligned} -i \sum_{\mathbf{p}} \tilde{\mathcal{G}}^<(\mathbf{r}, t) &\equiv -i \sum_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\mathcal{G}}^<(\mathbf{p}, \omega, \mathbf{r}, t) \\ &= \begin{pmatrix} n_{\uparrow}(\mathbf{r}, t) & \frac{\tilde{\Delta}(\mathbf{r}, t)}{U} \\ \frac{\tilde{\Delta}^*(\mathbf{r}, t)}{U} & 1 - n_{\downarrow}(\mathbf{r}, t) \end{pmatrix}. \end{aligned} \quad (54)$$

Note that the diagonal component is directly related to the particle density  $n_{\sigma}(\mathbf{r}, t)$  of the  $\sigma$ -spin component. Therefore, the order parameter dynamics and the stability of the steady-state solutions in the superfluid phase can be examined by analyzing the dynamics of Eq. (53).

Below, we consider the dynamics of the  $\omega$ -integrated (and Wigner-transformed) lesser Green's function,  $\tilde{\mathcal{G}}^<(\mathbf{r}, t)$ . Using the Dyson's equation, we arrive at [95,104] (see Appendix A for the derivation)

$$\begin{aligned} i\partial_t \tilde{\mathcal{G}}^<(\mathbf{r}, t) - [\xi_{\mathbf{p}} \tau_3, \tilde{\mathcal{G}}^<(\mathbf{r}, t)]_- - \left[ \frac{\mathbf{Q}^2}{8m} \tau_3, \tilde{\mathcal{G}}^<(\mathbf{r}, t) \right]_- + \left[ \frac{1}{8m} \tau_3, \nabla_r^2 \tilde{\mathcal{G}}^<(\mathbf{r}, t) \right]_- + \frac{i}{2} \left[ \frac{\mathbf{p}}{m} \tau_3, \nabla_r \tilde{\mathcal{G}}^<(\mathbf{r}, t) \right]_+ \\ + \frac{i}{2} \left[ \frac{\mathbf{Q}}{2m} \tau_0, \nabla_r \tilde{\mathcal{G}}^<(\mathbf{r}, t) \right]_+ = \mathcal{I}_{\mathbf{p}}(\mathbf{r}, t). \end{aligned} \quad (55)$$



In Eq. (55), the collision term  $\mathcal{I}_p(\mathbf{r}, t) = \mathcal{I}_p^{\text{int}}(\mathbf{r}, t) + \mathcal{I}_p^{\text{env}}(\mathbf{r}, t)$  consists of the interaction term

$$\mathcal{I}_p^{\text{int}}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\tilde{\Sigma}_{\text{int}}^{\text{R}} \circ \tilde{\mathcal{G}}^< - \tilde{\mathcal{G}}^< \circ \tilde{\Sigma}_{\text{int}}^{\text{A}} + \tilde{\Sigma}_{\text{int}}^< \circ \tilde{\mathcal{G}}^{\text{A}} - \tilde{\mathcal{G}}^{\text{R}} \circ \tilde{\Sigma}_{\text{int}}^<](\mathbf{p}, \omega, \mathbf{r}, t), \quad (56)$$

as well as the environment term

$$\mathcal{I}_p^{\text{env}}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\tilde{\Sigma}_{\text{env}}^{\text{R}} \circ \tilde{\mathcal{G}}^< - \tilde{\mathcal{G}}^< \circ \tilde{\Sigma}_{\text{env}}^{\text{A}} + \tilde{\Sigma}_{\text{env}}^< \circ \tilde{\mathcal{G}}^{\text{A}} - \tilde{\mathcal{G}}^{\text{R}} \circ \tilde{\Sigma}_{\text{env}}^<](\mathbf{p}, \omega, \mathbf{r}, t). \quad (57)$$

In Eqs. (56) and (57),  $[A \circ B](\mathbf{p}, \omega, \mathbf{r}, t)$  is the Wigner transformation of the convolution  $[A \circ B](1, 2)$  in Eq. (16), which is known to be represented by [95,96]

$$\begin{aligned} [A \circ B](\mathbf{p}, \omega, \mathbf{r}, t) &= A(\mathbf{p}, \omega, \mathbf{r}, t) e^{\frac{i}{2} [\overleftarrow{\partial}_\omega \overrightarrow{\partial}_t - \overleftarrow{\partial}_t \overrightarrow{\partial}_\omega + \overleftarrow{\partial}_r \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_r]} B(\mathbf{p}, \omega, \mathbf{r}, t) = A(\mathbf{p}, \omega, \mathbf{r}, t) B(\mathbf{p}, \omega, \mathbf{r}, t) \\ &+ \frac{i}{2} A(\mathbf{p}, \omega, \mathbf{r}, t) [\overleftarrow{\partial}_\omega \overrightarrow{\partial}_t - \overleftarrow{\partial}_t \overrightarrow{\partial}_\omega + \overleftarrow{\partial}_r \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_r] B(\mathbf{p}, \omega, \mathbf{r}, t) + \dots \end{aligned} \quad (58)$$

Here, the left (right) arrow on each differential operator means that it acts on the left (right) side of this operator. Since we are interested in the slowly varying dynamics both in terms of time and space, below, we will only retain up to the first order with respect to  $[\overleftarrow{\partial}_\omega \overrightarrow{\partial}_t - \overleftarrow{\partial}_t \overrightarrow{\partial}_\omega + \overleftarrow{\partial}_r \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_r]$ , which greatly simplifies the analysis.

Let us first evaluate further the interaction term  $\mathcal{I}_p^{\text{int}}(\mathbf{r}, t)$  of the collision term in Eq. (56). The self-energies  $\tilde{\Sigma}_{\text{int}}^{\text{R}, \text{A}, <}$  in the Wigner representation that enters  $\mathcal{I}_p^{\text{int}}(\mathbf{r}, t)$  also depend on  $t$  through  $\tilde{\Delta}(\mathbf{r}, t)$  as

$$\tilde{\Sigma}_{\text{int}}^{\text{R}}(\mathbf{p}, \omega, \mathbf{r}, t) = -[\tilde{\Delta}(\mathbf{r}, t)\tau_+ + \tilde{\Delta}^*(\mathbf{r}, t)\tau_-]s \equiv -\tilde{\Delta}(\mathbf{r}, t), \quad (59)$$

$$\tilde{\Sigma}_{\text{int}}^{\text{A}}(\mathbf{p}, \omega, \mathbf{r}, t) = -\tilde{\Delta}^\dagger(\mathbf{r}, t), \quad (60)$$

$$\tilde{\Sigma}_{\text{int}}^<(\mathbf{p}, \omega, \mathbf{r}, t) = 0. \quad (61)$$

Within the first-order gradient approximation of the Moyal product mentioned above, Eq. (56) is evaluated as

$$\begin{aligned} \mathcal{I}_p^{\text{int}}(\mathbf{r}, t) &\simeq -[\tilde{\Delta}(\mathbf{r}, t), \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_- - \frac{i}{2} [\nabla_r \tilde{\Delta}(\mathbf{r}, t), \partial_p \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_+ \\ &+ \frac{i}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\partial_t \tilde{\Delta}(\mathbf{r}, t), \partial_\omega \tilde{\mathcal{G}}^<(\mathbf{p}, \omega, \mathbf{r}, t)]_+ \\ &= -[\tilde{\Delta}(\mathbf{r}, t), \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_- - \frac{i}{2} [\nabla_r \tilde{\Delta}(\mathbf{r}, t), \nabla_p \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_+, \end{aligned} \quad (62)$$

where the last term in the first expression vanishes by the integration by parts.

For the environment part  $\mathcal{I}_p^{\text{env}}(\mathbf{r}, t)$  of the collision term in Eq. (57), we take advantage of the fact that the reservoirs are assumed to be huge compared to the main system and remain unchanged. Because of this property, the self energy  $\tilde{\Sigma}_{\text{env}}^<$  that enters  $\mathcal{I}_p^{\text{env}}(\mathbf{r}, t)$  can be expressed as

$$\tilde{\Sigma}_{\text{env}}^{\text{R}}(\mathbf{p}, \omega, \mathbf{r}, t) = -2i\gamma\tau_0, \quad (63)$$

$$\tilde{\Sigma}_{\text{env}}^{\text{A}}(\mathbf{p}, \omega, \mathbf{r}, t) = 2i\gamma\tau_0, \quad (64)$$

$$\tilde{\Sigma}_{\text{env}}^<(\mathbf{p}, \omega, \mathbf{r}, t) = 4i\gamma F(\omega)\tau_0, \quad (65)$$

which are actually the same as the steady-state counterpart given in Eq. (39). Substituting these into Eq. (57), we obtain, again within the first-order gradient expansion,

$$\begin{aligned} \mathcal{I}_p^{\text{env}}(\mathbf{r}, t) &\simeq -4i\gamma \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - 4\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega) \mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t) \\ &+ 4\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \partial_\omega F(\omega) \partial_t \mathcal{R}(\mathbf{p}, \omega, \mathbf{r}, t), \end{aligned} \quad (66)$$

where we introduced the spectral weight [95]

$$\mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t) = i[\tilde{\mathcal{G}}^{\text{R}} - \tilde{\mathcal{G}}^{\text{A}}](\mathbf{p}, \omega, \mathbf{r}, t), \quad (67)$$

and

$$\mathcal{R}(\mathbf{p}, \omega, \mathbf{r}, t) = \frac{1}{2} [\tilde{\mathcal{G}}^{\text{R}} + \tilde{\mathcal{G}}^{\text{A}}](\mathbf{p}, \omega, \mathbf{r}, t). \quad (68)$$

As shown in Appendix B, the last term in Eq. (66) vanishes in the present case. Thus, we obtain

$$\mathcal{I}_p^{\text{env}}(\mathbf{r}, t) = -4i\gamma \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - 4\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega) \mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t). \quad (69)$$

Using the expressions for  $\tilde{\mathcal{G}}^{\text{R}}(\mathbf{p}, \omega, \mathbf{r}, t)$  and  $\tilde{\mathcal{G}}^{\text{A}}(\mathbf{p}, \omega, \mathbf{r}, t)$  given, respectively, in Eqs. (B4) and (B5), one finds that the time-dependent spectral function  $\mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t)$  in Eq. (67) has the form

$$\mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t) = \sum_{\eta=\pm} \frac{4\gamma}{[\omega - E_{p,Q}^\eta(\mathbf{r}, t)]^2 + 4\gamma^2} \Xi_{p,Q}^\eta(\mathbf{r}, t), \quad (70)$$

where  $E_{p,Q}^\pm(\mathbf{r}, t)$  and  $\Xi_{p,Q}^\eta(\mathbf{r}, t)$  are given in Eqs. (B6) and (B7), respectively.

Substituting Eqs. (62) and (69) into Eq. (55), we obtain the desired QKE

$$\begin{aligned} i\partial_t \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) = & [\xi_p \tau_3 - \tilde{\Delta}(\mathbf{r}, t), \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_- + \left[ \frac{\mathcal{Q}^2}{8m} \tau_3, \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_- - \left[ \frac{1}{8m} \tau_3, \nabla_r^2 \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_- - \frac{i}{2} \left[ \frac{\mathbf{p}}{m} \tau_3, \nabla_r \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_+ \\ & - \frac{i}{2} \left[ \frac{\mathcal{Q}}{2m} \tau_0, \nabla_r \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_+ - 4i\gamma \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - 4\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega) \mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t) - \frac{i}{2} [\nabla_r \tilde{\Delta}(\mathbf{r}, t), \nabla_p \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_+. \end{aligned} \quad (71)$$

In fact, the last term does not actually affect the time evolution of the superfluid order parameter  $\tilde{\Delta}(\mathbf{r}, t)$ . To see this, we recall that  $\tilde{\Delta}(\mathbf{r}, t)$  is related to the (gauge-transformed) lesser Green's function as

$$\tilde{\Delta}(\mathbf{r}, t) = -iU \sum_p \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)_{12}. \quad (72)$$

Thus, the equation of motion for the superfluid order parameter  $\tilde{\Delta}(\mathbf{r}, t)$  is obtained from the  $\mathbf{p}$  summation of the (12) component of Eq. (71). In the resulting equation, the contribution coming from the last term in Eq. (71) vanishes. Since our stability analysis only needs the time evolution of  $\tilde{\Delta}(\mathbf{r}, t)$ , the last term in Eq. (71) may be ignored for our purpose [105].

We also note that the first term on the right-hand side in Eq. (71) represents the unitary time evolution. When we only retain this term and further assume a uniform superfluid  $\tilde{\Delta}(\mathbf{r}, t) = \tilde{\Delta}(t)$ , the QKE (71) is reduced to

$$i\partial_t \tilde{\mathcal{G}}_p^<(t) = [\xi_p \tau_3 - \tilde{\Delta}(t), \tilde{\mathcal{G}}_p^<(t)]. \quad (73)$$

This is equivalent to the so-called time-dependent Bogoliubov–de Gennes (TDBdG) equation [106,107], which was widely used in studying the dynamics of a closed Fermi condensate. In this sense, Eq. (71) may be interpreted as an extension of TDBdG theory to the open Fermi system shown in Fig. 1(a).

### B. Stability analysis of nonequilibrium superfluid steady states

Using the QKE scheme discussed in Sec. III.A, we are now in the position to study the stability of the obtained steady states. We compute the time evolution of the superfluid order parameter  $\tilde{\Delta}(\mathbf{r}, t)$  in the situation where the initial condition is prepared arbitrarily close to the steady-state solution  $\Delta_0$ . We can then judge the stability of the solution by checking whether the deviation converges to zero or amplifies even more. It is useful to consider the deviation of the superfluid order parameter from the steady-state value

$$\delta \tilde{\Delta}(\mathbf{r}, t) = \tilde{\Delta}(\mathbf{r}, t) - \Delta_0 = -iU \sum_p \delta \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)_{12}. \quad (74)$$

Here,  $\delta \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \equiv \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - \tilde{\mathcal{G}}_{p,\text{NESS}}^<$  is the deviation of the lesser Green's function from the (nonequilibrium) mean-field value, where

$$\tilde{\mathcal{G}}_{p,\text{NESS}}^< = \int \frac{d\omega}{2\pi} \tilde{\mathcal{G}}_{p,\text{NESS}}^<(\omega) \quad (75)$$

is the  $\omega$ -integrated lesser Green's function in the nonequilibrium steady state, with  $\tilde{\mathcal{G}}^<(\mathbf{p}, \omega)$  being given in Eq. (46).  $\delta \tilde{\Delta}(\mathbf{r}, t)$  decays (amplifies) as a function of time if the steady-state solution is (un)stable.

To derive the equation for the dynamics of  $\delta \tilde{\Delta}(\mathbf{r}, t)$ , it is useful to linearize the QKE (71) in terms of  $\delta \tilde{\Delta}(\mathbf{r}, t)$  and  $\delta \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)$ . Carrying out the Fourier transformation with respect to  $\mathbf{r}$ , one has

$$\begin{aligned} i\partial_t \delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t) = & [\xi_p \tau_3 - \tilde{\Delta}_0, \delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t)]_- - [\delta \tilde{\Delta}(\mathbf{q}, t), \tilde{\mathcal{G}}_{p,\text{NESS}}^<]_- - \frac{\mathcal{Q}^2 - q^2}{8m} [\tau_3, \delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t)]_- \\ & + \frac{\mathbf{p} \cdot \mathbf{q}}{2m} [\tau_3, \delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t)]_+ + \frac{\mathcal{Q} \cdot \mathbf{q}}{4m} [\tau_0, \delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t)]_+ - 4i\gamma \delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t) - 4\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega) \delta \mathcal{A}(\mathbf{p}, \omega, \mathbf{q}, t), \end{aligned} \quad (76)$$

where  $\tilde{\Delta}_0 = \Delta_0 \tau_1$  and

$$\delta \tilde{\mathcal{G}}_p^<(\mathbf{q}, t) = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \delta \tilde{\mathcal{G}}_p^<(\mathbf{r}, t), \quad (77)$$

$$\delta \tilde{\Delta}(\mathbf{q}, t) = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \delta \tilde{\Delta}(\mathbf{r}, t) \equiv \delta \tilde{\Delta}(\mathbf{q}, t) \tau_+ + \delta \tilde{\Delta}^*(\mathbf{q}, t) \tau_- . \quad (78)$$

Here,

$$\delta \mathcal{A}(\mathbf{p}, \omega, \mathbf{q}, t) = \sum_{\eta=\pm} \left[ \frac{4\gamma}{[\omega - \eta E_{\mathbf{p},\mathcal{Q}}^{0,\eta}]^2 + 4\gamma^2} \delta \Xi_{\mathbf{p},\mathcal{Q}}^{\eta}(\mathbf{q}, t) + \frac{8\gamma [\omega - \eta E_{\mathbf{p},\mathcal{Q}}^{0,\eta}]}{[[\omega - \eta E_{\mathbf{p},\mathcal{Q}}^{0,\eta}]^2 + 4\gamma^2]^2} \Xi_{\mathbf{p},\mathcal{Q}}^{0,\eta} \delta E_{\mathbf{p},\mathcal{Q}}(\mathbf{q}, t) \right] \quad (79)$$

is the linearized time-dependent spectral function, which is obtained by linearizing  $\mathcal{A}(\mathbf{p}, \omega, \mathbf{r}, t)$  in Eq. (70) with respect to  $\delta\bar{\Delta}(\mathbf{r}, t)$ . We also introduce here

$$\delta\Xi_{\mathbf{p},\mathbf{Q}}^+(\mathbf{q}, t) = -\delta\Xi_{\mathbf{p},\mathbf{Q}}^-(\mathbf{q}, t) = -\frac{1}{2[E_{\mathbf{p},\mathbf{Q}}^0]^2} \begin{pmatrix} \xi_{\mathbf{p},\mathbf{Q}}^{(s)}\delta E_{\mathbf{p},\mathbf{Q}}(\mathbf{q}, t) & E_{\mathbf{p},\mathbf{Q}}^0\delta\bar{\Delta}(\mathbf{q}, t) + \Delta_0\delta E_{\mathbf{p},\mathbf{Q}}(\mathbf{q}, t) \\ E_{\mathbf{p},\mathbf{Q}}^0\delta\bar{\Delta}^*(\mathbf{q}, t) + \Delta_0\delta E_{\mathbf{p},\mathbf{Q}}(\mathbf{q}, t) & -\xi_{\mathbf{p},\mathbf{Q}}^{(s)}\delta E_{\mathbf{p},\mathbf{Q}}(\mathbf{q}, t) \end{pmatrix}, \quad (80)$$

and

$$\delta E_{\mathbf{p},\mathbf{Q}}(\mathbf{q}, t) = \frac{\Delta_0}{E_{\mathbf{p},\mathbf{Q}}^0} \text{Re}[\delta\bar{\Delta}(\mathbf{q}, t)]. \quad (81)$$

Since Eq. (76) does not involve any mode-coupling term, we may safely focus on a particular value of the momentum  $\mathbf{q}$  ( $\equiv \bar{\mathbf{q}}$ ) in considering how the initial deviation of the superfluid order parameter from the mean-field value evolves over time. Keeping this in mind, we take the following initial condition for the quantum kinetic equation (76):

$$\delta\bar{\Delta}(\bar{\mathbf{q}}, t=0) = [\Delta_0 + \delta|\bar{\Delta}(\bar{\mathbf{q}}, t=0)|]e^{i\bar{\mathbf{q}}\cdot\mathbf{r}} - \Delta_0. \quad (82)$$

Here,  $\delta|\bar{\Delta}(\bar{\mathbf{q}}, t=0)|$  and  $\bar{\mathbf{q}} \cdot \mathbf{r}$  physically have the meanings of amplitude and phase deviations from the mean-field value  $\Delta_0$ , respectively.

We numerically solve Eq. (76) in the weak-coupling regime  $(p_F a_s)^{-1} = -1$ , by using the fourth-order implicit Runge-Kutta method with small time steps. At each time step, the deviation  $\delta\bar{\Delta}(\bar{\mathbf{q}}, t)$  in the right-hand side of this equation is evaluated from  $\delta\tilde{\mathcal{G}}_{\mathbf{p}}^<(\bar{\mathbf{q}}, t)$ , to proceed to the next time step. Because our QKE approach uses the gradient expansion as explained in Sec. III A, we set the initial condition so as to satisfy  $\delta|\bar{\Delta}(\bar{\mathbf{q}}, t=0)| \ll \Delta_0$  and  $|\bar{\mathbf{q}}| \ll p_F$  [108,109].

#### IV. NONEQUILIBRIUM SUPERFLUID STEADY STATES IN DRIVEN-DISSIPATIVE FERMI GAS

We now explore nonequilibrium superfluid steady states in a driven-dissipative Fermi gas. In Sec. IV.A, we first look for possible mean-field solutions of the gap equation (47), under the vanishing current condition in Eq. (49). As we emphasized in the previous section, the solution of the gap equation (47) does not necessarily mean the stability of this pairing state. To assess which solutions are physical, we apply the stability analysis explained in Sec. III B to each mean-field solution in Sec. IV B.

##### A. Nonequilibrium superfluid steady states

Figure 5(a) summarizes the nonequilibrium superfluid steady-state solutions of the gap equation (47) under the vanishing current condition in Eq. (49), in the weak-coupling regime  $[(p_F a_s)^{-1} = -1]$  of a driven-dissipative Fermi gas at  $T_{\text{env}} = 0$  and  $\gamma \rightarrow +0$  [110]. As seen in this figure, four self-consistent solutions are obtained under the ansatz in Eq. (29). Among them, the nonequilibrium BCS state (NBCS) and nonequilibrium interior gap state (NIG) are isotropic superfluid states ( $\mathbf{Q} = 0$ ). In particular, NBCS is reduced to the ordinary BCS state in the thermal equilibrium limit  $\delta\mu \rightarrow 0$  [solid circle in Fig. 5(a)], so that it may be viewed as an extension of the ordinary BCS state to the nonequilibrium steady state. On the other hand, NIG is similar to the

so-called interior gap state [111,112] that arises in aspin-imbalanced system, as we argue in the following. In the next subsection, we will show further that NBCS (NIG) is a (un)stable state, which is in parallel to the known results in equilibrium that the ordinary BCS state (interior gap state)

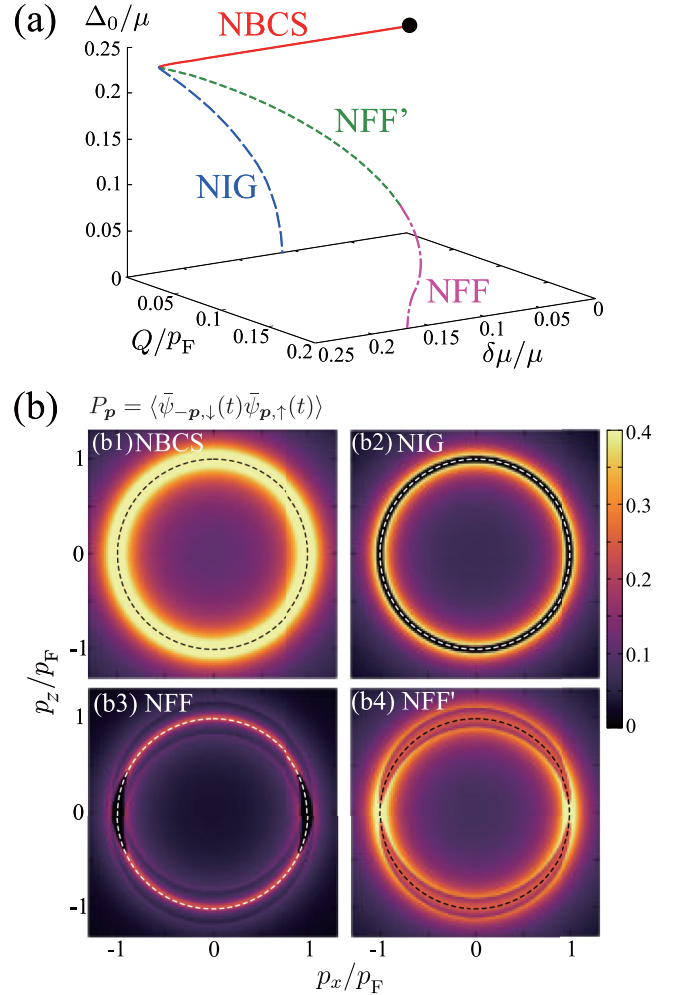


FIG. 5. (a) Nonequilibrium superfluid solutions of the gap equation (47) in the weak-coupling regime  $(p_F a_s)^{-1} = -1$  of a driven-dissipative Fermi gas. We set  $T_{\text{env}} = 0$  and  $\gamma \rightarrow +0$ , and impose the vanishing current condition in Eq. (49). Among the four mean-field solutions, NBCS and NIG (nonequilibrium interior gap state) are uniform superfluid states ( $\mathbf{Q} = 0$ ). NFF and NFF' are FF-like nonuniform states ( $\mathbf{Q} \neq 0$  and  $\mathbf{Q} \parallel p_z$ ). The solid circle is at the BCS state in the thermal equilibrium case ( $\delta\mu = 0$ ). (b) Calculated intensity of the pair amplitude of each state when  $\delta\mu = 0.145\mu$ . The dotted line in each panel shows the position at  $p = p_F = \sqrt{2m\mu}$ . We will show in Sec. IV B that NBCS and NFF are stable solutions, while NIG and NFF' are unstable solutions.

TABLE I. Summary of the stability analysis for  $T_{\text{env}} = 0$  and  $\gamma = 0.005\mu$ . Among the superfluid solutions, the underlined states are only stable.

	Region	Ssuperfluid solutions
Region I	$(0 < \delta\mu \leq 0.111\mu)$	<u>NBCS</u>
Region II	$(0.111\mu < \delta\mu \leq 0.135\mu)$	<u>NBCS, NIG</u>
Region III	$(0.135\mu < \delta\mu \leq 0.152\mu)$	<u>NBCS, NIG, NFF, NFF'</u>
Region IV	$(0.152\mu < \delta\mu \leq 0.183\mu)$	<u>NBCS, NIG, NFF'</u>

is (un)stable against superfluid fluctuations (see Table I). The remaining two solutions, NFF and NFF' (nonequilibrium Fulde-Ferrell states), are anisotropic superfluid states with  $\mathbf{Q} \neq 0$ . We briefly note that the existence of two kinds of FF states was also pointed out in a thermal-equilibrium spin-imbalanced Fermi gas [38]. Actually, NFF (NFF') turns out to be a (un)stable state, as we will discuss in the next subsection, which is again parallel to the equilibrium case (see Table I).

To grasp the character of each state, we conveniently consider the pair amplitude

$$P_p \equiv \langle \bar{\psi}_{-p,\downarrow}(t) \bar{\psi}_{p,\uparrow}(t) \rangle, \quad (83)$$

which physically describes the pairing structure in momentum space. Here,

$$\bar{\psi}_{p,\sigma}(t) = \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \bar{\psi}_{\sigma}(\mathbf{r}, t) \quad (84)$$

is the Fourier transformation of the gauge-transformed field operator

$$\bar{\psi}_{\sigma}(\mathbf{r}, t) = e^{-i\chi(\mathbf{r}, t)} \psi_{\sigma}(\mathbf{r}, t), \quad (85)$$

where the phase  $\chi$  is given in Eq. (32). Within the present mean-field scheme,  $P_p$  in Eq. (83) can be evaluated from the lesser Green's function  $\tilde{G}^<(\mathbf{p}, \omega)$  in Eq. (46) as

$$\begin{aligned} P_p &= -i \int \frac{d\omega}{2\pi} \tilde{G}_{12}^<(\mathbf{p}, \omega) \\ &= -\frac{\Delta_0}{E_{p,\mathbf{Q}}^0} \sum_{\eta=\pm} \eta \int \frac{d\omega}{2\pi} F(\omega) \frac{2\gamma}{[\omega - \eta E_{p,\mathbf{Q}}^{0,\eta}]^2 + 4\gamma^2}. \end{aligned} \quad (86)$$

In the small damping limit  $\gamma \rightarrow +0$  [which is the case of Fig. 5(a)], one may carry out the  $\omega$  integration in Eq. (86), giving

$$P_p = \frac{\Delta_0}{2E_{p,\mathbf{Q}}^0} [F(-E_{p,\mathbf{Q}}^{0,-}) - F(E_{p,\mathbf{Q}}^{0,+})]. \quad (87)$$

In the NBCS and NIG cases ( $\mathbf{Q} = 0$ ), Eq. (87) is reduced to

$$P_p = \frac{\Delta_0}{2E_p} [1 - 2F(E_p)], \quad (88)$$

where  $E_p$  is given below Eq. (48). In the thermal equilibrium BCS limit ( $\delta\mu \rightarrow 0$ ),  $F(E_p) = f(E_p)$  vanishes at  $T_{\text{env}} = 0$ . Then, as is well known in the ordinary BCS theory, the pair amplitude

$$P_p = \frac{\Delta_0}{2\sqrt{(\varepsilon_p - \mu)^2 + \Delta_0^2}}, \quad (89)$$

has a large intensity around the Fermi momentum  $p_F = \sqrt{2m\mu}$ , indicating that Cooper pairs are dominantly formed around the Fermi surface. We also find from the definition of  $F(\omega)$  in Eq. (45) that, even when  $\delta\mu > 0$ , Eq. (89) still holds, as far as  $\Delta_0 \geq \delta\mu$ . This is only for the NBCS case. Indeed, as is shown in Fig. 5(b1), the calculated NBCS pair amplitude has a large intensity around  $p_F = \sqrt{2m\mu}$  (although  $p_F$  no longer has the meaning of the Fermi momentum when  $\delta\mu = 0.145\mu > 0$ ).

For NIG, although the pair amplitude  $P_p$  is also isotropic, it vanishes around the ‘‘Fermi momentum’’  $p = p_F$  [see Fig. 5(b2)]. In this pairing state, the superfluid order parameter  $\Delta_0$  is not as large as the NBCS case [see Fig. 5(a)]. When  $\Delta_0 < \delta\mu$ , one obtains

$$P_p = \frac{\Delta_0}{2\sqrt{(\varepsilon_p - \mu)^2 + \Delta_0^2}} \Theta\left(\sqrt{(\varepsilon_p - \mu)^2 + \Delta_0^2} - \delta\mu\right). \quad (90)$$

Equation (90) immediately explains the vanishing pairing amplitude around  $p = p_F$  seen in Fig. 5(b2) (because the step function vanishes there). In addition, the region where  $P_p > 0$  is given by  $p \leq \tilde{p}_{F1}, \tilde{p}_{F2} \leq p$ , where

$$\tilde{p}_{F1} = \sqrt{2m\left[\mu - \sqrt{\delta\mu^2 - \Delta_0^2}\right]}, \quad (91)$$

$$\tilde{p}_{F2} = \sqrt{2m\left[\mu + \sqrt{\delta\mu^2 - \Delta_0^2}\right]}. \quad (92)$$

Particularly in the limiting case,  $\delta\mu \gg \Delta_0$ , one finds

$$\tilde{p}_{F1} \simeq \sqrt{2m[\mu - \delta\mu]} = p_{F1}, \quad (93)$$

$$\tilde{p}_{F2} \simeq \sqrt{2m[\mu + \delta\mu]} = p_{F2}, \quad (94)$$

that coincide with the positions of two edges imprinted on the Fermi momentum distribution  $n_{p,\sigma}$  by the two reservoirs [54] (see Fig. 1). When we simply regard these edges as two ‘‘Fermi surfaces’’ with different sizes, Fig. 5(b2) indicates that NIG Cooper pairs are formed around the ‘‘Fermi surfaces’’ at  $p_{F1}$  and  $p_{F2}$ . This pairing structure is similar to the Sarma(-Liu-Wilczek) state [111,112] (which is also referred to as the interior gap state in the literature) discussed in thermal equilibrium superconductivity under an external magnetic field, as well as in a spin-imbalanced Fermi gas.

We next consider the anisotropic NFF and NFF' states with  $\mathbf{Q} \neq 0$ . To grasp their pairing structures, it is useful to rewrite Eq. (87) in the form

$$P_p = \frac{1}{2} [P_p^{\text{FF}}(\mathbf{Q}, \delta\mu) + P_{-p}^{\text{FF}}(\mathbf{Q}, -\delta\mu)], \quad (95)$$

where

$$P_p^{\text{FF}}(\mathbf{Q}, \delta\mu) = \frac{\Delta_0}{2E_{p,\mathbf{Q}}^0} [1 - f(\bar{E}_{p,\mathbf{Q}}^+(\delta\mu)) - f(\bar{E}_{p,\mathbf{Q}}^-(\delta\mu))]. \quad (96)$$

In Eq. (96),  $\bar{E}_{p,\mathbf{Q}}^{\pm}(\delta\mu)$  is given in Eq. (43) where  $\xi_{\pm p+\mathbf{Q}/2}$  in  $\xi_{p,\mathbf{Q}}^{(a)}$  is replaced by

$$\bar{\xi}_{\pm p+\mathbf{Q}/2} = \varepsilon_{\pm p+\mathbf{Q}/2} - \mu - \delta\mu. \quad (97)$$

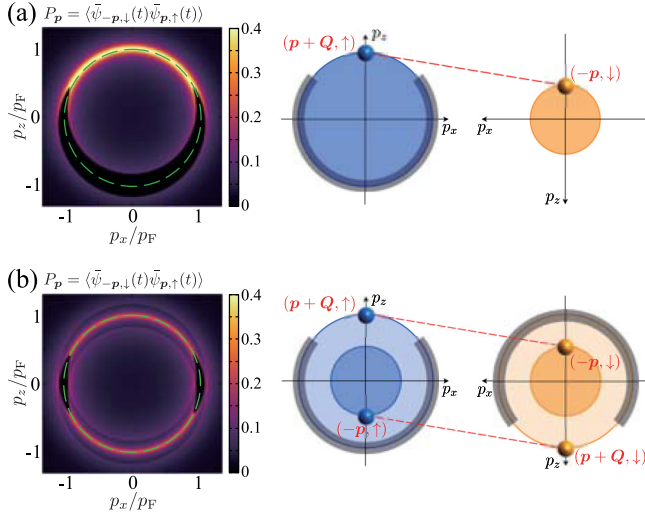


FIG. 6. Calculated intensity of the pair amplitude and schematic pictures of pair-formation. (a) FF state. (b) NFF state. Fermions in the shaded regions (blocking regions) do not contribute to the pair formation.

Equation (96) is just the same form as the pair amplitude in the thermal equilibrium FF state [35,36], when one regards  $\delta\mu$  as an *external magnetic field*. Thus, the first (second) term in Eq. (95) may be viewed as the pair amplitude in the FF state under an external magnetic field  $\delta\mu$  ( $-\delta\mu$ ), which just corresponds to the pairing (A) [(B)] in Fig. 1(b).

In the thermal equilibrium FF state, when  $\mathbf{Q}$  points to the  $z$  direction, the pair amplitude has large intensity around the Fermi surface in the region  $p_z > 0$ , as shown in Fig. 6(a). Thus, in Figs. 5(b3) and 5(b4), the anisotropic pair amplitude in the upper-half (lower-half) plane is dominated by the first term  $P_p^{\text{FF}}(\mathbf{Q}, \delta\mu)$  [second term  $P_{-p}^{\text{FF}}(\mathbf{Q}, -\delta\mu)$ ] in Eq. (95). On the other hand, the vanishing region (which is also referred to the blocking region in the superconductivity literature) spreads over the lower hemisphere in the thermal equilibrium FF state [see Fig. 6(a)]. Noting that NFF may be viewed as a mixture of two FF states with  $\mathbf{Q}$  and  $-\mathbf{Q}$  as shown in Figs. 6(a) and 6(b), we find that their blocking regions give the vanishing pair amplitude around the equator in Fig. 5(b3).

Because the two-edge structure of the Fermi momentum distribution  $n_{p,\sigma}$  at  $p_{F1}$  and  $p_{F2}$  are essentially important in obtaining NIG, NFF, and NFF' solutions, these states are expected to be suppressed when this structure is blurred with increasing the environment temperature  $T_{\text{env}}$ . This can be confirmed in Fig. 7, where one sees that only the NBCS state remains at  $T_{\text{env}} = 0.08\mu$ . Of course, NBCS also eventually disappears at higher  $T_{\text{env}}$  as in the thermal equilibrium case, although we do not explicitly show the result here.

We note that, while the chemical potential difference  $\delta\mu$  produces the two-edge structure  $p_{F1}$  and  $p_{F2}$  in  $n_{p,\sigma}$ , this structure may also be viewed as the smearing of the Fermi surface edge at  $p_F = \sqrt{2m\mu}$ . Because this is a similar effect to the thermal broadening, all four solutions eventually disappear when  $\delta\mu$  is large to some extent, as shown in Figs. 5(a) and 7.

We also note that the same depairing mechanism also works when the damping rate  $\gamma$  becomes large. Figure 8(a)

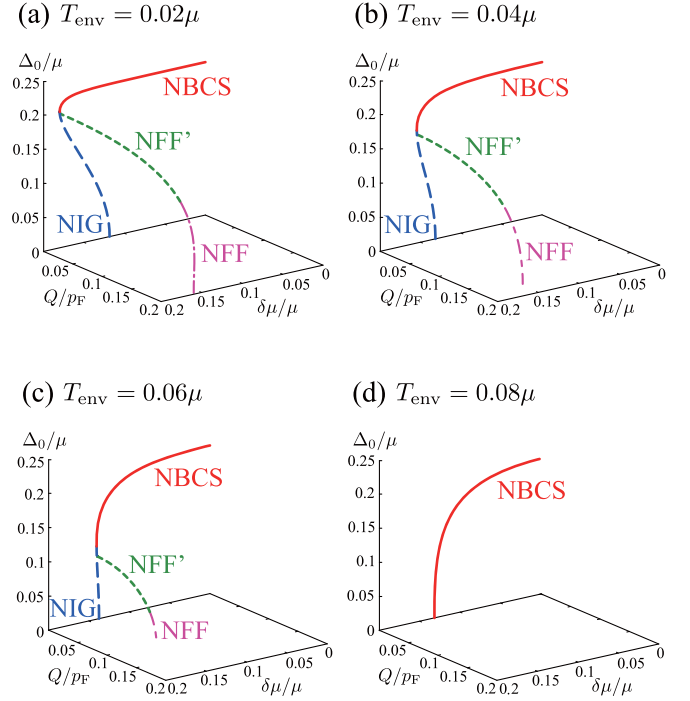


FIG. 7. Same plots as Fig. 5(a) for nonzero environment temperatures  $T_{\text{env}}$ .

shows the superfluid phase transition line in the  $\gamma$ - $\delta\mu$  plane ( $T_{\text{env}} = 0$ ), determined from the pole condition of the particle-particle scattering vertex  $\chi(\mathbf{Q}, \nu)$  in Eq. (52). Because the damping  $\gamma$  makes the two-step structure in  $n_{p,\sigma}$  obscure [54], the superfluid instability of the NFF ( $\mathbf{Q} \neq 0$ ) changes to the BCS-type phase transition with  $\mathbf{Q} = 0$ , when  $\gamma/\mu \gtrsim 0.04$  [see Figs. 8(b1) and 8(b2)]. As one further increases  $\gamma$ , the two steps in  $n_{p,\sigma}$  are completely smeared out and the overall structure becomes similar to the thermal equilibrium case at

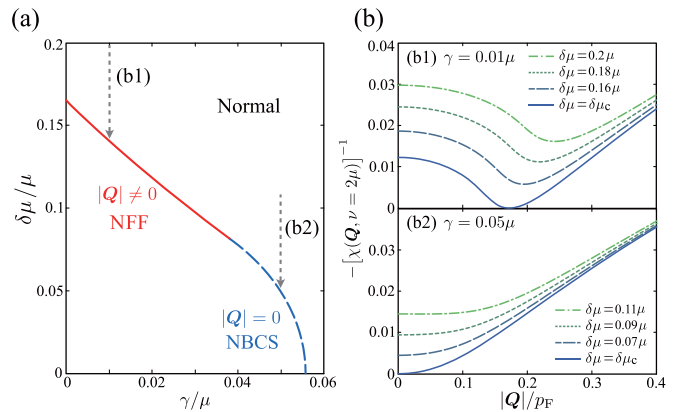


FIG. 8. (a) Calculated  $\delta\mu$  at the superfluid phase transition in the model driven-dissipative Fermi gas in Fig. 1(a). We take  $T_{\text{env}} = 0$  and  $(p_{Fa_s})^{-1} = -1$ . The system exhibits the NFF (NBCS) superfluid instability on the solid (dashed) line. (b) Inverse particle-particle scattering vertex  $[\chi(\mathbf{Q}, \nu = 2\mu)]^{-1}$  in Eq. (52), as a function of  $\mathbf{Q}$ . Upper (lower) panel shows the result along the path (b1) [path (b2)] in (a).

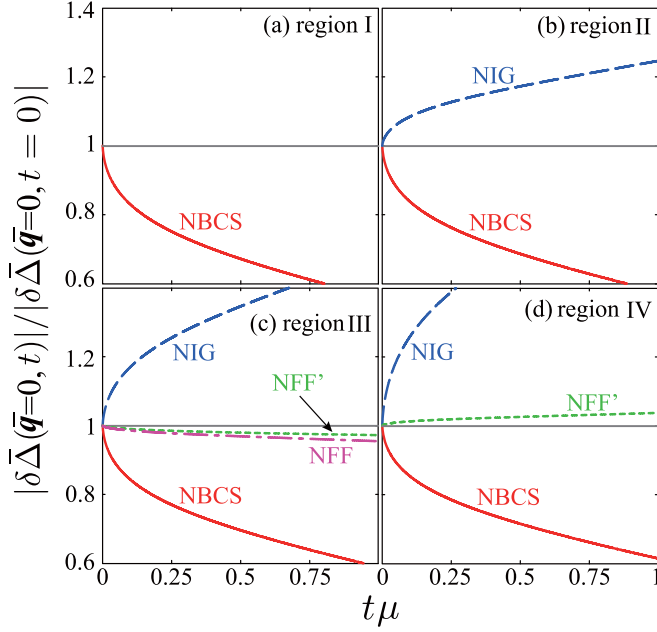


FIG. 9. Calculated time evolution of the deviation  $|\delta\bar{\Delta}(\bar{q}=0, t)|$  of the superfluid order parameter from the mean-field value. We set  $T_{\text{env}} = 0$ ,  $\gamma = 0.005\mu$ , and  $|\delta\bar{\Delta}(\bar{q}=0, t=0)| = 0.001\mu$ . (a)  $\delta\mu = 0.05\mu$  (region I). (b)  $\delta\mu = 0.13\mu$  (region II). (c)  $\delta\mu = 0.145\mu$  (region III). (d)  $\delta\mu = 0.16\mu$  (region IV).

high temperatures. As a result, the main system is in the normal state, when  $\gamma/\mu \gtrsim 0.056$  in Fig. 8(a).

### B. Stability analysis of nonequilibrium superfluid solutions

We next assess the stability of the four nonequilibrium superfluid solutions (NBCS, NIG, NFF, and NFF') obtained in Sec. IV A, by solving the linearized quantum kinetic equation (76) under the initial condition in Eq. (82). Table I summarizes our result for  $T_{\text{env}} = 0$  and  $\gamma = 0.005\mu$ . For finite bath temperature results, see also Fig. 2(a), which shows the same sets of the solution to be stable. We find that BCS and NFF are stable steady-state solutions in all regions, while NIG and NFF' are unstable.

We report below on the stability analysis result that confirms these results. Figure 9 shows the time evolution of the deviations of the superfluid order parameter from the mean-field value when the amplitude deviation is only considered at  $t = 0$  [ $\bar{q} = 0$  and  $|\delta\bar{\Delta}(\bar{q}=0, t=0)| \neq 0$ ]. In the NBCS case, we see in this figure that the deviation  $|\delta\bar{\Delta}(\bar{q}=0, t)|$  decays over time in all the regions I to IV. This means that NBCS is stable against small perturbation in terms of the amplitude of the superfluid order parameter. The same conclusion is also obtained in the presence of phase deviation ( $\bar{q} \neq 0$ ). As an example, we show in Fig. 10(a) the result in region III. Thus, we judge that NBCS is a *stable* nonequilibrium superfluid steady state.

In contrast to NBCS, NIG exhibits the opposite behavior, as seen in Figs. 9(b) to 9(d): The deviation  $|\delta\bar{\Delta}(\bar{q}=0, t)|$  grows over time, indicating that, although NIG is one of the four mean-field solutions, it is actually destroyed by this small perturbation. The instability of this state is also seen when

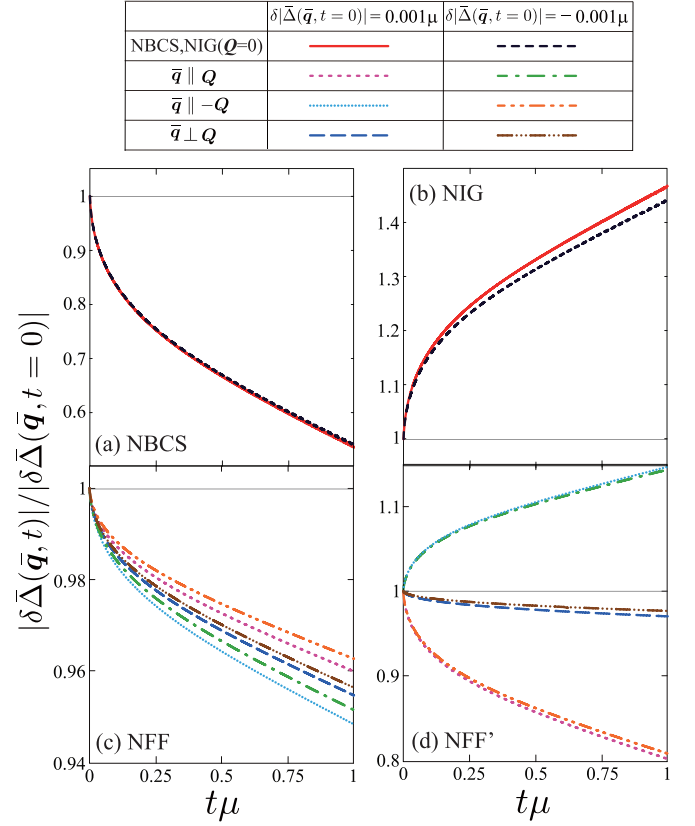


FIG. 10. Calculated time evolution of  $|\delta\bar{\Delta}(\bar{q}, t)|$  in the region III ( $\delta\mu = 0.145\mu$ ), when  $|\bar{q}| = 0.001p_F > 0$ . We take  $T_{\text{env}} = 0$ ,  $\gamma = 0.005\mu$ . (a) NBCS. (b) NIG. (c) NFF. (d) NFF'. In this figure, “ $\bar{q} \parallel \pm Q$ ” show the cases when  $\bar{q}$  is parallel and points to  $\pm Q$ . Because NBCS and NIG are isotropic with  $Q = 0$ , the results shown in panels (a) and (b) do not depend on the direction of  $\bar{q}$ .

$\bar{q} \neq 0$ , as shown in Fig. 10(b). These results conclude that NIG is *unstable*.

For the anisotropic solutions with  $Q \neq 0$ , Figs. 9(c) and 10(c) conclude that NFF is a *stable* FF-type superfluid state in region III. For NFF', as far as the perturbation with  $\bar{q} = 0$  is considered, while it is unstable in region IV, it is stable in region III [see Figs. 9(d) and 9(c), respectively]. However, even in the later region, Fig. 10(d) shows that this FF-type state cannot be always stable against the initial deviation with  $\bar{q} \neq 0$ . In this sense, we classify NFF' as an *unstable* superfluid state.

The above conclusions hold true even in the case with finite bath temperature  $T_{\text{env}} > 0$  and  $\gamma > 0$ : We show in Fig. 11 the effects of the environment temperature  $T_{\text{env}}$  (Fig. 11(a)) and damping rate  $\gamma$  [Fig. 11(b)] on the time evolution of  $|\delta\bar{\Delta}(\bar{q}=0, t)|$  in the NBCS case: Initial deviations are found to always decay over time, irrespective of the values of these environment parameters.

We also find from Fig. 11 that the relaxation time (timescale to recover the mean-field solution) depends on the environment's parameters,  $T_{\text{env}}$  and  $\gamma$ . For the damping rate  $\gamma$ , we find from Fig. 11(b) that it would be preferable to make the value of  $\gamma/\mu$  as small as possible for realizing the NFF. If our proposed setup is implemented by using current experimental

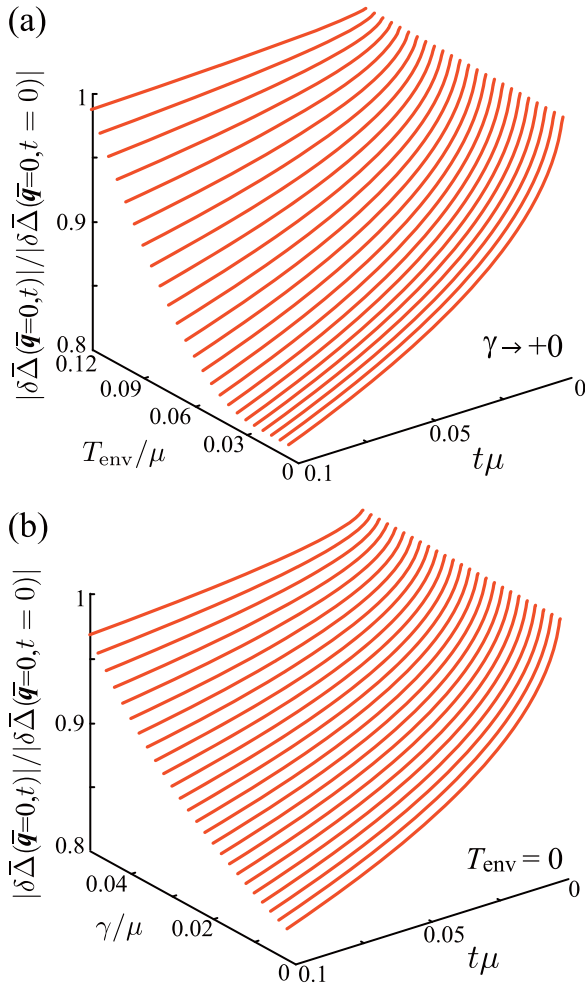


FIG. 11. Time evolution of the deviation  $|\delta\bar{\Delta}(\mathbf{q} = 0, t)|$  from the NBCS mean-field order parameter and effects of (a) environment temperature  $T_{\text{env}}$  and (b) damping rate  $\gamma$ . We take  $\delta\mu = 0.05\mu$ , and  $\delta|\bar{\Delta}(\mathbf{q} = 0, t = 0)| = 0.001\mu$ . In panels (a) and (b) we set  $\gamma \rightarrow +0$  and  $T_{\text{env}} = 0$ , respectively.

techniques in a cold atomic system, we expect that  $\gamma$  can be reduced to about  $0.01 \mu\text{K}$  and  $\mu$  can be increased to about  $1 \mu\text{K}$ , thereby keeping the value of  $\gamma/\mu$  below  $0.01$ , which is small enough to realize the NFF [see also Fig. 2(a)]. How to estimate these values and more detailed discussions are given in Appendix C.

Summarizing the above-mentioned stability analyses for various parameter sets  $(T_{\text{env}}, \gamma, \delta\mu)$ , we obtain the phase diagram in Figs. 2(a) and 2(b). Among the four candidates (NBCS, NIG, NFF, NFF'), NBCS and NFF only survive as stable nonequilibrium superfluid steady states. In the superfluid region, NBCS is always stable. Thus, in the region where NFF is stable, the bistability occurs. We also see in Figs. 2(a) and 2(b) the existence of the other bistability region (region III) where NBCS and the normal state are both stable.

Because the energetic consideration, which is useful in determining the *thermodynamically stable state*, does not work in the present nonequilibrium case, one cannot immediately identify which state is realized in the bistability region. The answer to this question is considered to depend on how to tune

the environment parameters: As  $\delta\mu$  increases adiabatically from  $\delta\mu = 0$ , the NBCS would be maintained both in regions II and III. As we decreases  $\delta\mu$  from region IV, on the other hand, the phase transition from the normal state to NFF occurs at the boundary between regions II and III [113]. As a result, the superfluid order parameter  $\Delta_0$  would exhibit the hysteresis behavior shown in Fig. 2(c). We note that the authors of Refs. [114–116] clarified that a voltage-driven superconductor (where the momentum distribution of conduction electrons is highly nonthermal and exhibits a two-step structure as in the nonequilibrium case we consider in this paper) shows the same kind of bistability, although the possibility of an FF-like state has not been discussed. Our result is consistent with these previous works and the NFF predicted in this paper would also be expected in such a voltage-driven superconductor.

Finally, we comment on the experimental observational advantage of the NFF state compared to conventional FF-type superfluid or superconducting states in thermal equilibrium spin-imbalanced Fermi gases [38–43] and metallic superconductors under an external magnetic field [33–37]. Although a spin-imbalanced Fermi gas is simpler than the setup proposed in this paper, which makes, at a glance, the previous to be an ideal system for the realization of FF-type superfluid states, no clear observation of this exotic state has been reported so far. One reason for this difficulty is that the realization of an FF-type superfluid state in a spin-imbalanced Fermi gas always suffers from the phase separation (where the BCS state and the normal state coexist). Indeed, in previous experiments, only phase separation was observed [117–119]. This happens because the experimental system is done with a fixed number of particles  $N_{\uparrow}$  and  $N_{\downarrow}$ , while the chemical potential  $\mu_{\uparrow}$  and  $\mu_{\downarrow}$  are not fixed [120,121]. On the other hand, in our proposed setup, the main system would be controlled by the fixed chemical potential of the two reservoirs  $\mu_L$  and  $\mu_R$  while the number of particles is *not* fixed. Moreover, the NFF state realized by the Fermi-surface reservoir-engineering does *not* need any spin imbalance. Thus, our proposal can avoid the occurrence of the unwanted phase separation phenomenon, which is a clear experimental observational advantage compared to the conventional proposal in a thermal equilibrium cold atomic system.

A metallic superconductors under an external magnetic field is also considered to be an ideal system for observing the FF state and has been vigorously studied. However, unambiguous experimental evidence for the pure FF state is still lacking in experiments. A major obstacle to observing the pure FF state in it is the orbital pair-breaking effect, which leads to a mixing of the FF state and the Abrikosov vortex state [122,123]. However, using our proposed Fermi-surface reservoir-engineering instead of an external magnetic field, one can clearly avoid this problem: As discussed in Sec. II A, a voltage-biased superconducting wire or thin film is also a promising experimental setup to observe the NFF state. Since there is no external field coupled to the spatial motion of electrons in these setups, in principle, we can realize a pure FF-type superconducting state in any (clean) superconducting metals by using our proposed engineering scheme.

Thus, while the previous thermal equilibrium approaches do not succeed in observing clear FF-type superfluid or superconducting state, we expect that the Fermi-surface

reservoir-engineering is a promising alternative route to reach this inhomogeneous pairing state.

## V. SUMMARY

To summarize, we proposed an idea to process the Fermi momentum distribution  $n_{p,\sigma}$ , by using reservoirs with different chemical potentials. Although we expect our scheme to work for generic Fermi systems, as a paradigmatic example, we discussed nonequilibrium superfluid steady states and their stability in a driven-dissipative two-component Fermi gas. Using the Nambu-Keldysh Green's function technique, we extended the BCS theory developed in the thermal equilibrium state to the nonequilibrium steady state. To examine the stability of the steady-state solutions obtained from this nonequilibrium BCS scheme, we also derived a quantum kinetic equation to examine the time evolution of the superfluid order parameter.

By solving the nonequilibrium gap equation, we obtained four superfluid steady-state solutions: Among them, one of them (NBCS state) may be viewed as an extension of the ordinary thermal equilibrium BCS state to the nonequilibrium case and another one (NIG state) is similar to the Sarma(-Liu-Wilczek) interior gap state. While these are isotropic uniform states, the remaining two are FF-like anisotropic and nonuniform superfluid states (NFF and NFF' states), even though the present system has no spin imbalance. Analyzing their pair amplitudes, we found that the last three superfluid solutions originated from the two-edge structure of the nonequilibrium Fermi momentum distribution, which was produced by the coupled two reservoirs with different chemical potentials. We also pointed out that each FF-like state may be viewed as the superposition of the thermal equilibrium FF state under an external magnetic field  $h = \delta\mu$  and that under an external magnetic field  $h = -\delta\mu$ .

We then studied the stability of these four superfluid steady-state solutions by solving the linearized quantum kinetic equation. This concluded that only the BCS-type state (NBCS) and one of the two FF-type states (NFF) were stable, in the sense that the initial deviation of the superfluid order parameter always decayed over time. The other two, NIG and NFF', are unstable because small perturbations were amplified over time. These stability analyses lead to the phase diagram of a driven-dissipative Fermi gas shown in Figs. 2(a) and 2(b).

Our proposed Fermi-surface reservoir-engineering can be applied not only to the Fermi gas system, but also to various many-body Fermi systems. Particularly in lattice systems, the combination of the band structure and multistep structure on

the Fermi momentum distribution may trigger unconventional ordered phases, such as spin- and charge-density wave-like states. The stability of such unconventional ordered phases can be assessed by evaluating the time evolution of fluctuations around the steady-state value, in the same manner as this paper. The search for unconventional ordered phases in nonequilibrium systems is currently one of the most exciting challenges in condensed matter physics, and our results would contribute to the further development of this research field.

## ACKNOWLEDGMENTS

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## APPENDIX A: DERIVATION OF EQ. (55)

We first introduce two inverse Green's functions  $\overrightarrow{\mathcal{G}}_0^{-1}(1)$  and  $\overleftarrow{\mathcal{G}}_0^{-1}(2)$  that obey

$$\overrightarrow{\mathcal{G}}_0^{-1}(1)\mathcal{G}_0^{\text{R(A)}}(1,2) = \delta(1-2)\tau_0, \quad (\text{A1})$$

$$\mathcal{G}_0^{\text{R(A)}}(1,2)\overleftarrow{\mathcal{G}}_0^{-1}(2) = \delta(1-2)\tau_0, \quad (\text{A2})$$

where  $\mathcal{G}_0^{\text{R(A)}}(1,2)$  is the retarded (R) or advanced (A) component of the bare Green's function in Eq. (17),  $\delta(1-2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2)$ , and the left (right) arrow on each differential operator means that it acts on the left (right) side of this operator. From the Heisenberg equation of the field operator, these inverse Green's functions are found to have the forms,

$$\overrightarrow{\mathcal{G}}_0^{-1}(1) = \begin{pmatrix} i\overrightarrow{\partial}_{t_1} - h_0(-i\overrightarrow{\nabla}_{r_1}) & 0 \\ 0 & i\overrightarrow{\partial}_{t_1} - h_0(-i\overrightarrow{\nabla}_{r_1}) \end{pmatrix}, \quad (\text{A3})$$

$$\overleftarrow{\mathcal{G}}_0^{-1}(2) = \begin{pmatrix} -i\overleftarrow{\partial}_{t_2} - h_0(i\overleftarrow{\nabla}_{r_2}) & 0 \\ 0 & -i\overleftarrow{\partial}_{t_2} - h_0(i\overleftarrow{\nabla}_{r_2}) \end{pmatrix}, \quad (\text{A4})$$

where  $h_0(-i\nabla_r) = (-i\nabla_r)^2/(2m)$ . Carrying out the gauge transformation discussed in Eqs. (30) to (32), one has

$$\overrightarrow{\mathcal{G}}_0^{-1}(1) \equiv e^{-i\chi(1)\tau_3}\overrightarrow{\mathcal{G}}_0^{-1}(1)e^{i\chi(1)\tau_3} = \begin{pmatrix} i\overrightarrow{\partial}_{t_1} - h_0(-i\overrightarrow{\nabla}_{r_1} + \mathbf{Q}/2) + \mu & 0 \\ 0 & i\overrightarrow{\partial}_{t_1} + h_0(-i\overrightarrow{\nabla}_{r_1} + \mathbf{Q}/2) + \mu \end{pmatrix}, \quad (\text{A5})$$

$$\overleftarrow{\mathcal{G}}_0^{-1}(2) \equiv e^{-i\chi(2)\tau_3}\overleftarrow{\mathcal{G}}_0^{-1}(2)e^{i\chi(2)\tau_3} = \begin{pmatrix} -i\overleftarrow{\partial}_{t_2} - h_0(i\overleftarrow{\nabla}_{r_2} + \mathbf{Q}/2) + \mu & 0 \\ 0 & -i\overleftarrow{\partial}_{t_2} + h_0(i\overleftarrow{\nabla}_{r_2} + \mathbf{Q}/2) + \mu \end{pmatrix}. \quad (\text{A6})$$

Operating  $\overrightarrow{\mathcal{G}}_0^{-1}(1)$  and  $\overleftarrow{\mathcal{G}}_0^{-1}(2)$  to the Dyson equation of the (gauge-transformed) lesser Green's function [95,96]

$$\tilde{\mathcal{G}}^<(1,2) = [\tilde{\mathcal{G}}^{\text{R}} \circ \tilde{\Sigma}^< \circ \tilde{\mathcal{G}}^{\text{A}}](1,2), \quad (\text{A7})$$



from the left and the right, we have, respectively,

$$\vec{\mathcal{G}}_0^{-1}(1)\tilde{\mathcal{G}}^<(1, 2) = [\tilde{\Sigma}^< \circ \tilde{\mathcal{G}}^A + \tilde{\Sigma}^R \circ \tilde{\mathcal{G}}^<](1, 2), \quad (\text{A8})$$

$$\tilde{\mathcal{G}}^<(1, 2)\overleftarrow{\mathcal{G}}_0^{-1}(2) = [\tilde{\mathcal{G}}^R \circ \tilde{\Sigma}^< + \tilde{\mathcal{G}}^< \circ \tilde{\Sigma}^A](1, 2). \quad (\text{A9})$$

In obtaining these equations, we used

$$\vec{\mathcal{G}}_0^{-1}(1)\tilde{\mathcal{G}}^{\text{R(A)}}(1, 2) = \delta(1-2)\tau_0 + [\tilde{\Sigma}^{\text{R(A)}} \circ \tilde{\mathcal{G}}^{\text{R(A)}}](1, 2), \quad (\text{A10})$$

$$\tilde{\mathcal{G}}^{\text{R(A)}}(1, 2)\overleftarrow{\mathcal{G}}_0^{-1}(2) = \delta(1-2)\tau_0 + [\tilde{\mathcal{G}}^{\text{R(A)}} \circ \tilde{\Sigma}^{\text{R(A)}}](1, 2). \quad (\text{A11})$$

Equations (A8) and (A9) yield the Kadanoff-Baym (KB) equation [95,104]

$$[\vec{\mathcal{G}}_0^{-1}\tilde{\mathcal{G}}^< - \tilde{\mathcal{G}}^<\overleftarrow{\mathcal{G}}_0^{-1}](1, 2) = [\tilde{\Sigma}^R \circ \tilde{\mathcal{G}}^< - \tilde{\mathcal{G}}^< \circ \tilde{\Sigma}^A + \tilde{\Sigma}^< \circ \tilde{\mathcal{G}}^A - \tilde{\mathcal{G}}^R \circ \tilde{\Sigma}^<](1, 2). \quad (\text{A12})$$

Carrying out the Wigner transformation, which is followed by the  $\omega$  integration, one finds that the KB equation (A12) becomes

$$\vec{\mathcal{G}}_0^{-1}\tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)\overleftarrow{\mathcal{G}}_0^{-1} = \mathcal{I}_p(\mathbf{r}, t), \quad (\text{A13})$$

where  $\mathcal{I}_p(\mathbf{r}, t)$  is given in the sum of Eqs. (56) and (57). Since the left-hand side of Eq. (A13) is evaluated as

$$\begin{aligned} \vec{\mathcal{G}}_0^{-1}\tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)\overleftarrow{\mathcal{G}}_0^{-1} &= i\partial_t\tilde{\mathcal{G}}_p^<(\mathbf{r}, t) - [\xi_p\tau_3, \tilde{\mathcal{G}}_p^<(\mathbf{r}, t)]_- - \left[ \frac{\mathbf{Q}^2}{8m}\tau_3, \tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_- \\ &+ \left[ \frac{1}{8m}\tau_3, \nabla_r^2\tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_- + \frac{i}{2} \left[ \frac{\mathbf{p}}{m}\tau_3, \nabla_r\tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_+ + \frac{i}{2} \left[ \frac{\mathbf{Q}}{2m}\tau_0, \nabla_r\tilde{\mathcal{G}}_p^<(\mathbf{r}, t) \right]_+, \end{aligned} \quad (\text{A14})$$

we obtain Eq. (55).

## APPENDIX B: VANISHMENT OF THE LAST TERM IN EQ. (66)

To evaluate  $\mathcal{R}(\mathbf{p}, \omega, \mathbf{r}, t)$  in Eq. (68), we carry out the Wigner transformation of Eqs. (A10) and (A11). Then, retaining the Moyal product to the first-order gradient expansion explained below Eq. (58), we have

$$\vec{\mathcal{G}}_0^{-1}\tilde{\mathcal{G}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t) = \tau_0 + [\tilde{\Sigma}^{\text{R(A)}} \circ \tilde{\mathcal{G}}^{\text{R(A)}}](\mathbf{p}, \omega, \mathbf{r}, t) \simeq \tau_0 + \tilde{\Sigma}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t)\tilde{\mathcal{G}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t), \quad (\text{B1})$$

$$\tilde{\mathcal{G}}^{\text{R(A)}}\overleftarrow{\mathcal{G}}_0^{-1}(\mathbf{p}, \omega, \mathbf{r}, t) = \tau_0 + [\tilde{\mathcal{G}}^{\text{R(A)}} \circ \tilde{\Sigma}^{\text{R(A)}}](\mathbf{p}, \omega, \mathbf{r}, t) \simeq \tau_0 + \tilde{\mathcal{G}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t)\tilde{\Sigma}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t). \quad (\text{B2})$$

The sum of these equations gives

$$\left[ \omega\tau_0 - \xi_p\tau_3 - \frac{\mathbf{Q}^2}{8m}\tau_3 - \frac{1}{2}\frac{\mathbf{p}}{m} \cdot \mathbf{Q}\tau_0 - \tilde{\Sigma}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t), \tilde{\mathcal{G}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t) \right]_+ = 2\tau_0. \quad (\text{B3})$$

Here,  $\tilde{\Sigma}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t) = \tilde{\Sigma}_{\text{int}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t) + \tilde{\Sigma}_{\text{env}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t)$ . This equation yields

$$\tilde{\mathcal{G}}^{\text{R}}(\mathbf{p}, \omega, \mathbf{r}, t) = \sum_{\eta=\pm} \frac{1}{\omega + 2i\gamma - E_{\mathbf{p},\mathbf{Q}}^{\eta}(\mathbf{r}, t)} \Xi_{\mathbf{p},\mathbf{Q}}^{\eta}(\mathbf{r}, t), \quad (\text{B4})$$

$$\tilde{\mathcal{G}}^{\text{A}}(\mathbf{p}, \omega, \mathbf{r}, t) = \sum_{\eta=\pm} \frac{1}{\omega - 2i\gamma - E_{\mathbf{p},\mathbf{Q}}^{\eta}(\mathbf{r}, t)} \Xi_{\mathbf{p},\mathbf{Q}}^{\eta}(\mathbf{r}, t), \quad (\text{B5})$$

where

$$E_{\mathbf{p},\mathbf{Q}}^{\pm}(\mathbf{r}, t) = \sqrt{(\xi_{\mathbf{p},\mathbf{Q}}^{(s)})^2 + |\tilde{\Delta}(\mathbf{r}, t)|^2} \pm \xi_{\mathbf{p},\mathbf{Q}}^{\text{a}} \equiv E_{\mathbf{p},\mathbf{Q}}(\mathbf{r}, t) \pm \xi_{\mathbf{p},\mathbf{Q}}^{(\text{a})}, \quad (\text{B6})$$

$$\Xi_{\mathbf{p},\mathbf{Q}}^{\pm} = \frac{1}{2} \left[ \tau_0 \pm \frac{\xi_{\mathbf{p},\mathbf{Q}}^{(s)}}{E_{\mathbf{p},\mathbf{Q}}(\mathbf{r}, t)} \tau_3 \mp \frac{\tilde{\Delta}(\mathbf{r}, t)}{E_{\mathbf{p},\mathbf{Q}}(\mathbf{r}, t)} \right]. \quad (\text{B7})$$

Equation (B4) may be viewed as an extension of the Green's function in the nonequilibrium steady state in Eq. (40) to the

case when the superfluid order parameter  $\tilde{\Delta}(\mathbf{r}, t)$  depends on  $t$  and  $\mathbf{r}$  as given in Eq. (54).

Subtracting Eqs. (B1) from (B2), one obtains

$$\begin{aligned} i\partial_t\tilde{\mathcal{G}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t) \\ = \left[ \xi_p\tau_3 + \frac{\mathbf{Q}^2}{8m}\tau_3 + \tilde{\Sigma}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t), \tilde{\mathcal{G}}^{\text{R(A)}}(\mathbf{p}, \omega, \mathbf{r}, t) \right]. \end{aligned} \quad (\text{B8})$$

Because  $\tilde{G}^R(\mathbf{p}, \omega, \mathbf{r}, t)$  and  $\tilde{G}^A(\mathbf{p}, \omega, \mathbf{r}, t)$  are, respectively, given by Eqs. (B4) and (B5), the right-hand side of Eq. (B8) is found to vanish, which immediately proves  $\partial_t \mathcal{R}(\mathbf{p}, \omega, \mathbf{r}, t) = 0$  in the last term in Eq. (66).

### APPENDIX C: DETAILED PROPOSALS FOR EXPERIMENTS

In Sec. II A, we gave concrete proposals for experiments to realize our predicted nonequilibrium FF-type superfluid state (NFF). To achieve the NFF state, however, we foresee two experimental challenges. The first challenge is to achieve the parameters which can actually realize NFF. In particular, in addition to cooling down the left and right reservoirs to low-enough temperature  $T_{L,R} \ll T_F$  (where  $T_F = [3\pi^2(n_\uparrow + n_\downarrow)]^{2/3}$  is the Fermi temperature of the main system), it is necessary for the linewidth  $\gamma$  arising from the reservoir-system coupling to be kept below  $0.01\mu$  [see Figs. 2(a) and 2(b)]. The second challenge (for ultracold atomic systems) is to maintain the system in the timescale of relaxation to the steady state. Since, realistically, both the main system and the reservoirs are isolated and finite, the chemical potential bias between the reservoirs initially present would eventually relax to its chemical equilibrium  $\mu_L = \mu_R$ . It is, therefore, important for the system to achieve the (quasi) steady state much faster than the change of the chemical potential.

In the following, we argue that current state-of-the-art experimental techniques of ultracold Fermi systems enable us to overcome these challenges. We first argue that it is possible to achieve  $\gamma/\mu$  around 0.01 in our proposed cold atomic setup. Then, we explain that the relaxation to the steady state is expected to occur on a timescale sufficiently fast compared to the relaxation timescale of the chemical potential bias to allow us to observe the NFF.

First, to address the first challenge, let us estimate the magnitude of the parameter  $\gamma$ . In the experiment of cold atomic transport measurements in a two-terminal configuration [67–72], quantized conductance is observed by sweeping the gate potential (corresponding to the gate voltage) at the quantum point contact (QPC) between the reservoirs. In the idealized situation with no broadening effects, the conductance is expected to change discontinuously from one plateau to the next when the gate potential is swept over a (transverse-)energy mode in the QPC. Realistically, however, the line broadening with the magnitude of about  $0.01 \mu\text{K}$  is observed [69,70]. This step broadening is considered to be mainly attributed to the following two factors: (i) finite temperature effects and (ii) broadening of the energy modes in the QPC due to the coupling with the reservoirs. The parameter  $\gamma$  in our model is related to the latter. However, it is known that the broadening of the plateau is dominated by the finite temperature effect (i) [70–72] (where it shows good agreement with the experimental results even though the theoretical prediction in [70] neglects the influence of (ii)). Thus, we can reasonably estimate that  $\gamma$  can be less than  $0.01 \mu\text{K}$ .

Next, we estimate the value of  $\mu$ . To make the model parameter  $\gamma/\mu$  small, it is preferable to make the chemical potential  $\mu$  as large as possible. Experimentally, we expect  $\mu$

to be able to increase to about  $1 \mu\text{K}$ . In the recent transport experiments on a  ${}^6\text{Li}$  Fermi gas in a two-terminal configuration, the Fermi temperature of the atomic cloud is set from  $300 \text{ nK}$  to  $1 \mu\text{K}$  [71]. Thus, the Fermi temperature of the main system, which is comparable to  $\mu$ , is also estimated to be possible to set to about  $1 \mu\text{K}$ . (Note that the Fermi temperature is determined by the particle density and the density of the reservoirs and the main systems are comparable in our model.)

From the above-mentioned analyses, we reasonably expect that it is possible to reduce the model parameter  $\gamma/\mu$  to less than 0.01 when our proposed experimental setup is implemented in cold atomic systems. This enables the nonequilibrium distribution to show a clear two-step structure, which is a necessary ingredient to realize a stable NFF.

We next discuss the second challenge for the implementation of our proposal: the timescale of the chemical potential bias decay and the relaxation to a steady state. In a typical cold atomic transport measurement, it is known that the relaxation of the chemical potential bias occurs on a timescale of a few seconds (for example, about 8 s in Ref. [70]). On the other hand, the timescale of the relaxation to a steady state is considered to be dominated by interatomic scattering when the quasiparticle damping due to the system-reservoir couplings is sufficiently small ( $\gamma \ll \mu$ ). The thermalization of incident particles from the QPC to the reservoir occurs on the scale of the interatomic scattering time  $\tau = (1/n\sigma v_F)(T/T_F) \sim 400 \text{ ms}$ , where  $n$  is the particle density,  $\sigma$  is the scattering cross-section for interparticle collisions, and  $v_F$  is the Fermi velocity [70]. As a result, our proposed setup is also expected to relax to a steady state in a few hundred milliseconds due to interatomic scattering. We note that the larger  $n$  and  $v_F$  are, the shorter  $\tau$  becomes and the more quickly the steady state can be achieved. This is compatible with the requirement that  $\mu$  should be as large as possible in order to keep  $\gamma/\mu$  small.

To summarize, the relaxation to the nonequilibrium steady state we are interested in is expected to occur on a fast-enough timescale ( $\sim$  a few hundred milliseconds) compared to the timescale of the chemical potential bias decay ( $\sim$  a few seconds), which indicates that it is quite promising to observe the NFF in our proposed cold atomic setup.

So far, we discussed the cold atomic setup, but voltage-biased superconductors would also satisfy the necessary condition for observing the NFF. Since a nonequilibrium quasiparticle distribution with a clear two-step structure was experimentally observed in a voltage-biased electron system [84–87], we can reasonably expect that it is possible to make  $\gamma/\mu$  small enough to realize the NFF. Actually, for the electron system, the damping due to impurity scattering would become more essential for the realization of NFF, rather than the damping  $\gamma$  due to the system-reservoir couplings. Since impurities also have similar effects as  $\gamma$  in our model, the two-step structure of the nonequilibrium distribution is smeared, which is detrimental to NFF, as the impurity concentration increases (indeed, it was observed that the two-step structure of a nonequilibrium distribution becomes blurred as the impurity concentration increases in mesoscopic wires [86]). This suggests that it is preferable to use clean (ballistic) samples to realize the NFF in voltage-biased electron systems.

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space reads  $\partial_{\mathbf{p}} = p_F^{-1} \partial_{p_\theta} \hat{\mathbf{e}}_\theta + (p_F \sin \theta)^{-1} \partial_{p_\phi} \hat{\mathbf{e}}_\phi$ , and one can see that the  $\mathbf{p}$  derivative is on the order of  $p_F^{-1}$  (see [109] for a more general and longer discussion). Thus, the terms arising from  $\overleftarrow{\partial}_r \cdot \overrightarrow{\partial}_{\mathbf{p}}$  are of order  $|\vec{q}|/p_F$ , and the first-order gradient approximation explained below Eq. (58) will hold well when  $|\vec{q}|/p_F \ll 1$  is satisfied.

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*Correction:* The previously published order of authors was presented incorrectly and has been fixed.