

Algorithmic cooling for resolving state preparation and measurement errors in quantum computing

Raymond Laflamme,^{1,2} Junan Lin^{1,*}, and Tal Mor³

¹*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*

²*Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2L 2Y5*

³*Computer Science Department, Technion - Israel Institute of Technology, Technion City, Haifa 3200003, Israel*



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State preparation and measurement errors are commonly regarded as indistinguishable. The problem of distinguishing state preparation errors from measurement errors is important to the field of characterizing quantum processors. In this work, we propose a method to separately characterize state preparation and measurement errors using a different type of algorithmic cooling protocol called measurement-based algorithmic cooling (MBAC). MBAC assumes the ability to perform (potentially imperfect) projective measurements on individual qubits, which is available on many modern quantum processors. We demonstrate that MBAC can significantly reduce state preparation error under realistic assumptions, with a small overhead that can be upper bounded by measurable quantities. Thus, MBAC can be a valuable tool not only for benchmarking near-term quantum processors but also for improving the performance of quantum processors in an algorithmic manner.

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I. INTRODUCTION

Quantum computers are believed to solve some problems more efficiently [1,2], and sometimes exponentially more efficiently [3,4], than classical computers. On the other hand, various types of noise make it hard to design and build useful quantum computers and perform quantum algorithms. The main types of noise include state preparation errors, gate errors, dephasing and relaxation during computation, and measurements errors, along also with qubit-correlated errors. Among these, state preparation and measurement (SPAM) errors are usually studied as a whole, because they are fundamentally hard to distinguish. For example, if one tries to prepare a target state $|0\rangle$, immediately measures it, and gets an outcome of $|1\rangle$, it is not clear whether the state preparation or the measurement contributes more to the inconsistency.

Algorithmic cooling (AC), or more accurately heat-bath AC [5], is a method that employs relaxation with a bath for cooling spins. We show here that an extended form of AC, which we call measurement-based algorithmic cooling, can be a useful tool for distinguishing state preparation errors from measurement errors. Most commonly, various AC algorithms [6–15] work under the assumptions that the system consists of qubits and that the state preparation noise is thermal. In addition, it is commonly assumed that the system is an ensemble of many identical subsystems, such as in NMR quantum computing. In ensemble quantum computing, some AC algorithms are sometimes performed prior to the standard quantum computing process, as part of a prolonged state preparation process. These algorithms redistribute the entropy among the

qubits, so that the state of a subset of target qubits after AC is more pure, as if they were put in a lower temperature than the initial temperature of the original thermal state.

In the ensemble, a vital assumption is the limitation on the quantum computing abilities: it is assumed that operations cannot be applied to individual qubits, but only to the same qubits in each of the many parallel quantum computers. Consequently, one is limited to ensemble measurements that estimate expectation values of certain operators. On the other hand, AC in other quantum computing systems can be less restrictive: if projective measurements are allowed, a trivial way to cool a qubit is to measure it. If the outcome is 0, the system is in the $|0\rangle$ state immediately after the measurement, corresponding to a zero temperature (discussed later). If the outcome is 1, then one can perform a Pauli X gate to flip it to $|0\rangle$, achieving the same result. However, there are two major issues with this approach: first, in some existing gate-based quantum computing architectures, the measurement process only takes place at the final stage of the computation. No computation can proceed on the measured qubit and a new round of computation must be initiated. In other words, the measurement “destroys” the system of interest, so this naive cooling method is not so useful for doing further computations. Second, this method works only if measurement errors are absent and there is a perfect correspondence between the measurement result and the postmeasurement state; if this is not true, we cannot safely deduce the postmeasurement state.

In this work, we propose an AC protocol under the assumption that (potentially imperfect) measurements can be applied on individual qubits, which resolves the above issues. We call this alternative class “measurement-based AC” (MBAC) since measurement is used as a resource to cool. The outline of this paper is as follows. In Sec. II A we give a review of

*junan.lin@uwaterloo.ca

conventional AC protocols, which motivate the alternative MBAC protocols. In Sec. II B we describe k -qubit MBAC assuming ideal measurements on the ancillary qubits and compare it with the optimal reversible AC scheme. In Sec. II C we describe a procedure to separately characterize SPAM errors using MBAC. In Sec. II D we relax the assumption made in Sec. II B and analyze MBAC when the ancillary qubits have finite measurement errors, and we derive a lower bound for its performance. In Sec. II E we study the number of trials needed to cool down a target qubit by a desired multiplicative factor, and we illustrate the practical usefulness of MBAC despite its probabilistic nature. Finally in Sec. III we conclude our paper and point to some future research directions.

II. RESULTS

A. Review of conventional AC

We first describe AC protocols, notations, and goals, in a way that leads more naturally to our alternative type of AC, the MBAC. Below, we denote the Pauli matrices X , Y , and Z by σ_x , σ_y , and σ_z , and the 2×2 identity matrix by σ_I . The ideal state preparation step should initialize a single qubit to the $|0\rangle$ state. We assume that due to imperfect state preparation processes, a bit-flip (equivalently, σ_x) error occurs with probability δ , so that the actual initial state is described by the following mixed state density matrix:

$$\rho = (1 - \delta)|0\rangle\langle 0| + \delta|1\rangle\langle 1|. \quad (1)$$

This represents a state preparation error on the quantum processor. Throughout this work we assume that $0 \leq \delta < 1/2$, where $\delta = 0$ corresponds to the pure state $|0\rangle$ and $\delta \rightarrow 1/2$ corresponds to the completely mixed state.

Algorithmic cooling procedures are designed to reduce δ on the target qubit towards 0. The term ‘‘cooling’’ comes from viewing ρ as a thermal state [16],

$$\rho = \frac{1}{Z} e^{-\beta \hat{H}} = \frac{1}{Z} \begin{pmatrix} e^{-\frac{\hbar\omega}{2k_B T}} & 0 \\ 0 & e^{-\frac{\hbar\omega}{2k_B T}} \end{pmatrix}, \quad (2)$$

where $\hat{H} = -\frac{1}{2}\hbar\omega\sigma_z$ is the bare qubit Hamiltonian, k_B is the Boltzmann constant, T is an effective temperature, and Z is the partition function so that ρ is normalized. From Eqs. (1) and (2) we can identify $\delta = (e^{\frac{\hbar\omega}{k_B T}} + 1)^{-1}$. Solving for T gives $k_B T = \hbar\omega / \ln(\frac{1}{\delta} - 1)$, so that T is closer to 0 when δ is closer to 0, for a fixed ω . Therefore, reducing its effective temperature increases its probability of being in the ground state.

AC has two main variants, namely, the Schulman-Vazirani reversible scheme [17] and the heat-bath scheme [5,6]. In the reversible variant, the relaxation time is assumed to be infinite or extremely large relative to the computing time. In the heat-bath variant, one assumes that, in addition to computational (or, target) qubits that relax very slowly, there are also ancillary ‘‘reset’’ qubits that can relax back to thermal equilibrium much faster (through interaction with a heat bath) than the target qubits. We first review the heat-bath scheme using a minimal example with $m = 3$ qubits [6], which inspires our measurement-based protocol. We review the reversible variant later in Sec. II B when we compare MBAC with conventional AC.

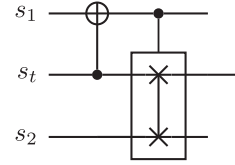


FIG. 1. The circuit for three-qubit BCS aiming to cool down the target spin s_t . Within the box is a SWAP gate, which is controlled by spin s_1 .

Let us now assume that there is a target spin to be cooled called s_t and two ancillary spins s_1 and s_2 . Starting with the system in a state $\rho_1 \otimes \rho_t \otimes \rho_2$ where all ρ 's are described by Eq. (1) with the same value of δ , we can apply a basic compression subroutine (BCS) to reduce δ on s_t (see Fig. 1). The BCS involves essentially two steps: first, a controlled-NOT (CNOT) operation controlled by s_t and targeted on s_1 ; second, a controlled-SWAP (CSWAP) operation controlled by s_1 and targeted on s_t and s_2 . To see why this protocol works, first consider the evolution of s_1 and s_t after the CNOT gate. We replace the noisy states of s_t and s_1 by assuming an error model is applied on pure qubit states. The error model is such that a σ_x channel occurs independently on each qubit after the preparation. Referring to Fig. 2, this corresponds to the following four possibilities where U equals

- (i) $\sigma_I \otimes \sigma_I$, $p = (1 - \delta)^2$;
- (ii) $\sigma_I \otimes \sigma_x$, $p = \delta(1 - \delta)$;
- (iii) $\sigma_x \otimes \sigma_I$, $p = \delta(1 - \delta)$;
- (iv) $\sigma_x \otimes \sigma_x$, $p = \delta^2$,

with the corresponding probabilities listed after each case.

In the above the left error-operator is applied to s_1 and the right operator to s_t . From Fig. 2 and the mapping rule of Pauli operators under the CNOT gate, the four cases can be written equivalently where U' equals

- (i) $\sigma_I \otimes \sigma_I$, $p = (1 - \delta)^2$;
- (ii) $\sigma_x \otimes \sigma_x$, $p = \delta(1 - \delta)$;
- (iii) $\sigma_x \otimes \sigma_I$, $p = \delta(1 - \delta)$;
- (iv) $\sigma_I \otimes \sigma_x$, $p = \delta^2$.

Since s_1 and s_t both start from $|0\rangle$, the CNOT gate does nothing and can be removed. Thus, the output states are

- (i) $|0\rangle \otimes |0\rangle$, $p = (1 - \delta)^2$;
- (ii) $|1\rangle \otimes |1\rangle$, $p = \delta(1 - \delta)$;
- (iii) $|1\rangle \otimes |0\rangle$, $p = \delta(1 - \delta)$;
- (iv) $|0\rangle \otimes |1\rangle$, $p = \delta^2$.

It is now clear that s_1 and s_t become correlated, since the probability of them being in the same state (cases 1 and 2) is higher than that of being in different states (cases 3 and 4), for all $0 \leq \delta < 1/2$. When s_1 is in $|0\rangle$, s_t is more likely to be in $|0\rangle$ (case 1) than in $|1\rangle$ (case 4), so we keep this purified spin

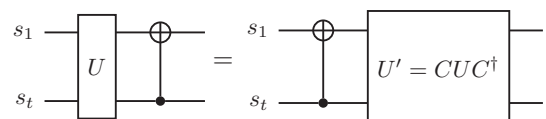


FIG. 2. Two equivalent circuits where the order of two unitary gates are exchanged, and the second gate on the right-hand side is replaced by the original gate conjugated by the first gate. C stands for the CNOT gate in this case.

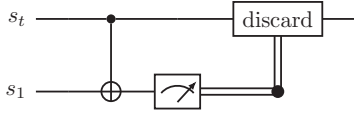


FIG. 3. A circuit for two-qubit measurement-based algorithmic cooling which increases the bias of the target qubit s_t . The double-line notation means “controlled on classical outcome”; i.e., s_t is kept for further computations when the measurement outcome is 0 and discarded if the outcome is 1.

s_t in the second step of Fig. 1. When s_1 is in $|1\rangle$, s_t is equally likely to be in $|0\rangle$ or $|1\rangle$ (cases 2 and 3), which corresponds to the completely mixed state and an effective temperature $T \rightarrow \infty$. In this case s_t has been heated up, so we can restore its original state by swapping s_t and s_2 in the second step of Fig. 1. Overall, the density matrix of s_t becomes closer to $|0\rangle$ at the output end.

We can calculate the exact reduced state of s_t at the output by taking an average over the aforementioned four cases (after applying the CSWAP step). Denote the probability of being in $|1\rangle$ for s_t after a BCS round as δ'_t . For cases 1 and 4, s_1 is in $|0\rangle$ so no SWAP gate is applied, and the probability of s_t being in $|1\rangle$ is 0 and 1, respectively. For cases 2 and 3, s_t is swapped with s_2 , so the probability of s_t being in $|1\rangle$ is restored to δ in both cases. Averaging over these four cases yields

$$\delta'_t = \delta^2 + 2\delta^2(1 - \delta) = 3\delta^2 - 2\delta^3 \leq \delta, \forall 0 \leq \delta < \frac{1}{2}. \quad (3)$$

For small δ , it is reduced to order $O(\delta^2)$. In the more general case where the initial δ 's on each spin can be different, we have

$$\delta'_t = \delta_1\delta_t + \delta_2\delta_t + \delta_1\delta_2 - 2\delta_1\delta_t\delta_2. \quad (4)$$

If we further assume that the ancillas s_1 and s_2 can relax much faster to their original states than s_t , so that we can effectively repeat this BCS round (where s_t now has error δ'_t), then s_t can be further purified. In the limit of performing infinitely many rounds, s_t arrives at the steady state, whose error can be calculated by setting $\delta'_t = \delta_t = \delta_t^\infty$ in Eq. (4), which gives

$$\delta_t^\infty = \frac{\delta_1\delta_2}{1 - \delta_1 - \delta_2 + 2\delta_1\delta_2}. \quad (5)$$

B. MBAC

Based on the previous analysis, one can see that the second CSWAP gate (and consequently, s_2 as well) is not needed if we can learn the state of s_1 after the first CNOT gate. If s_1 is in $|0\rangle$, we know that s_t has a higher probability of being in $|0\rangle$. If we discard the cases where s_1 is in $|1\rangle$ and only keep the ones where it is in $|0\rangle$, we can reduce the error δ on the target qubit. The working principle behind this method is similar to an error *detecting* code, where the effective noise level is reduced by accepting the cases where no error occurs and discarding those with an error occurring.

Referring to the circuit in Fig. 3, measuring $|0\rangle$ on s_1 updates the state of s_t to

$$\rho'_t = \frac{\langle 0|_1\tau|0\rangle_1}{\langle 0|\text{Tr}_t[\tau]|0\rangle}, \quad (6)$$

where, throughout this work, we use τ to denote the state of the full system right before the measurement. Starting from two states with initial error rates δ_1 and δ_t , the final error rate on s_t upon measuring $|0\rangle$ on s_1 becomes

$$\delta'_t = \frac{\delta_1\delta_t}{1 - \delta_1 - \delta_t + 2\delta_1\delta_t}. \quad (7)$$

Then, taking $\delta_t = \delta_1 = \delta$ again for simplicity, we see that in the small- δ limit, the error is also reduced to $O(\delta^2)$ to leading order. Importantly, the target qubit s_t is not being measured, which resolves the first issue we raised in Sec. I. Assuming that the ancillary qubit s_1 can maintain sufficiently long coherence, we can store them in the quantum computer and measure them at the end of the computation, along with other qubits. We then postselect only those measurement outcomes where s_1 measures to 0. Interestingly, comparing with Eq. (5), we see that applying the above protocol once achieves the same polarization on s_t as applying infinitely many rounds of BCS, if initially $\delta_t = \delta_2$.

The above two-qubit protocol, which we call MBAC-2, forms a basis for analyzing expanded versions of MBAC. For conventional AC protocols there are two major ways to expand them, either by using more ancillary qubits or by repeating the protocol for multiple rounds. Since we have assumed that the ancillary qubits cannot be reused once they have been measured, MBAC protocols cannot be expanded by repeating them for multiple rounds. However, it is feasible to use more ancillary qubits to achieve better cooling. Specifically, imagine expanding s_1 to $k - 1$ qubits in Fig. 3. We apply $k - 1$ CNOT gates controlled by s_t and targeted on s_i , $i = 1, \dots, k - 1$, and we measure all s_1, \dots, s_{k-1} at the end. The target s_t is kept only if *all* measurement outcomes are 0, and it is discarded otherwise. We call the above protocol MBAC- k , which serves as an expanded version of MBAC-2.

To analyze MBAC- k , observe that it is equivalent to repeating $k - 1$ “rounds” of MBAC-2, if the ancilla in MBAC-2 is allowed to return to its original state after being measured and can be reused again. The evolution of noise in a single “round” is already given in Eq. (7), so the evolution for MBAC- k can be recursively calculated from Eq. (7). Furthermore, by assuming again $\delta_t = \delta_1 = \delta$ initially and observing the first few terms, we can infer the analytic solution for this series, given by

$$\delta_t[k] = \frac{\delta^k}{\delta^k + (1 - \delta)^k}, \quad (8)$$

where from now on we use $\delta_t[k]$ to denote “SP error on s_t after applying MBAC- k .” In particular, $\delta_t[1]$ is used to denote the initial SP error on s_t . The solution in Eq. (8) can be readily verified by plugging it back into Eq. (7). From this solution one can see how δ_t decreases approximately exponentially in k , especially in the small- δ limit.

To make a fair comparison between MBAC and conventional AC, we now briefly review the reversible scheme first proposed by Schulman and Vazirani [17]. The main step in reversible AC is called entropy compression, where through a unitary map \mathcal{U} the entropy is extracted from some subset of qubits and transferred to another subset of qubits. If we constrain the system to start and end in diagonal states, and the goal is to cool down only one qubit, then it can be shown

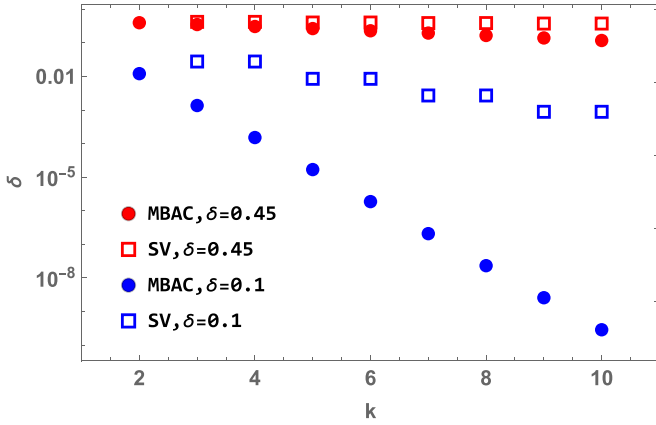


FIG. 4. Simulated evolution of δ_t between MBAC- k and SV- k , plotted on a semi-log scale. The initial errors (red: $\delta = 0.45$; blue: $\delta = 0.1$) are assumed to be the same on all qubits. The effect of decoherence is assumed negligible and gates are assumed to be ideal.

that the optimal unitary \mathcal{U} is to perform a descending sort on the diagonal elements of the full system's density matrix [7,18]. We thus consider this to be an upper bound on the performance of Schulman-Vazirani type of cooling, and we call this scheme SV- k if it uses a total of k qubits to cool one qubit. In Fig. 4, we compare MBAC- k (in circles) and SV- k (in squares) starting from two initial noise levels, $\delta = 0.1$ (in blue) and $\delta = 0.45$ (in red). The advantage of allowing projective measurement into the task of cooling, compared to the optimal reversible scheme, is clearly visible.

C. SPAM error characterization

In Sec. II A, we have already seen how the parameter δ can be interpreted as a measure of state preparation (SP) error. We now formally define what SP and M errors are, and then we demonstrate how MBAC can be used to characterize SPAM errors. Our definition here will closely follow the one given in Ref. [19]. The ideal SPAM operators are denoted by a density operator ρ and a two-outcome positive operator-valued measure (POVM) $M = \{M_0, M_1\}$, which satisfy the physicality constraints that ρ and M_i are positive-semidefinite Hermitian operators, and $M_0 + M_1 = \sigma_I$. We assume that ideally, $\rho = M_0 = |0\rangle\langle 0|$. The SPAM error is defined as the probability of obtaining an incorrect outcome when measuring the initial state; that is,

$$\delta_{\text{SPAM}} := 1 - \text{Tr}[\rho M_0]. \quad (9)$$

We now define the SP error, denoted as δ_{SP} , to be equal to δ_{SPAM} with an ideal measurement operator. Similarly, the M error (denoted as δ_{M}) is defined as δ_{SPAM} with an ideal input state. While the total SPAM error δ_{SPAM} is a measurable quantity, δ_{SP} and δ_{M} are not, when both state preparation and measurement errors are present. However, one may design algorithms that allow one to isolate the contributions to the total error from state preparation or measurement processes.

We now demonstrate how one can separately estimate δ_{SP} and δ_{M} , based on the concept of AC. For simplicity, we first assume that both the imperfect state and measurement opera-

tors are of the same form

$$\rho = \begin{pmatrix} 1 - \delta_{\text{SP}} & 0 \\ 0 & \delta_{\text{SP}} \end{pmatrix}, \quad M_0 = \begin{pmatrix} 1 - \delta_{\text{M}} & 0 \\ 0 & \delta_{\text{M}} \end{pmatrix}, \quad (10)$$

so that the total SPAM error is given [from Eq. (9)] by

$$\delta_{\text{SPAM}} = \delta_{\text{SP}} + \delta_{\text{M}} - 2\delta_{\text{SP}}\delta_{\text{M}}. \quad (11)$$

As before, we also assume that $\delta_{\text{SP}}, \delta_{\text{M}} \in [0, 0.5)$. Our goal is to separately estimate $\delta_{\text{SP},t}$ and $\delta_{\text{M},t}$ on s_t . For now we also assume, for simplicity, that measurement operations on all *ancillary* qubits are *ideal*, and the only noisy measurement is the one on s_t . This allows us to directly apply the previous calculations. This assumption is relaxed in the next section.

Since $\delta_{\text{SPAM},t}$ can be obtained from directly measuring s_t , the problem of separately characterizing SPAM errors is then reduced to estimating either $\delta_{\text{SP},t}$ or $\delta_{\text{M},t}$: from the symmetry between the two, once either is known, the other can also be calculated from the second equation.

It is now possible to intuitively see how AC can be used to resolve SPAM errors. We saw in Eq. (8) and Fig. 4 that MBAC can quickly reduce the error δ_t on s_t close to 0. Now imagine two separate experiments where in the first one we measure s_t directly and obtain $\delta_{\text{SPAM},t}$. In the second one, we first apply multiple rounds of MBAC until the final bias on the target is sufficiently close to 1 (we later show how this can be determined), and then we measure the target qubit. Since the measurement operation is independent of the qubit state, $\delta_{\text{M},t}$ is directly obtained from the measurement result, since the input state is now ideal. From here, $\delta_{\text{SP},t}$ can be easily calculated from Eq. (11). We have thus separately estimated SPAM errors by first eliminating the SP error, determining the measurement error, and then inferring the SP error from the total $\delta_{\text{SPAM},t}$.

We have so far focused on the case of diagonal state and measurement operators. To justify this, below we describe an averaging technique to convert arbitrary one-qubit SPAM elements to this simpler case. Begin by noting that we can generally write

$$\rho = \frac{1}{2}(\sigma_I + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z), \\ M_0 = \frac{1}{2}(m_x\sigma_I + m_x\sigma_x + m_y\sigma_y + m_z\sigma_z), \quad (12)$$

where the s 's and m 's are unknown parameters ($s_i = 1$ because $\text{Tr}[\rho] = 1$). Assuming ideal quantum gates, we can obtain an effective initial state with $s_x = s_y = 0$ for an arbitrary one-qubit circuit as follows. We perform two separate experiments, where in the first we apply the original circuit, and in the second we apply a σ_z gate immediately after the state preparation and then carry out the same circuit. Due to linearity of quantum operations, the average of measurement outcomes from the two experiments is then equivalent to one where $\rho_{\text{eff}} = \frac{1}{2}(\rho + \sigma_z\rho\sigma_z^\dagger) = \frac{1}{2}(\sigma_I + s_z\sigma_z)$. Similarly, we can also make $m_x = m_y = 0$ by averaging the results from the original experiment and one where a σ_z gate is applied immediately before the measurement. To set $m_i = 1$, we can average the original circuit with one where a σ_x gate is applied immediately before the measurement and the outcomes 0 and 1 are relabeled (so that outcome 0 corresponds to the POVM element M_1 and *vice versa*). This reduces the SPAM operators to the ones described by Eq. (10).

D. MBAC with measurement errors

We now deal with the second issue raised in Sec. I and study the performance of MBAC when the measurement error is nonzero on all *ancillary* qubits as well, thereby relaxing the assumption made in the previous section. We show that MBAC still performs well if δ_{SPAM} on each ancilla is not very large, thereby relaxing the most crucial assumption in SPAM characterization. Again, in this section we assume that quantum gates are ideal, which means that the SPAM averaging processes are also ideal, and the state and measurement operators can each be described by a single parameter.

The problem setup is as follows. We assume that each qubit i in a quantum computer has an independent state preparation error $\delta_{\text{SP},i}$ and measurement error $\delta_{\text{M},i}$, where i labels the qubit. The target qubit is labeled by t as usual. The goal is again to learn $\delta_{\text{SP},t}$ and $\delta_{\text{M},t}$. But as we show later, similar arguments can be used to learn all $\delta_{\text{SP},i}$ and $\delta_{\text{M},i}$ if desired.

Consider performing one successful round of the MBAC-2 protocol with two qubits s_t and s_1 . With an imperfect measurement, we generalize the case of projective measurement in Eq. (6) to a POVM measurement, so that performing one successful round of MBAC updates the state of s_t to [20]

$$\rho'_t = \frac{\text{Tr}_1[\tau(I \otimes M_0)]}{\text{Tr}[\text{Tr}_t[\tau]M_0]}, \quad (13)$$

where, again, τ denotes the state of the full system immediately before measurement. Using Eq. (13), following again the circuit in Fig. 3 and the parametrization in Eq. (10), we calculate the SP error on s_t after one round of MBAC to be

$$\begin{aligned} \delta'_{\text{SP},t} &= \delta_{\text{SP},t} \frac{2(\delta_{\text{SP},1} + \delta_{\text{M},1} - 2\delta_{\text{SP},1}\delta_{\text{M},1})}{1 + (1 - 2\delta_{\text{SP},1})(1 - 2\delta_{\text{SP},t})(1 - 2\delta_{\text{M},1})} \\ &= \delta_{\text{SP},t} \frac{2\delta_{\text{SPAM},1}}{1 + (1 - 2\delta_{\text{SP},1})(1 - 2\delta_{\text{SP},t})(1 - 2\delta_{\text{M},1})}, \end{aligned} \quad (14)$$

where the second equality comes from Eq. (11). The ratio

$$\frac{\delta_{\text{SP},t}}{\delta'_{\text{SP},t}} = \frac{1 + (1 - 2\delta_{\text{SP},1})(1 - 2\delta_{\text{SP},t})(1 - 2\delta_{\text{M},1})}{2\delta_{\text{SPAM},1}} \quad (15)$$

is a measure of the improvement of the SP error on s_t after one round of MBAC, which is better when larger. For example, a ratio of 100 implies that $\delta_{\text{SP},t}$ has been reduced by a factor of 100. Intuitively, the improvement should be more significant when there is less error on s_1 : indeed, if $\delta_{\text{SP},1} = \delta_{\text{M},1} = 0$ in Eq. (14), then $\delta'_{\text{SP},t} = 0$ and s_t will always be projected to $|0\rangle\langle 0|$ when the measurement outputs 0 on s_1 . In the more general case where the SPAM error on s_1 is present, we observe that the numerator on the right-hand side of Eq. (15) is always ≥ 1 , in the relevant region where $\delta_{\text{SP},1}, \delta_{\text{SP},t}, \delta_{\text{M},1} \in [0, 1/2]$. Therefore,

$$\frac{\delta_{\text{SP},t}}{\delta'_{\text{SP},t}} \geq \frac{1}{2\delta_{\text{SPAM},1}}. \quad (16)$$

Furthermore, in the limit where $\delta_{\text{SP},1}, \delta_{\text{SP},t}$, and $\delta_{\text{M},1}$ are all $\ll 1$, the bound in Eq. (16) can be improved by approximately a factor of 2 to simply $1/\delta_{\text{SPAM},1}$.

Next, recall from Sec. IIB that MBAC- k is equivalent to repeating $k - 1$ rounds of MBAC-2. Using mathematical induction, we see that by applying one successful run of

MBAC- k , the final SP-error on s_t is upper bounded by

$$\delta_{\text{SP},t}[k] \leq \delta_{\text{SP},t} \prod_{i=1}^{k-1} (2\delta_{\text{SPAM},i}). \quad (17)$$

Recall that $\delta_{\text{SPAM},i}$ is a measurable quantity obtained by measuring the initial state on ancilla s_i . The product simply corresponds to the probability of directly measuring all ancillary qubits s_1, \dots, s_{k-1} after they are prepared and getting the output 1 on all qubits. Therefore, given $\delta_{\text{SPAM},i}$ on each qubit (which can be obtained before the experiment, during the calibration step), Eq. (17) guarantees that $\delta_{\text{SP},t}$ from the output of a successful run of MBAC- k is at least reduced by $\prod_{i=1}^{k-1} (2\delta_{\text{SPAM},i})$. This shows that finite measurement errors on the ancillary qubits do not pose a fundamental limitation to the cooling power of MBAC. In particular, as long as each $\delta_{\text{SPAM},i} < 1/2$ (which we assume from now on), the output state is guaranteed to be more pure than the input state. Moreover, if both $\delta_{\text{SP},t}$ and all $\delta'_{\text{SPAM},i}$ s are upper bounded by a constant δ , then $\delta_{\text{SP},t}[k]$ is simply upper bounded by δ^k .

E. Number of trials needed in MBAC- k

Next, we study whether the probabilistic nature of MBAC hinders its usefulness in practice. We see that it remains practically useful for a wide range of experimentally relevant SPAM error rates. To illustrate the problem, first ignore any measurement error and consider a run of MBAC-2. The post-measurement state upon measuring 1 (which implies a failed run) can be computed by changing all 0's to 1's in Eq. (6), leading to

$$\delta'_{t,\text{fail}} = \frac{\delta_t(1 - \delta_1)}{\delta_t + \delta_1 - 2\delta_t\delta_1} = \frac{1}{2} - \frac{\delta_1 - \delta_t}{2(\delta_t + \delta_1 - 2\delta_t\delta_1)}. \quad (18)$$

One sees that, if $\delta_1 = \delta_t$ initially, then obtaining a measurement outcome of 1 on s_1 will heat up the state to a completely mixed one. The probability of failure is

$$p_{\text{fail}} = \delta_1 + \delta_t - 2\delta_1\delta_t. \quad (19)$$

For MBAC- k , since we need all the $k - 1$ measurements to succeed, the rate of success decreases exponentially with k , and it becomes increasingly difficult to get a successful run.

Fortunately, recall from Eq. (17) that the target polarization also improves exponentially fast with increasing number of ancillas. Therefore, the intuition is that one does not need a large k value to achieve significant cooling, which in turn will not have a vanishingly small success probability. Below we make this intuition more concrete. Specifically, we ask the following question: if we would like to reduce $\delta_{\text{SP},t}$ by a factor of r , i.e., we want $\delta_{\text{SP},t}/\delta_{\text{SP},t}[k] = r$, how many runs are needed to achieve this cooling ratio? To answer this we start from Eq. (17) and first derive a relation between k and r . Note that because each $\delta_{\text{SPAM},i}$ can be different, the most general expression will involve all $\delta'_{\text{SPAM},i}$ s. In order to obtain an expression in k , we now make the assumption that all $k - 1$ ancillary qubits have the same δ_{SPAM} , which we denote as $\delta_{\text{SPAM},a}$ where a stands for the word ‘‘ancillary.’’ The case where the $\delta'_{\text{SPAM},i}$ s are different can be bounded similarly by setting $\delta_{\text{SPAM},a}$ to the highest among all $\delta_{\text{SPAM},i}$.

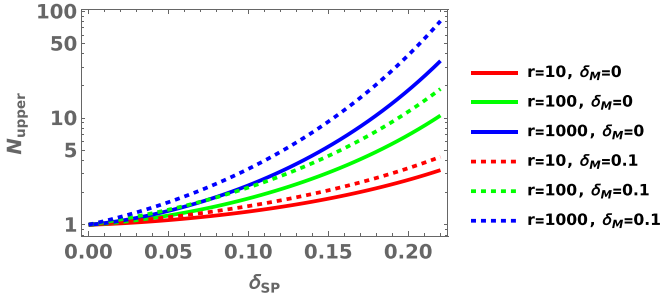


FIG. 5. Upper bound N_{upper} on the expected number of runs required for different values of r [the improvement ratio defined in Eq. (20)] versus the initial SP error δ_{SP} (assumed to be the same on the target and all ancillary qubits). Solid and dashed lines represent the cases of $\delta_{M,a} = 0$ and $\delta_{M,a} = 0.1$, respectively.

With this assumption, we obtain the following inequality between k and r :

$$r = \frac{\delta_{\text{SP},t}}{\delta_{\text{SP},t}[k]} \geq (2\delta_{\text{SPAM},a})^{-(k-1)}. \quad (20)$$

Taking the logarithm of both sides and rearranging the terms [note that $\ln(2\delta_{\text{SPAM},a}) < 0$] results in

$$k - 1 \leq \frac{\ln(r)}{-\ln(2\delta_{\text{SPAM},a})}. \quad (21)$$

This allows the experimentalist to determine the total number of ancillary qubits needed in order to cool to the desired noise level, based on their hardware specifications (i.e., the $\delta_{\text{SPAM},a}$ on their hardware). Importantly, this upper bound scales logarithmically with r .

Next, we compute the expectation value of the total number of runs needed before having a successful run, in order to achieve a cooling ratio r . We show in the Appendix that the expected number of runs is upper bounded by a function,

$$N_{\text{upper}}(r) = (r)^{\frac{\ln(A)}{\ln(B)}}, \quad (22)$$

where

$$\begin{aligned} A &= (1 - \delta_{\text{SP},t}[1] - \delta_{\text{SP},a} + 2\delta_{\text{SP},t}[1]\delta_{\text{SP},a})(1 - \delta_{M,a}), \\ B &= 2\delta_{\text{SPAM},a}. \end{aligned} \quad (23)$$

We see that Eq. (22) scales polynomially in r . Note that A is simply the success probability (i.e., measuring 0) of doing the first run of MBAC-2, so the exponent is a measurable quantity. Thus, the upper bound can be calculated given a target r .

To understand the behavior of $N_{\text{upper}}(r)$ more concretely, we can further simplify Eq. (22) by assuming again $\delta_{\text{SP},t}[1] = \delta_{\text{SP},a} := \delta_{\text{SP}}$, i.e., both the target and the ancillas have the same initial error δ_{SP} . The exponent then becomes [recall δ_{SPAM} from Eq. (11)]

$$\frac{\ln(A)}{\ln(B)} = \frac{\ln[(1 - 2\delta_{\text{SP}} + 2\delta_{\text{SP}}^2)(1 - \delta_{M,a})]}{\ln[2(\delta_{\text{SP}} + \delta_{M,a} - 2\delta_{\text{SP}}\delta_{M,a})]}. \quad (24)$$

Shown in Fig. 5 are plots of $N_{\text{upper}}(r)$ in Eq. (22) as a function of δ_{SP} , after making the simplifying assumption in Eq. (24), for a few chosen values of r and $\delta_{M,a}$. As the initial SP error rate approaches 0.5 (the theoretical maximum), MBAC fails since the measurement is simply giving random outputs. In this case, $N_{\text{upper}}(r)$ diverges as expected. On the other hand, for reasonably low values of δ_{SP} , we see that

the expected number of runs before achieving a successful one is rather low. For example, for $\delta_{\text{SP}} = 0.1$, $\delta_{M,a} = 0$, and $r = 1000$, we expect to obtain a successful run in about two trials. For $\delta_{\text{SP}} = 0.1$, $\delta_{M,a} = 0.1$, and $r = 1000$, the expected number of trials is approximately 3. Note that this corresponds to a case of 10% SP error rate plus 10% measurement error rate on the ancillas, combining to almost 20% of the total SPAM error rate. Many modern quantum computing platforms [21–23] can now achieve SPAM error rates below this level. In these cases, MBAC will be a useful and easy tool to significantly improve the quality of state preparation in quantum computers.

III. CONCLUSION

We introduce a different variant of the algorithmic cooling (AC) protocol based on the ability to perform imperfect measurements on individual qubits, which we call measurement-based AC (MBAC). Using this method, we develop a simple way to separately characterize state preparation and measurement errors, by eliminating the former using MBAC and directly obtaining the latter. Our approach is applicable to many current quantum computing platforms and significantly outperforms the optimal reversible AC protocol in the absence of measurement errors. Moreover, its cooling ability can still be lower bounded by some experimentally measurable quantities when measurement errors are present. Despite the probabilistic nature of MBAC, we have shown that it remains practically useful for many current quantum processors. We believe our method can be a helpful tool for benchmarking and improving current noisy, intermediate-scale quantum (NISQ) processors, and can provide further insights to measurement as a resource for cooling.

A few open questions remain after our work. First, while we have shown that measurement errors do not pose a fundamental challenge for MBAC, it remains an interesting question to also study the effect of errors in the quantum gates used [19,24]. Second, it would be useful to investigate the performance of MBAC with fully general SPAM operators, without the averaging techniques used in our work. Third, it can be very interesting to integrate the concept of MBAC into other algorithmic cooling protocols (see, e.g., Refs. [5–7,13–15,25]; also see the reviews [26,27]), which goes far beyond the BCS subroutine mentioned in Sec. II B, to develop new protocols that may have better performances than these current protocols. Last, our work suggests that measurements may be regarded as a resource in quantum thermodynamic theories. It would be interesting to explore the possibility of integrating measurement operations into the current quantum resource theory framework.

The data supporting the findings of this work are available from the corresponding author upon request.

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All authors contributed extensively to the presented work. R.L. and T.M. conceived the original ideas and supervised the project. J.L. performed analytical studies and numerical simulations and generated different versions of the manuscript. R.L. and T.M. verified the calculations, proofread the manuscript, and applied many insightful updates.

APPENDIX: CALCULATING THE EXPECTED NUMBER OF RUNS BEFORE A SUCCESSFUL ONE

Here we calculate the expected number of failed MBAC- k runs before having a successful one. The scenario is that one will continue running the experiment, until a successful round of MBAC- k occurs. Assume that one would like to reduce $\delta_{SP,t}$ by a factor of r (e.g., $r = 100$ or $r = 1000$ can be set by the experimentalist), by using MBAC- k . According to Eq. (14) in the main text, the final SP error is exponentially suppressed in terms of successful cooling rounds.

We again think about MBAC- k as a $(k - 1)$ -step protocol, where each step corresponds to measuring an ancilla and tracing it out after the measurement. The probability of measuring 0 on the i th step is

$$p_{0,i} = (1 - \delta_{SP,t}[i] - \delta_{SP,a} + 2\delta_{SP,t}[i]\delta_{SP,a})(1 - \delta_{M,a}). \quad (A1)$$

The measurements are independent for each round, so the probability of measuring all 0's is given by the product of $p_{0,i}$ from each round. Since $p_{0,i}$ improves as i increases, we can lower bound all of them by $p_{0,1}$, i.e., the probability of getting 0 in the first run of MBAC-2.

Thus, we see that the expected total number of runs is upper bounded by a simpler case, where n independent Bernoulli trials are conducted in series, each having success probability $p_{0,1}$. The probability of having all trials successful, which we will call the success probability, is $(p_{0,1})^{k-1}$. Note that for a Bernoulli trial that has success probability p , the expected number of tests n can be calculated to be

$$\begin{aligned} \mathbb{E}(n) &= \sum_i n_i p_i(n_i) \\ &= p + 2p(1 - p) + 3p(1 - p)^2 + \dots \end{aligned}$$

$$\begin{aligned} &= p \sum_{n=0}^{\infty} (n + 1)(1 - p)^n \\ &= \frac{p}{p^2} = \frac{1}{p} \end{aligned} \quad (A2)$$

as expected. Thus, if we denote the expected number of tests *in reality* by N , we have

$$N \leq \frac{1}{(p_{0,1})^{k-1}}, \quad (A3)$$

where $p_{0,1}$ is defined in Eq. (A1). Next, recall that Eq. (21) gives an upper bound on the required number of ancillary qubits, $k - 1$. Because $p_{0,1} \in (0, 1)$, $p_{0,1}^x$ decreases with x for $x \geq 0$, so that $(1/p_{0,1})^x$ increases with x for $x \geq 0$. Combining this with Eq. (A3), we have

$$N \leq \left(\frac{1}{p_{0,1}}\right)^{k-1} \leq \left(\frac{1}{p_{0,1}}\right)^{\frac{\ln(r)}{-\ln(2\delta_{SPAM,a})}} := A^{\frac{\ln(r)}{\ln(B)}}, \quad (A4)$$

where

$$A = p_{0,1}, \quad B = 2\delta_{SPAM,a}. \quad (A5)$$

We now define the expression on the right-hand side of Eq. (A4) as an upper bound on N , i.e.,

$$N_{\text{upper}}(r) := A^{\frac{\ln(r)}{\ln(B)}}. \quad (A6)$$

Note that this upper bound is only a function of r and some measurable quantities that are specific to the machine. Taking the log on both sides of Eq. (A6) gives

$$\ln[N_{\text{upper}}(r)] = \frac{\ln(r)}{\ln(B)} \ln(A) = \ln(r) \frac{\ln(A)}{\ln(B)} \quad (A7)$$

and taking the exponential on both sides again gives

$$N_{\text{upper}}(r) = (r)^{\frac{\ln(A)}{\ln(B)}}, \quad (A8)$$

which is polynomial in r , as claimed.

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