Quantifying coherence relative to channels via metric-adjusted skew information

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In terms of the metric-adjusted skew information (an important and versatile class of quantum Fisher information), which generalizes the seminal Wigner-Yanase skew information arising naturally from the study of quantum measurement, we propose a family of coherence measures of states relative to quantum channels, and reveal their basic properties such as unitary covariance, convexity, and monotonicity. Furthermore, we evaluate these coherence measures of states relative to several prototypical quantum channels, and make a comparative study for this family of coherence measures with relative entropy of coherence. This provides a general approach to coherence of states relative to quantum channels, which also captures decoherence on the states caused by quantum channels and asymmetry of states relative to quantum channels.

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I. INTRODUCTION

Quantum coherence, as a physical consequence of the superposition principle, is a cornerstone of quantum mechanics and plays a key role in quantum information processing and quantum technology. Although coherence has been extensively and intensively studied as a recurring theme ever since the inception of quantum mechanics, quantitative and axiomatic investigations of coherence from the resource perspective are a quite recent subject starting from the seminal work of Herbut [1] (which curiously remained largely unnoticed), Åberg [2], Levi and Mintert [3], and Baumgratz *et al.* [4]. In the influential framework of resource theory initiated in Ref. [4], various measures of coherence have been introduced and studied [5–30]. See Refs. [12,20] for reviews and a vast literature therein.

Motivated by the well-established fact that measurements described by positive operator-valued measures (POVMs) have operational advantages over the conventional von Neumann measurements in many scenarios, the resource theory of coherence of a quantum state relative to a basis (equivalently, a von Neumann measurement) was generalized to that relative to a POVM. The block coherence, i.e., the coherence relative to a projective measurement (not necessarily of rank 1), was studied in Refs. [2,15,22]. Now there are increasing interests in the study of coherence relative to a general measurement and associated applications [22–28]. The coherence of a state relative to a quantum channel (henceforth abbreviated as a channel) was explored in terms of the generalized skew information [16,28], and was applied to investigating quantum correlations [31], quantum interference [32], quantum metrology [29,33], and asymmetry [34,35], etc. Compared with POVMs, channels retain more phase information and many physical processes are described by channels beyond POVMs. Consequently, coherence of a state relative to a channel is an interesting issue worth studying.

Fisher information is a crucial concept in signal detection and parameter estimation (quantum metrology) [36–50], since it is used to provide an intrinsic lower bound for the precision of parameter estimation by virtue of the celebrated Cramér-Rao inequality. In the classical setup of probability theory, Fisher information is unique in the sense that it is the only Riemannian metric possessing the contractive property under coarse graining (classical channels) [45,46]. However, in the quantum scenario, the metrics satisfying the contractivity under channels are not unique, which means that quantum extensions of the classical Fisher information are not unique. There are infinitely many versions of quantum Fisher information, among which two prominent ones are the Wigner-Yanase skew information [51] (or more generally the Wigner-Yanase-Dyson skew information [51–54]) and the quantum Fisher information based on the symmetric logarithmic derivative [36–39,41]. Each version of quantum Fisher information may have its own merits in special contexts.

Explicit characterization of the metrics with contractivity in the quantum scenario was studied by Morozova and Chentsov [46], Petz [47–49], Petz and Ghinea [50], and Hansen [43], among others. In particular, by virtue of the operator monotone metrics, Hansen identified a class of quantum Fisher information as the metric-adjusted skew information [44], which encapsulates the well-known Wigner-Yanase skew information and the quantum Fisher information based on the symmetric logarithmic derivative as two prominent

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special instances, and generalizes them considerably. As a general and important quantity of information, the metric-adjusted skew information has found wide applications in many contexts, such as uncertainty relations [55–66], quantum interference [67], and asymmetry [68].

In this paper, we will quantify coherence of a state relative to a channel via the metric-adjusted skew information. The rest of the paper is arranged as follows. In Sec. II, after reviewing the metric-adjusted skew information, we introduce a family of coherence measures of a state relative to a channel via the metric-adjusted skew information, and investigate their basic properties. In Sec. III, we evaluate these coherence measures for several important qubit channels. We make an extensive comparative study between our coherence measures and the relative entropy of coherence in Sec. IV. Finally, we conclude with a summary in Sec. V.

II. COHERENCE VIA METRIC-ADJUSTED SKEW INFORMATION

Quantum Fisher information, and in particular, metric-adjusted skew information, as a generalization of classical Fisher information in mathematical statistics, is an important concept in quantum metrology and plays a crucial role in quantum parameter estimation. In this section, after some preliminary discussion on motivation and background of metric-adjusted skew information and coherence relative to a channel, we recall the definition and basic properties of metric-adjusted skew information induced by any operator monotone function, and further generalize it to any reference operator (not necessarily Hermitian). Then we introduce a family of coherence measures of a state relative to a channel in terms of metric-adjusted skew information, and illuminate their basic properties.

A. Motivation and background

A wide family of quantum Fisher information is the metric-adjusted skew information characterized by the celebrated Morozova-Chenstov functions [44]. This is based on the information contents introduced in Ref. [51], the geometrical formulation of quantum statistics formulated in Refs. [45,46], and the monotone metrics classified in Ref. [47]. Different versions of quantum Fisher information may have different merits and different uses. The general form of the quantum Cramér-Rao inequality based on metric-adjusted skew information provides a family of lower bounds for the precision of the parameter estimation [48], and as such plays a central role in quantum metrology. Given the importance of metric-adjusted skew information, it is desirable to further apply this information quantity to quantifying coherence of a state relative to a channel and explore their applications in quantum metrology. Just like different versions of entropy (such as the Tsallis entropy and the Rényi entropy) beyond the convenient von Neumann entropy are useful in studying quantum information, the approach to coherence via metric-adjusted skew information may provide a more complete picture of the properties of coherence, which in turn may be useful in theoretical investigations of quantum information processing.

Traditionally, when talking about coherence of a state, we need to specify an orthonormal basis as the reference basis, which is equivalent to specifying the corresponding von Neumann measurement with the basis elements as measurement operators. In this sense, the traditional notion of coherence of a state has two seemingly different but actually intrinsically related meanings: (1) (potential) coherence of the state before the von Neumann measurement and (2) (realized) decoherence of the state after the von Neumann measurement. With this second meaning of coherence of a state relative to a von Neumann measurement as the quantifier of decoherence of the state caused by the von Neumann measurement, it is natural to consider decoherence of a state caused by a channel (a more general physical process than POVM), which is a key issue in quantum measurement and open system dynamics. This motivates our study of coherence of a state relative to a channel. This quantity, apart from characterizing certain aspects of the interplay between states and channels, has further applications in quantifying asymmetry, quantum interference, quantumness, etc.

We remark that some coherence measures of a state relative to a channel based on two special instances of metric-adjusted skew information, i.e., Wigner-Yanase skew information and quantum Fisher information involving the symmetric logarithmic derivative, have been used to study quantum correlations, interference, and quantum metrology. For example, in Ref. [31], we introduced a quantifier of correlations (relative to a local channel) as the coherence difference in terms of Wigner-Yanase skew information and proved that both product states and some natural classical-quantum states can be operationally characterized in terms of local channels. In Ref. [32], two of us used the coherence of a state relative to a unitary channel parametrized by the interfering paths and phase shifts in multipath interference to quantify interference. In Refs. [29,33], the authors introduced the coherence measure via quantum Fisher information involving the symmetric logarithmic derivative and provided an operational meaning in quantum metrology. In Ref. [67], Gibilisco *et al.* built a unifying information-geometric framework to quantify quantum correlations in terms of metric-adjusted skew information. They extended the physically meaningful definition of local quantum uncertainty to a more general class of information measures and proved that metric-adjusted quantum correlation quantifiers possess a set of desirable properties which make them robust information measures. These studies illustrate certain applications and significance of metric-adjusted skew information. In this paper, we pursue an application of metric-adjusted skew information in quantifying coherence of a state relative to a general channel.

B. Metric-adjusted skew information

In their seminal study of quantum measurement, Wigner and Yanase introduced the information quantity [51]

$$I(\rho, X) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, X]^2$$

of a state ρ relative to the observable X (which served as a conserved observable). Here the square bracket denotes the commutator between operators, and tr denotes the operator (matrix) trace. This quantity is later termed the

Wigner-Yanase skew information, and turns out to be the first version of quantum Fisher information. After the suggestion of Dyson, Wigner and Yanase also introduced the quantity [51]

$$I_s(\rho, X) = -\tfrac{1}{2}\mathrm{tr}[\rho^s, X][\rho^{1-s}, X], \quad 0 < s < 1$$

which is now termed the Wigner-Yanase-Dyson skew information. Its convexity relative to ρ (the Wigner-Yanase-Dyson conjecture), first established by Lieb [52], is a deep and elegant result with extensive applications in information theory. In particular, it plays an innovative role in the path to strong subadditivity of von Neumann entropy (which is equivalent to monotonicity of quantum relative entropy). The skew information has been extensively used to quantify coherence, quantum uncertainty, asymmetry, and correlations [13–16,55–71].

Metric-adjusted skew information is a considerable generalization of the above skew information along the lines of quantum Fisher information. To illuminate this, we review some basic notions.

(a) Operator monotone functions: A function $f : \mathbb{R} \to \mathbb{R}$ is called operator monotone if $X \leqslant Y$ implies that $f(X) \leqslant f(Y)$ for any Hermitian operators (observables) X and Y on any finite-dimensional Hilbert space [47].

(b) Morozova-Chentsov functions: For an operator monotone function $f : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ satisfying $f(0) > 0$ and $xf(1/x) = f(x)$ (a kind of symmetry), the associated Morozova-Chentsov function is defined as [47]

$$c_f(x, y) = \frac{1}{yf(x/y)}. \tag{1}$$

The corresponding generalized mean is defined as

$$m_f(x, y) = \frac{1}{c_f(x, y)} = yf(x/y).$$

We remark that from the symmetry constraint $xf(1/x) = f(x)$, we may interpret $0f(x/0)$ as

$$\begin{aligned} 0f(x/0) &= \lim_{y\to 0} yf(x/y) \\ &= \lim_{y\to 0} x(y/x)f[1/(y/x)] \\ &= \lim_{y\to 0} xf(y/x) \\ &= xf(0). \end{aligned}$$

This will be used in the subsequent Eqs. (10) and (12).

(c) Monotone metrics: Given a state ρ and an operator monotone function $f : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ satisfying $f(0) > 0$ and $xf(1/x) = f(x)$ with the associated Morozova-Chentsov function c_f defined by Eq. (1), the monotone metric associated with ρ and f is defined as [47]

$$\langle A, B\rangle_{\rho,f} = \mathrm{tr}[A^\dagger c_f(L_\rho, R_\rho)(B)] \tag{2}$$

for any operators A and B on the system Hilbert space. Here

$$L_\rho(A) = \rho A, \quad R_\rho(A) = A\rho$$

are the left and right multiplications by ρ, respectively. Since the superoperators L_ρ and R_ρ commute, the superoperators $c_f(L_\rho, R_\rho)$ and $m_f(L_\rho, R_\rho)$ are well defined by functional calculus.

(d) Metric-adjusted skew information: For any quantum state ρ and any observable X, the metric-adjusted skew information of ρ relative to X is defined as [44]

$$\begin{aligned} F_f(\rho, X) &= \frac{f(0)}{2}\langle i[\rho, X], i[\rho, X]\rangle_{\rho,f} \\ &= -\frac{f(0)}{2}\mathrm{tr}\{[\rho, X]c_f(L_\rho, R_\rho)([\rho, X])\}, \end{aligned} \tag{3}$$

which is precisely the monotone metric defined by Eq. (2) specialized to $A = B = i[\rho, X]$.

By defining

$$\tilde{f}(x) = \frac{1}{2}\left((x+1) - (x-1)^2\frac{f(0)}{f(x)}\right), \tag{4}$$

which is still an operator monotone function, one further obtains

$$F_f(\rho, X) = \mathrm{tr}(\rho X^2) - \mathrm{tr}[Xm_{\tilde{f}}(L_\rho, R_\rho)(X)]$$

with $m_{\tilde{f}}(x, y) = y\tilde{f}(x/y)$.

Among the metric-adjusted skew information, there are two distinguished ones.

(a) Taking the operator monotone function

$$f(x) = \left(\frac{1+\sqrt{x}}{2}\right)^2, \tag{5}$$

then the corresponding metric-adjusted skew information is reduced to the Wigner-Yanase skew information [51]

$$I(\rho, X) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, X]^2.$$

(b) Taking the operator monotone function

$$f(x) = \frac{1+x}{2}, \tag{6}$$

then the corresponding metric-adjusted skew information is reduced to the quantum Fisher information [36–38]

$$F(\rho, X) = \tfrac{1}{4}\mathrm{tr}(\rho L^2) \tag{7}$$

based on the symmetric logarithmic derivative. Here L is the symmetric logarithmic derivative determined by

$$i[\rho, X] = \tfrac{1}{2}(L\rho + \rho L).$$

These quantities play a key role in quantum estimation theory [36–38]. In addition, the quantum Fisher information $F(\rho, X)$ and the Wigner-Yanase skew information $I(\rho, X)$ have the following relation [40]:

$$I(\rho, X) \leqslant F(\rho, X) \leqslant 2I(\rho, X). \tag{8}$$

Furthermore, Gibilisco *et al.* extended the above inequality relation to any metric-adjusted skew information as [72]

$$F_f(\rho, X) \leqslant F(\rho, X) \leqslant \frac{1}{2f(0)}F_f(\rho, X),$$

and showed that the constant $1/[2f(0)]$ is optimal.

In this context, we clarify the concept of *quantum Fisher information*, which may have different meanings for different people in different contexts.

(i) In some earlier literature [36–39], the parameter-independent version of quantum Fisher information simply refers to the quantity defined by Eq. (7), which is a special instance of metric-adjusted skew information determined by the symmetric logarithmic derivative.

(ii) In parameter estimation, quantum Fisher information refers to quantities for a family of parametrized states ρ_θ which are contracting under channels. When $\rho_\theta = e^{-i\theta X}\rho e^{i\theta X}$ is a translation family determined by some observable X, these quantities are independent of the parameter θ and are completely determined by ρ and X. For the multiparameter case, one naturally comes to quantum Fisher information matrices.

(iii) In the language of information geometry, following the classification scheme of Petz [47], quantum Fisher information (as a metric) refers to the sesquilinear form (i.e., a monotone metric indexed by the Morozova-Chentsov function) defined by Eq. (2). In this setting, metric-adjusted skew information is interpreted as a special parameter-independent version of the most general quantum Fisher information.

Since non-Hermitian operators arise naturally in the Kraus representations of channels, it is desirable to extend the metric-adjusted skew information defined by Eq. (3) to any operator K (not necessarily Hermitian). For this purpose, by substituting any operator K for the observable X in Eq. (3), we naturally obtain a generalization of the metric-adjusted skew information as

$$F_f(\rho, K) = \frac{f(0)}{2}\langle i[\rho, K], i[\rho, K]\rangle_{\rho,f}. \tag{9}$$

Direct manipulation shows that

$$F_f(\rho, K) = \tfrac{1}{2}\mathrm{tr}[\rho(K^\dagger K + KK^\dagger)] - \mathrm{tr}[K^\dagger m_{\tilde{f}}(L_\rho, R_\rho)(K)].$$

In terms of the spectral decomposition

$$\rho = \sum_i \lambda_i|\phi_i\rangle\langle\phi_i|$$

of the state ρ with $\{|\phi_i\rangle\}$ an orthonormal basis (some λ_i may possibly be zero), and noting the fact

$$m_{\tilde{f}}(L_\rho, R_\rho) = \sum_{ij} m_{\tilde{f}}(\lambda_i, \lambda_j)L_{|\phi_i\rangle\langle\phi_i|}R_{|\phi_j\rangle\langle\phi_j|},$$

$F_f(\rho, K)$ can be explicitly calculated as

$$F_f(\rho, K) = \frac{f(0)}{2}\sum_{ij}\frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)}|\langle\phi_i|K|\phi_j\rangle|^2. \tag{10}$$

As a closely related quantity, we also introduce the generalized variance

$$V(\rho, K) = \tfrac{1}{2}\mathrm{tr}[\rho(K^\dagger K + KK^\dagger)] - |\mathrm{tr}(\rho K)|^2$$

of the general operator (not necessarily Hermitian) K in the state ρ.

The generalized metric-adjusted skew information $F_f(\rho, K)$ defined by Eq. (9) has the following properties, which follow from Refs. [44,73], or can be directly verified.

(i) $0 \leqslant F_f(\rho, K) \leqslant V(\rho, K)$. Moreover, $F_f(\rho, K) = 0$ if and only if $[\rho, K] = 0$, and $F_f(\rho, K) = V(\rho, K)$ (the generalized variance) if ρ is a pure state. In particular, $F_f(\rho, K)$ is independent of f for any pure state ρ.

(ii) $F_f(U\rho U^\dagger, UKU^\dagger) = F_f(\rho, K)$ for any unitary operator U.

(iii) $F_f(\rho, K)$ is convex in ρ.

(iv) $F_f(\rho, K)$ is additive under tensoring in the sense that

$$F_f(\rho\otimes\sigma, K\otimes \mathbf{1}^b + \mathbf{1}^a\otimes J) = F_f(\rho, K) + F_f(\sigma, J)$$

for any quantum states ρ and σ, and any operators K and J on parties a and b, respectively. In particular,

$$F_f(\rho\otimes\sigma, K\otimes\mathbf{1}^b) = F_f(\rho, K).$$

(v) $F_f(\rho, K)$ is additive under direct sum in the sense that

$$F_f\left(\bigoplus_i p_i\rho_i, \bigoplus_i K_i\right) = \sum_i p_i F_f(\rho_i, K_i)$$

for any quantum states ρ_i, any operators K_i, and any probability distribution $\{p_i\}$.

(vi) For any bipartite state ρ^{ab} on the composite system ab and any operator K^a on party a,

$$F_f(\rho^{ab}, K^a\otimes\mathbf{1}^b) \geqslant F_f(\rho^a, K^a)$$

with $\rho^a = \mathrm{tr}_b(\rho^{ab})$ the reduced state on party a of the bipartite state ρ^{ab} shared by parties a and b.

C. Coherence via metric-adjusted skew information

Let ρ be a quantum state on a d-dimensional system Hilbert space H, and let

$$\Phi(\rho) = \sum_l K_l\rho K_l^\dagger$$

be a channel with the same input and output system. K_l are called the Kraus operators of the channel and satisfy $\sum_l K_l^\dagger K_l = \mathbf{1}$. The coherence of ρ relative to the channel Φ via the generalized metric-adjusted skew information is defined as

$$F_f(\rho, \Phi) = \sum_l F_f(\rho, K_l). \tag{11}$$

Let $\rho = \sum_i \lambda_i|\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of ρ, then

$$F_f(\rho, \Phi) = \frac{f(0)}{2}\sum_{ij}\frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)}\langle\phi_i|\Phi(|\phi_j\rangle\langle\phi_j|)|\phi_i\rangle. \tag{12}$$

From the above expression, it is obvious that $F_f(\rho, \Phi)$ is independent of the choice of the Kraus operators K_l of the channel Φ. Furthermore, $F_f(\rho, \Phi)$ satisfies the following properties which make it a rational coherence measure.

(i) Non-negativity: $F_f(\rho, \Phi) \geqslant 0$ and $F_f(\rho, \Phi) = 0$ if and only if $[\rho, K_l] = 0$ for any l.

(ii) Unitary covariance: For any unitary operator U on the system Hilbert space, we have

$$F_f(U\rho U^\dagger, U\Phi U^\dagger) = F_f(\rho, \Phi),$$

where $U\Phi U^\dagger(\rho) = \sum_l (UK_lU^\dagger)\rho(UK_lU^\dagger)^\dagger$ for $\Phi(\rho) = \sum_l K_l\rho K_l^\dagger$.

(iii) Convexity: $F_f(\rho, \Phi)$ is convex in ρ.

(iv) Linearity: $F_f(\rho, \Phi)$ is positive-real-linear in the channel Φ in the sense that

$$F_f(\rho, c_1\Phi_1 + c_2\Phi_2) = c_1F_f(\rho, \Phi_1) + c_2F_f(\rho, \Phi_2)$$

for any channels Φ_k and constants $c_k \geqslant 0, c_1 + c_2 = 1$.

(v) Ancillary independence:

$$F_f(\rho^a\otimes\rho^b,\Phi^a\otimes\mathcal{I}^b)=F_f(\rho^a,\Phi^a),$$

where ρ^a and ρ^b are states of parties a and b, respectively, Φ^a is a channel on party a, and $\mathcal{I}^b$ is the identity channel on party b.

(vi) Decreasing under partial trace:

$$F_f(\rho^{ab},\Phi^a\otimes\mathcal{I}^b)\geqslant F_f(\rho^a,\Phi^a),$$

where ρ^{ab} is any bipartite state shared by two parties a and b.

(vii) Monotonicity: If a channel $\mathcal{E}(\rho)=\sum_j E_j\rho E_j^\dagger$ does not disturb the channel Φ in the sense that $[E_j,K_l]=0$ and $[E_j,K_l^\dagger]=0$ for any j,l, then

$$F_f(\mathcal{E}(\rho),\Phi)\leqslant F_f(\rho,\Phi).$$

(viii) Strong monotonicity: Let the channel $\mathcal{E}(\rho)=\sum_j E_j\rho E_j^\dagger$ be as in item (vii), then

$$\sum_j p_jF_f(\rho_j,\Phi)\leqslant F_f(\rho,\Phi)$$

with $p_j=\mathrm{tr}(E_j\rho E_j^\dagger)$, $\rho_j=E_j\rho E_j^\dagger/p_j$.

(ix) Superadditivity:

$$F_f(\rho^{ab},\Phi^a\otimes\mathcal{I}^b+\mathcal{I}^a\otimes\Phi^b)\geqslant F_f(\rho^a,\Phi^a)+F_f(\rho^b,\Phi^b),$$

where ρ^{ab} is a bipartite state with $\rho^a=\mathrm{tr}_b\rho^{ab}$ and $\rho^b=\mathrm{tr}_a\rho^{ab}$ the corresponding reduced states, and Φ^a and Φ^b are channels on parties a and b, respectively.

Now we proceed to the proof of the above properties.

Items (i)–(iii), (v), and (vi) can be directly verified from the corresponding properties of the generalized metric-adjusted skew information $F_f(\rho,K)$ and the coherence measure $F_f(\rho,\Phi)$ defined by Eq. (11).

Item (iv) is obvious from Eq. (12).

To establish item (vii), we first recall the concept of quasientropy. For non-negative definite operators ρ_1 and ρ_2, any operator A and function $g:\mathbb{R}^+\to\mathbb{R}$, the quasientropy is defined as [74–76]

$$S_g^A(\rho_1|\rho_2)=\langle A,m_g(L_{\rho_1},R_{\rho_2})(A)\rangle$$

with $\langle A,B\rangle=\mathrm{tr}(A^\dagger B)$ the Hilbert-Schmidt inner product between operators A and B. In particular, if g is an operator monotone function with $g(0)\geqslant 0$, then [74–76]

$$S_g^A[\alpha^\dagger(\rho_1)|\alpha^\dagger(\rho_2)]\geqslant S_g^{\alpha(A)}(\rho_1|\rho_2)$$

for any unital Schwarz mapping α [that is, $\alpha(\mathbf{1})=\mathbf{1}$ and $\alpha(B^\dagger B)\geqslant\alpha(B^\dagger)\alpha(B)$ for any operator B]. For the channel $\mathcal{E}$, its dual mapping $\mathcal{E}^\dagger$ is obviously a unital and completely positive mapping, which is actually a unital Schwarz mapping [77,78]. Thus for any Kraus operator K_l and the operator monotone function $\tilde f$ defined by (4), we have

$$S_{\tilde f}^{K_l}[\mathcal{E}(\rho)|\mathcal{E}(\rho)]\geqslant S_{\tilde f}^{\mathcal{E}^\dagger(K_l)}(\rho|\rho).$$

By the commutativity $[E_j,K_l]=0$ and $[E_j,K_l^\dagger]=0$ for any j,l, we further have

$$\mathrm{tr}[\mathcal{E}(\rho)K_lK_l^\dagger]=\mathrm{tr}(\rho K_lK_l^\dagger)$$

and

$$\mathcal{E}^\dagger(K_l)=K_l$$

for any l. Thus

$$S_{\tilde f}^{K_l}[\mathcal{E}(\rho)|\mathcal{E}(\rho)]\geqslant S_{\tilde f}^{K_l}(\rho|\rho).$$

Then the desired monotonicity follows from

$$\begin{aligned}F_f(\mathcal{E}(\rho),\Phi)&=\frac{1}{2}\left\{1+\sum_l\mathrm{tr}[\mathcal{E}(\rho)K_lK_l^\dagger]\right\}-\sum_l S_{\tilde f}^{K_l}[\mathcal{E}(\rho)|\mathcal{E}(\rho)]\\&\leqslant\frac{1}{2}\left\{1+\sum_l\mathrm{tr}(\rho K_lK_l^\dagger)\right\}-\sum_l S_{\tilde f}^{K_l}(\rho|\rho)\\&=F_f(\rho,\Phi).\end{aligned}$$

Next, we prove item (viii). Let R be an auxiliary system with Hilbert space H_R and let $\{|j\rangle: j=0,1,\cdots,m-1\}$ be an orthonormal basis of H_R. Here m is the number of the Kraus operators for the channel $\mathcal{E}$. Consider the channel

$$\Lambda(\tau)=\sum_{j=0}^{m-1}\mathcal{V}_j\otimes\mathcal{E}_j(\tau)\tag{13}$$

on the composite system space $H_R\otimes H$. Here

$$\mathcal{V}_j(\xi)=V_j\xi V_j^\dagger,\quad V_j=\sum_{k=0}^{m-1}|k+j\rangle\langle k|,$$
$$\mathcal{E}_j(\sigma)=E_j\sigma E_j^\dagger,$$

for $j=0,1,\cdots,m-1$, and the sum $k+j$ is understood as mod m. It is easy to show that

$$[V_j\otimes E_j,\mathbf{1}\otimes K_l]=V_j\otimes[E_j,K_l]=0$$

and

$$[V_j\otimes E_j,\mathbf{1}\otimes K_l^\dagger]=V_j\otimes[E_j,K_l^\dagger]=0$$

for any j,l. Thus Λ commutes with $\mathcal{I}\otimes\Phi$. By items (v) and (vii), we have

$$\begin{aligned}F_f(\rho,\Phi)&=F_f(|0\rangle\langle 0|\otimes\rho,\mathcal{I}\otimes\Phi)\\&\geqslant F_f(\Lambda(|0\rangle\langle 0|\otimes\rho),\mathcal{I}\otimes\Phi)\\&=F_f\left(\sum_{j=0}^{m-1}p_j|j\rangle\langle j|\otimes\rho_j,\mathcal{I}\otimes\Phi\right)\\&=\sum_{j=0}^{m-1}p_jF_f(|j\rangle\langle j|\otimes\rho_j,\mathcal{I}\otimes\Phi)\\&=\sum_{j=0}^{m-1}p_jF_f(\rho_j,\Phi),\end{aligned}$$

where the inequality follows from the monotonicity of $F_f(\tau,\mathcal{I}\otimes\Phi)$ under the channel Λ. We remark here that the monotonicity of $F_f(\rho,\Phi)$ can also be obtained directly from the strong monotonicity and the convexity of $F_f(\rho,\Phi)$.

Item (ix) follows from items (iv) and (vi).

We further specialize to two important cases.

(a) For $f(x)$ defined by Eq. (5), we have

$$F_{\mathrm{WY}}(\rho,\Phi)=F_f(\rho,\Phi)=\sum_l I(\rho,K_l),$$

which is precisely the coherence measure based on the Wigner-Yanase skew information [16].

(b) For $f(x)$ defined by Eq. (6), we have

$$F_{\rm SLD}(\rho,\Phi)=F_f(\rho,\Phi)=\sum_l F(\rho,K_l),$$

where $F(\rho,K)$ is the quantum Fisher information of ρ based on the symmetric logarithmic derivative. We notice that this coherence measure is different from that introduced in Ref. [29], which is defined as the sum of the quantum Fisher information relative to the corresponding measurement operators $K_l^\dagger K_l$, i.e., $\sum_l F(\rho,K_l^\dagger K_l)$.

It is interesting to note that the above two coherence measures are related as

$$F_{\rm WY}(\rho,\Phi)\leqslant F_{\rm SLD}(\rho,\Phi)\leqslant 2F_{\rm WY}(\rho,\Phi).$$

This follows directly from inequality relation (8) and the definitions of $F_{\rm WY}(\rho,\Phi)$ and $F_{\rm SLD}(\rho,\Phi)$.

Now we consider a special channel induced by a compact Lie group which provides an interpretation of coherence as asymmetry. For a compact Lie group G, let $\{U_g:g\in G\}$ be a unitary representation of G, then

$$T_G(\rho)=\int_G U_g\rho U_g^\dagger dg$$

is the twirling channel induced by the group G with dg the normalized Haar measure over the compact Lie group G. The coherence of ρ relative to the channel T_G is

$$F_f(\rho,T_G)=\frac{f(0)}{2}\sum_{ij}\frac{(\lambda_i-\lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)}\int_G|\langle\phi_i|U_g|\phi_j\rangle|^2dg$$

with $\rho=\sum_i\lambda_i|\phi_i\rangle\langle\phi_i|$ the spectral decomposition of ρ. Since $F_f(\rho,T_G)=0$ if and only if $[\rho,U_g]=0$ for any $g\in G$, $F_f(\rho,T_G)$ also quantifies the asymmetry of ρ relative to the group G in some sense. To illustrate this idea more clearly, we analyze two special unitary groups.

(i) For the full unitary group $U(H)$ on H, the twirling channel $T_{U(H)}$ is

$$T_{U(H)}(\rho)=\int_{U(H)}U\rho U^\dagger dU=\frac{1}{d}\mathbf{1},$$

which coincides with the completely depolarizing channel

$$\Phi_{\rm CDe}(\rho)=\frac{1}{d}\sum_{l=1}^{d^2}X_l\rho X_l=\frac{1}{d}\mathbf{1}$$

with $\{X_l:l=1,2,\cdots,d^2\}$ an orthonormal basis of the Hilbert space of all Hermitian operators on H and d the dimension of H. Thus the asymmetry of ρ relative to the full unitary group $U(H)$ may be quantified by the coherence of ρ relative to the completely depolarizing channel $\Phi_{\rm CDe}$ as

$$F_f(\rho,T_{U(H)})=F_f(\rho,\Phi_{\rm CDe})=\frac{f(0)}{2d}\sum_{ij}\frac{(\lambda_i-\lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)}.$$

(ii) For the block-diagonal unitary group

$$U_{\rm diag}=\left\{U_\theta=\sum_{l=1}^n e^{i\theta_l}\Pi_l:\theta=(\theta_1,\cdots,\theta_n)\in[0,2\pi)^n\right\}$$

with $\Pi=\{\Pi_l:l=1,2,\cdots,n\}$ the Lüders (projective) measurement, the twirling channel generated by the group $U_{\rm diag}$ is

$$T_{U_{\rm diag}}(\rho)=\int_0^{2\pi}\cdots\int_0^{2\pi}U_\theta\rho U_\theta^\dagger d\theta$$

with $d\theta=d\theta_1d\theta_2\cdots d\theta_n$, which coincides with the decohering channel

$$\Pi(\rho)=\sum_{l=1}^n\Pi_l\rho\Pi_l$$

induced by the Lüders measurement $\Pi=\{\Pi_l:l=1,2,\cdots,n\}$. Therefore the asymmetry of ρ relative to the group $U_{\rm diag}$ can be quantified by the coherence of ρ relative to the decohering channel Π as

$$\begin{aligned}F_f(\rho,T_{U_{\rm diag}})&=F_f(\rho,\Pi)\\&=\frac{f(0)}{2}\sum_{ijl}\frac{(\lambda_i-\lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)}|\langle\phi_i|\Pi_l|\phi_j\rangle|^2.\end{aligned}$$

III. EXAMPLES

In this section, we illustrate the coherence measure $F_f(\rho,\Phi)$ through several prototypical examples, which reveal some quantitative features of the channels from the perspective of coherence.

In the computational basis $\{|0\rangle,|1\rangle\}$ of a qubit system, any qubit state can be represented as

$$\rho=\frac{1}{2}\left(\mathbf{1}+\sum_{i=1}^3 r_i\sigma_i\right)=\frac{1}{2}\begin{pmatrix}1+r_3 & r_1-ir_2\\ r_1+ir_2 & 1-r_3\end{pmatrix},\quad(14)$$

where $r_i\in\mathbb{R}$, $r=\sqrt{r_1^2+r_2^2+r_3^2}\leqslant 1$, and σ_i are the Pauli matrices. The spectral decomposition of ρ reads

$$\rho=\lambda_1|\phi_1\rangle\langle\phi_1|+\lambda_2|\phi_2\rangle\langle\phi_2|$$

with the eigenvalues

$$\lambda_1=\tfrac{1}{2}(1+r),\quad \lambda_2=\tfrac{1}{2}(1-r)$$

and the corresponding eigenvectors

$$|\phi_1\rangle=\frac{(r_1-ir_2)|0\rangle-(r_3-r)|1\rangle}{\sqrt{2r(r-r_3)}},$$

$$|\phi_2\rangle=\frac{(r_1-ir_2)|0\rangle-(r_3+r)|1\rangle}{\sqrt{2r(r+r_3)}}.$$

From this it can be directly evaluated that

$$F_f(\rho,K)=\frac{r^2f(0)(|\langle\phi_1|K|\phi_2\rangle|^2+|\langle\phi_2|K|\phi_1\rangle|^2)}{(1+r)f\left(\frac{1-r}{1+r}\right)}$$

for any operator K on the qubit system. From the above expression, we see that the coherence measures associated with different operator monotone functions f are proportional to each other for any fixed state. Thus they have similar behaviors. Of course, this phenomenon is special to the qubit system.

To further illustrate the coherence measures based on metric-adjusted skew information, we evaluate coherence of a general state relative to several prototypical qubit channels.

In the following examples, ρ is a general qubit state defined by Eq. (14) with $r=\sqrt{r_1^2+r_2^2+r_3^2}$.

Example 0. For the unitary channel $\mathcal{U}(\rho)=U\rho U^\dagger$ with the unique Kraus operator

$$U=e^{i\alpha}\begin{pmatrix} u & -v \\ \bar{v} & \bar{u}\end{pmatrix},$$

where $\alpha\in[0,2\pi)$ and $u,v\in\mathbb{C}$ satisfying $|u|^2+|v|^2=1$, we have

$$F_f(\rho,\mathcal{U})=\frac{2f(0)\left[x^2+2r_3yz+z^2\left(r^2-r_3^2\right)+r_3^2|v|^2\right]}{(1+r)f\left(\frac{1-r}{1+r}\right)},$$

where $x=\mathrm{Re}[(r_1+ir_2)v]$, $y=\mathrm{Im}[(r_1+ir_2)v]$, and $z=\mathrm{Im}(u)$ with Re and Im denoting the real and imaginary parts of a complex number, respectively. In particular, let $\mathcal{U}_i(\rho)=\sigma_i\rho\sigma_i^\dagger$ with σ_i the Pauli matrices, then

$$F_f(\rho,\mathcal{U}_i)=\frac{2f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}\left(r^2-r_i^2\right),\quad i=1,2,3.$$

Clearly, due to the convexity in ρ, the maximal value $\max_\rho F_f(\rho,\mathcal{U}_i)$ is achieved by pure states (i.e., $r=1$), and $\max_\rho F_f(\rho,\mathcal{U}_i)=1$. Any pure state with $r_i=0$ achieves this maximal coherence for the unitary channel $\mathcal{U}_i$.

For the bit flip channel

$$\Phi_{\mathrm{BF}}(\rho)=p\rho+(1-p)\mathcal{U}_1(\rho)=\sum_{l=1}^{2}K_l\rho K_l^\dagger$$

with the Kraus operators $K_1=\sqrt{p}\mathbf{1}$ and $K_2=\sqrt{1-p}\sigma_1$ and $0\leqslant p\leqslant 1$, we have

$$F_f(\rho,\Phi_{\mathrm{BF}})=(1-p)F_f(\rho,\mathcal{U}_1),$$

and $\max_\rho F_f(\rho,\Phi_{\mathrm{BF}})=1-p$. Any pure state ρ with $r_1=0$ achieves this maximal coherence.

For the phase flip channel

$$\Phi_{\mathrm{PF}}(\rho)=p\rho+(1-p)\mathcal{U}_3(\rho)=\sum_{l=1}^{2}K_l\rho K_l^\dagger$$

with the Kraus operators $K_1=\sqrt{p}\mathbf{1}$ and $K_2=\sqrt{1-p}\sigma_3$, we have

$$F_f(\rho,\Phi_{\mathrm{PF}})=(1-p)F_f(\rho,\mathcal{U}_3),$$

and $\max_\rho F_f(\rho,\Phi_{\mathrm{PF}})=1-p$. Any pure state ρ with $r_3=0$ achieves this maximal coherence.

For the bit-phase flip channel

$$\Phi_{\mathrm{BPF}}(\rho)=p\rho+(1-p)\mathcal{U}_2(\rho)=\sum_{l=1}^{2}K_l\rho K_l^\dagger$$

with the Kraus operators $K_1=\sqrt{p}\mathbf{1}$ and $K_2=\sqrt{1-p}\sigma_2$, we have

$$F_f(\rho,\Phi_{\mathrm{BPF}})=(1-p)F_f(\rho,\mathcal{U}_2),$$

and $\max_\rho F_f(\rho,\Phi_{\mathrm{PF}})=1-p$. Any pure state ρ with $r_2=0$ achieves this maximal coherence.

Example 1. For the amplitude damping (spontaneous emission) channel $\Phi_{\mathrm{AD}}(\rho)=\sum_{l=1}^{2}K_l\rho K_l^\dagger$ with the Kraus operators

$$K_1=\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix},\quad K_2=\begin{pmatrix}0 & \sqrt{p}\\ 0 & 0\end{pmatrix},\quad 0\leqslant p\leqslant 1,$$

let $q=1-\sqrt{1-p}$, then we have

$$F_f(\rho,\Phi_{\mathrm{AD}})=\frac{f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}\left[\left(r^2-r_3^2\right)q+r_3^2p\right],$$

and $\max_\rho F_f(\rho,\Phi_{\mathrm{AD}})=p/2$. The pure states with $r_3=\pm1$ (i.e., $|0\rangle,|1\rangle$) achieve the maximal coherence.

Example 2. For the phase damping channel $\Phi_{\mathrm{PD}}(\rho)=\sum_{l=1}^{2}K_l\rho K_l^\dagger$ with

$$K_1=\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix},\quad K_2=\begin{pmatrix}0 & 0\\ 0 & \sqrt{p}\end{pmatrix},\quad 0\leqslant p\leqslant 1,$$

we have

$$F_f(\rho,\Phi_{\mathrm{PD}})=\frac{f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}\left(r^2-r_3^2\right)q. \tag{15}$$

and $\max_\rho F_f(\rho,\Phi_{\mathrm{AD}})=q/2$. The pure states with $r_3=0$ achieve the maximal coherence.

It is interesting to compare the above two damping channels. We see that $F_f(\rho,\Phi_{\mathrm{AD}})\geqslant F_f(\rho,\Phi_{\mathrm{PD}})$, and the states with the maximal coherence are quite different for the two damping channels Φ_{AD} and Φ_{PD}.

Example 3. For the depolarizing channel

$$\Phi_{\mathrm{De}}(\rho)=(1-3p)\rho+p\sum_{i=1}^{3}\sigma_i\rho\sigma_i,\quad 0\leqslant p\leqslant 1/3$$

with σ_i the Pauli matrices, we have

$$F_f(\rho,\Phi_{\mathrm{De}})=\frac{4r^2f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}p, \tag{16}$$

which is an increasing function of p and is in line with our intuitive understanding. Moreover, $\max_\rho F_f(\rho,\Phi_{\mathrm{De}})=2p$, and this maximal coherence is achieved by any pure state.

Example 4. For fixed $x\in[0,1/2]$, consider the channel

$$\Phi_{\mathrm{W}}(\rho)=K_x\rho K_x^\dagger+K_{1-x}\rho K_{1-x}^\dagger$$

induced by the qubit weak measurement with $K_x=\sqrt{1-x}|0\rangle\langle0|+\sqrt{x}|1\rangle\langle1|$. Here $\{|0\rangle\langle0|,|1\rangle\langle1|\}$ is a von Neumann measurement induced by the computational basis $\{|0\rangle,|1\rangle\}$. The coherence of ρ relative to Φ_{W} can be directly evaluated as

$$F_f(\rho,\Phi_{\mathrm{W}})=\frac{f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}\left(r^2-r_3^2\right)\left[1-2\sqrt{x(1-x)}\right].$$

It is a decreasing function of $x\in[0,1/2]$, which is consistent with our intuition, that is, the coherence of ρ relative to the von Neumann measurement $\{|0\rangle\langle0|,|1\rangle\langle1|\}$ with $x=0$ is maximal and that relative to the identity channel with $x=1/2$ is minimal. Moreover, $\max_\rho F_f(\rho,\Phi_{\mathrm{W}})=[1-2\sqrt{x(1-x)}]/2$. Any pure state with $r_3=0$ achieves this maximal coherence.

Example 5. Recall that the Hadamard channel (completely decoherent channel) in a qubit system is defined as [79,80]

$$\Phi_{\rm H}(\rho) = M \circ \rho,$$

where M is a non-negative definite matrix with all diagonal elements being 1, and $\circ$ denotes the Hadamard (entrywise) product of matrices. In general,

$$M = \begin{pmatrix} 1 & \alpha^* \\ \alpha & 1 \end{pmatrix}, \quad |\alpha| \leqslant 1.$$

It is easy to verify that the channel can be equivalently expressed as $\Phi_{\rm H}(\rho) = \sum_{l=1}^{3} K_l \rho K_l^\dagger$ with the Kraus operators

$$K_1 = \begin{pmatrix} \sqrt{1-|\alpha|} & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-|\alpha|} \end{pmatrix},$$
$$K_3 = \sqrt{|\alpha|}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

where $\alpha = |\alpha|e^{i\theta}$, $\theta \in [0, 2\pi)$. By direct calculations, we have

$$F_f(\rho, \Phi_{\rm H}) = \frac{f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}\left(r^2 - r_3^2\right)(1 - |\alpha|\cos\theta),$$

and $\max_\rho F_f(\rho, \Phi_{\rm H}) = (1 - |\alpha|\cos\theta)/2$. Any state with $r_3 = 0$ achieves this maximal coherence.

Example 6. For later comparison, consider the distorted trine channel [24]

$$\Phi_{\rm T}(\rho) = \sum_{l=1}^{3} K_l \rho K_l^\dagger$$

with the Kraus operators $K_l = \sqrt{2\alpha_l}|\psi_l\rangle\langle\psi_l|$ and

$$\alpha_1 = t, \quad \alpha_2 = \alpha_3 = \frac{1}{2}(1-t), \quad t \in [0, 1/3],$$
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{i\theta}|1\rangle),$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{-i\theta}|1\rangle), \quad \cos\theta = \frac{t}{1-t}.$$

We have

$$F_f(\rho, \Phi_{\rm T}) = \frac{f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}\left(r^2 - \frac{t}{1-t}r_1^2 - \frac{1-2t}{1-t}r_2^2\right),$$

and $\max_\rho F_f(\rho, \Phi_{\rm T}) = 1/2$. The pure states with $r_3 = \pm 1$ (i.e., $|0\rangle, |1\rangle$) achieve the maximal coherence. In particular, when $t = 0$, $\Phi_{\rm T}$ reduces to the unitary channel $\mathcal{U}_2$ in example 0, and when $t = 1/3$, $\Phi_{\rm T}$ reduces to the trine channel $\Phi_{\rm Trine}(\rho) = \sum_{l=1}^{3} K_l \rho K_l^\dagger$ with $K_l = \sqrt{\frac{2}{3}}|\psi_l\rangle\langle\psi_l|$ and $|\psi_l\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \omega^{l-1}|1\rangle)$, $\omega = e^{i4\pi/3}$. We have

$$F_f(\rho, \Phi_{\rm Trine}) = \frac{f(0)}{2(1+r)f\left(\frac{1-r}{1+r}\right)}\left(r^2 + r_3^2\right),$$

and $\max_\rho F_f(\rho, \Phi_{\rm Trine}) = 1/2$.

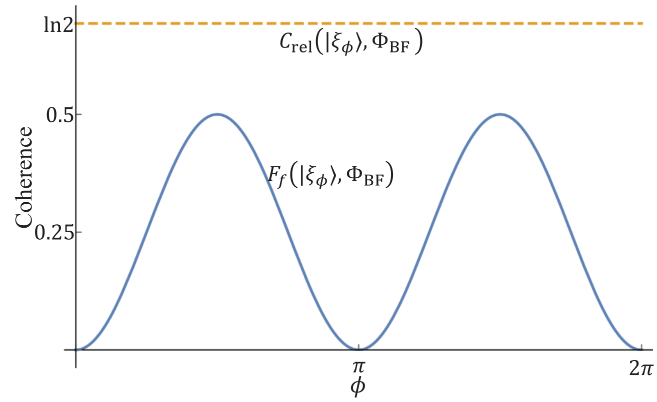

FIG. 1. Comparison between the two coherence measures $F_f(|\xi_\phi\rangle, \Phi_{\rm BF})$ and $C_{\rm rel}(|\xi_\phi\rangle, \Phi_{\rm BF})$ as functions of $\phi \in [0, 2\pi)$ for the bit flip channel $\Phi_{\rm BF}$ with $p = 1/2$ (example 0). Here $|\xi_\phi\rangle = (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$. We see that they display quite different behaviors in this special case: $F_f(|\xi_\phi\rangle, \Phi_{\rm BF})$ is oscillating in the parameter ϕ, while $C_{\rm rel}(|\xi_\phi\rangle, \Phi_{\rm BF}) = \ln 2$ is a constant independent of ϕ.

IV. COMPARISON

Several coherence measures of a state relative to a POVM were introduced in the literature [23–26], and it is desirable to compare our coherence measures with them. Recall that the quantity

$$C_{\rm rel}(\rho, M) = H(\{p_l\}) + \sum_l p_l S(\rho_l) - S(\rho) \tag{17}$$

was introduced in Refs. [23,24], which is defined as a coherence measure of a state ρ relative to a POVM $M = \{M_l : l = 1, 2, \cdots, m\}$. Here $p_l = {\rm tr}(M_l\rho)$, $\rho_l = \sqrt{M_l}\rho\sqrt{M_l}/p_l$, $H(\{p_l\}) = -\sum_l p_l \ln p_l$ is the Shannon entropy, while $S(\rho) = -{\rm tr}\rho\ln\rho$ is the von Neumann entropy. The logarithm refers to the natural base throughout this paper. This measure is based on relative entropy and has the operational meaning of entropy production caused by M. In contrast, our approach is based on metric-adjusted skew information rather than relative entropy and yields a whole family of coherence measures, which have the origin in quantum metrology due to the involvement of quantum Fisher information.

It is straightforward to generalize the quantity defined by Eq. (17) to a channel Φ as

$$C_{\rm rel}(\rho, \Phi) = H(\{p_l\}) + \sum_l p_l S(\rho_l) - S(\rho),$$

where $p_l = {\rm tr}(K_l\rho K_l^\dagger)$, $\rho_l = K_l\rho K_l^\dagger/p_l$, and $\Phi(\rho) = \sum_l K_l\rho K_l^\dagger$. In particular, when ρ is any pure state, then since ρ_l will also be pure the above coherence measure is reduced to $C_{\rm rel}(\rho, \Phi) = H(\{p_l\})$, which is simply the Shannon entropy of the probability distribution of the measurement outcomes.

In order to facilitate a comparison between the above coherence measure $C_{\rm rel}(\rho, \Phi)$ and our coherence measure $F_f(\rho, \Phi)$, we depict the graphs of coherence of parametrized pure states relative to the bit flip channel and amplitude damping channel in Figs. 1 and 2, respectively. We see that although $F_f(\rho, \Phi)$ and $C_{\rm rel}(\rho, \Phi)$ share many similar features, they also display remarkable differences. In general, they yield different orderings of coherence, as illustrated in Figs. 1 and 2.

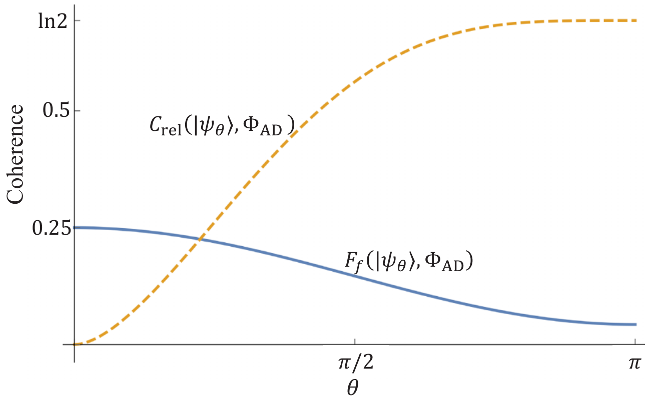

FIG. 2. Comparison between the two coherence measures $F_f(|\psi_\theta\rangle, \Phi_{\rm AD})$ and $C_{\rm rel}(|\psi_\theta\rangle, \Phi_{\rm AD})$ as functions of $\theta \in [0, \pi]$ for the amplitude damping channel $\Phi_{\rm AD}$ with $p = 1/2$ (example 1). Here $|\psi_\theta\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$. We see that they display radically different behaviors in this special case: $F_f(|\psi_\theta\rangle, \Phi_{\rm AD})$ is decreasing in the parameter $\theta \in [0, \pi]$, while $C_{\rm rel}(|\psi_\theta\rangle, \Phi_{\rm AD})$ is increasing in $\theta \in [0, \pi]$. Consequently, they yield different orderings of coherence of states relative to the same amplitude damping channel.

For the purpose of more detailed comparison, we evaluate $C_{\rm rel}(\rho, \Phi)$ for pure states and various channels studied in Sec. III, and list the values as well as those of our coherence measure in Table I. We further list the maximal values of coherence and the associated pure states achieving the maximal values in Table II. We see that in many cases $F_f(\rho, \Phi)$ is more sensitive than $C_{\rm rel}(\rho, \Phi)$. In particular, for the weak measurement channel $\Phi_{\rm W}$ (example 4), $F_f(\rho, \Phi_{\rm W})$ is more sensitive than $C_{\rm rel}(\rho, \Phi_{\rm W})$ in the sense that $\max_\rho F_f(\rho, \Phi_{\rm W})$ depends on the parameter (measurement strength) x, while $\max_\rho C_{\rm rel}(\rho, \Phi_{\rm W})$ is independent of x. Similarly, for the Hadamard channel $\Phi_{\rm H}$ (example 5), $F_f(\rho, \Phi_{\rm H})$ is more sensitive than $C_{\rm rel}(\rho, \Phi_{\rm H})$ in the sense that $\max_\rho F_f(\rho, \Phi_{\rm H})$ depends on the phase angle θ, while $\max_\rho C_{\rm rel}(\rho, \Phi_{\rm H})$ is independent of θ.

All the above comparisons are for pure states. It is desirable to also consider mixed states for comparison since the function f in metric-adjusted skew information plays a significant role only when mixed states are involved. For this purpose, we consider parametrized mixed states

$$\rho_\lambda = \lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|, \quad \lambda \in (0, 1)$$

and the following choices of the function f:

$$f_1(x) = \frac{1+x}{2},$$
$$f_2(x) = \left(\frac{1+\sqrt{x}}{2}\right)^2,$$
$$f_3(x) = \frac{3(x-1)^2}{16(x^{1/4}-1)(x^{3/4}-1)},$$

which are all operator monotone functions satisfying $f(0) > 0$ and $xf(1/x) = f(x)$, and thus can be employed to define metric-adjusted skew information $F_{f_i}(\rho, \Phi)$. We specify the channel Φ to the amplitude channel $\Phi_{\rm AD}$ in example 1 with $p = 1/2$. In this case, the various coherence measures can be evaluated as

$$C_{\rm rel}(\rho_\lambda, \Phi_{\rm AD}) = (1-\lambda)\ln 2,$$
$$F_{f_1}(\rho_\lambda, \Phi_{\rm AD}) = \frac{(2\lambda-1)^2}{4},$$
$$F_{f_2}(\rho_\lambda, \Phi_{\rm AD}) = \frac{(\sqrt{\lambda}-\sqrt{1-\lambda})^2}{4},$$
$$F_{f_3}(\rho_\lambda, \Phi_{\rm AD}) = \frac{(\lambda^{1/4}-(1-\lambda)^{1/4})(\lambda^{3/4}-(1-\lambda)^{3/4})}{4}.$$

To gain an intuitive understanding of the comparison, we depict the graphs of the above coherence measures in Fig. 3. We see that while our coherence measures yield qualitatively similar behaviors with respect to the parameter λ (decreasing and then increasing), the relative entropy of coherence $C_{\rm rel}(\rho_\lambda, \Phi)$ is a monotonically decreasing function of λ. This further highlights some differences between our coherence measures based on metric-adjusted skew information and the coherence measure $C_{\rm rel}(\rho, \Phi)$ in the literature.

In Refs. [25,26], several coherence measures involving optimization were introduced. Since it is usually difficult to perform the optimization, these quantities are in general difficult to calculate. Of course, each coherence measure may have its own merits and usage in different contexts, and they

TABLE I. Comparison between the two coherence measures $C_{\rm rel}(\rho, \Phi)$ and $F_f(\rho, \Phi)$ of pure states ρ relative to various channels $\Phi(\rho) = \sum_l K_l \rho K_l^\dagger$ in Sec. III. Here the pure states $\rho = \frac{1}{2}(\mathbf{1} + \sum_{i=1}^3 r_i\sigma_i)$ are completely determined by the Bloch parameters r_i with $r = \sqrt{r_1^2 + r_2^2 + r_3^2} = 1$. The measurement induced by the channel Φ yields the outcome probability $p_l = {\rm tr}(K_l\rho K_l^\dagger)$, $H(\{p_i\}) = -\sum_i p_i \ln p_i$ is the Shannon entropy, and $q = 1 - \sqrt{1-p}$.

Channels Φ	Coherence $F_f(\rho, \Phi)$	Coherence $C_{\rm rel}(\rho, \Phi)$
$\Phi_{\rm BF}$	$(1-p)(1-r_1^2)$	
$\Phi_{\rm PF}$	$(1-p)(1-r_3^2)$	$H(\{p, 1-p\})$
$\Phi_{\rm BPF}$	$(1-p)(1-r_2^2)$	
$\Phi_{\rm AD}$	$(q+(p-q)r_3^2)/2$	
$\Phi_{\rm PD}$	$(1-r_3^2)q/2$	$H(\{p(1-r_3)/2, 1-p(1-r_3)/2\})$
$\Phi_{\rm De}$	$2p$	$H(\{p, p, p, 1-3p\})$
$\Phi_{\rm W}$	$(1-r_3^2)[1-2\sqrt{x(1-x)}]/2$	$H(\{[1+(1-2x)r_3]/2, [1-(1-2x)r_3]/2\})$
$\Phi_{\rm H}$	$(1-r_3^2)(1-\lvert\alpha\rvert\cos\theta)/2$	$H(\{\lvert\alpha\rvert, (1+r_3)(1-\lvert\alpha\rvert)/2, (1-r_3)(1-\lvert\alpha\rvert)/2\})$
$\Phi_{\rm T}$	$[1-t(1+r_1^2)-(1-2t)r_2^2]/(2-2t)$	$H(\{t(1+r_1), [1-t(1+r_1) \pm \sqrt{1-2t}r_2]/2\})$

TABLE II. Comparison between the maximal coherence as quantified by $\max_\rho F_f(\rho, \Phi)$ and $\max_\rho C_{\rm rel}(\rho, \Phi)$ for prototypical channels. The maximum is taken over all pure states $\rho = \frac{1}{2}(\mathbf{1} + \sum_{i=1}^3 r_i\sigma_i)$, which are completely determined by the Bloch parameters r_i with $r = \sqrt{r_1^2 + r_2^2 + r_3^2} = 1$. Note that, due to the convexity, the maximum of both the coherence measures $F_f(\rho, \Phi)$ and $C_{\rm rel}(\rho, \Phi)$ can be achieved by pure states. $H(\{p_i\}) = -\sum_i p_i \ln p_i$ is the Shannon entropy, and $q = 1 - \sqrt{1-p}$.

Channels Φ	$\max_\rho F_f(\rho, \Phi)$	$\max_\rho C_{\rm rel}(\rho, \Phi)$	$\mathrm{argmax}_\rho F_f(\rho, \Phi)$	$\mathrm{argmax}_\rho C_{\rm rel}(\rho, \Phi)$
$\Phi_{\rm BF}$			$r_1 = 0$	
$\Phi_{\rm PF}$	$1 - p$	$H(\{p, 1-p\})$	$r_3 = 0$	Any pure state
$\Phi_{\rm BPF}$			$r_2 = 0$	
$\Phi_{\rm AD},\ p < 1/2$	$p/2$	$H(\{p, 1-p\})$	$r_3 = \pm 1$	$r_3 = -1$
$\Phi_{\rm AD},\ p \geqslant 1/2$		$\ln 2$		$r_3 = 1 - 1/p$
$\Phi_{\rm PD},\ p < 1/2$	$q/2$	$H(\{p, 1-p\})$	$r_3 = 0$	$r_3 = -1$
$\Phi_{\rm PD},\ p \geqslant 1/2$		$\ln 2$		$r_3 = 1 - 1/p$
$\Phi_{\rm De}$	$2p$	$H(\{p, p, p, 1-3p\})$	Any pure state	Any pure state
$\Phi_{\rm W}$	$1/2 - \sqrt{x(1-x)}$	$\ln 2$	$r_3 = 0$	$r_3 = 0$
$\Phi_{\rm H}$	$(1 - \lvert\alpha\rvert\cos\theta)/2$	$H(\{\lvert\alpha\rvert, (1-\lvert\alpha\rvert)/2, (1-\lvert\alpha\rvert)/2\})$		
$\Phi_{\rm T},\ 0 \leqslant t < 1/6$	$1/2$	$H(\{2t, (1-2t)/2, (1-2t)/2\})$	$r_3 = \pm 1$	$r_1 = 1$
$\Phi_{\rm T},\ 1/6 \leqslant t \leqslant 1/3$		$\ln 3$		$r_1 = 1/(3t) - 1, r_2 = 0$

complement each other in providing a more complete picture of coherence.

V. SUMMARY

We have introduced a family of coherence measures of a state relative to a channel via metric-adjusted skew information, and have revealed their basic properties. This generalizes the coherence of a state relative to an orthonormal basis (von Neumann measurement) to a general channel beyond POVMs. For a compact Lie group, we have shown that the coherence of a state relative to the twirling channel associated with the group can be used to characterize the asymmetry of this state relative to the group. To illustrate these coherence measures, we have evaluated them for several qubit channels and have found that the effects of a qubit channel for the coherence of a state exhibit qualitatively similar behaviors for all operator monotone functions. This is a special effect for qubit systems.

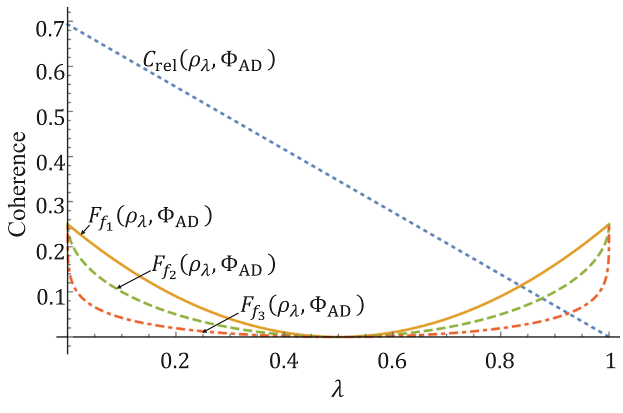

FIG. 3. Comparison between the coherence measures $F_{f_i}(\rho_\lambda, \Phi_{\rm AD})$ and $C_{\rm rel}(\rho_\lambda, \Phi_{\rm AD})$, as functions of the mixing parameter $\lambda \in (0, 1)$ for the amplitude damping channel $\Phi_{\rm AD}$ with $p = 1/2$ (example 1). Here $\rho_\lambda = \lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|$. We see that $F_{f_i}(\rho_\lambda, \Phi_{\rm AD})$ share similar properties for $i = 1, 2, 3$ (all are special instances of coherence measures defined via metric-adjusted skew information). In sharp contrast, $F_{f_i}(\rho_\lambda, \Phi_{\rm AD})$ and $C_{\rm rel}(\rho_\lambda, \Phi_{\rm AD})$ are radically different in the sense that they display different monotonicity: $F_{f_i}(\rho_\lambda, \Phi_{\rm AD})$ is decreasing in $\lambda \in (0, 1/2]$ and increasing in $\lambda \in [1/2, 1)$, but $C_{\rm rel}(\rho_\lambda, \Phi_{\rm AD})$ is monotonically decreasing in $\lambda \in (0, 1)$. $F_{f_i}(\rho_\lambda, \Phi_{\rm AD})$ and $C_{\rm rel}(\rho_\lambda, \Phi_{\rm AD})$ yield different orderings of coherence.

We have made a rather detailed comparison between our coherence measures and the relative entropy of coherence in the literature. We have illuminated some remarkable difference between them. In general, they yield quite different orderings of coherence. They capture coherence from different angles and have their own advantages in different contexts.

One may further consider coherence via more general information quantities and related issues. What is the advantage of metric-adjusted skew information over other quantities related to quantum Fisher information? General quantum Fisher information is characterized by monotonicity under coarse graining (channels), as such, it is rather abstract in general. The metric-adjusted skew information has the advantage that it is a sufficiently large family including many important information quantities such as the Wigner-Yanase-Dyson skew information and the most celebrated quantum Fisher information involving the symmetric logarithmic derivative, and in the meantime has an explicit form for manipulation and calculation. However, other coherence measures relative to channels via generalized quantum Fisher information and various entropies such as the relative Tsallis entropy are certainly worth investigating. A more detailed comparative study of different coherence measures is also desirable.

In view of the information-theoretic meaning of the coherence measures via the metric-adjusted skew information, it is desirable to further investigate their theoretical implications and experimental applications. One may try to apply these quantifiers to studying correlations, quantum interference, quantum metrology, and quantum chaos, which are left for further investigations. We hope these coherence measures based on the metric-adjusted skew information may shed light on the structure of quantum coherence in particular and quantum information processing in general.

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