Self-protected adiabatic quantum computation

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Recent experiments show the existence of collective decoherence in quantum systems. We study the possibility of quantum computation in a decoherence-free subspace which is robust against such kinds of decoherence processes. This passive protection protocol can be especially advantageous for continuous quantum computation such as quantum annealers. As an example we propose to use decoherence-protected adiabatic quantum computation for the Grover search problem.

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I. INTRODUCTION

Decoherence is one of the main obstacles to building a scalable quantum computer [1], and is often understood in terms of independent or individual error models. Recent experiments unexpectedly show that there exists *collective decoherence* in nature [2,3]. This newly discovered phenomenon is triggered by high-energy cosmic rays which produce long lifetime phonons in a substrate. These high-energy phonons can affect multiple qubits coherently. Thus, it will be important and timely to develop well-tailored protection schemes for quantum computers against such a type of decoherence.

There are several proposals for decoherence suppression and error correction in quantum computation (QC) in the literature, such as dynamical decoupling [4] and quantum error corrections [5,6]. In general, one can divide the decoherence suppression and error correction protocols to active ones which involve external pulses, and passive ones which explore the symmetry of the system-environment interaction. One of the solutions for passive protection against collective decoherence is to process quantum information in the decoherence-free subspace (DFS) [7–11].

Different from the gate-based QC initially proposed by Deutsch [12], where error correction procedures may be implemented easily, it is inconvenient to interrupt continuous QC [13] or adiabatic quantum computation (AQC) [14] in order to make such corrections. Therefore, it is natural to use passive decoherence protection for continuous QC, for instance, for the D-Wave system [15]. In addition, several self-protection protocols against specific noises have also been proposed for quantum algorithms [16] or reported for geometric phases [17,18].

In this paper we show that it is possible to combine DFS with the continuous QC as well as gate-based QC. We illustrate the proposed protocols with the Grover search problem. We also aim to use Hamiltonians directly available for spin chains [19], as there are recent developments in the control of spin chains and individual spins [20–22], which can be made either in a solid state or generalized directly to trapped-ion systems [23,24]. State-of-the-art technology [19] also allows for the preparation of an initial pure quantum state in spin chains, which is essential for a realistic quantum process.

II. CONSTRUCTION OF DFS

We consider a system consisting of n qubits (spins). The general form of coupling to the common environment can be described by the Hamiltonian

$$H = H_s(t) \otimes \mathbb{I} + \mathbb{I} \otimes H_B + Z_t \otimes B, \tag{1}$$

where H_s refers to the system, H_B is the environment Hamiltonian, $Z_t = \sum_i^n Z_i$ is the total Z operator, and B is some operator acting in the bath Hilbert space. This Hamiltonian can be assumed as an approximation for real dephasing.

We chose the system Hamiltonian to have some symmetry such that

$$[H_s(t), Z_t] = 0. \tag{2}$$

The next natural assumption is that the initial system-bath state is separable: $|\Psi(0)\rangle = |\psi(0)\rangle \otimes |\chi(0)\rangle$, where $|\psi\rangle$ ($|\chi\rangle$) is the system (bath) state. Let us write the evolution governed by (1),

$$U(T) = \exp\left[-i(\mathbb{I} \otimes H_B + Z_t \otimes B)T\right] \times \mathcal{T} \exp\left(-i \int_0^T H_s(t') \otimes \mathbb{I} dt'\right), \tag{3}$$

where $\mathcal{T}$ is the time-ordering operator, and T is the evolution time. The factorization of (3) is possible since $[H_s(t) \otimes \mathbb{I}, \mathbb{I} \otimes$

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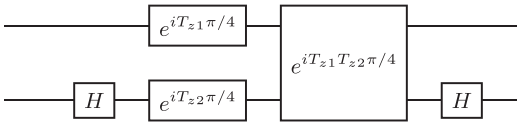

FIG. 1. Realization of a CNOT gate in DFS. Each wire corresponds to a logical qubit. An entangling gate is realized via the Hamiltonian acting on two physical qubits.

$H_B + Z_t \otimes B] = 0$, which follows from (2). The result of acting $U(T)$ on the initial state is

$$U(T)\,|\Psi(0)\rangle = e^{-i(\mathbb{I}\otimes H_B + Z_t\otimes B)T}\,|\psi(T)\rangle \otimes |\chi(0)\rangle\,,$$

where $|\psi(T)\rangle = \mathcal{T}\exp\left(-i\int_0^T H_s(t')dt'\right)|\psi(0)\rangle$. We assume $|\psi(T)\rangle$ is one of the vectors from the computational basis, which means it is also an eigenstate of Z_t. This allows us to write $U(T)\,|\Psi(0)\rangle = |\psi(T)\rangle \otimes |\chi(T)\rangle$, where $|\chi(T)\rangle = \exp\left[-i(H_B + \lambda B)T\right]|\chi(0)\rangle$, and $Z_t\,|\psi(T)\rangle = \lambda\,|\psi(T)\rangle$. Thus we can see that the system and bath remain disentangled.

Let us assume we have an even number of spins. The computational basis is chosen to be the protected subspace corresponding to $\langle Z_t\rangle = 0$. For example, in the case $n = 4$ we have the following basis vectors in this subspace: $\{|\downarrow\downarrow\uparrow\uparrow\rangle, |\downarrow\uparrow\downarrow\uparrow\rangle, |\downarrow\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\downarrow\uparrow\rangle, |\uparrow\downarrow\uparrow\downarrow\rangle, |\uparrow\uparrow\downarrow\downarrow\rangle\}$. It is easy to see that the dimension of this subspace is

$$N \overset{\text{def}}{=} \dim \text{DFS} = C_n^{n/2} = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \approx \sqrt{\frac{2}{\pi n}}2^n. \tag{4}$$

The simplest system which has a DFS is a pair of spins. In such a case the DFS is $\{|0\rangle_L, |1\rangle_L\}$, where we denote the states of logical qubit as $|0\rangle_L = |\uparrow\downarrow\rangle$, and $|1\rangle_L = |\downarrow\uparrow\rangle$. As was shown [25], we can organize single-qubit gates in this subspace with these generators of the SU(2) group:

$$T_x = \frac{X_1X_2 + Y_1Y_2}{2}, \tag{5}$$

$$T_y = \frac{Y_1X_2 - X_1Y_2}{2}, \tag{6}$$

$$T_z = \frac{Z_1 - Z_2}{2}. \tag{7}$$

If we consider a set of pairs of spins as logical qubits we can rewrite the requirement (2) in the following way: $[H_l(t), Z_{2l-1} + Z_{2l}] = 0$, where $l = 1, 2, 3, \ldots, n/2$ is the number of spin pairs, and $H_l(t)$ acts in the Hilbert space of this spin pair. A controlled operation between two logical qubits can be made by $T_{z1}T_{z2} = -Z_2Z_3$, where we assume spins 1 and 2 (3 and 4) belong to the first (second) logical qubit [25]. This allows us to construct an Ising gate or more generally an $\exp(iT_{z1}T_{z2}\theta)$ entangling gate acting on a *pair of logical qubits* and thus it is possible to realize other two-qubit gates such as a controlled-NOT (CNOT) gate (see Fig. 1 and Refs. [26,27]). From the above one can conclude that QC in DFS can be universal and gate based with known quantum algorithms.

In this paper we propose different models of quantum computation satisfying requirement (2) which we describe in the next sections.

III. CONTINUOUS GROVER SEARCH IN DFS

We consider the Grover search problem [28] as an example of using DFS. Following Farhi *et al.* [13] we define the oracle Hamiltonian as $H_w = -|w\rangle\langle w|$, where $|w\rangle$ is the unknown state residing in the DFS. The next step is introducing an equal superposition of all basis vectors in the DFS:

$$|s\rangle = \frac{1}{\sqrt{N}}\sum_{m=1}^{N}|m\rangle\,. \tag{8}$$

In order to organize quantum computation we consider the total Hamiltonian which consists of an oracle and driving term,

$$H = H_d + H_w = -|s\rangle\langle s| - |w\rangle\langle w|\,. \tag{9}$$

The driving part in (9) can also be written as

$$H_d = -|s\rangle\langle s| = -\sum_{k=1}^{n/2}\sum_{m_1<m_2<\cdots m_{2k}}^{n}\sigma_{m_1}^{\dagger}\cdots\sigma_{m_k}^{\dagger}\sigma_{m_{k+1}}\cdots\sigma_{m_{2k}} + \text{H.c.} \tag{10}$$

Here, we use $\sigma_i = |\uparrow\rangle_i\langle\downarrow|_i$. The initial state of the system is set to be $|\psi_0\rangle = |s\rangle$. The result of unitary evolution governed by (9) can be written as follows,

$$|\psi(t)\rangle = e^{-iHt}\,|s\rangle = e^{it}\{[x\cos(xt) + i\sin(xt)]\,|w\rangle + \sqrt{1-x^2}\cos(xt)\,|r\rangle\}, \tag{11}$$

where $|r\rangle = (|s\rangle - x\,|w\rangle)/\sqrt{1-x^2}$, and $x = 1/\sqrt{N}$. As can be seen from (11) after time $T = \pi\sqrt{N}$ the state of the system is $|\psi(T)\rangle \approx |w\rangle$. It is important to note $|\psi(t)\rangle$ is evolving in the DFS all the time.

The way we constructed Hamiltonian (9) on the one hand allows us to use already known results from the continuous Grover algorithm, and on the other hand we have a *self-protection* of quantum computation from collective dephasing. The driving Hamiltonian (10) contains many-body interactions which can be hard to realize in practice. Therefore, we propose a more feasible way of QC in DFS in the next section.

IV. ADIABATIC GROVER SEARCH IN DFS

As was discussed above, it is possible to use a pair of spins as a logical qubit in DFS, and use existing quantum algorithms with such qubits. However, gate-based QC requires complicated control such as precise switching off and on interactions, and fields over singles and pairs of logical qubits.

Thus, as a proof of concept we propose to implement adiabatic quantum computation in DFS. In AQC we chose the time-dependent system Hamiltonian

$$H_s(t) = [1 - s(t)]H_i + s(t)H_f, \tag{12}$$

where H_i (H_f) is the initial (final) Hamiltonian, and $s(0) = 0$, $s(T) = 1$. In simple case linear switching $s(t) = t/T$. The initial state $|\psi(0)\rangle$ is a ground state of the H_i, and in the limit $T \to \infty$ the final state $|\psi(T)\rangle$ is guaranteed to be a ground state of H_f. Obtaining the ground state of H_f is the goal of AQC.

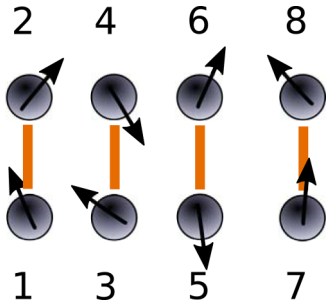

FIG. 2. Independent pairs of spins. Each pair is an antiferromagnetic XXX spin chain.

Let us assume for the moment the initial Hamiltonian describes a set of independent XX ferromagnetic spin chains, each of them containing only two spins (Fig. 2). Therefore the Hamiltonian of each chain is just T_x from (5),

$$H_i = -\sum_{l=1}^{n/2} (X_{2l-1}X_{2l} + Y_{2l-1}Y_{2l}). \tag{13}$$

The spectra of a single chain are $\{-2, 0, 0, 2\}$, while there are two states in DFS: $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} = (|0\rangle_L + |1\rangle_L)/\sqrt{2}$ with eigenenergy -2, and $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ with eigenenergy 2. Thus we see that the ground state of a set of chains is nondegenerate and is equal to $|s\rangle$ [see (8)]. Moreover, the gap between the ground state and the first excited state does not depend on n. Assuming $H_f = -|w\rangle\langle w|$, where $|w\rangle$ is the unknown state from DFS we achieve AQC in DFS which can be described in the same way as was made in Ref. [14]. However, Hamiltonian (13) with a suppressed ZZ interaction can be difficult to realize. Nevertheless, we can note that adding the ZZ interaction into (13) does not change the eigenstates. Moreover, it is well known that an antiferromagnetc XXX spin chain is a natural consequence of a half-filled Hubbard fermionic model in the limit of a strong on-site interaction [29]. Thus, instead of (13) we propose the following initial Hamiltonian,

$$H_i = J\sum_{l=1}^{n/2} (X_{2l-1}X_{2l} + Y_{2l-1}Y_{2l} + Z_{2l-1}Z_{2l}), \tag{14}$$

where $J > 0$. The spectra of a single chain k are $\{-3J, J, J, J\}$, where the nondegenerate ground state is a singlet $|\varphi_g\rangle_k \equiv (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} = (|0\rangle_L - |1\rangle_L)/\sqrt{2}$. Note, the Hamiltonian of a single chain in DFS subspace reads $H_L = -J\mathbb{I}_L + 2JX_L$ (with $X_L = |0\rangle_L\langle 1|_L + |1\rangle_L\langle 0|_L$). As we can see, the ground of the total system is nondegenerate, but it is no longer equal to $|s\rangle$. We can write the ground state of (14) as

$$|\Psi_0\rangle = |\varphi_g\rangle_1 \otimes |\varphi_g\rangle_2 \otimes \cdots |\varphi_g\rangle_{n/2}$$
$$= N^{-1/2}\left(\sum_{\substack{\text{number of } |1\rangle_L \\ \text{in } |n\rangle \text{ is even}}} |n\rangle - \sum_{\substack{\text{number of } |1\rangle_L \\ \text{in } |n\rangle \text{ is odd}}} |n\rangle\right), \tag{15}$$

where $|n\rangle$ are the states in the computational DFS basis as $|0\rangle_L \otimes |0\rangle_L \otimes |1\rangle_L$, etc., and $N = \dim \mathrm{DFS} = 2^{n/2}$. Despite the ground state not being an equal superposition as in Ref. [14], we can use it as the initial state in the AQC Grover search, due to the correspondence given by the following unitary transformation,

$$U = \sum_{\substack{\text{number of } |1\rangle_L \\ \text{in } |n\rangle \text{ is even}}} |n\rangle\langle n| - \sum_{\substack{\text{number of } |1\rangle_L \\ \text{in } |n\rangle \text{ is odd}}} |n\rangle\langle n|. \tag{16}$$

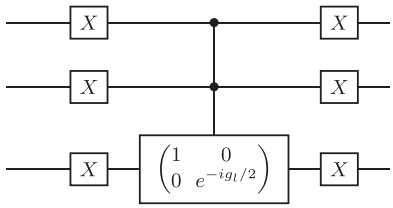

FIG. 3. Possible realization of the oracle unitary for Trotterized AQC in the DFS algorithm for $|w\rangle = |000\rangle_L$. Each wire corresponds to a logical qubit. Note, this oracle depends on the number of steps l in (17). A complicated controlled-phase operation CCPHASE can be made by using Tofolli gates, which in its turn can be decomposed by using CNOT gates (see Fig. 1 and Ref. [1]) and single-qubit gates. Thus only two-body interactions are needed for realizing quantum computation in DFS.

As shown in different experimental setups, we need to test the feasibilities of our proposal in state-of-the-art experiments. The oracle Hamiltonian $-|w\rangle\langle w|$ contains many-body interactions such as $Z_1Z_2\cdots Z_{n/2}$ which usually do not appear in available experiments. Thus, for proof-of-concept purposes we propose to realize AQC in DFS via Trotterization [30–33], i.e., dividing the evolution time T into small parts $\Delta t = T/M$ with $M \gg 1$ and an approximate smooth adiabatic evolution by a sequence

$$U(T) \approx \prod_{l=1}^{M} (e^{-iH_f g_l/2K} e^{-iH_i f_l/K} e^{-iH_f g_l/2K})^K, \tag{17}$$

where for the linear switching (12) $f_l = (1 - \Delta t \cdot l/T)\Delta t$, $g_l = \Delta t^2 \cdot l/T$. Such a decomposition is valid when $\Delta t||H_i - H_f|| \ll 1$. Also, the parameter K should be big enough $K \gg M\Delta t^3$ (see Ref. [33]). However, in our numerical examples we use $K \gtrsim 1$, which provides relatively good output fidelity for small systems, and reduces the number of gates, which is crucial for a proof-of-concept implementation on near-future quantum computers.

Each Trotter step consists of two different types of unitary operations governed by the H_i and H_f Hamiltonian. We already know how to implement an oracle in a circuit model. Thus, the operation $e^{-iH_f g_l/2}$ (which is a *phase oracle*) can be made by using a set of gates in DFS (5)–(7) and controlled-Z [25] (see Refs. [1,34] and Fig. 3 for the scheme of the oracle). The operation $e^{-iH_i f_l/2}$ is just an evolution governed by Hamiltonian (14) and does not require the implementation of additional gates such as a Grover diffusion operator in a circuit model. Here, we have a similar situation to the one described in Ref. [35] where the standard Grover diffusion operator was changed to $\exp(-it\sum_n X_n)$, corresponding to a transverse field acting individually on each qubit. Here, we have evolution governed by a Hamiltonian of independent dimers which in turn can be considered as a transverse field

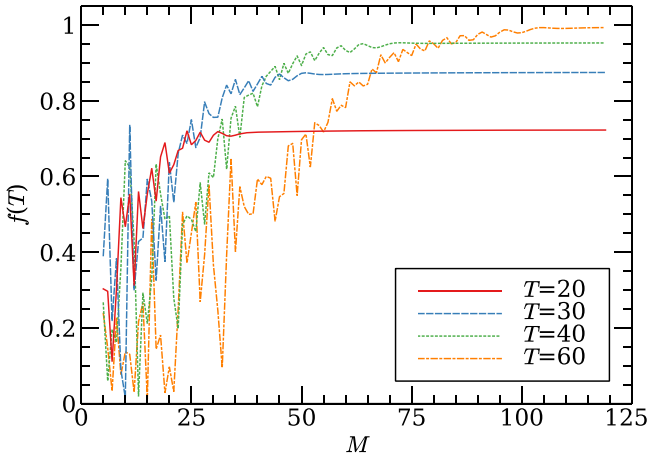

FIG. 4. Fidelity of computation as a function of a number of Trotter steps for different total times of computation. Here, we use a system of six spins with $J = 1$ and dim DFS $= 8$.

in the logic subspace. Therefore, our proposal can be used for covering the algorithm [35] in DFS. Thus, despite using Trotterization and unitary gates instead of a direct realization of the oracle Hamiltonian, the proposed method has an advantage over the circuit search model, because there is no need to organize a set of gates for the diffusion operator. This can help to avoid control errors. As a numerical example we use a system of six spins, which corresponds to three logical qubits with dim DFS $= 8$. In Fig. 4 we show the resulting fidelity $f(T) = |\langle w|U(T)|\psi(0)\rangle|^2$ of the computation as a function of the number of steps M for different times $T = 20, 30, 40, 60$. The saturation of curves corresponds to continuous adiabatic evolution governed by (12).

Linear switching in (12) is not optimal and does not provide a quantum speedup [14]. Following Ref. [36] we can improve the efficiency by using the following changes in (17): $f_l = (1 - s)\Delta t$, $g_l = s\Delta t$, where for each step l we can find the corresponding value of s by solving the following equation,

$$l\,\Delta t = T\,\frac{\int_0^s \frac{ds'}{[E_1(s')-E_0(s')]^2}}{\int_0^1 \frac{ds'}{[E_1(s')-E_0(s')]^2}}, \tag{18}$$

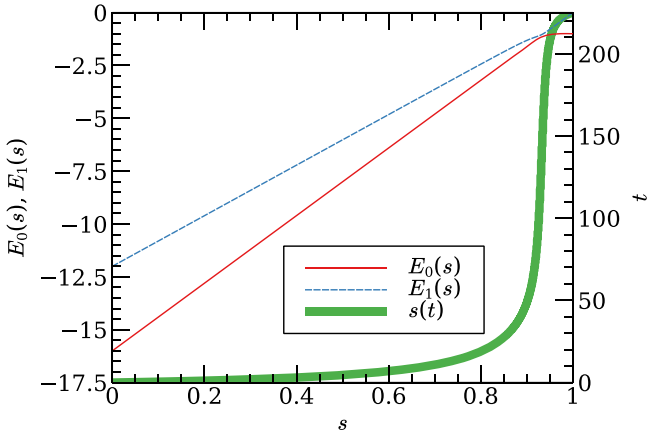

FIG. 5. Two lowest eigenenergies E_0, E_1 of Hamiltonian (12) and optimized adiabatic path $s(t)$ for $T = 225$ and $n_L = 7$.

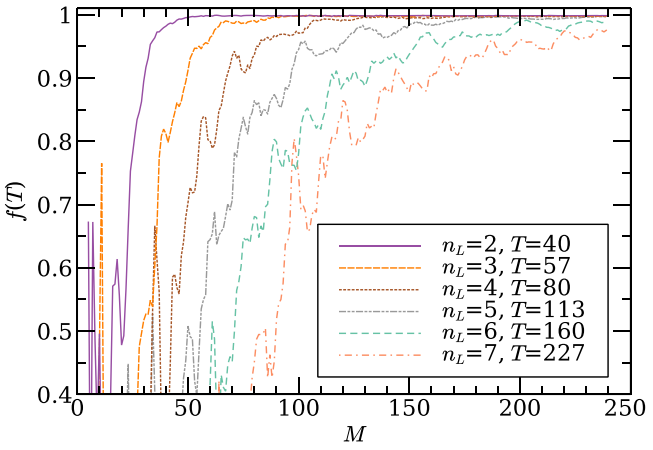

FIG. 6. Fidelity of computation as a function of a number of steps M for different total times of computation and different sizes of the system ($n_L = n/2$ is the number of logical qubits). Here, we use optimized adiabatic switching with a Trotter parameter $K = 1$.

where $E_{0,1}(s')$ are the ground and next after eigenenergies of the Hamiltonian $H(s')$ (12). Expression (18) is written in such a way as to satisfy $M\Delta t = T$. In Fig. 5 we show an example of the dependence $E_{0,1}(s)$ and $s(t)$ for the case $n = 14$ and $T = 225$ (dim DFS $= 2^7$). In Fig. 6 we show the numerical simulation by using optimized switching from H_i to H_f. As can be seen from Fig. 6, it is enough that $K = 1$ in (17) for a system with <5 logical qubits. The increasing Trotter parameter K improves the output fidelity. In Fig. 6 we show the output fidelity with $K = n_L$, where $n_L = n/2$ is the number of logical qubits. In both Figs. 6 and 7 we chose the evolution time to increase by a factor of $\sqrt{2}$, i.e., $T \to \sqrt{2}T$ for each increment of the number of logical qubits $n_L \to n_L + 1$.

In addition to increasing the Trotter steps M or parameter K, both of which increase the number of gates required, it is also possible to increase the output fidelity by further optimizing the switching function $s(t)$. One such way to optimize the switching function is by means of the Krotov method [37–39]. The switching function obtained from Eq. (18) can be used as an initial guess for the Krotov method. Since the monotonic convergence of the Krotov method is only guaranteed

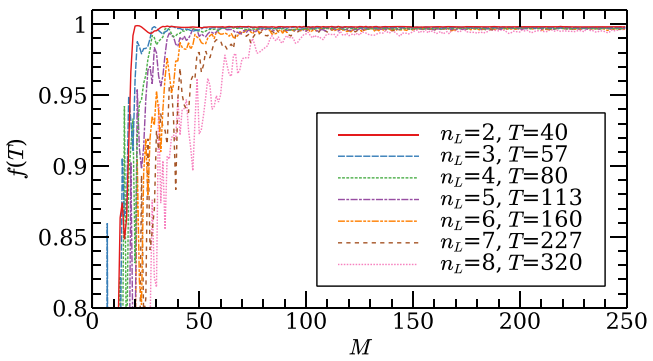

FIG. 7. Fidelity of computation as a function of a number of steps M for different total times of computation and different sizes of the system ($n_L = n/2$ is the number of logical qubits). Here, we use optimized adiabatic switching with a Trotter parameter $K = n_L$.

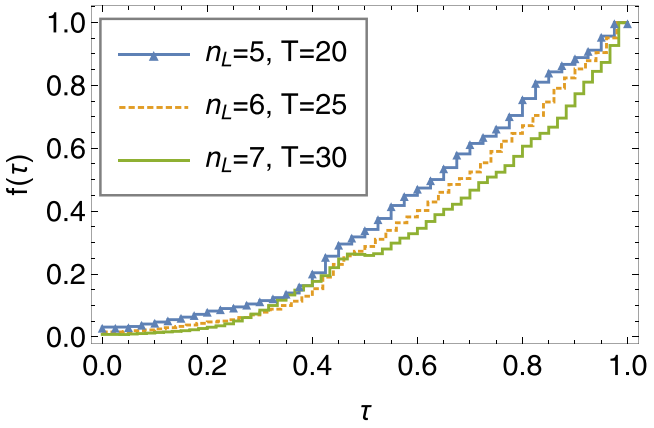

FIG. 8. Using the Krotov method to optimize the switching function, we show the output fidelity f as a function of the scaled time $\tau = t/T$, for various numbers of logic qubits. The final output fidelity is $\{0.999\,683, 0.999\,368, 0.999\,069\}$ up from $\{0.854\,497, 0.789\,701, 0.691\,594\}$ using the vanilla switching function obtained from Eq. (18), for $n_L = 5$ (solid blue line marked by ▲), $n_L = 6$ (dashed yellow line), and $n_L = 7$ (solid green line), respectively. Here, we have used $M = 2T$ and the Trotter parameter $K = 1$.

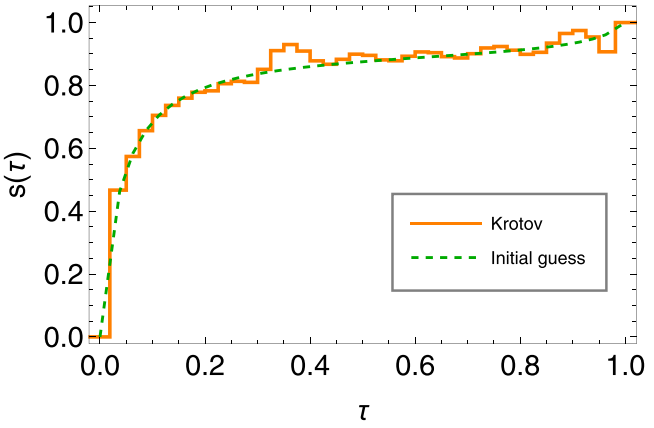

FIG. 9. The switching function $s(\tau)$ as a function of the scaled time $\tau = t/T$ for $n_L = 5$ logic qubits; the parameters are the same as in Fig. 8. The orange line is the optimized result using the Krotov method, and the dashed green line is obtained via Eq. (18) which serves as an initial guess for the Krotov method.

in the continuous control limit, the parameters for the Krotov method have also been appropriately chosen to account for the coarse time step. In Fig. 8 we show the fidelity as a function of the scaled time $\tau = t/T$ for $n_L = 5, 6, 7$ logic qubits. We can see that the final output fidelity can all reach 0.999, which is a quite noticeable increase compared with the vanilla switching function Eq. (18) under the same parameters. It is worth pointing out that this is achieved by optimizing the switching function under a relatively short runtime and a small number of Trotter steps, which reduces the number of gates required to carry out the search algorithm. In Fig. 9, we show the optimized switching function and the vanilla switching function obtained from Eq. (18) as a function of the scaled time $\tau = t/T$ for five logical qubits. It can be seen that the two agree well initially, when the energy gap between the two lowest eigenenergies is large, and small corrections are made when the energy gap becomes smaller.

Finally, we heuristically analyze the effects of possible imperfections in the system-bath Hamiltonian (1). For simplicity we assume $H_s = 0$. Let us consider a small addition to (1), $\epsilon X_t \otimes B_1$, where $X_t = \sum_i^n X_i$, and $\epsilon \to 0$. Using a first-order expansion of the evolution operator we change (3) to $U(T)(\mathbb{I} - i\epsilon TW)$ [where $W = \int_0^1 U^\dagger(\theta T) X_t \otimes B_1 U(\theta T) d\theta$]. Therefore, the probability of error $P_{\text{error}} \propto \epsilon^2 T^2$. For the runtime of the Grover algorithm we have $T \propto 2^{n_L/2}$, and in order to avoid errors in computation one should have $\epsilon \ll 2^{-n_L/2}$ scaling. This shows that, as expected, for larger databases one should have more precise tuning of the parameters (see Ref. [40]). Note, the imperfections in Hamiltonian (1) lead not only to decoherence, but also to leakage from DFS. This fact can be used for checking the result of computation after the final measurement.

V. CONCLUSIONS

We propose to use DFS as a computational space for continuous QC. At first we show that the Farhi [13,14] proposals of continuous and adiabatic QC can be implemented in DFS. Next, we show that it is possible to achieve QC in DFS via both continuous and gate-based QC, where each logical protected qubit consists of two physical qubits with an XXX interaction between each. This passive protection does not require application of complicated external pulses. Also, to realize this protection only two-body interactions are necessary.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, UK, 2010).

[2] C. D. Wilen, S. Abdullah, N. A. Kurinsky, C. Stanford, L. Cardani, G. D'Imperio, C. Tomei, L. Faoro, L. B. Ioffe, C. H. Liu, A. Opremcak, B. G. Christensen, J. L. DuBois, and R. McDermott, Nature (London) **594**, 369 (2021).

[3] M. McEwen, L. Faoro, K. Arya, A. Dunsworth, T. Huang, S. Kim, B. Burkett, A. Fowler, F. Arute, J. C. Bardin, A. Bengtsson, A. Bilmes, B. B. Buckley, N. Bushnell, Z. Chen, R. Collins, S. Demura, A. R. Derk, C. Erickson, M. Giustina *et al.*, Nat. Phys. **18**, 107 (2022).

[4] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

[5] P. W. Shor, Phys. Rev. A **52**, R2493(R) (1995).
[6] A. Steane, Proc. R. Soc. London, Ser. A **452**, 2551 (1996).
[7] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
[8] P. Zanardi, Phys. Rev. A **63**, 012301 (2000).
[9] J. Kempe, D. Bacon, D. A. Lidar, and K. B. Whaley, Phys. Rev. A **63**, 042307 (2001).
[10] L. Viola, E. M. Fortunato, M. A. Pravia, E. Knill, R. Laflamme, and D. G. Cory, Science **293**, 2059 (2001).
[11] D. Lidar, L.-A. Wu, and A. Blais, Encoding and error suppression for superconducting quantum computers, U.S. Patent No. 7,307,275 B2 (11 December 2007).
[12] D. Deutsch, Proc. R. Soc. London, Ser. A **425**, 73 (1989).
[13] E. Farhi and S. Gutmann, Phys. Rev. A **57**, 2403 (1998).
[14] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, arXiv:quant-ph/0001106.
[15] M. W. Johnson, M. H. S. Amin, S. Gildert, T. Lanting, F. Hamze, N. Dickson, R. Harris, A. J. Berkley, J. Johansson, P. Bunyk, E. M. Chapple, C. Enderud, J. P. Hilton, K. Karimi, E. Ladizinsky, N. Ladizinsky, T. Oh, I. Perminov, C. Rich, M. C. Thom *et al.*, Nature (London) **473**, 194 (2011).
[16] L.-A. Wu and M. S. Byrd, Quantum Inf. Process. **8**, 1 (2009).
[17] D.-W. Luo, J. Q. You, H.-Q. Lin, L.-A. Wu, and T. Yu, Phys. Rev. A **98**, 052117 (2018).
[18] D.-W. Luo, H.-Q. Lin, J. Q. You, L.-A. Wu, R. Chatterjee, and T. Yu, Phys. Rev. A **100**, 062112 (2019).
[19] D.-J. Choi, N. Lorente, J. Wiebe, K. von Bergmann, A. F. Otte, and A. J. Heinrich, Rev. Mod. Phys. **91**, 041001 (2019).
[20] R. Toskovic, R. van den Berg, A. Spinelli, I. S. Eliens, B. van den Toorn, B. Bryant, J.-S. Caux, and A. F. Otte, Nat. Phys. **12**, 656 (2016).
[21] S. Baumann, W. Paul, T. Choi, C. P. Lutz, A. Ardavan, and A. J. Heinrich, Science **350**, 417 (2015).
[22] P. N. Jepsen, J. Amato-Grill, I. Dimitrova, W. W. Ho, E. Demler, and W. Ketterle, Nature (London) **588**, 403 (2020).
[23] J. Simon, W. S. Bakr, R. Ma, M. E. Tai, P. M. Preiss, and M. Greiner, Nature (London) **472**, 307 (2011).
[24] B. Neyenhuis, J. Zhang, P. W. Hess, J. Smith, A. C. Lee, P. Richerme, Z.-X. Gong, A. V. Gorshkov, and C. Monroe, Sci. Adv. **3**, e1700672 (2017).
[25] L.-A. Wu and D. A. Lidar, Phys. Rev. A **65**, 042318 (2002).
[26] Y. Makhlin, Quantum Inf. Process. **1**, 243 (2002).
[27] N. Schuch and J. Siewert, Phys. Rev. A **67**, 032301 (2003).
[28] L. K. Grover, Phys. Rev. Lett. **80**, 4329 (1998).
[29] P. Fazekas, *Lecture Notes on Electron Correlation and Magnetism* (World Scientific, Singapore, 1999).
[30] S. Lloyd, Science **273**, 1073 (1996).
[31] L.-A. Wu, M. S. Byrd, and D. A. Lidar, Phys. Rev. Lett. **89**, 057904 (2002).
[32] A. Smith, M. S. Kim, F. Pollmann, and J. Knolle, npj Quantum Inf. **5**, 106 (2019).
[33] F. Tacchino, A. Chiesa, S. Carretta, and D. Gerace, Adv. Quantum Technol. **3**, 1900052 (2020).
[34] C. Figgatt, D. Maslov, K. A. Landsman, N. M. Linke, S. Debnath, and C. Monroe, Nat. Commun. **8**, 1918 (2017).
[35] Z. Jiang, E. G. Rieffel, and Z. Wang, Phys. Rev. A **95**, 062317 (2017).
[36] J. Roland and N. J. Cerf, Phys. Rev. A **65**, 042308 (2002).
[37] S. E. Sklarz and D. J. Tannor, Phys. Rev. A **66**, 053619 (2002).
[38] V. Krotov, *Global Methods in Optimal Control Theory* (CRC Press, Boca Raton, FL, 1995).
[39] X.-P. Zhang, B. Shao, S. Hu, J. Zou, and L.-A. Wu, Ann. Phys. **375**, 435 (2016).
[40] N. Shenvi, K. R. Brown, and K. B. Whaley, Phys. Rev. A **68**, 052313 (2003).