Quantum steering as resource of quantum teleportation

Yi Fan, Chuanlei Jia, and Liang Qiu*
School of Materials Science and Physics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China

(Received 3 March 2022; accepted 13 July 2022; published 25 July 2022)

Quantum steering, which lies between Bell nonlocality and entanglement, is a characteristic quantum correlation. Its application in quantum information tasks is worth pursuing. In this paper, we investigate the application of quantum steering as resource of quantum teleportation. It is found that two-qubit states violating the three-setting linear steering inequality are always useful for teleportation. Some steerable states which obey the Bell–Clauser-Horne-Shimony-Holt (CHSH) inequality could be used for teleportation. Two-qubit states that are optimal for quantum teleportation for a fixed amount of steering are present. The optimal states achieve the maximal average fidelity and also exhibit zero fidelity deviation.

DOI: 10.1103/PhysRevA.106.012433

I. INTRODUCTION

Quantum teleportation [1] is a fundamental protocol to transmit quantum information from one object to another object using shared entanglement, where the spatially separated sender and receiver are only allowed to perform local operations and communicate among themselves via a classical channel [2]. In the primitive teleportation scheme, a class of maximally entangled states is required. In reality, however, the available states are typically mixed entangled, and teleportation will not be perfect due to the interactions between the transmitted qubit and the environment during sharing of quantum entanglement or imperfection of preparation. The noisy state is of little significance for information tasks, and moreover, it will not provide any better transmission fidelity than that of an ordinary classical communication channel if the noisy state is mixed too much [3,4]. The average fidelity, which is a measure of the expected closeness between the input and output states, is the standard figure of merit for quantum teleportation [5–9]. The average fidelity not only could answer the question whether the entangled states can offer nonclassical fidelity within the standard protocol supplemented by local unitary operations, but also can be used to indicate how a given entangled state is useful for teleportation. Nevertheless, the average fidelity cannot give information on fidelity fluctuations. The standard deviation of fidelity over all input states is introduced to appropriately quantify such fluctuations and is named as fidelity deviation [10,11].

Quantum steering, as a notion introduced by Schrödinger in 1935 [12,13], is formalized from the perspective of quantum information theory [14,15]. Steering captures the fact that the local measurements on one side can remotely steer the state on the other side when both sides share an entangled state. Steering criteria, which are obtained using correlations, state assemblages, and full information, are employed to detect steering. In particular, quantum steering can be certified by the violation of steering inequality [16–27]. Steerable states have an important application in randomness generation [28], subchannel discrimination [29], quantum information tasks [30], and one-sided device-independent processing in quantum key distribution [31].

An attempt to make some general statement about steering and teleportation was given in Ref. [32], which indicated that two-qudit quantum states with large-enough fully entangled fraction must be steerable. However, rather large, fully entangled fraction values were needed. Very recently, the problem of characterizing two-qubit states that are optimal for quantum teleportation for a fixed amount of purity, Bell nonlocality, and entanglement was considered [9]. The optimal states achieved the maximal average fidelity and exhibited zero fidelity deviation for a fixed amount of some state property [5,9,33]. The maximal average fidelity was the maximal value of the average fidelity achievable within the standard teleportation protocol and local operations [5]. The maximal average fidelity could judge how well an input state, on average, was teleported, while the fidelity deviation was a measure of dispersion. It was more effective to combine them together to serve as a better performance measure than only one of them [11,34]. For example, to select the best-performing states from a set of states with the same maximal average fidelity, the minimum fidelity deviation could act as a supplementary term if the fidelity deviation is expected to be as small as possible. As quantum steering is an important resource lying between entanglement and Bell nonlocality [35], the question to characterize the optimal two-qubit states of quantum teleportation for a fixed amount of steering is interesting and left open.

In this paper, we investigate the application of quantum steering in teleportation. First, we show that the two-qubit states violating the three-setting linear steering inequality are useful for teleportation. Subsequently, we characterize the optimal two-qubit states of quantum teleportation for a fixed amount of steering. The rest of the paper is arranged as follows. In Sec. II, the definitions of the maximal average fidelity, fidelity deviation, and steering are reviewed. The usefulness of the steerable states in teleportation is investigated in Sec. III.

*cumtlqiu@163.com

Then, in Sec. IV, the optimal two-qubit states of teleportation for a fixed amount of steering are considered. A discussion and conclusion part is given in Sec. V.

II. PRELIMINARIES

In this section, we review the concepts and definitions of the average fidelity, the fidelity deviation, and quantum steering.

In the Hilbert-Schmidt decomposition, a two-qubit state shared by the sender and the receiver can be represented as

$$\rho = \frac{1}{4}\left(I \otimes I + \vec{a}\cdot\vec{\sigma}\otimes I + I\otimes\vec{b}\cdot\vec{\sigma} + \sum_{i,j=1}^{3} r_{i,j}\sigma_i\otimes\sigma_j\right), \tag{1}$$

where $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$, $\vec{a}(\vec{b})\cdot\vec{\sigma} = \sum_{i=1}^{3} a_i(b_i)\sigma_i$ with σ_i being the Pauli matrix. $r_{i,j} = \mathrm{Tr}(\rho\sigma_i\otimes\sigma_j)$ is the element of the correlation matrix R.

According to the results given in Ref. [36,37], a general two-qubit state can always be reduced to following form by local unitary transformations:

$$\rho' = \frac{1}{4}\left(I \otimes I + \vec{a}'\cdot\vec{\sigma}\otimes I + I\otimes\vec{b}'\cdot\vec{\sigma} + \sum_{i=1}^{3} r_i\sigma_i\otimes\sigma_i\right), \tag{2}$$

where the correlation matrix of ρ' is $R' = \mathrm{diag}(r_1, r_2, r_3)$. One could choose appropriate unitary transformations such that $r_i < 0 (i = 1, 2, 3)$ if $\det R \leqslant 0$. The transformed state ρ' is the canonical form of ρ [9,33,34,36,37]. Without loss of generality, we employ the canonical form for our studies as the properties considered in this paper are invariant under local unitary transformations.

The average teleportation fidelity, or average fidelity for brevity, for a two-qubit state ρ is defined as $f(\rho) = \int\langle\psi|\rho_{\rm out}|\psi\rangle d\psi$, where the integral is over a uniform distribution $d\psi$ of input state $|\psi\rangle$ and $\int d\psi = 1$. $\langle\psi|\rho_{\rm out}|\psi\rangle$ is the fidelity between the input and output states.

One of the conditions that a two-qubit state is optimal for teleportation is that the average fidelity reaches its maximal value over all strategies within the standard protocol and local unitary operations. The maximal average fidelity for the state ρ given in Eq. (1) is

$$F(\rho) = \frac{1}{2}\left(1 + \frac{1}{3}\mathrm{Tr}\sqrt{R^T R}\right). \tag{3}$$

Here, the superscript T denotes the transpose of the correlation matrix. A two-qubit state is useful for quantum teleportation iff $F(\rho) > 2/3$ as the maximum fidelity achieved classically is 2/3. Based on the results present in Refs. [7,33,34], two-qubit states with $\det R \geqslant 0$ are useless for teleportation because $F \leqslant 2/3$ for these states. Thus, it suffices to focus only on the states with $\det R < 0$, which will lead to $r_i \leqslant 0$. And for these states, the maximal average fidelity can be simplified as

$$F(\rho') = \frac{1}{2}\left(1 + \frac{1}{3}\sum_{i=1}^{3}|r_i|\right). \tag{4}$$

Fidelity deviation is used to measure the fidelity fluctuations and is defined as the standard deviation of fidelity over all input states [10,11,34]

$$\delta(\rho) = \sqrt{\int\langle\psi|\rho_{\rm out}|\psi\rangle^2 d\psi - f(\rho)^2}. \tag{5}$$

For two-qubit states with $\det R < 0$, $\delta(\rho)$ reduces to

$$\delta(\rho') = \frac{1}{3\sqrt{10}}\sqrt{\sum_{i<j=1}^{3}(|r_i| - |r_j|)^2}. \tag{6}$$

The states with maximal average fidelity and zero fidelity deviation are called the optimal states in this paper. Thus, the optimal states form the subset of the set of states with the maximal average fidelity.

The three-setting linear steering inequality was introduced by Cavalcanti *et al.* to detect the steerability of a state [17] and was also considered as a very useful criteria to quantify steering. For a two-qubit state, it can be expressed as

$$\frac{1}{\sqrt{3}}\left|\sum_{m=1}^{3}\mathrm{Tr}(A_m\cdot B_m\rho)\right| \leqslant 1, \tag{7}$$

where the Hermitian operators acing on qubits A and B are expressed as $A_m = \vec{a}_m\cdot\vec{\sigma}$ and $B_m = \vec{b}_m\cdot\vec{\sigma}$, respectively. Here, $m = 1, 2, 3$. $\vec{a}_m, \vec{b}_m \in \mathbb{R}^3$ are two unit vectors, and $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are orthogonal vectors. Violation of the inequality (7) implies that the state ρ is steerable and the steering observable is given by the maximum violation

$$S = \max_{\{A_m,B_m\}}\frac{1}{\sqrt{3}}\left|\sum_{m=1}^{3}\mathrm{Tr}(A_m\otimes B_m\rho)\right|. \tag{8}$$

Since the states given in Eqs. (1) and (2) are local unitary equivalent, the steering observable for two-qubit states in the Hilbert-Schmidt as well as the canonical form is

$$S(\rho) = S(\rho') = \sqrt{\mathrm{Tr}(R^T R)} = \sqrt{\mathrm{Tr}(R'^T R')} = \sqrt{\sum_{i=1}^{3} r_i^2}. \tag{9}$$

Assuming $A_m = \vec{a}_m\cdot\vec{\sigma} = \sum_{j=1}^{3} a_{mj}\sigma_j$, $B_m = \vec{b}_m\cdot\vec{\sigma} = \sum_{k=1}^{3} b_{mk}\sigma_k$, Eq. (9) could be proved as follows;

$$\begin{aligned}
S(\rho') &= \max_{\{A_m,B_m\}}\frac{1}{\sqrt{3}}\left|\sum_{m=1}^{3}\mathrm{Tr}(A_m\otimes B_m\rho')\right| \\
&= \frac{1}{\sqrt{3}}\left|\sum_{m,i=1}^{3}\mathrm{Tr}\left(\frac{1}{4}\sum_{j,k=1}^{3} a_{mj}b_{mk}r_i(\sigma_j\otimes\sigma_k)(\sigma_i\otimes\sigma_i)\right)\right| \\
&= \frac{1}{\sqrt{3}}\left|\sum_{m,j,k,i=1}^{3}\frac{1}{4}a_{mj}b_{mk}r_i\mathrm{Tr}(\sigma_j\sigma_i)\mathrm{Tr}(\sigma_k\sigma_i)\right| \\
&= \frac{1}{\sqrt{3}}\left|\sum_{i=1}^{3}\left(\sum_{m=1}^{3} a_{mi}b_{mi}\right)r_i\right| \\
&= \frac{1}{\sqrt{3}}\left|\sum_{i=1}^{3} q_i r_i\right| \leqslant \frac{1}{\sqrt{3}}\sqrt{\sum_{i=1}^{3} q_i^2\sum_{i=1}^{3} r_i^2}
\end{aligned} \tag{10}$$

$$\leqslant \frac{1}{\sqrt{3}}\sqrt{\sum_{i=1}^{3}\left(\sum_{m=1}^{3}a_m^2\sum_{m=1}^{3}b_m^2\right)\sum_{i=1}^{3}r_i^2}$$
$$=\frac{1}{\sqrt{3}}\sqrt{3\sum_{i=1}^{3}r_i^2}=\sqrt{\sum_{i=1}^{3}r_i^2},$$

where $q_i=\sum_{m=1}^{3}a_{mi}b_{mi}$, $\mathrm{Tr}(\sigma_i\sigma_j)=\delta_{ij}$. In the proof, the Cauchy-Schwarz inequality was used twice.

The steering observable $S(\rho)\leqslant 1$ implies that the two-qubit states obey the three-setting linear steering inequality. However, the states violate the inequality and are steerable if $S(\rho)>1$. This condition is just a sufficient criterion to check steerability because there are steerable states with $S(\rho)<1$ [38].

III. TWO-QUBIT STATES VIOLATING THE THREE-SETTING LINEAR STEERING INEQUALITY BEING USEFUL FOR TELEPORTATION

In this section, the problem whether the two-qubit states violating the three-setting linear steering inequality are useful for teleportation is considered. An affirmative answer to this problem is given. Here, the fact that the state is useful for standard teleportation protocol implies that the maximal average fidelity is always larger than $2/3$.

Theorem 1. If a two-qubit state violates the three-setting linear steering inequality, i.e., $S(\rho)>1$, it is useful for teleportation.

Proof. If the eigenvalues of R^TR are assumed to be $E_i(i=1,2,3)$, $\mathrm{Tr}\sqrt{R^TR}=\sum_{i=1}^{3}\sqrt{E_i}$, which is always larger than $\sqrt{\sum_{i=1}^{3}E_i}=\sqrt{\mathrm{Tr}(R^TR)}$. Based on the maximal average fidelity given in Eq. (3), we have

$$F(\rho)=\frac{1}{2}\left(1+\frac{1}{3}\mathrm{Tr}\sqrt{R^TR}\right)\geqslant\frac{1}{2}\left(1+\frac{1}{3}\sqrt{\mathrm{Tr}(R^TR)}\right)$$
$$=\frac{1}{2}\left(1+\frac{1}{3}S(\rho)\right). \tag{11}$$

Obviously, the states with $S(\rho)>1$ will make the maximal average fidelity $F(\rho)>2/3$. Thus, all two-qubit states that violate the three-setting linear steering inequality are useful for teleportation.

In Ref. [5], it was shown that two-qubit mixed states which violate the Bell–Clauser-Horne-Shimony-Holt (CHSH) inequality were useful for teleportation. The result in Ref. [39] indicated that all two-qubit states violating CHSH-type steering inequalities were Bell nonlocal. Theorem 1 tells us that two-qubit mixed states which violate the three-setting linear steering inequality are useful for teleportation, and thus it neatly ties into the previous two results. While steering lies between Bell nonlocality and entanglement, there are some states which violate the three-setting linear steering inequality and obey Bell-CHSH inequality are useful for teleportation. Now let's give an example to illustrate this result in detail.

A class of two-qubit maximally entangled states (MEMS) was constructed by Ishizaka *et al.* [40,41] and the construction was [42]

$$\rho_{\mathrm{M}}=\lambda_1|\psi^-\rangle\langle\psi^-|+\lambda_2|00\rangle\langle 00|$$
$$+\lambda_3|\psi^+\rangle\langle\psi^+|+\lambda_4|11\rangle\langle 11|, \tag{12}$$

where $|\psi^\pm\rangle=(|01\rangle\pm|10\rangle)/\sqrt{2}$ were the maximally entangled states. λ_i $(i=1,\dots,4)$ were the eigenvalues of the state ρ_{M}.

The presence of Bell nonlocality could be indicated by the violation of the Bell-CHSH inequality. A two-qubit state ρ violates the Bell-CHSH inequality if and only if $M(\rho)>1$. Here, the Bell-CHSH observable $M(\rho)=\max_{i>j}\{r_i^2+r_j^2\}$.

By choosing $\lambda_1=(1+3p)/4$, $\lambda_2=\lambda_3=\lambda_4=(1-p)/4$ with $p\in[0,1]$, the MEMS state reduces to the Werner state [43], which is a state of rank 4. By simply calculation, the steering observable, the Bell-CHSH observable and the maximal average fidelity are $\sqrt{3p^2}$, $2p^2$ and $(1+p)/2$, respectively. From this it is found that the Werner state violates the three-setting linear steering inequality and is steerable for $p>\sqrt{3}/3$. The steerable Werner state is useful for teleportation as the maximal average fidelity is larger than $2/3$ if $p\in(\sqrt{3}/3,1]$. However, the condition that the Werner state violates the Bell-CHSH inequality requires $p>\sqrt{2}/2$. Therefore, the Werner state with $p\in(\sqrt{3}/3,\sqrt{2}/2]$ violates the three-setting linear inequality and obeys Bell-CHSH inequality. In this region, the Werner state is also useful for teleportation.

If one chooses $\lambda_1=(1+2p)/3$, $\lambda_2=\lambda_3=(1-p)/3$, and $\lambda_4=0$, the MEMS state reduces to the state with rank 3. The steering observable, the Bell-CHSH observable and the maximal average fidelity are $\sqrt{(1+4p+22p^2)/9}$, $\max\{2p^2,(1+4p+13p^2)/9\}$ and $(5+4p)/9$, respectively. The state violates the three-setting linear steering inequality if $p>(3\sqrt{5}-1)/11$, which will ensure the state being useful for teleportation because the maximal average fidelity is larger than $2/3$. The state violates the Bell-CHSH inequality if $p>2(3\sqrt{3}-1)/13$. Thus, the state with $p\in((3\sqrt{5}-1)/11,2(3\sqrt{3}-1)/13]$ is steerable and useful for teleportation, while it also obeys the Bell-CHSH inequality.

When taking $\lambda_1=(1+p)/2$, $\lambda_2=(1-p)/2$, $\lambda_3=\lambda_4=0$ into consideration, the MEMS state reduces to the state with rank 2. For this the steering observable, the Bell-CHSH observable, and the maximal average fidelity are $\sqrt{(1+2p+3p^2)/2}$, $\max\{(1+p)^2/2,(1+2p+5p^2)/4\}$, and $(2+p)/3$, respectively. It is easily found that the state with $p\in(1/3,\sqrt{2}-1]$ is steerable and useful for teleportation. However, the state in this region obeys the Bell-CHSH inequality.

IV. OPTIMAL TWO-QUBIT STATES FOR A FIXED AMOUNT OF STEERING

In the previous section, it aws found that two-qubit states which violate the three-setting linear steering inequality, i.e., $S(\rho)>1$, were useful for teleportation. However, the connection between steerability and the ability of a state to perform quantum teleportation as well as other quantum information tasks has not been explored in a comprehensive manner, especially in the context of optimality of resource states. In this section, we investigate the optimal two-qubit states of teleportation for a fixed amount of steering. Similar to the explanation present in the previous section, a necessary condition that two-qubit states are useful for teleportation is $\det R < 0$. For these states, the maximal average fidelity and steering are given in Eqs. (4) and (9), respectively. Moreover, it should be noted that in this section, we only consider the steerable states with $S(\rho) > 1$. Therefore, the constrained optimization problem, i.e., the optimal two-qubit states for a fixed amount of steering, can be expressed as

$$\text{maximize} \quad \sum_{i=1}^{3} |r_i|, \tag{13}$$

$$\text{such that} \quad \sqrt{\sum_{i=1}^{3} r_i^2} = \mathcal{S}. \tag{14}$$

Based on Eq. (14), one can parametrize r_i as

$$r_1 = \mathcal{S} \sin\alpha \cos\beta, \tag{15}$$

$$r_2 = \mathcal{S} \sin\alpha \sin\beta, \tag{16}$$

$$r_3 = \mathcal{S} \cos\alpha, \tag{17}$$

where $\alpha \in (0, 2\pi)$ and $\beta \in (0, 2\pi)$. Now, the maximization problem present in Eq. (13) reduces to find the maximization of the function

$$f(\alpha, \beta) = \mathcal{S}(|\sin\alpha \cos\beta| + |\sin\alpha \sin\beta| + |\cos\alpha|). \tag{18}$$

By simpe calculation and noting the condition $\det R < 0$, the maxima is obtained at two critical points: $\alpha = \cos^{-1}(-\sqrt{3}/3)$, $\beta = 5\pi/4$ and $\alpha = 2\pi - \cos^{-1}(-\sqrt{3}/3)$ and $\beta = \pi/4$. The corresponding value of r_i is

$$r_i = -\frac{\mathcal{S}}{\sqrt{3}}, \; i = 1, 2, 3. \tag{19}$$

The maximal average fidelity for a fixed amount of steering $\mathcal{S}$ can be obtained from Eq. (4)

$$\mathcal{F} = \frac{1}{2}\left(1 + \frac{\mathcal{S}}{\sqrt{3}}\right). \tag{20}$$

Noting the fact that the maximal average fidelity reaches the maxima for the case of $r_i = r_j$, the fidelity deviation equals to zero.

The maximal average fidelity $\mathcal{F} > 2/3$ only requires $\mathcal{S} > \sqrt{3}/3$. All two-qubit states violating the three-setting linear steering inequality have the steering observable $S(\rho) > 1$, and they are optimal for teleportation. Therefore, for a fixed amount of steering $\mathcal{S}$, the optimal two-qubit states for teleportation are those with $r_i = -\mathcal{S}/\sqrt{3}$.

The canonical form of the optimal two-qubit states can be expressed as

$$\rho_{\text{opt}} = \frac{1}{4}\left(I \otimes I + \vec{a}' \cdot \vec{\sigma} \otimes I + I \otimes \vec{b}' \cdot \sigma - \frac{\mathcal{S}}{\sqrt{3}} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i\right). \tag{21}$$

Correspondingly, the eigenvalues of the correlation matrix of the optimal two-qubit states in the Hilbert-Schmidt decomposition are the same and equal to $-\mathcal{S}/\sqrt{3}$. It is noted that no particular condition is being imposed on the local vectors.

From the above discussion, one can make the following conclusion. Let ρ be a two-qubit steerable state with steering $\mathcal{S} > 1$, $F(\rho) = \mathcal{F}$, and $\delta(\rho) = 0$ if and only if $r_i = -\mathcal{S}/\sqrt{3}$. In other words, these states have the maximal average fidelity as well as zero fidelity deviation, and they are optimal. For a fixed amount of steering, the set of optimal states is identical to the set of the maximal average fidelity states. The optimal states for a fixed amount of purity and Bell nonlocality also have similar characteristics [9]. However, the optimal states form a strict subset of the maximal average fidelity states for a fixed amount of concurrence [9]. The states with vanishing fidelity deviation are of special interest because they are said to satisfy the universality condition that all input states are teleported equally well [9–11,34].

V. DISCUSSION AND CONCLUSION

The proposal to quantify the degree of steering of two-qubit states based on the violation of the three-setting linear steering inequality has been put forward [25]. The rationale behind the proposal is as follows. A state that violates more the inequality and is more robust under noisy channels, is said to be more steering. The degree of steering is given on the basis of the steering observable $S(\rho)$ as $S_3(\rho) = \max\{0, [S(\rho) - 1]/(\sqrt{3} - 1)\}$. The definition is intuitively related to the notion of steering robustness [29]. Thus, our result based on the steering observable is also related to steering robustness.

Steering is an important quantum resource in quantum information tasks. In practical situations, it is useful to find the best performing quantum states with available resource. We address the question within the set of two-qubit states in this paper. The introduction of the fluctuations makes us closer to the use of teleportation in practice. In Refs. [9,44], teleportation was used as an intermediate step in a quantum circuit and was later processed by some gates which are typically sensitive to fluctuations of their inputs. In this case, the fact that one just considers the average fidelity is not enough. Thus, the results obtained in this paper will be beneficial for the application of quantum teleportation.

Steering is a characteristic quantum correlation and quantum resource lying between entanglement and Bell nonlocality. Its application in quantum information tasks deserves studying. In this paper, we investigate the application of steering as quantum resource in teleportation. First, it is found that the states violating the three-setting linear steering inequality are useful for teleportation by the fact that the maximal average fidelity is larger than the maximum fidelity achieved classically. An explicit example is given to show that some steerable states obey the Bell-CHSH inequality but are useful for teleportation. Subsequently, the problem of the optimal two-qubit states for teleportation with a fixed amount of steering is considered. The form of the optimal two-qubit states are present, and they are the maximal average fidelity states because these states have zero fidelity deviation. The results as well as those given in Ref. [9] indicate that the state properties can have an essential role in characterizing optimal states. We hope the results in this paper would help us to better understand the relation between quantum teleportation and the properties of the resource states. An open question is whether the main results can be generalized to the bipartite states of higher dimensions.

ACKNOWLEDGMENT

We appreciate the anonymous reviewers for their meaningful advice. The work was supported by the Fundamental Research Funds for the Central Universities under Grant No. 2020ZDPYMS03.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting An Unknown Quantum State Via dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[2] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels, Phys. Rev. Lett. **76**, 722 (1996).

[3] S. Popescu, Bell's Inequalities Versus Teleportation: What is Nonlocality? Phys. Rev. Lett. **72**, 797 (1994).

[4] L. Qiu, G. Tang, X. Q. Yang, and A. M. Wang, Enhancing teleportation fidelity by means of weak measurements or reversal, Ann. Phys.(NY) **350**, 137 (2014).

[5] R. Horodecki, M. Horodecki, and P. Horodecki, Teleportation, Bell's inequalities and inseparability, Phys. Lett. A **222**, 21 (1996).

[6] N. Gisin, Nonlocality criteria for quantum teleportation, Phys. Lett. A **210**, 157 (1996).

[7] M. Horodecki, P. Horodecki, and R. Horodecki, General teleportation channel, singlet fraction, and quasidistillation, Phys. Rev. A **60**, 1888 (1999).

[8] F. Verstraete and H. Verschelde, Optimal Teleportation with a Mixed State of Two Qubits, Phys. Rev. Lett. **90**, 097901 (2003).

[9] A. Ghosal, D. Das, S. Roy, and S. Bandyopadhyay, Optimal two-qubit states for quantum teleportation vis-à-vis state properties, Phys. Rev. A **101**, 012304 (2020).

[10] J. Bang, S.-W. Lee, H. Jeong, and J. Lee, Procedures for realizing an approximate universal-NOT gate, Phys. Rev. A **86**, 062317 (2012).

[11] J. Bang, J. Ryu, and D. Kaszlikowski, Fidelity deviation in quantum teleportation, J. Phys. A: Math. Theor. **51**, 135302 (2018).

[12] E. Schrödinger, Discussion of Probability Relations between Separated Systems, Proc. Cambridge Philos. Soc. **31**, 553 (1935).

[13] E. Schrödinger, Probability relations between separated systems, Proc. Cambridge Philos. Soc. **32**, 446 (1936).

[14] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. **98**, 140402 (2007).

[15] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering, Phys. Rev. A **76**, 052116 (2007).

[16] M. D. Reid, A. Acín, and N. Brunner, Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification, Phys. Rev. A **40**, 913 (1989).

[17] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Phys. Rev. A **80**, 032112 (2009).

[18] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Revealing Hidden Einstein-Podolsky-Rosen Nonlocality, Phys. Rev. Lett. **106**, 130402 (2011).

[19] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Einstein-Podolsky-Rosen steering inequalities from entropic uncertainty relations, Phys. Rev. A **87**, 062103 (2013).

[20] J.-L. Chen, X.-J. Ye, C. Wu, H.-Y. Su, A. Cabello, L. C. Kwek, and C. H. Oh, All-versus-nothing proof of Einstein-Podolsky-Rosen steering, Sci. Rep. **3**, 2143 (2013).

[21] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, and G. Adesso, Hierarchy of Steering Criteria Based on Moments for All Bipartite Quantum Systems, Phys. Rev. Lett. **115**, 210401 (2015).

[22] E. G. Cavalcanti, C. J. Foster, M. Fuwa, and H. M. Wiseman, Analog of the Clauser-Horne-Shimony-Holt inequality for steering, J. Opt. Soc. Am. B **32**, A74 (2015).

[23] M. Żukowski, A. Dutta, and Z. Yin, Geometric Bell-like inequalities for steering, Phys. Rev. A **91**, 032107 (2015).

[24] S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, Einstein-Podolsky-Rosen steering and the steering ellipsoid, J. Opt. Soc. Am. B **32**, A40 (2015).

[25] A. C. S. Costa, and R. M. Angelo, Quantification of Einstein-Podolski-Rosen steering for two-qubit states, Phys. Rev. A **93**, 020103(R) (2016).

[26] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, Rev. Mod. Phys. **92**, 015001 (2020).

[27] T. Z. Dai, Y. Fan, and L. Qiu, Complementary relation between tripartite entanglement and the maximum steering inequality violation, Phys. Rev. A **105**, 022425 (2022).

[28] Y. Z. Law, L. P. Thinh, J.-D. Bancal, and V. Scarani, Quantum randomness extraction for various levels of characterization of the devices, J. Phys. A: Math. Theor. **47**, 424028 (2014).

[29] M. Piani and J. Watrous, Necessary and Sufficient Quantum Information Characterization of Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. **114**, 060404 (2015).

[30] C. Branciard and N. Gisin, Quantifying the Nonlocality of Greenberger-Horne-Zeilinger Quantum Correlations by a Bounded Communication Simulation Protocol, Phys. Rev. Lett. **107**, 020401 (2011).

[31] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, Phys. Rev. A **85**, 010301(R) (2012).

[32] C.-Y. Hsieh, Y.-C. Liang, and R.-K. Lee, Quantum steerability: Characterization, quantification, superactivation, and unbounded amplification, Phys. Rev. A **94**, 062120 (2016).

[33] P. Badzia¸g, M. Horodecki, P. Horodecki, and R. Horodecki, Local environment can enhance fidelity of quantum teleportation, Phys. Rev. A **62**, 012311 (2000).

[34] A. Ghosal, D. Das, S. Roy, and S. Bandyopadhyay, Fidelity deviation in quantum teleportation with a two-qubit state, J. Phys. A: Math. Theor. **53**, 145304 (2020).

[35] M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Inequivalence of entanglement, steering, and Bell nonlocality for general measurements, Phys. Rev. A **92**, 032107 (2015).

[36] S. L. Luo, Quantum discord for two-qubit systems, Phys. Rev. A **77**, 042303 (2008).

[37] S. L. Luo and S. S. Fu, Quantifying correlations via the Wigner-Yanase skew information, Phys. Rev. A **85**, 032117 (2012).

[38] B. Paul and K. Mukherjee, Shareability of quantum steering and its relation with entanglement, Phys. Rev. A **102**, 052209 (2020).

[39] P. Girdhar and E. G. Cavalcanti, All two-qubit states that are steerable via Clauser-Horne-Shimony-Holt-type correlations are Bell nonlocal, Phys. Rev. A **94**, 032317 (2016).

[40] S. Ishizaka and T. Hiroshima, Maximally entangled mixed states under nonlocal unitary operations in two qubits, Phys. Rev. A **62**, 022310 (2000).

[41] F. Verstraete, K. Audenaert, and B. De Moo, Maximally entangled mixed states of two qubits, Phys. Rev. A **64**, 012316 (2001).

[42] K. G. Paulson, and S. V. M. Satyanarayana, Bounds on mixedness and entanglement of quantum teleportation resources, Phys. Lett. A **381**, 1134 (2017).

[43] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).

[44] L. H. Pedersen, N. M. Møller, and K. Mølmer, The distribution of quantum fidelities, Phys. Lett. A **372**, 7028 (2008).