Adaptive procedures for discriminating between arbitrary tensor-product quantum states

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Discriminating between quantum states is a fundamental task in quantum information theory. Given two quantum states ρ_+ and ρ_-, the Helstrom measurement distinguishes between them with minimal probability of error. However, finding and experimentally implementing the Helstrom measurement can be challenging for quantum states on many qubits. Due to this difficulty, there is great interest in identifying local measurement schemes which are close to optimal. In the first part of this work, we generalize previous work by Acin *et al.* [Phys. Rev. A **71**, 032338 (2005)] and show that a locally greedy scheme using Bayesian updating can optimally distinguish between any two states that can be written as a tensor product of arbitrary pure states. We then show that the same algorithm cannot distinguish tensor products of mixed states with vanishing error probability (even in a large subsystem limit), and introduce a modified locally greedy scheme with strictly better performance. In the second part of this work, we compare these simple local schemes with a general dynamic programming approach which finds both the optimal series of local measurements as well as the optimal order in which subsystems are measured.

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I. INTRODUCTION

Measurement lies at the heart of quantum mechanics. Since the exact state of a quantum system cannot be directly observed, measurement is the primary means of understanding real quantum systems [1–6]. However, due to the inherent uncertainty in quantum systems it is impossible to design a quantum measurement capable of perfectly discriminating between two non-orthogonal quantum states [7,8]. The optimal measurement for state discrimination was described by Helstrom [9]. However, for composite quantum systems, the Helstrom measurement is impractical to implement experimentally because it typically requires simultaneously measuring all subsystems.

Several works in the literature have investigated techniques that use only local operations to distinguish between two possible qubit states [10–16]. Given N copies of the state, their aim is to achieve or approximate the Helstrom probability of success. Such algorithms measure one local subsystem in each round and determine parameters of the next measurement as a function of the past measurement results. The simplest strategy, a naïve "majority vote," has been shown to have probability of error which approaches zero exponentially fast in N [10,11]. For N copies of mixed qubit states, tight bounds on the error rate of the best locally adaptive protocol can be found in Ref. [12]. Dynamic programming has also been utilized to recursively minimize the expected future error over all possible allowed measurements, and thus compute the optimal adaptive strategy for any given family of measurements [11]. Furthermore, for the special case where the states are tensor powers of qubit pure states, it has been shown that a greedy adaptive strategy, involving Bayesian updates of the prior after each measurement result, is optimal and achieves the same success probability as the collective Helstrom measurement [10]. Finally, several works have investigated alternate hypothesis testing problems such as unambiguous state discrimination [17–19] and discriminating between more than two candidate states [20–23].

In this paper, we generalize known results and consider the problem of discriminating between two arbitrary tensor product quantum states (TPQSs) and also provide discussion of

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higher-dimensional quantum subsystems such as qutrits. More specifically, we suppose that we are given either ρ_+ or ρ_- with prior probability q and $1-q$, respectively, where $\rho_\pm = \rho_\pm^{(1)} \otimes \cdots \otimes \rho_\pm^{(N)}$ and $\rho_\pm^{(j)}$ is potentially different for each $j \in \{1, \ldots, N\}$. This problem is of practical interest in quantum communications, where we might modulate a classical binary codeword into a TPQS in order to transmit information through multiple uses of the channel, and each subsystem could experience a (slightly) different channel parameter. The received codeword set then consists of non-orthogonal noisy TPQSs.

When distinct subsystems are in different states, the optimal measurement order for the subsystems depends on the measurement outcome of the previous subsystems. We prove that, if all of the systems are pure states, then the order of measurement does not matter and the Bayesian update-based strategy with locally greedy measurements is optimal. This generalizes the result in Ref. [10] mentioned above.

When the states are mixed, the locally greedy algorithm is no longer optimal and in fact performs worse than most nonadaptive local strategies in the limit as $N \to \infty$. We show that this poor asymptotic performance arises from the local Helstrom measurement becoming noninformative for sufficiently imbalanced priors. To overcome this, we introduce a modified locally greedy adaptive strategy with strictly better performance. A similar phenomena was observed and discussed in Ref. [24], where it was demonstrated that for the case of noisy qubit copies the error rate of the standard locally greedy strategy is nonzero even in the limit of infinite copies. We show that this modified algorithm is asymptotically optimal as the number of subsystems approaches infinity, and conjecture that this algorithm is optimal or near-optimal when there are a finite number of subsystems corresponding to identical copies of depolarized states.

We also discuss a dynamic programming-based strategy that finds the optimal locally adaptive strategy, generalizing the technique introduced in Ref. [11] to include optimizing over the order in which subsystems are measured. This dynamic programming approach is the optimal locally adaptive technique subject to some simple constraints and includes the locally greedy techniques as a special case of itself.

Finally, we consider higher dimensional (qutrit) subsystems and compare the performance of ternary versus binary projective measurements. We show that, in general, multiple-outcome measurements are needed for optimality. Numerical results are provided for all these scenarios and the source code used to generate them is available online [25].

II. NOTATION

Following the same notation as above, ρ is the random variable representing the given state, so that either $\rho = \rho_+ = \rho_+^{(1)} \otimes \cdots \otimes \rho_+^{(N)}$ or $\rho = \rho_- = \rho_-^{(1)} \otimes \cdots \otimes \rho_-^{(N)}$, and we refer to N as the number of subsystems. We additionally require that each $\rho_\pm^{(j)}$ be a real matrix. For qubit states, if this condition is not satisfied one can always create a state discrimination problem which is unitarily equivalent and where the density matrices are real-valued. Namely, there exists a unitary $U = U^{(1)} \otimes \cdots \otimes U^{(n)}$ such that $U\rho_+U^\dagger$ and $U\rho_-U^\dagger$ are real matrices. A similar argument holds for higher-dimensional subsystems and pure candidate states. However, numerical results suggest that for the most general state discrimination setup of higher-dimensional subsystems and mixed candidate states, there may not always be a unitarily equivalent real-valued problem.

In the qubit case, we use the parametrization $\rho_\pm^{(j)} \triangleq (1-\gamma)|\theta_{\pm,j}\rangle\langle\theta_{\pm,j}| + \frac{\gamma}{2}\mathbb{I}$, where $|\theta\rangle \triangleq \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ and $j \in \{1, 2, \ldots, N\}$ denotes the subsystem index. Without loss of generality, we set $\theta_{+,j} = -\theta_{-,j}$.

The prior probability of state ρ_+ is denoted by $q \triangleq \mathbb{P}[\rho = \rho_+]$. The permutation $\sigma \in \mathcal{S}_N$, where $\mathcal{S}_N$ is the symmetric group on N elements, is unknown at the beginning of the protocol, and is defined progressively in each round (index by index) when the algorithm determines the next subsystem to measure (assuming no grouping of subsystems, i.e., $m = 1$). At round $j \in \{1, \ldots, N\}$, we determine the next subsystem $\sigma(j)$. Let $A_{\sigma(j)} \in \mathcal{A}$ be the random variable corresponding to the action in the jth round which takes values $\mathbf{a}_{\sigma(j)} \in \mathcal{A}$. The measurement result upon executing the action is represented by the random variable $D_{\sigma(j)} \in \mathcal{D}$ whose realization is denoted $d_{\sigma(j)}$. Here $\mathcal{A}$ is a generic action set which is specified by the type of measurements in any specific scheme, and $\mathcal{D}$ is the space containing possible outcomes for the chosen action set. For example, if $\mathcal{A}$ contains projective measurements on qubits, then $\mathcal{D} = \{\pm 1\}$. For a natural number n, define $[n] \triangleq \{1, \ldots, n\}$. Then at round j, the past actions and results are recorded into the vectors $\mathbf{a}^\sigma_{[j-1]} = (\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(j-1)})$ and $\mathbf{d}^\sigma_{[j-1]} = (d_{\sigma(1)}, \ldots, d_{\sigma(j-1)})$, respectively.

III. RESULTS

A. Locally greedy algorithm

First, we describe a simple extension of the locally greedy algorithm, which was introduced as the "locally optimal locally adaptive" algorithm for identical qubit copies in Ref. [11]. For $m = 1$, at round $j \in [N]$, the algorithm updates the probability that the state is ρ_+ based on the results of past measurements. The algorithm does not consider any nontrivial ordering of subsystems, so $\sigma(j) = j$ for all $j \in [N]$. Once the prior is updated at round j, it performs the Helstrom measurement on the subsystem j according to the given $\rho_\pm^{(j)}$ and this updated prior. To formally describe this process and later generalize it to the dynamic-programming algorithm in the next section, we begin by defining the conditional state probability at round j for a nontrivial permutation σ on the N subsystems.

Definition 1. The *conditional state probability (CSP)* $C_j^\sigma(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]})$ is defined as the probability that $\rho = \rho_+$ given that the starting prior was q, that the first j rounds of measurement were executed with ordering σ and actions $\mathbf{a}^\sigma_{[j]}$, and that the results were $\mathbf{d}^\sigma_{[j]}$. Therefore, the updated prior at round j is the corresponding CSP

$$C_j^\sigma\left(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]}\right) \triangleq \mathbb{P}\left(\rho = \rho_+ \middle| A^\sigma_{[j]} = \mathbf{a}^\sigma_{[j]}, D^\sigma_{[j]} = \mathbf{d}^\sigma_{[j]}\right). \quad (1)$$

Thus, when $j = 0$ we recover the initial prior as $C_0^\sigma(q) \triangleq q$. The dependence of the conditional probability on the initial prior q is left implicit in the above definition and in the following.

Then, the CSP can be computed using past actions and results as

$$C_j^\sigma\left(q, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j]}^\sigma\right) = \frac{\mathbb{P}\left(\rho_+, d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)}{\mathbb{P}\left(d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)} = \frac{\mathbb{P}(d_{\sigma(j)}|\rho_+, \mathbf{a}_{\sigma(j)})C_{j-1}^\sigma}{\mathbb{P}(d_{\sigma(j)}|\rho_+, \mathbf{a}_{\sigma(j)})C_{j-1}^\sigma + \mathbb{P}(d_{\sigma(j)}|\rho_-, \mathbf{a}_{\sigma(j)})\left(1 - C_{j-1}^\sigma\right)}, \tag{2}$$

where in the above, C_{j-1}^σ is an abbreviation for $C_{j-1}^\sigma(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma)$ (see Appendix A for further details of computing the CSP).

We can simplify the notation by defining two quantities:

$$\mathcal{L}(p, \mathbf{a}, d) \triangleq \mathbb{P}(d|\rho_+, \mathbf{a})p + \mathbb{P}(d|\rho_-, \mathbf{a})(1-p), \tag{3}$$

$$\mathcal{P}(p, \mathbf{a}, d) \triangleq \frac{\mathbb{P}(d|\rho_+, \mathbf{a})p}{\mathcal{L}(p, \mathbf{a}, d)}. \tag{4}$$

The naming follows from observing that they represent a likelihood and a posterior, respectively. Thus we can write

$$\mathbb{P}\left(d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) = \mathcal{L}\left(C_{j-1}^\sigma, \mathbf{a}_{\sigma(j)}, d_{\sigma(j)}\right), \tag{5}$$

$$C_j^\sigma\left(q, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j]}^\sigma\right) = \mathcal{P}\left(C_{j-1}^\sigma, \mathbf{a}_{\sigma(j)}, d_{\sigma(j)}\right), \tag{6}$$

where C_{j-1}^σ is an abbreviation for $C_{j-1}^\sigma(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma)$. This completes the description of the Bayesian update in the locally greedy algorithm.

Next we discuss the performance of this algorithm when the N subsystems are identical copies of qubits. In this case, the ordering of the subsystems is clearly immaterial. At round j, the locally greedy algorithm uses the CSP $C_{j-1}^\sigma(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma)$ and applies the (optimal) Helstrom measurement on the jth subsystem. This measurement is defined by the projector

$$\Pi(p, j) \triangleq \sum_{|v\rangle \in \mathcal{V}(p,j)} |v\rangle\langle v|, \tag{7}$$

where, for $M \triangleq (1-p)\rho_-^{(j)} - p\rho_+^{(j)}$,

$$\mathcal{V}(p, j) \triangleq \{|v\rangle |\, M|v\rangle = \lambda|v\rangle;\ \lambda \geqslant 0\}, \tag{8}$$

and $p = C_{j-1}^\sigma(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma)$. Since $\rho_\pm^{(i)} = \rho_\pm^{(j)}$ for all $i, j \in [N]$, the projector $\Pi(p, j)$ changes at every round only because of the changing prior p. The outcome probabilities for a given measurement element Π are given by

$$\mathbb{P}(d|\rho_\pm^{(j)}, \Pi) = \begin{cases} 1 - \mathrm{Tr}(\Pi\rho_\pm^{(j)}) & \text{if } d = +1 \\ \mathrm{Tr}(\Pi\rho_\pm^{(j)}) & \text{if } d = -1. \end{cases} \tag{9}$$

Thus, the probability of error for a decision after round j of the locally greedy algorithm is given by

$$P_{\mathrm{err},j} = (1-p)[1 - \mathrm{Tr}(\Pi(p, j)\rho_-^{(j)})] \tag{10}$$

$$+ p\mathrm{Tr}(\Pi(p, j)\rho_+^{(j)}). \tag{11}$$

Under the locally greedy algorithm, the probability of successfully distinguishing between states ρ_+ and ρ_- is given by

$$P_{\mathrm{lg}}(q, \rho_\pm) \triangleq 1 - P_{\mathrm{err},N}. \tag{12}$$

For the case where $\rho_\pm$ is a tensor product of arbitrary pure states, we now prove that the locally greedy algorithm (and hence the MOODY algorithm) achieves the optimal Helstrom probability of success.

Theorem 2. Let $P_{\mathrm{h}}(q, \rho_\pm)$ and $P_{\mathrm{lg}}(q, \rho_\pm)$ denote the probabilities of successful state discrimination, given initial prior $\mathbb{P}(\rho = \rho_+) = q$, using the joint N-system Helstrom measurement and the locally greedy measurement technique, respectively. If ρ_+ and ρ_- are pure states, i.e., $\rho_\pm^{(j)} = |\pm\theta_j\rangle\langle\pm\theta_j|$ where $|\theta\rangle \triangleq \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$, for some $\theta_j \in (0, 2\pi)$ for every $j \in [N]$, then

$$P_{\mathrm{h}}(q, \rho_\pm) = P_{\mathrm{lg}}(q, \rho_\pm) \tag{13}$$

$$= \frac{\left(1 + \sqrt{1 - 4q(1-q)\Pi_{j=1}^N \cos^2(\theta_j)}\right)}{2}. \tag{14}$$

Proof. See Appendix B. The idea is to prove the result for $N = 2$ and then extend via induction for arbitrary N. ■

1. Plateau with locally greedy algorithm

We now demonstrate that the locally greedy algorithm exhibits a plateau in performance for mixed (depolarized) states using the following experimental setup. For now, we assume the prior $q = \frac{1}{2}$ in all cases unless specified otherwise:

(1) We choose $\mathcal{S}_{\mathrm{dep}} = \{0.01, 0.05, 0.1, 0.3\}$ to be our set of allowed depolarizing parameters (i.e., possible values of γ.) This set thus includes the case of minimal depolarization ($\gamma = 0.01$) and moderate depolarization ($\gamma = 0.3$) as well as intermediate values. We set the number of trials to be $n_{\mathrm{trial}} = 1000$.

(2) Generate $\theta_\pm^{(t)} \in (0, 2\pi)$ uniformly, where $t \in [n_{\mathrm{trial}}]$ denotes the trial index.

(3) For each $\gamma \in \mathcal{S}_{\mathrm{dep}}$, define the corresponding qubit quantum states $\rho_\pm(\gamma, t) \triangleq (1-\gamma)|\theta_\pm^{(t)}\rangle\langle\theta_\pm^{(t)}| + \frac{\gamma}{2}I$, where

$$|\theta\rangle \triangleq \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle. \tag{15}$$

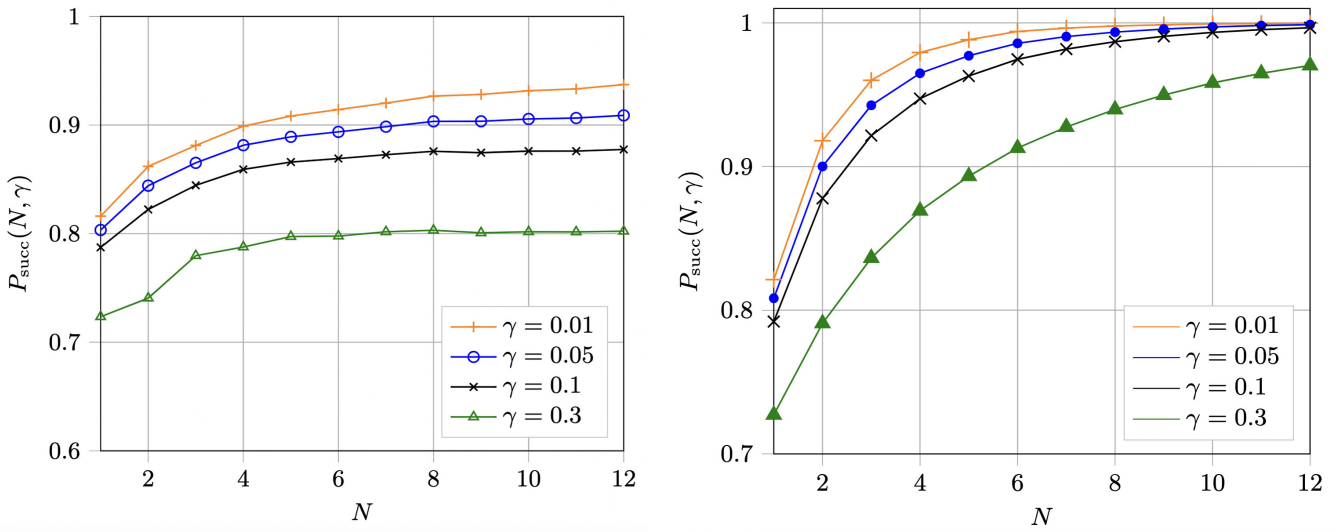
Note that the subscript $\pm$ in $\theta_\pm^{(t)}$ is used to represent that the angles are chosen independently for the ρ_+ and ρ_- states.

(4) For all $\gamma \in \mathcal{S}_{\mathrm{dep}}$ and all $N = 1, 2, \ldots, 12$, we define the candidate TPQS generated by the random sampling as denoted by

$$P_{\mathrm{succ}}(N, \gamma) = \frac{1}{n_{\mathrm{trial}}} \sum_{t=1}^{n_{\mathrm{trial}}} P_{\mathrm{lg}}(\rho_\pm(\gamma, t)^{\otimes N}), \tag{16}$$

where $P_{\mathrm{lg}}(\rho_\pm)$ is the success probability for the locally greedy algorithm and candidate states $\rho_\pm(\gamma, t)^{\otimes N}$.

In the above, we randomly sample a set of pure states $\{|\theta_\pm^{(t)}\rangle\langle\theta_\pm^{(t)}|\}|_{t=1}^{n_{\mathrm{trial}}}$ and generate the corresponding set of candidate states $\{\rho_\pm(\gamma, t)^{\otimes N}\}|_{t=1}^{n_{\mathrm{trial}}}$ for each N and γ. Thus, $P_{\mathrm{succ}}(N, \gamma)$ represents the Monte Carlo average of performance for fixed N and γ.



(a) Locally Greedy (LG) (b) Modified Locally Greedy (MLG)

FIG. 1. Comparison of the probability of success as a function of the number of available systems for the case of identical copies. As the depolarizing parameter γ increases, the LG probability of success levels off for large N while the MLG probability of success converges to 1.

We plot the results of this computational experiment in Fig. 1(a). We observe that the average probability of success (asymptotically) approaches a value strictly less than 1 when the depolarizing parameter is sufficiently high. In the limit as $\gamma = 0$, the probability of success must approach 1 with increasing N because the locally greedy approach recovers the optimal Helstrom performance (see Theorem 2). Next, we prove a result that explains the performance plateau in Fig. 1(a) and then define a modified locally greedy approach that overcomes this suboptimality.

Lemma 3. For two d-dimensional qudit states ρ_+ and ρ_-, suppose we are given ρ_+ with probability q where $q \leqslant \frac{1}{2}$. Let the depolarized versions of $\rho_\pm$ be denoted by

$$\rho_\pm^{\text{dep}} := (1-\gamma)\rho_\pm + \frac{\gamma}{d} I. \tag{17}$$

Let the probability of distinguishing ρ_+ and ρ_- be P_{succ}. If

$$\frac{\gamma}{1-\gamma}\frac{1-2q}{d}$$

is less than the magnitude of the largest negative eigenvalue of $(1-q)\rho_- - q\rho_+$, then the probability of distinguishing ρ_+^{dep} and ρ_-^{dep} is given by

$$P_{\text{succ}}^{\text{dep}} = \gamma q + \frac{\gamma(1-2q)k}{d} + (1-\gamma)P_{\text{succ}}, \tag{18}$$

where k is the rank of the Helstrom projector distinguishing ρ_+ and ρ_-.

Proof. See Appendix C. ■

In the case of qubits, this lemma implies the following corollary:

Corollary 4. Consider the problem of distinguishing between two distinct single qubit states ρ_+^{dep} and ρ_-^{dep} with prior probabilities q and $1-q$ respectively. Assume that ρ_+^{dep} and ρ_-^{dep} are depolarized such that there exist pure states $|\psi_\pm\rangle$ where

$$\gamma_\pm \in [0,1] \quad \text{and} \quad \rho_\pm^{\text{dep}} \triangleq (1-\gamma_\pm)|\psi_\pm\rangle\langle\psi_\pm| + \frac{\gamma_\pm}{2} I.$$

For any choice of $\gamma_\pm$, $q \in [0,1]$ the probability of correctly distinguishing ρ_+^{dep} and ρ_-^{dep}, is denoted by $P_{\text{succ}}^{\text{dep}}$ and satisfies

$$P_{\text{succ}}^{\text{dep}} \leqslant \max\left\{1-q, q, 1-\frac{\gamma_{\min}}{2}\right\} \tag{19}$$

where $\gamma_{\min} \triangleq \min(\gamma_+, \gamma_-)$.

Proof. See Appendix D. ■

Assume with loss of generality that $q \leqslant \frac{1}{2}$. Then, observe that $1-q \geqslant 1-\frac{\gamma}{2}$ implies $\gamma \geqslant 2q$ and therefore $\frac{\gamma}{1-\gamma}\frac{(1-2q)}{qd} \geqslant 1$ $(d=2)$. In the notation of Lemma 3, set $\rho_\pm = |\psi_\pm\rangle\langle\psi_\pm|$. Since the spectrum of $[(1-q)|\psi_-\rangle\langle\psi_-| - q|\psi_+\rangle\langle\psi_+|]$ lies in the interval $[-1,1]$, Eq. (C1) implies that the smallest eigenvalue of $[(1-q)\rho_-^{\text{dep}} - q\rho_+^{\text{dep}}]$ will now be non-negative and hence $\Pi_{\text{Hel},\rho_\pm} = I$ will be trivial. In Appendix I we show that such trivial measurements cause the local greedy method to exhibit plateaus in success probability for general TPQS.

This result provides motivation for us to modify the conventional locally greedy method discussed above (first introduced by Refs. [10] and [11]).

B. Modified locally greedy algorithm

Like the locally greedy algorithm, the MLG algorithm updates the prior after each measurement round. Then, it performs the modified Helstrom measurement according to the new prior, with the modified Helstrom measurement defined by

$$\Pi^*(p,j) \triangleq \begin{cases} \Pi(p,j) & \text{if } \Pi(p,j) \notin \{\mathbb{I}, 0\} \\ |v_{\lambda_{\max}}\rangle\langle v_{\lambda_{\max}}| & \text{if } \Pi(p,j) = 0 \\ \mathbb{I} - |v_{\lambda_{\min}}\rangle\langle v_{\lambda_{\min}}| & \text{if } \Pi(p,j) = \mathbb{I}, \end{cases} \tag{20}$$

where, for $M = (1-p)\rho_-^{(j)} - p\rho_+^{(j)}$, we define

$$\lambda_{\max} \triangleq \max_{\lambda} \{\lambda | M | v_\lambda\rangle = \lambda | v_\lambda\rangle\}, \tag{21}$$

$$\lambda_{\min} \triangleq \min_{\lambda} \{\lambda | M | v_\lambda\rangle = \lambda | v_\lambda\rangle\}, \tag{22}$$

and where the final state is decoded as $\hat{\rho} = \rho_+$ if $C_N^\sigma(q, \mathbf{a}_{[N]}^\sigma, \mathbf{d}_{[N]}^\sigma) \geqslant \frac{1}{2}$ and as $\hat{\rho} = \rho_-$ otherwise.

The key idea is that, in the case where the Helstrom measurement is trivial, the MLG algorithm creates an informative measurement by partitioning projectors based on the *ordering* of their eigenvalues rather than the sign of the eigenvalues. Thus, it separates out the projector that is most strongly predictive of the less-likely candidate state.

Denote by $P_{\text{s,MLG}}(q, \rho_\pm)$ the success probability of distinguishing $\{\rho_+, \rho_-\}$ with initial prior q using the MLG algorithm. We now show that the MLG method exhibits the desired asymptotic behavior in the limit of large N.

Lemma 5. For any $\rho_\pm$ where $\rho_+^{(j)} \neq \rho_-^{(j)}$ for all subsystems j, then in the limit $N \to \infty$, $P_{\text{s,MLG}}(q, \rho_\pm) = 1$.

Proof. See Appendix E. ∎

From the above, we can conclude that $P_{\text{s,MLG}}(\rho_\pm) \geqslant P_{\text{s,LG}}(\rho_\pm)$ as the MLG and LG methods are equivalent whenever the Helstrom measurement is nontrivial. When the Helstrom measurement is trivial, it follows that the MLG method does strictly better. The improved asymptotic behavior of the MLG algorithm is depicted in Fig. 1(b), where we repeat the previous experimental setup with the MLG algorithm, and plot the resulting $P_{\text{succ}}(N, \gamma) = \frac{1}{n_{\text{trial}}} \sum_{t=1}^{n_{\text{trial}}} P_{\text{s,LG}}(\rho_\pm(\gamma, t)^{\otimes N})$.

In the next section, we generalize to a dynamic programming based algorithm capable of optimizing over the order of subsystem measurement as well as the measurement performed on each subsystem.

IV. DYNAMIC-PROGRAMMING ALGORITHMS

A. Order-optimized modified locally greedy algorithm

In the order-optimized MLG algorithm, we choose $\sigma(j)$ carefully in each round j. To do this, we recursively compute an *expected future risk* function $R_S : [0, 1] \to [0, 1]$, where S denotes the set of subsystem indices that are yet to be measured and where the domain is the updated prior. Formally, at round j,

$$S \triangleq [N] \setminus \sigma([j-1]). \tag{23}$$

(i) For the base case $S = \emptyset$, one can make a hard decision on $C_N^\sigma(q, \mathbf{a}_{[N]}^\sigma, \mathbf{d}_{[N]}^\sigma)$, i.e., by comparing it to 0.5. Namely, upon setting $p = C_N^\sigma(q, \mathbf{a}_{[N]}^\sigma, \mathbf{d}_{[N]}^\sigma)$, we have

$$R_\emptyset(p) = \min(p, 1-p) \tag{24}$$

for $p \in [0, 1]$.

(ii) For the general case $S \neq \emptyset$ and $j = N - |S| + 1$, $N - |S|$ measurements have been performed. The goal is to choose the best subsystem $\sigma(j)$ to be measured next with the modified Helstrom measurement in order to minimize the expected error probability over the remaining measurements, i.e.,

$$\begin{aligned} &R_S\big[C_{N-|S|}^\sigma\big(q, \mathbf{a}_{[N-|S|]}^\sigma, \mathbf{d}_{[N-|S|]}^\sigma\big)\big] \\ &\quad = \min_{k \in S} \sum_{d_k \in \mathcal{D}} \mathbb{P}\big(d_k | q, \big(\mathbf{a}_{[N-|S|]}^\sigma, \mathbf{a}_k\big), \mathbf{d}_{[N-|S|]}^\sigma\big) \\ &\qquad R_{S\setminus\{k\}}\big[C_{N-|S|+1}^\sigma\big(q, \big(\mathbf{a}_{[N-|S|]}^\sigma, \mathbf{a}_k\big), \big(\mathbf{d}_{[N-|S|]}^\sigma, d_k\big)\big)\big], \end{aligned} \tag{25}$$

where $\mathbf{a}_k = \Pi^*(p, k)$ is the modified Helstrom measurement, with $p = C_{N-|S|}^\sigma(q, \mathbf{a}_{[N-|S|]}^\sigma, \mathbf{d}_{[N-|S|]}^\sigma) = C_{j-1}^\sigma(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma)$.

This expression can be written as a function of $p \in [0, 1]$ as

$$R_S(p) = \min_{k \in S} \sum_{d_k \in \mathcal{D}} \mathcal{L}(p, \Pi^*(p, k), d_k) \tag{26}$$

$$R_{S\setminus\{k\}}[\mathcal{P}(p, \Pi^*(p, k), d_k)]. \tag{27}$$

The next mapping for σ at round j can be defined as

$$\sigma(j) \triangleq \operatorname*{argmin}_{k \in S} \sum_{d_k \in \mathcal{D}} \mathcal{L}(p, \Pi^*(p, k), d_k) \tag{28}$$

$$R_{S\setminus\{k\}}[\mathcal{P}(p, \Pi^*(p, k), d_k)], \tag{29}$$

where $\mathcal{L}$ and $\mathcal{P}$ are defined in (11).

Similar to the case of identical copies, the measurement outcome probabilities are given by (9) with $\Pi(p, j)$ replaced by $\Pi^*(p, j)$, the probability of error at round j is given by (10), and the overall probability of success of the order-optimized locally greedy algorithm is given by (12) with N replaced by $\sigma(N)$.

B. Measurement- and order-optimized dynamic algorithm

The measurement- and order-optimized dynamic (MOODY) algorithm is a generalization of the order-optimized locally greedy algorithm described above for the distinct subsystems scenario. During execution of round j, the algorithm optimizes over all choices of $\sigma(j)$ as well as the measurement actions that could be performed over the chosen subsystem $\sigma(j)$. Hence, the expected future risk function is given by

$$R_S(p) = \min_{(k, \mathbf{a}_k) \in S \times \mathcal{A}} \sum_{d_k \in \mathcal{D}(\mathcal{A})} \mathcal{L}(p, \mathbf{a}_k, d_k) R_{S\setminus\{k\}}[\mathcal{P}(p, \mathbf{a}_k, d_k)]. \tag{30}$$

An optimal choice for the next subsystem, $k \in S$, and the optimal action to be performed on that subsystem, $\mathbf{a}_k \in \mathcal{A}$, are given by the minimizer $\hat{A}_S(p) = (k, \mathbf{a}_k)$ of the above function:

$$\hat{A}_S(p) \triangleq \operatorname*{argmin}_{(k, \mathbf{a}_k) \in S \times \mathcal{A}} \sum_{d_k \in \mathcal{D}(\mathcal{A})} \mathcal{L}(p, \mathbf{a}_k, d_k) R_{S\setminus\{k\}}[\mathcal{P}(p, \mathbf{a}_k, d_k)]. \tag{31}$$

Therefore, during round j of the algorithm, we have $j = N - |S| + 1$ and we choose

$$(\sigma(j), \mathbf{a}_{\sigma(j)}) = \hat{A}_S\big(C_{j-1}^\sigma\big(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\big)\big). \tag{32}$$

The MOODY algorithm can be summarized as below. Once the DP subroutine is completed, we have a set of

expected future error functions $\{R_S|S \subseteq [N]\}$ and a set of best measurement action functions $\{A_S|S \subseteq [N]\}$. Setting $p_0 = \mathbb{P}(\hat{\rho} = \hat{\rho}_+) = q$ and $S_0 = \{1, \ldots, N\}$, we then have for $i = 0, \ldots, N-1$:

$$P_i^{\text{err}} = R_{S_i}(p_i), \tag{33a}$$

$$\hat{A}_{S_i}(p_i) = (\sigma(i+1), \mathbf{a}_{\sigma(i+1)}), \tag{33b}$$

$$\mathcal{L}(p_i, \mathbf{a}_{\sigma(i+1)}, d) = \mathbb{P}(d_{\sigma(i+1)} = d|p_i, \mathbf{a}_{\sigma(i+1)}), \tag{33c}$$

$$p_{i+1} = \mathcal{P}(p_i, \mathbf{a}_{\sigma(i+1)}, d_{\sigma(i+1)}), \tag{33d}$$

$$S_{i+1} = S_i \setminus \{\sigma(i+1)\}. \tag{33e}$$

Finally, after N rounds of measurements one makes the decision to decode ρ as $\hat{\rho}$ for

$$\hat{\rho} = \begin{cases} \rho_+, & \text{if } p_N > 0.5 \\ \rho_-, & \text{if } p_N < 0.5 \\ \text{random guess}, & \text{if } p_N = 0.5, \end{cases} \tag{34}$$

and $P_{\text{succ}} = \max(p_N, 1 - p_N)$.

For more details about implementation and complexity, see Appendix F.

C. Results for qubits and qutrits

We demonstrate that the order in which subsystems are measured can generally affect the success probability. We first show this analytically by considering candidate states of the form

$$\rho_+ = \begin{pmatrix} 1-x & 0 \\ 0 & x \end{pmatrix} \otimes |\theta\rangle\langle\theta|,$$

$$\rho_- = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} \otimes |-\theta\rangle\langle\theta|.$$

Measuring the subsystems in the best order (diagonal matrices first followed by $|\pm\theta\rangle\langle\pm\theta|$) yields a success probability of $P_{\text{succ,best}} = \frac{1}{2}\{1 + [1 - 4(1-x)x\cos^2(2\theta)]^{1/2}\}$, whereas measuring in the reverse order yields a probability of success of $P_{\text{succ,worst}} = \max\{x, 1-x, \frac{1}{2}(1 + [1 - \frac{1}{2}\cos^2(2\theta)]^{1/2})\}$.

We now provide numerical experiments illustrating the difference between "best" and "worst" ordering: results for the qubit case are discussed in Appendix G where we demonstrate that there is a small difference in success probability between the best and worst case ordering. Additionally, this difference persists even when using the MOODY algorithm.

We now discuss the qutrit case.

D. Qutrit results

We begin our qutrit results by investigating whether binary projectors are sufficient for qutrit (3-dimensional) quantum subsystems.

Definition 6. Action space $\mathcal{A}$ is *sufficient* for state space $\mathcal{H}$ if and only if for all $\rho_\pm \in \mathcal{H}$ and $q \in [0, 1]$,

$$P_{\text{succ},\mathcal{A}}(q, \rho_\pm) = P_{\text{succ},\mathcal{A}_{\text{all}}}(q, \rho_\pm),$$

where $\mathcal{A}_{\text{all}}$ is the set of all quantum measurements of dimension $\dim(\rho_\pm)$ and $P_{\text{succ},\mathcal{A}}(q, \rho_\pm)$ is the probability of success of the order-optimized MOODY algorithm for a given action space $\mathcal{A}$.

For pure states, Theorem 2 confirms that binary projectors are sufficient, and by the definition of the Helstrom measurement, binary projectors are additionally sufficient whenever $N = 1$.

We show by example that binary projective measurements are not sufficient for general state spaces. To this aim, we define $\mathcal{H}_{\text{qutrit}}$ to be the space of depolarized, real qutrit states and define the action space of real binary (ternary) measurements as $\mathcal{A}_{\text{b}}$ ($\mathcal{A}_{\text{t}}$):

$$\mathcal{A}_{\text{b}} \triangleq \Big\{\big\{\Pi_j^{\text{b}}\big\}_{j=1}^2 \Big| \Pi_j^{\text{b}}\Pi_{j'}^{\text{b}} = \delta_{j,j'}\Pi_j^{\text{b}}\ \forall\ j, j' \in \{1, 2\}, \tag{35}$$

$$\text{rank}\big(\Pi_1^{\text{b}}\big) = 2,\ \text{rank}\big(\Pi_2^{\text{b}}\big) = 1\Big\}, \tag{36}$$

$$\mathcal{A}_{\text{t}} \triangleq \Big\{\big\{\Pi_j^{\text{t}}\big\}_{j=1}^3 \Big| \Pi_j^{\text{t}}\Pi_{j'}^{\text{t}} = \delta_{j,j'}\Pi_j^{\text{t}}, \tag{37}$$

$$\text{rank}\big(\Pi_j^{\text{t}}\big) = 1\ \forall\ j, j' \in \{1, 2, 3\}\Big\}. \tag{38}$$

Parametrizing the action spaces is equivalent to generating (with some quantization) all orthonormal bases $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ and defining the corresponding ternary positive operator-valued measure (POVM) $\Pi^{\text{t}}(\{|u_1\rangle, |u_2\rangle, |u_3\rangle\})$ and three corresponding binary POVMs $\Pi^{\text{b},k}(\{|u_1\rangle, |u_2\rangle, |u_3\rangle\})$, for $k \in \{1, 2, 3\}$ as follows:

$$\Pi^{\text{t}}\big(\{|u_j\rangle\}|_{j=1}^3\big) \triangleq \{|u_1\rangle\langle u_1|,\ |u_2\rangle\langle u_2|,\ |u_3\rangle\langle u_3|\}, \tag{39}$$

$$\Pi^{\text{b},k}\big(\{|u_j\rangle\}|_{j=1}^3\big) \triangleq \left\{\sum_{l\neq k} |u_l\rangle\langle u_l|,\ |u_k\rangle\langle u_k|\right\}. \tag{40}$$

We define $|u_j\rangle = R(\phi, \theta)\vec{v}_j(\omega)$, where

$$\vec{v}_1(\omega) = \begin{bmatrix} \cos(\omega) \\ \sin(\omega) \\ 0 \end{bmatrix},$$

$$\vec{v}_2(\omega) = \begin{bmatrix} -\sin(\omega) \\ \cos(\omega) \\ 0 \end{bmatrix}, \quad \vec{v}_3(\omega) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and

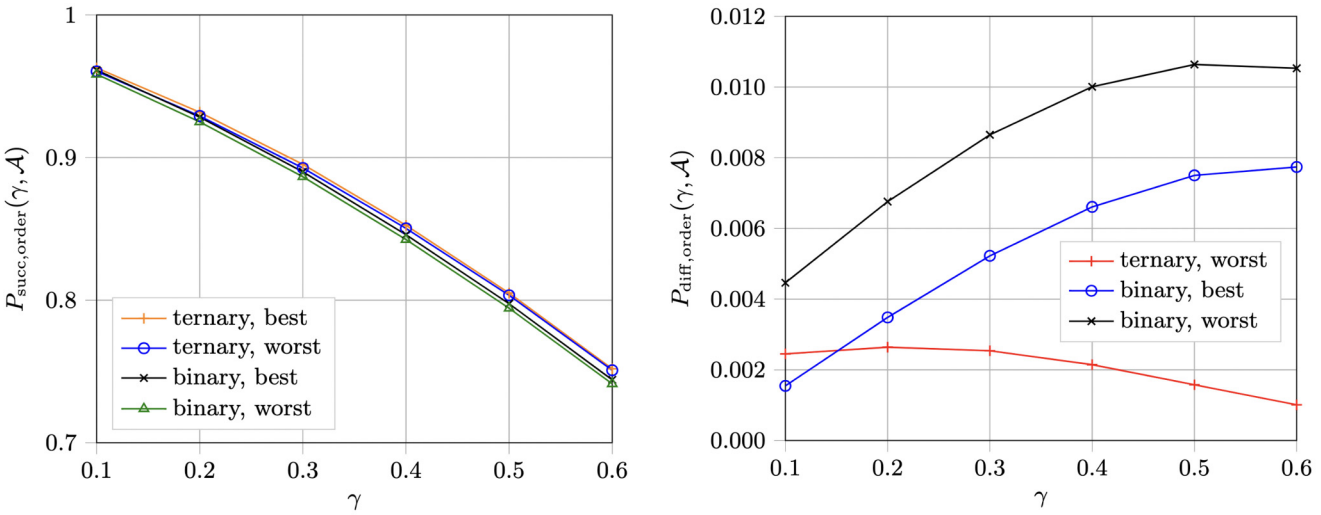
$$R(\phi, \theta) = \begin{pmatrix} -\sin(\phi) & \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) \\ \cos(\phi) & \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

For details of quantization of (ϕ, θ, ω) and sufficiency of using real-valued quantum measurements, see Appendix H.

Using the following procedure, we demonstrate that for general real depolarized qutrit states $\{\hat{\rho}_+, \hat{\rho}_-\}$, $P_{\text{succ},\mathcal{A}_{\text{b}}}(\frac{1}{2}, \hat{\rho}_\pm) < P_{\text{succ},\mathcal{A}_{\text{t}}}(\frac{1}{2}, \hat{\rho}_\pm)$, and hence binary projective measurements are not sufficient.

(1) Set $N = 3$, $\mathcal{S}_{\text{dep}} = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$, and $n_{\text{trial}} = 1000$.

(2) Generate $\alpha_\pm^{(t,j)}, \beta_\pm^{(t,j)} \in (0, 1)$ uniformly, where $t \in [n_{\text{trial}}]$ denotes the trial index, and $j = 1, 2, \ldots, N$ denotes



(a) Average Success Probability (b) Difference in Success Probability

FIG. 2. Left-hand side depicts the average probability of success for the best and worst ordering using both ternary and binary projective measurements for qutrit product states when $N = 3$. Results are averaged over 1000 trials. Right-hand side depicts difference in average success probability for the various methods, namely, $P_{\text{diff,order}}(\gamma, \mathcal{A})$ as a function of γ when $N = 3$. Results are averaged over 1000 trials.

the subsystem index. Set $\phi_{\pm}^{(t,j)} = 2\pi\alpha_{\pm}^{(t,j)}$ and $\theta = \arccos(1 - 2\beta_{\pm}^{(t,j)})$

(3) For each $\gamma \in \mathcal{S}_{\text{dep}}$, define the corresponding qutrit quantum states

$$\rho_{\pm}(\gamma, t, N) \triangleq \bigotimes_{j=1}^{N} \left((1-\gamma) \left| v\left(\phi_{\pm}^{(t,j)}, \theta_{\pm}^{(t,j)}\right) \right\rangle \right. \\ \left. \times \left\langle v\left(\phi_{\pm}^{(t,j)}, \theta_{\pm}^{(t,j)}\right) \right| + \frac{\gamma}{3} I \right). \tag{41}$$

(4) For each $\rho_{\pm}(\gamma, t, N)$ perform the MOODY algorithm for $\mathcal{A}_{\text{b}}$ and $\mathcal{A}_{\text{t}}$ for both best ordering and worst ordering.

(5) For $\gamma \in \mathcal{S}_{\text{dep}}$, given an order of "best" or "worst," and given an action space $\mathcal{A} \in \{\mathcal{A}_{\text{b}}, \mathcal{A}_{\text{t}}\}$, denote

$$P_{\text{s,order}}(\gamma, \mathcal{A}) = \frac{1}{n_{\text{trial}}} \sum_{t=1}^{n_{\text{trial}}} P_{\text{order}}(\rho_{\pm}(\gamma, t, N), \mathcal{A}), \tag{42}$$

where $P_{\text{order}}(\rho_{\pm}, \mathcal{A})$ indicates that we perform the MOODY algorithm over action space $\mathcal{A}$ with corresponding ordering on states $\rho_{\pm}$.

We plot the results of all four methods in Fig. 2(a), and compare the difference of the remaining three methods to the ternary, best ordering method $[P_{\text{diff,order}}(\gamma, \mathcal{A}) = P_{\text{s,best}}(\gamma, \mathcal{A}_t) - P_{\text{s,order}}(\gamma, \mathcal{A})]$ in Fig. 2(b). We observe that the best ternary ordering is better than best binary ordering, and affects performance even in the MOODY algorithm. We conjecture that, for any action space and any adaptive approach, there is a candidate state set such that the order of subsystem measurement will affect the success probability. It remains an open question whether it is always sufficient to consider d rank-1 orthogonal projectors for d-dimensional real quantum subsystems.

V. CONCLUSION

In this work, we investigated simple locally greedy and modified locally greedy algorithms as well as more general dynamic programming-based algorithms for quantum state discrimination when the given states are tensor products of N arbitrary qubit or qutrit states. We analytically proved that when the individual subsystems are pure states the simple locally greedy algorithm achieves the optimal performance of the joint N-system Helstrom measurement. For the scenario where each subsystem contains identical copies of arbitrary qubit states, we demonstrated a plateau in the probability of success attained by the locally greedy algorithm with increasing N and introduced a modified locally greedy algorithm with strictly better performance whose state discrimination became perfect in the large-N limit.

For the general MOODY algorithm, we show that ordering of subsystems continues to affect the performance when the individual subsystems have distinct states. Finally, for qutrit states, we showed that binary projective measurements are inadequate to achieve optimal performance. Our work has potential implications for developing locally adaptive measurement protocols in more general quantum hypothesis settings.

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APPENDIX A: CONDITIONAL STATE PROBABILITY

The CSP can be computed using past actions and results as

$$C_j^\sigma\left(q, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j]}^\sigma\right) = \frac{\mathbb{P}\left(\rho_+, d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)}{\mathbb{P}\left(d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)} \tag{A1}$$

$$= \frac{\mathbb{P}\left(\rho_+, d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)}{\mathbb{P}\left(\rho_+, d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) + \mathbb{P}\left(\rho_-, d_{\sigma(j)}|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)} \tag{A2}$$

$$= \frac{\mathbb{P}\left(d_{\sigma(j)}|\rho_+, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)\mathbb{P}\left(\rho_+|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)}{\sum_{x\in\{+,-\}}\mathbb{P}\left(d_{\sigma(j)}|\rho_x, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)\mathbb{P}\left(\rho_x|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)}, \tag{A3}$$

where, in the second equality, we have marginalized over the possible values of ρ in the denominator. Now we make some observations that will allow us to simplify this expression. Since all measurements are performed on different subsystems, the outcome of the jth measurement does not depend on the previous $j-1$ measurements once the jth action is given, i.e.,

$$\left[d_{\sigma(j)} \perp \left(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right)\right]|\mathbf{a}_{\sigma(j)} \tag{A4}$$

$$\implies \mathbb{P}\left(d_{\sigma(j)}|\rho_\pm, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) = \mathbb{P}(d_{\sigma(j)}|\rho_\pm, \mathbf{a}_{\sigma(j)}). \tag{A5}$$

Executing a measurement action without knowing the outcome does not help with inference, namely,

$$\mathbb{P}\left(\rho = \rho_+|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) = \mathbb{P}\left(\rho = \rho_+|\mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) \tag{A6}$$

$$= C_{j-1}^\sigma\left(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right), \tag{A7}$$

and

$$\mathbb{P}\left(\rho = \rho_-|\mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) = \mathbb{P}\left(\rho = \rho_-|\mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right) \tag{A8}$$

$$= 1 - C_{j-1}^\sigma\left(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma\right). \tag{A9}$$

For two different permutations σ and τ, if for all $i = 1, \ldots, j$ we have $\sigma(i) = \tau(i)$, $\mathbf{a}_{\sigma(i)} = \mathbf{a}_{\tau(i)}$, and $d_{\sigma(i)} = d_{\tau(i)}$, then $C_j^\sigma(q, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j]}^\sigma) = C_j^\tau(q, \mathbf{a}_{[j]}^\tau, \mathbf{d}_{[j]}^\tau)$.

Applying these observations, we find that $C_j^\sigma(q, \mathbf{a}_{[j]}^\sigma, \mathbf{d}_{[j]}^\sigma)$ is equivalent to

$$\frac{\mathbb{P}(d_{\sigma(j)}|\rho_+, \mathbf{a}_{\sigma(j)})C_{j-1}}{\mathbb{P}(d_{\sigma(j)}|\rho_+, \mathbf{a}_{\sigma(j)})C_{j-1}^\sigma + \mathbb{P}(d_{\sigma(j)}|\rho_-, \mathbf{a}_{\sigma(j)})\left(1 - C_{j-1}^\sigma\right)}. \tag{A10}$$

where in the above, C_{j-1}^σ is an abbreviation for $C_{j-1}^\sigma(q, \mathbf{a}_{[j-1]}^\sigma, \mathbf{d}_{[j-1]}^\sigma)$.

APPENDIX B: PROOF OF THEOREM 2

We prove the statement by induction, first considering the base case where $N = 2$ and additionally specifying $q = \frac{1}{2}$. We leave q implicit in the following until stated otherwise. Then the probability of success can be written as

$$P_{\text{lg}}\left(\frac{1}{2}, \rho_\pm\right) = \sum_{d_1\in\{+,-\}}\mathbb{P}(\rho = \rho_+)\mathbb{P}(d_1|\rho_+)\mathbb{P}(d_2 = +|\rho_+, d_1) + \sum_{d_1\in\{+,-\}}\mathbb{P}(\rho = \rho_-)\mathbb{P}(d_1|\rho_-)\mathbb{P}(d_2 = -|\rho_-, d_1)$$

$$= \frac{1}{2}\sum_{d_1\in\{+,-\}}[\mathbb{P}(d_1|\rho_+)\mathbb{P}(d_2 = +|\rho_+, d_1) \tag{B1}$$

$$+ \mathbb{P}(d_1|\rho_-)\mathbb{P}(d_2 = -|\rho_-, d_1)] \tag{B2}$$

$$= \mathbb{P}(d_1 = +|\rho_+)\mathbb{P}(d_2 = +|\rho_+, d_1 = +) \tag{B3}$$

$$+ \mathbb{P}(d_1 = +|\rho_-)\mathbb{P}(d_2 = -|\rho_-, d_1 = +), \tag{B4}$$

where the final equality follows from symmetry of probability outcomes for pure states, i.e.,

$$\mathbb{P}(d_j = +|\rho_+, d_1 = \pm) = \mathbb{P}(d_j = -|\rho_-, d_1 = \mp).$$

From the definition of the Helstrom measurement, we observe that

$$\mathbb{P}(d_1 = \pm|\rho_\pm) = \tfrac{1}{2}[1 + \sin(\theta_1)], \quad \mathbb{P}(d_1 = \pm|\rho_\mp) = \tfrac{1}{2}[1 - \sin(\theta_1)]. \tag{B5}$$

Hence the updated prior is $p_1(d_1) = \frac{1}{2}[1 + d_1 \sin(\theta_1)]$. Then according to the locally greedy algorithm,

$$\mathbb{P}(d_2|\rho_\pm, d_1) = \begin{cases} 1 - \mathrm{Tr}[\Pi(p_1(d_1), j = 1)\rho_\pm] & \text{if } d_2 = + \\ \mathrm{Tr}[\Pi(p_1(d_1), j = 1)\rho_\pm] & \text{if } d_2 = -. \end{cases} \tag{B6}$$

Equations 2.13 and 2.14 from Ref. [9] provide a solution for $\mathbb{P}(d_2|\rho_\pm, d_1)$. Upon simplifying, we observe

$$\mathbb{P}(d_2 = \pm|\rho_+, d_1 = \pm) = \frac{1}{2}\left(1 \pm \frac{\sin^2\theta_2 \pm \cos^2\theta_2 \sin\theta_1}{\sqrt{\cos^2(\theta_2)\sin^2(\theta_1) + \sin^2(\theta_2)}}\right). \tag{B7}$$

Again using the symmetry property we have

$$\mathbb{P}(d_2 = \pm|\rho_-, d_1 = \pm) = \frac{1}{2}\left(1 \mp \frac{\sin^2\theta_2 \pm \cos^2\theta_2 \sin\theta_1}{\sqrt{\cos^2(\theta_2)\sin^2(\theta_1) + \sin^2(\theta_2)}}\right). \tag{B8}$$

Upon substitution we obtain

$$\begin{aligned} P_{\mathrm{lg}}\left(\tfrac{1}{2}, \rho_\pm\right) &= \mathbb{P}(d_1 = +|\rho_+)\mathbb{P}(d_2 = +|\rho_+, d_1 = +) + \mathbb{P}(d_1 = +|\rho_-)\mathbb{P}(d_2 = -|\rho_-, d_1 = +) &\text{(B9)}\\ &= \tfrac{1}{2}(1 + \sqrt{\cos^2(\theta_2)\sin^2(\theta_1) + \sin^2(\theta_2)}) \\ &= \tfrac{1}{2}(1 + \sqrt{1 - \cos^2(\theta_1)\cos^2(\theta_2)}). &\text{(B10)}\end{aligned}$$

For the inductive step, we define a new variable $\tilde{\theta} \in [0, \pi]$ such that $\cos^2(\tilde{\theta}) \triangleq \Pi_{i=1}^{N-1}\cos^2(\theta_i)$. Then by assumption

$$P_{\mathrm{lg}}\left(\tfrac{1}{2}, \rho_\pm^{(1,\ldots,N-1)}\right) = \tfrac{1}{2}\left(1 + \sqrt{1 - \Pi_{i=1}^{N-1}\cos^2(\theta_i)}\right) = \tfrac{1}{2}(1 + \sqrt{1 - \cos^2(\tilde{\theta})}). \tag{B11}$$

We then apply the previously shown statement for $N = 2$, letting the first subsystem now be the combined subsystems $1, 2, \ldots, N-1$, i.e., $\rho_\pm^{(1,\ldots,N-1)}$.

$$\begin{aligned} P_{\mathrm{lg}}\left(\tfrac{1}{2}, \rho_\pm\right) &= \tfrac{1}{2}(1 + \sqrt{1 - \cos^2(\tilde{\theta})\cos^2(\theta_N)}) = \tfrac{1}{2}\left(1 + \sqrt{1 - \Pi_{i=1}^{N-1}\cos^2(\theta_i)\cos^2(\theta_N)}\right) &\text{(B12)}\\ &= \tfrac{1}{2}\left(1 + \sqrt{1 - \Pi_{i=1}^{N}\cos^2(\theta_i)}\right). &\text{(B13)}\end{aligned}$$

Now we consider the case of general priors. We can artificially rearrange this problem so that it is mathematically equivalent to a new quantum state discrimination problem between two transformed states $\hat{\rho}'_\pm$. We start by defining θ_0 such that $q = \frac{1}{2}[1 + \sin(\theta_0)]$. For pure states, we have $P_{\mathrm{s,h}}(q, \rho_\pm) = P_{\mathrm{s,h}}(1 - q, \rho_\pm)$, so

$$\begin{aligned} P_{\mathrm{h}}(q, \rho_\pm) &= \frac{1}{2}\left[P_{\mathrm{s,h}}\left(\frac{1 + \sin\theta_0}{2}, \rho_\pm\right) + P_{\mathrm{h}}\left(\frac{1 - \sin\theta_0}{2}, \rho_\pm\right)\right] = P_{\mathrm{s,LG}}\left(\frac{1}{2}, \rho_\pm^{(0)} \otimes \rho_\pm\right) \\ &= \frac{1}{2}[\mathbb{P}(d_0 = +|\rho^{(0)} = \rho_+^{(0)})\mathbb{P}(d_N = +|d_0 = +, \rho = \rho_+) + \mathbb{P}(d_0 = +|\rho^{(0)} = \rho_-^{(0)})\mathbb{P}(d_N = -|d_0 = +, \rho = \rho_-) \\ &\quad + \mathbb{P}(d_0 = -|\rho^{(0)} = \rho_+^{(0)})\mathbb{P}(d_N = +|d_0 = -, \rho = \rho_+) + \mathbb{P}(d_0 = -|\rho^{(0)} = \rho_-^{(0)})\mathbb{P}(d_N = -|d_0 = -, \rho = \rho_+)], \end{aligned}$$

where we define the newly appended states $\rho_\pm^{(0)} \triangleq |\theta_{0,\pm}\rangle\langle\theta_{0,\pm}|$ such that $|\langle\theta_{0,+}|\theta_{0,-}\rangle|^2 = \cos^2(\theta_0)$. Here $\mathbb{P}(d_N|d_0, \text{state})$ denotes the probability of obtaining d_N as the measurement result on the Nth subsystem given the updated prior $P_1(q_0 = \frac{1}{2}, d_0)$ and state, with all local measurements determined by the locally greedy algorithm.

Since we have restricted all quantum subsystems to be in pure states, we can simplify through symmetry as follows:

$$\begin{aligned} \mathbb{P}(d_N = +|d_0 = -, \rho = \rho_+) &= \mathbb{P}(d_N = -|d_0 = +, \rho = \rho_-) \\ \mathbb{P}(d_N = -|d_0 = -, \rho = \rho_-) &= \mathbb{P}(d_N = +|d_0 = +, \rho = \rho_+). \end{aligned}$$

Substituting these properties we obtain

$$\begin{aligned} P_{\mathrm{h}}(q, \rho_\pm) &= \mathbb{P}(d_0 = +|\rho^{(0)} = \rho_+^{(0)})\mathbb{P}(d_N = +|d_0 = +, \rho = \rho_+) + \mathbb{P}(d_0 = +|\rho^{(0)} = \rho_-^{(0)})\mathbb{P}(d_N = -|d_0 = +, \rho = \rho_-) \\ &= q\mathbb{P}(d_N = +|d_0 = +, \rho = \rho_+) + (1 - q)\mathbb{P}(d_N = -|d_0 = +, \rho = \rho_-) = P_{\mathrm{lg}}(q, \rho_\pm). \end{aligned}$$

The last equality follows from noting that the updated prior $P_1(q_0 = \frac{1}{2}, +) = \frac{1}{2}[1 + \sin(\theta_0)] = q$. Thus, the probability of success is equivalent under both the joint N-system Helstrom measurement and the locally greedy method for pure states.

APPENDIX C: PROOF OF LEMMA 3

The Helstrom measurement is given by the orthogonal projector onto the positive eigenspace of the operator $[(1-q)\rho_- - q\rho_+]$. More explicitly, it is given by the orthogonal projector onto the vector space spanned by all eigenstates $|v\rangle$ such that

$$\langle v|[(1-q)\rho_- - q\rho_+]|v\rangle \geqslant 0.$$

Let us denote this projector as Π_{Hel}. Using this orthogonal projector, the probability of success is given by

$$P_{\text{succ}} = q\text{Tr}((I-\Pi_{\text{Hel}})\rho_+) + (1-q)\text{Tr}(\Pi_{\text{Hel}}\rho_-) = q + \text{Tr}(\Pi_{\text{Hel}}[(1-q)\rho_- - q\rho_+]).$$

Now let us consider the optimal measurement for distinguishing ρ_+^{dep} and ρ_-^{dep}. Calculating the analogous operator for ρ_+^{dep} and ρ_-^{dep} gives

$$(1-q)\rho_-^{\text{dep}} - q\rho_+^{\text{dep}} = (1-q)\left[(1-\gamma)\rho_- + \frac{\gamma}{d}I\right] - q\left[(1-\gamma)\rho_+ + \frac{\gamma}{d}I\right] = \frac{\gamma(1-2q)}{d}I + (1-\gamma)[(1-q)\rho_- - q\rho_+].$$

Since $\gamma < 1$, we can divide by $1-\gamma$ without changing the positive eigenspace. Therefore, the Helstrom optimal measurement projects onto the space of eigenstates $|v\rangle$ such that the following is positive:

$$\langle v|\left(\frac{\gamma}{1-\gamma}\frac{1-2q}{d}I + [(1-q)\rho_- - q\rho_+]\right)|v\rangle = \langle v|[(1-q)\rho_- - q\rho_+]|v\rangle + \frac{\gamma}{1-\gamma}\frac{1-2q}{d}. \tag{C1}$$

Therefore, if γ is sufficiently small, then the optimal projector distinguishing ρ_+^{dep} and ρ_-^{dep} is Π_{Hel}. Hence, for γ sufficiently small, we have

$$\begin{aligned} P_{\text{succ}}^{\text{dep}} &= q + \text{Tr}(\Pi_{\text{Hel}}[(1-q)\rho_-^{\text{dep}} - q\rho_+^{\text{dep}}]) \\ &= q + \frac{\gamma(1-2q)}{d}\text{Tr}(\Pi_{\text{Hel}}) + (1-\gamma)\text{Tr}(\Pi_{\text{Hel}}[(1-q)\rho_- - q\rho_+]) = \gamma q + \frac{\gamma(1-2q)k}{d} + (1-\gamma)P_{\text{succ}}. \end{aligned}$$

APPENDIX D: PROOF OF COROLLARY 4

Let us denote the Helstrom measurement for $\{|\psi_+\rangle\langle\psi_+|, |\psi_-\rangle\langle\psi_-|\}$ by $\Pi_{\text{Hel},|\psi_\pm\rangle\langle\psi_\pm|}$ and the Helstrom measurement for $\{\rho_+^{\text{dep}}, \rho_-^{\text{dep}}\}$ by $\Pi_{\text{Hel},\rho_\pm}$. Since $\rho_\pm^{\text{dep}}$ are qubit states, $\text{rank}(\Pi_{\text{Hel},\rho_\pm})$ is 0, 1, or 2.

If $\text{rank}(\Pi_{\text{Hel},\rho_\pm}) = 0$, then $\Pi_{\text{Hel},\rho_\pm} = 0$ and

$$P_{\text{succ}}^{\text{dep}} = q + \text{Tr}[\Pi_{\text{Hel},\rho_\pm}((1-q)\rho_-^{\text{dep}} - q\rho_+^{\text{dep}})] = q.$$

If $\text{rank}(\Pi_{\text{Hel},\rho_\pm}) = 2$, then $\Pi_{\text{Hel},\rho_\pm} = I$ and

$$P_{\text{succ}}^{\text{dep}} = q + \text{Tr}[\Pi_{\text{Hel},\rho_\pm}((1-q)\rho_-^{\text{dep}} - q\rho_+^{\text{dep}})] = 1-q.$$

Finally, consider the case where $\text{rank}(\Pi_{\text{Hel},\rho_\pm}) = 1$. The state discrimination problem between $\{\rho_+^{\text{dep}}, \rho_-^{\text{dep}}\}$ is physically equivalent to a black box which outputs one of the following four separate discrimination problems:

$$\left\{\{|\psi_+\rangle\langle\psi_+|, |\psi_-\rangle\langle\psi_-|\}, \left\{|\psi_+\rangle\langle\psi_+|, \frac{\mathbb{I}}{2}\right\}, \left\{\frac{\mathbb{I}}{2}, |\psi_-\rangle\langle\psi_-|\right\}, \left\{\frac{\mathbb{I}}{2}, \frac{\mathbb{I}}{2}\right\}\right\},$$

with probabilities

$$\{p_1, p_2, p_3, p_4\} \triangleq \{(1-\gamma_+)(1-\gamma_-), (1-\gamma_+)\gamma_-, \gamma_+(1-\gamma_-), \gamma_+\gamma_-\},$$

respectively. (This follows from viewing $\rho_\pm^{\text{dep}}$ as corresponding to a quantum system prepared in state $|\psi_\pm\rangle\langle\psi_\pm|$ with probability $1-\gamma_\pm$ and prepared in state $\frac{\mathbb{I}}{2}$ with probability $\gamma_\pm$.)

We denote by $P_{\text{succ}}(\rho_+, \rho_-, \Pi)$ the probability of successfully discriminating between $\{\rho_+, \rho_-\}$ given measurement $\{\Pi, \mathbb{I}-\Pi\}$ where the prior is implicitly defined as q. Then we can upper-bound the success probability as

$$\begin{aligned} P_{\text{succ}}^{\text{dep}} &\leqslant p_1 \max_{|\psi_+\rangle,|\psi_-\rangle,\Pi} P_{\text{succ}}(|\psi_+\rangle\langle\psi_+|,\ |\psi_-\rangle\langle\psi_-|,\ \Pi) + p_2 \max_{|\psi_+\rangle,\Pi} P_{\text{succ}}\left(|\psi_+\rangle\langle\psi_+|,\ \frac{\mathbb{I}}{2},\ \Pi\right) \\ &\quad + p_3 \max_{|\psi_-\rangle,\Pi} P_{\text{succ}}\left(\frac{\mathbb{I}}{2},\ |\psi_-\rangle\langle\psi_-|,\ \Pi\right) + p_4 \times \frac{1}{2} \\ &= p_1 P_{\text{succ}}(|0\rangle\langle 0|,\ |1\rangle\langle 1|,\ \Pi_{\text{Hel},\{|0\rangle\langle 0|,|1\rangle\langle 1|\}}) + p_2 P_{\text{succ}}\left(|0\rangle\langle 0|,\ \frac{\mathbb{I}}{2},\ \Pi_{\text{Hel},\{|0\rangle\langle 0|,|1\rangle\langle 1|\}}\right) \end{aligned}$$

$$+ p_3 P_{\text{succ}}\left(\frac{\mathbb{I}}{2},\ |1\rangle\langle 1|,\ \Pi_{\text{Hel},\{|0\rangle\langle 0|,|1\rangle\langle 1|\}}\right) + \frac{p_4}{2}$$

$$= P_{\text{succ}}\left((1-\gamma_+)|0\rangle\langle 0| + \frac{\gamma_+}{2}\mathbb{I},\ (1-\gamma_-)|1\rangle\langle 1| + \frac{\gamma_-}{2}\mathbb{I},\ \Pi_{\text{Hel},\{|0\rangle\langle 0|,|1\rangle\langle 1|\}}\right).$$

Thus, the success probability for $\rho_\pm^{\text{dep}}$ is upper bounded by the success probability when $|\psi_+\rangle$ and $|\psi_-\rangle$ are orthogonal (with loss of generality we have set $|\psi_+\rangle = |0\rangle$ and $|\psi_-\rangle = |1\rangle$). Upon solving for $P_{\text{succ}}(|0\rangle\langle 0|^{\text{dep}},\ |1\rangle\langle 1|^{\text{dep}},\ \ \Pi_{\text{Hel},\{|0\rangle\langle 0|,|1\rangle\langle 1|\}})$, it immediately follows that

$$P_{\text{succ}}^{\text{dep}} \leqslant \left(1 - \frac{\gamma_+}{2}\right)q + \left(1 - \frac{\gamma_-}{2}\right)(1-q) \leqslant 1 - \frac{\gamma_{\min}}{2}.$$

APPENDIX E: PROOF OF LEMMA 5

We begin with a lemma demonstrating that, when the Helstrom measurement is trivial, any measurement is locally optimal.

Lemma 7. Let $\rho_\pm^{(j)}$ and p be such that $\Pi(p, j) = \mathbb{I}$ or $\Pi(p, j) = 0$. Then

$$\max_{\Pi}\left(\frac{p\text{Tr}[\rho_+^{(j)}\Pi]}{\text{Tr}[\Pi((1-p)\rho_-^{(j)} + p\rho_+^{(j)})]}\right) < \frac{1}{2}$$

or

$$\min_{\Pi}\left(\frac{p\text{Tr}[\rho_+^{(j)}\Pi]}{\text{Tr}[\Pi((1-p)\rho_-^{(j)} + p\rho_+^{(j)})]}\right) \geqslant \frac{1}{2},$$

respectively. Namely, any local measurement is optimal given posterior-based decoding.

Proof. Define $M \triangleq (1-p)\rho_- - p\rho_+$ and let the resulting projector be $\Pi_h(p, \rho_\pm) = \mathbb{I}$. Then the eigenvalues of M satisfy $\lambda_j > 0\ \forall\ j$, and as M is Hermitian the eigenvectors $\{|v_j\rangle\}$ are orthogonal and form a basis. Any projector diagonal in this basis may be defined $\Pi_S \equiv \sum_{j\in S}|v_j\rangle\langle v_j|$ for some set of indices S. It follows that $\text{Tr}[M\Pi_s] = \sum_{j\in S}\lambda_j > 0$, so $\text{Tr}[\Pi_S p\rho_+] < \text{Tr}[\Pi_S(1-p)\rho_-]$. Then, the updated prior upon obtaining measurement corresponding to Π_S is

$$p' = \frac{\text{Tr}[\Pi_S\rho_+ p]}{\text{Tr}[\Pi_S p\rho_+] + \text{Tr}[\Pi_S(1-p)\rho_-]} < \frac{1}{2}\ \forall\ S.$$

Now suppose the projector is diagonal in an arbitrary basis $\{|w_k\rangle\}$ s.t. $|w_k\rangle = \sum_j \alpha_{k,j}|v_j\rangle$ where $\{\alpha_{k,j}\}$ form the entries of some unitary operator. Then it is sufficient to show that $\text{Tr}[M|w_k\rangle\langle w_k|] > 0$ for all k, since then $\text{Tr}[M\sum_{k\in S}|w_k\rangle\langle w_k|] > 0$ for all S. We observe

$$\text{Tr}[M|w_k\rangle\langle w_k|] = \sum_{j,j'}\alpha_{k,j}\alpha_{k,j'}^*\lambda_j\text{Tr}[|v_j\rangle\langle v_{j'}|] = \sum_j|\alpha_{k,j}|^2\lambda_j > 0.$$

Similarly, for any basis $\{|w_k\rangle\}$, then $\text{Tr}[M|w_k\rangle\langle w_k|] \leqslant 0$ if $\Pi_h(p, \rho_\pm) = 0$. ∎

Now we prove Lemma 4.

It is sufficient to show that for all $j \in \{0, 1, \ldots, N\}$ and for all $p_j \in (0, 1)$ we have

$$f_+(p_j) \triangleq \mathbb{E}[p_{j+1}|\rho = \rho_+^{(j)}, \Pi^*(p_j, j)] > p_j$$

and

$$f_-(p_j) \triangleq \mathbb{E}[p_{j+1}|\rho = \rho_-^{(j)}, \Pi^*(p_j, j)] < p_j,$$

and that $f_\pm(p_j)$ is continuous with no fixed points other than $p_j = 0$ or 1. For simplicity, we drop the superscript on $\rho_\pm^{(j)}$ in the following whenever the subsystem index is unambiguous. We denote the modified Helstrom measurement as $\Pi = \Pi^*(p_j, j)$, such that by definition $\text{Tr}[\Pi\rho_-] > \text{Tr}[\Pi\rho_+]$.

Let $x \triangleq \text{Tr}[\Pi\rho_-] - \text{Tr}[\Pi\rho_+] = \text{Tr}[\Pi^\perp\rho_+] - \text{Tr}[\Pi^\perp\rho_-]$ s.t. $x \in (0, 1]$. Then, there exists $y \in [\frac{x}{2}, 1 - \frac{x}{2}]$ such that the conditional measurement probabilities may be represented as follows:

$$\text{Tr}[\Pi^\perp\rho_\pm] = y \pm \frac{x}{2},\ \ \text{Tr}[\Pi\rho_\pm] = 1 - y \mp \frac{x}{2}.$$

Finally, we calculate $f_+(p_j)$ as follows:

$$f_+(p_j) = \text{Tr}[\Pi\rho_+]\left(\frac{p_j\text{Tr}[\Pi\rho_+]}{\text{Tr}[\Pi(p_j\rho_+ + (1-p_j)\rho_-)]}\right) + \text{Tr}[\Pi^\perp\rho_+]\left(\frac{p_j\text{Tr}[\Pi^\perp\rho_+]}{\text{Tr}[\Pi^\perp(p_j\rho_+ + (1-p_j)\rho_-)]}\right)$$

$$= p_j\left(\frac{\left(1-y-\frac{x}{2}\right)^2}{p_j\left(1-y-\frac{x}{2}\right) + (1-p_j)\left(1-y+\frac{x}{2}\right)} + \frac{\left(y+\frac{x}{2}\right)^2}{p_j\left(y+\frac{x}{2}\right) + (1-p_j)\left(y-\frac{x}{2}\right)}\right) > p_j,$$

where the final line follows from solving symbolically for the range $p_j \in (0, 1)$, $x \in (0, 1)$, $y \in [\frac{x}{2}, 1 - \frac{x}{2}]$. We then check for any fixed points $p_j^* = f_+(p_j)$. This results in the condition $p_j^*[-x^2 + 2p_j^* x^2 - (p_j^* x)^2] = 0$ so the only fixed points are $p_j^* = 0$ or 1. Additionally, it is clear that $f_+(p_j)$ is continuous in p_j.

A similar argument holds for the case $\rho = \rho_-$. Thus, the probability of success converges to 1 under the MLG algorithm.

APPENDIX F: IMPLEMENTATION AND COMPLEXITY

We compute the functions R_S and $\hat{A}_S$ using dynamic programming (DP), which requires mapping the states $(q, \mathbf{a}^\sigma_{[N-|S|]}, \mathbf{d}^\sigma_{[N-|S|]}) \in [0, 1] \times \mathcal{A}^{N-|S|} \times \mathcal{D}^{N-|S|}$ to $C^\sigma_{N-|S|}(q, \mathbf{a}^\sigma_{[N-|S|]}, \mathbf{d}^\sigma_{[N-|S|]}) \in [0, 1]$. Moreover, these functions can be stored for later use in problems with same states but different initial priors or if one needs to run MOODY on a larger system where the current system is a subsystem of it.

Since the interval $[0, 1]$ and action space $\mathcal{A}$ are not discrete, they are quantized to give a tractable implementation. The computational complexity and memory requirements of DP are highly dependent on this quantization. In our implementation, we again apply a quantized version of the above DP where the input p is quantized into Q_p equi-spaced points over $[0, 1]$ and the measurement action space $\mathcal{A}$ is quantized into a size Q_a set to make the minimization over $\mathcal{A}$ tractable. To store expected future error functions $\{R_S | S \subseteq [N]\}$ and a set of best measurement action functions $\{A_S | S \subseteq [N]\}$, the memory complexity is $\mathrm{O}(2^N Q_p)$. Besides, each value is obtained from a minimization over $S \times \mathcal{A}$, so the total computation is of complexity $\mathrm{O}(2^N Q_p N Q_a)$. The number of DP functions A_S and R_S is 2^N because the order of measurement matters in general. However, DP still represents a speedup over the case of naive exhaustion for all possible orders, which has complexity $N!$.

If all the qubits are identical copies, then the ordering is immaterial, and the different subsets S with same size correspond to the same case. Therefore, in this scenario the memory complexity is $\mathrm{O}(NQ_p)$ and the computation complexity is $\mathrm{O}(NQ_pQ_a)$.

Since there are only two possible states ρ_+ and ρ_-, we can also use log-likelihood ratios (LLR) to describe the probabilities. This parametrization simplifies the computation of CSPs. For $j \in [N]$, define

$$\ell_0^\sigma(q) \triangleq \ln\left(\frac{q}{1-q}\right), \tag{F1}$$

$$\ell_j^\sigma\left(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]}\right) \triangleq \ln\left(\frac{C_j^\sigma\left(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]}\right)}{1 - C_j^\sigma\left(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]}\right)}\right), \tag{F2}$$

$$\tilde{\ell}_{\sigma(j)}(\mathbf{a}_{\sigma(j)}, d_{\sigma(j)}) \triangleq \ln\left(\frac{\mathbb{P}(d_{\sigma(j)}|\rho_+, \mathbf{a}_{\sigma(j)})}{\mathbb{P}(d_{\sigma(j)}|\rho_-, \mathbf{a}_{\sigma(j)})}\right). \tag{F3}$$

It is easy to check that

$$\begin{aligned}
\exp\left[\ell_j^\sigma(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]})\right] &= \frac{C_j^\sigma(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]})}{1 - C_j^\sigma\left(q, \mathbf{a}^\sigma_{[j]}, \mathbf{d}^\sigma_{[j]}\right)} \\
&= \frac{\mathbb{P}(d_{\sigma(j)}|\rho_+, \mathbf{a}_{\sigma(j)})\, C_{j-1}^\sigma\left(q, \mathbf{a}^\sigma_{[j-1]}, \mathbf{d}^\sigma_{[j-1]}\right)}{\mathbb{P}(d_{\sigma(j)}|\rho_-, \mathbf{a}_{\sigma(j)})\left[1 - C_{j-1}^\sigma\left(q, \mathbf{a}^\sigma_{[j-1]}, \mathbf{d}^\sigma_{[j-1]}\right)\right]} \\
&= \exp[\tilde{\ell}_{\sigma(j)}(\mathbf{a}_{\sigma(j)}, d_{\sigma(j)})] \exp\left[\ell_{j-1}^\sigma\left(q, \mathbf{a}^\sigma_{[j-1]}, \mathbf{d}^\sigma_{[j-1]}\right)\right].
\end{aligned}$$

This yields the simplified recursive equation

$$\ell_j^\sigma\left(q, \mathbf{a}^\sigma_{[j]}, d^\sigma_{[j]}\right) = \tilde{\ell}_{\sigma(j)}(\mathbf{a}_{\sigma(j)}, d_{\sigma(j)}) + \ell_{j-1}^\sigma\left(q, \mathbf{a}^\sigma_{[j-1]}, \mathbf{d}^\sigma_{[j-1]}\right). \tag{F4}$$

APPENDIX G: RESULTS FOR BEST AND WORST ORDERING: QUBIT CASE

We then perform experiments when $\hat{\rho}_\pm^{(j)}$ are all real qubit states. The set of measurements $\mathcal{A}_{\text{qubit}}$ is taken to be the standard action space of real orthogonal projectors [11]

$$\mathcal{A}_{\text{qubit}} \triangleq \left\{\{|\phi\rangle\langle\phi|, |\phi^\perp\rangle\langle\phi^\perp|\} : \phi \in \left[0, \frac{\pi}{2}\right]\right\}, \tag{G1}$$

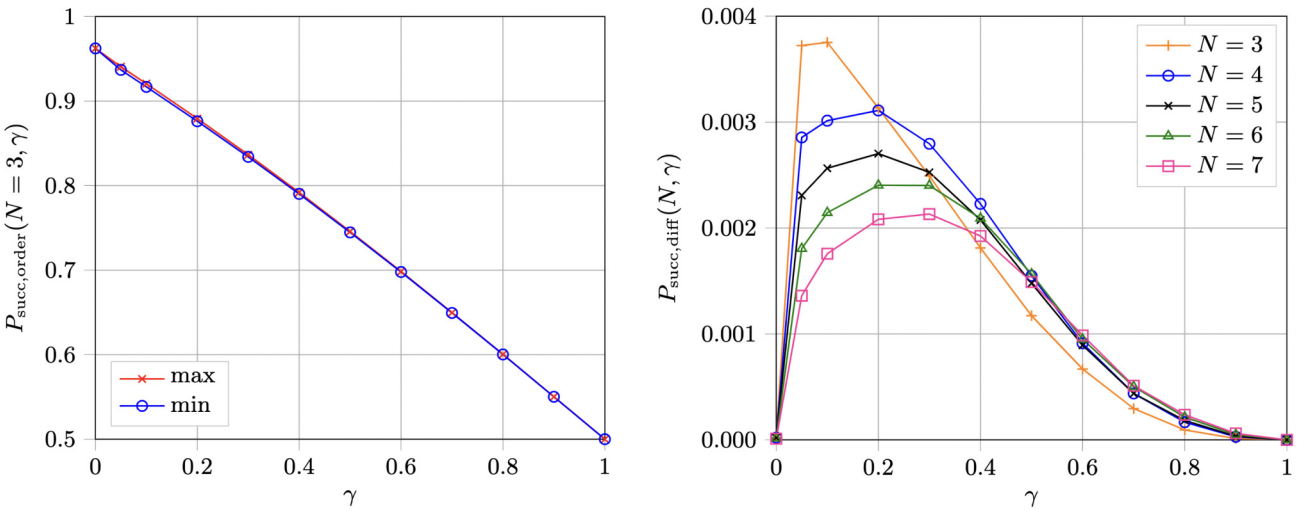
where we quantize ϕ into $Q_\phi = 128$ equally spaced points. The experimental setup is as follows:

(1) Choose a set of depolarizing parameters $\mathcal{S}_{\text{dep}} = \{0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ and number of trials $n_{\text{trial}} = 1000$.

(2) Generate $\theta_\pm^{(t,j)} \in (0, 2\pi)$ uniformly, where $t \in [n_{\text{trial}}]$ denotes the trial index, and $j = 1, 2, \ldots, 7$ denotes the subsystem index.

(3) For each $\gamma \in \mathcal{S}_{\text{dep}}$ and $N \in \{3, 4, 5, 6, 7\}$, define the corresponding qubit quantum states:

$$\rho_\pm(\gamma, t, N) \triangleq \bigotimes_{j=1}^{N}\left((1-\gamma)|\theta_\pm^{(t,j)}\rangle\langle\theta_\pm^{(t,j)}| + \frac{\gamma}{2}I\right). \tag{G2}$$



(a) Success Probability (b) Difference in Success Probability

FIG. 3. Left-hand side depicts probabilities $P_{\text{s,best}}(N=3,\gamma)$ and $P_{\text{s,worst}}(N=3,\gamma)$ as a function of the depolarising parameter γ over 1000 trials. Although $P_{\text{s,best}}(N=3,\gamma) \neq P_{\text{s,worst}}(N=3,\gamma)$, the relative difference is small. Right-hand side depicts difference in maximum and minimum probability of success, $P_{\text{succ,diff}}(N,\gamma)$, as a function of the depolarizing parameter γ over 1000 trials for $N=3,4,5,6,7$.

(4) For each $\rho_{\pm}(\gamma,t,N)$ perform two separate optimizations corresponding to the best and worst ordering respectively, where the corresponding future risk functions are

$$R_{S,\text{best}}(p,\{\rho_{\pm}(\gamma,t,N)\}) \triangleq \min_{(k,\mathbf{a}_k)\in S\times\mathcal{A}} \sum_{d_k\in\{+,-\}} \mathcal{L}(p,\mathbf{a}_k,d_k) R_{S\setminus\{k\}}[\mathcal{P}(p,\mathbf{a}_k,d_k)], \tag{G3}$$

$$R_{S,\text{worst}}[p,\{\rho_{\pm}(\gamma,t,N)\}] \triangleq \max_{k\in S} \min_{\mathbf{a}_k\in\mathcal{A}} \sum_{d_k\in\{+,-\}} \mathcal{L}(p,\mathbf{a}_k,d_k) R_{S\setminus\{k\}}[\mathcal{P}(p,\mathbf{a}_k,d_k)]. \tag{G4}$$

(5) For $\gamma\in\mathcal{S}_{\text{dep}}$ and $N\in\{3,4,5,6,7\}$, given an order of "best" or "worst," denote:

$$P_{\text{S,order}}(N,\gamma) \triangleq \frac{1}{n_{\text{trial}}} \sum_{t=1}^{n_{\text{trial}}} P_{\text{s,order}}(\rho_{\pm}(\gamma,t,N)), \tag{G5}$$

where $P_{\text{s,order}}(\rho_{\pm})$ indicates that we perform the MOODY algorithm with the specified ordering on states $\rho_{\pm}$.

We plot $P_{\text{s,order}}(N=3,\gamma)$ as a function of γ in Fig. 3(a) and we also compare the difference $P_{\text{succ,diff}}(N,\gamma) \triangleq P_{\text{s,best}}(N,\gamma) - P_{\text{s,worst}}(N,\gamma)$ for $N\in\{3,4,5,6,7\}$ in Fig. 3(b). From these results, we observe that the difference in probability of success with respect to ordering is quite small and persists even when using the MOODY algorithm.

APPENDIX H: DETAILS OF QUTRIT ACTION SPACE QUANTIZATION

First, we note that it is sufficient to consider real quantum measurements, as $\text{Tr}[\rho\,\Pi] = \text{Tr}[\rho\,\text{Re}(\Pi)]$ for any Hermitian projector Π and so the resulting statistics will be invariant upon taking only the real part of any measurement set. Subject to the constraint that each subsystem may be measured only once, any ternary set of orthogonal projectors may be chosen to have all elements rank 1 because any rank 2 or 3 element can be viewed as grouping the corresponding rank 1 projectors postmeasurement.

Now we implement the quantization of such real-valued qutrit measurements using the following steps:

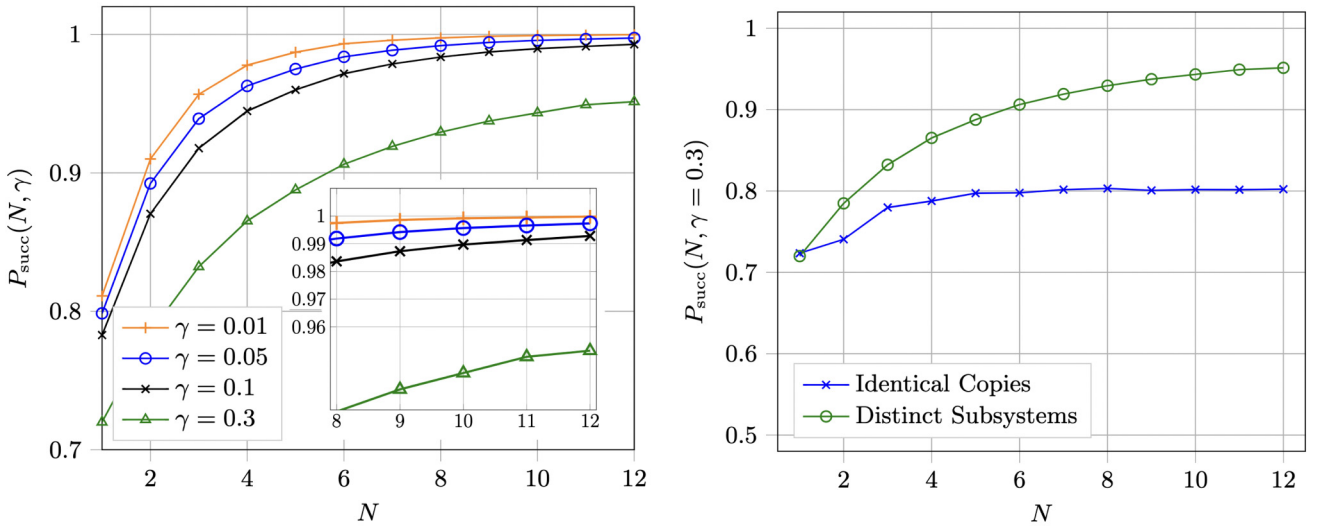
(1) We quantize the unit sphere by subdividing an icosahedron for T steps according to vector $\vec{r}=[r_1,\ldots,r_T]$ such that at the jth step we subdivide each segment by r_j. Then according to the Euler characteristic of convex polyhedrons, the number of vertices after all subdivisions are complete is given by $10\prod_{i=1}^{T} r_i^2 + 2$, and we denote this set of vertices as $\text{Sub}(\vec{r}\,)$.

(2) For each point (x,y,z) in $\text{Sub}([2,2,2])$, convert to polar coordinates (ϕ,θ) according to

$$x=\sin(\theta)\cos(\phi), \quad y=\sin(\theta)\sin(\phi), \quad z=\cos(\theta).$$

(3) For each pair (ϕ,θ), define the rotation matrix $R(\phi,\theta)$ as

$$R(\phi,\theta) \triangleq \begin{bmatrix} -\sin(\phi) & \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) \\ \cos(\phi) & \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$



(a) Success Probability (b) Difference in Success Probability

FIG. 4. Left-hand side depicts probability of success for varying γ in the distinct subsystems scenario as a function of the number of available systems, N. Results are average over 1000 trials. Right-hand side compares probability of success as a function of the number of available systems, N, for depolarizing parameter $\gamma = 0.3$. Results are averaged over 1000 trials.

(4) Choose Q as the resolution on the equatorial plane, let $\omega \in \{\frac{\pi q}{2Q}\}_{q=0}^{Q-1}$ and define

$$\vec{u}_1(\phi,\theta,\omega) = R(\phi,\theta)\begin{bmatrix}\cos(\omega)\\ \sin(\omega)\\ 0\end{bmatrix}, \quad \vec{u}_2(\phi,\theta,\omega) = R(\phi,\theta)\begin{bmatrix}-\sin(\omega)\\ \cos(\omega)\\ 0\end{bmatrix}, \quad \vec{u}_3(\phi,\theta,\omega) = R(\phi,\theta)\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

APPENDIX I: FURTHER PROPERTIES OF THE LOCALLY GREEDY METHOD

We show the plateau remains in the order-optimized locally greedy algorithm by generalizing the experiment from identical copies to the case where the subsystems are distinct. The primary change is that we now sample states parametrized by $\theta_{\pm}^{(t,j)}$ so that each subsystem in both ρ_+ and ρ_- can have (potentially) distinct copies. Also, the vector of success probabilities is altered accordingly.

(1) Choose a set of depolarizing parameters and number of trials. Again we set $\mathcal{S}_{\text{dep}} = \{0.01, 0.05, 0.1, 0.3\}$ and $n_{\text{trial}} = 1000$.

(2) Generate $\theta_{\pm}^{(t,j)} \in (0, 2\pi)$ uniformly, where $t \in [n_{\text{trial}}]$ denotes the trial index, and $j = 1, 2, \ldots, 12$ denotes the subsystem index.

(3) For each $\gamma \in \mathcal{S}_{\text{dep}}$ and $N = 1, 2, \ldots, 12$, define the corresponding qubit quantum states

$$\rho_{\pm}(\gamma,t,N) \triangleq \bigotimes_{j=1}^{N}\left((1-\gamma)\left|\theta_{\pm}^{(t,j)}\right\rangle\!\left\langle\theta_{\pm}^{(t,j)}\right| + \frac{\gamma}{2}I\right). \tag{I1}$$

(4) For all $\gamma \in \mathcal{S}_{\text{dep}}$ and all $N = 1, 2, \ldots, 12$, denote

$$P_{\text{succ}}(N,\gamma) = \frac{1}{n_{\text{trial}}}\sum_{t=1}^{n_{\text{trial}}} P_{\text{lg}}(\rho_{\pm}(\gamma,t,N)), \tag{I2}$$

where $P_{\text{lg}}(\rho_{\pm})$ indicates that we perform the locally greedy algorithm on states $\rho_{\pm}$.

We plot the results of this experiment in Fig. 4(a). When the subsystems are not identical copies, we notice that the plateau is much higher when compared with the case of identical copies. At first glance this appears to violate the bound obtained in Corollary 4, since here the pair of states in each subsystem are pure states depolarized with the same parameter γ, as required in the hypothesis of Corollary 4. However, the reason for the higher plateau is as follows. The algorithm orders the subsystems in such a way that the credulity can be updated to be as close to $1 - \frac{\gamma}{2}$ as possible (where we assume the states $\rho_{\pm}$ may be relabeled at any step to ensure the credulity is always greater than $\frac{1}{2}$). In the next round, it is still possible to obtain one more nontrivial measurement, after which either the updated credulity exceeds $1 - \frac{\gamma}{2}$ and all subsequent rounds are trivial, or the credulity is lowered below the threshold and another measurement is permitted until the updated credulity again exceeds $1 - \frac{\gamma}{2}$. This permitted "jump" in credulity due to the final measurement explains why the value appearing as the plateau in Fig. 4(a) can be larger than $1 - \frac{\gamma}{2}$.

The best "jump" beyond $1-\frac{\gamma}{2}$ is obtained when the states in that subsystem are an orthogonal pair of pure states subjected to the depolarizing channel and a measurement result which increases the credulity is attained, as formalized in the following lemma.

Lemma 8. Suppose that we are given one of two quantum states $\rho_+ = \bigotimes_{j=1}^N [(1-\gamma)|\theta_{j,+}\rangle\langle\theta_{j,+}| + \frac{\gamma}{2}\mathbb{I}]$ and $\rho_- = \bigotimes_{j=1}^N [(1-\gamma)|\theta_{j,-}\rangle\langle\theta_{j,-}| + \frac{\gamma}{2}\mathbb{I}]$ where γ is a fixed depolarizing parameter. Then, an upper bound on the probability of success using the locally greedy method is given by

$$P_{\mathrm{s,LG}}\left(\frac{1}{2}, \rho_\pm\right) \leqslant P_{\mathrm{bound}}(\gamma) \equiv \frac{\left(1-\frac{\gamma}{2}\right)^2}{\left(1-\frac{\gamma}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2}.$$

Proof. We show that the probability of success is upper bounded by the maximal attainable credulity after N measurements, and proceed by induction. For details, see Appendix J. ■

To illustrate the predictive value of this bound, we list the observed numerical asymptotic values found when $N = 12$ for nonidentical subsystems $[P_{\mathrm{obs}}(\gamma)]$ and the predicted upper bound for $\gamma = 0.1, 0.3, 0.4, 0.5$, respectively:

$$\{(P_{\mathrm{obs}}(0.1) = 0.9943, P_{\mathrm{bound}}(0.1) = 0.9972),\ (P_{\mathrm{obs}}(0.3) = 0.9549, P_{\mathrm{bound}}(0.3) = 0.9698),$$
$$(P_{\mathrm{obs}}(0.4) = 0.9198, P_{\mathrm{bound}}(0.4) = 0.9412),\ (P_{\mathrm{obs}}(0.5) = 0.8732, P_{\mathrm{bound}}(0.5) = 0.9000)\}.$$

Finally, we compare the two scenarios for the specific value of the depolarizing parameter $\gamma = 0.3$ in Fig. 4(b). This plot shows the nontrivial advantage obtained from subsystems being distinct rather than copies of each other, which is the case most considered in the literature. For the special case of $\gamma = 0$, we have shown in Theorem 2 that the order of subsystems does not matter and that the simple locally greedy algorithm itself achieves the optimal performance obtained with the joint N-system Helstrom measurement.

APPENDIX J: PROOF OF LEMMA 8

Given the sequence of measurement results $d_{[N]}$, the probability of success is $\max(p_N, 1-p_N)$ where $p_N = C_N^\sigma(q, \mathbf{a}_{[N]}^\sigma, \mathbf{d}_{[N]}^\sigma)$ [and where $\mathbf{a}_{\sigma(j)} = \Pi(j, p_j)$]. We suppose that at any step we may swap the labels of the composite states to enforce $p_j \geqslant \frac{1}{2}\ \forall\ j$. Then, the probability of success is upper bounded by the maximal attainable probability, namely $P_{\mathrm{s,LG}}(\frac{1}{2}, \rho_\pm) \leqslant \max_{\mathbf{d}_{[N]}^\sigma \in \mathcal{D}^N}[C_N^\sigma(\frac{1}{2}, \mathbf{d}_{[N]}^\sigma)]$.

We then show by induction that

$$p_j \leqslant \frac{\left(1-\frac{\gamma}{2}\right)^2}{\left(1-\frac{\gamma}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2}$$

$\forall\ j \in \{0, 1, \ldots, N\}$. For the base case, $j = 0$ and the statement is trivially true as $p_0 = q = \frac{1}{2}$. For the inductive step, we assume that the statement holds for $j \in \{0, 1, \ldots, N-1\}$ and show that it then also holds for $j+1$. First, consider the case where the updated prior at the jth step exceeds the critical value $1-\frac{\gamma}{2} \leqslant p_j$. Then, all future measurements are trivial and $p_k = p_j$ for all $k \geqslant j$. Thus, the inductive hypothesis again holds.

Next, consider the case where $p_j \in [\frac{1}{2}, 1-\frac{\gamma}{2}]$. For simplicity, we define $p_{j+1}(p_j, \mathbf{d}_{j+1}^\sigma) \triangleq C_{j+1}^\sigma(q, \mathbf{a}_{[j+1]}^\sigma, \mathbf{d}_{[j+1]}^\sigma)$ with $a_{\sigma(j)} = \Pi(j, p_j)$. If the Helstrom measurement $\Pi_{hel} \equiv \Pi(j+1, p_j)$ is trivial, then $p_j = p_{j+1}$ and the inductive hypothesis holds. In the case where Π_{hel} is nontrivial, the prior will increase when $d_{j+1} = +$ and the new maximal credulity p_{j+1}^* is defined as follows:

$$\begin{aligned} p_{j+1}^* &= \max\left(p_{j+1}(p_j,\ \mathbf{d}_{j+1}^\sigma = +),\ p_{j+1}(p_j,\ \mathbf{d}_{j+1}^\sigma = -)\right) = p_{j+1}(p_j,\ \mathbf{d}_{j+1}^\sigma = +) \\ &= \frac{p_j \mathrm{Tr}\left[(\mathbb{I}-\Pi_h)\left((1-\gamma)|\theta_{j,+}\rangle\langle\theta_{j,+}| + \frac{\gamma}{2}\mathbb{I}\right)\right]}{\mathrm{Tr}\left[(\mathbb{I}-\Pi_h)\left(p_j(1-\gamma)|\theta_{j,+}\rangle\langle\theta_{j,+}| + (1-p_j)(1-\gamma)|\theta_{j,-}\rangle\langle\theta_{j,-}| + \frac{\gamma}{2}\mathbb{I}\right)\right]} \\ &= \frac{(1-\gamma)p_j \mathrm{Tr}[(\mathbb{I}-\Pi_h)|\theta_{j,+}\rangle\langle\theta_{j,+}|] + \frac{p_j\gamma}{2}}{(1-\gamma)\mathrm{Tr}[(\mathbb{I}-\Pi_h)(p_j|\theta_{j,+}\rangle\langle\theta_{j,+}| + (1-p_j)|\theta_{j,-}\rangle\langle\theta_{j,-}|)] + \frac{\gamma}{2}} = \frac{(1-\gamma)p_j x_+ + \frac{p_j\gamma}{2}}{(1-\gamma)(p_j x_+ + (1-p_j)x_-) + \frac{\gamma}{2}}, \end{aligned}$$

for $x_\pm \equiv \mathrm{Tr}[(\mathbb{I}-\Pi_h)|\theta_{j,\pm}\rangle\langle\theta_{j,\pm}|] \in [0,1]$. The third line follows from substituting into Bayes' law and simplifying. In the following, we derive an upper bound on p_{j+1}^* and thus the success probability by optimizing over x_+, x_-, p_j without placing any restrictions on whether the optimal set $\{x_+^*, x_-^*, p_j^*\}$ is actually physically realizable.

$$\begin{aligned} p_{j+1}^* &\leqslant \max_{x_\pm \in [0,1]} \max_{p_j \in [\frac{\gamma}{2}, 1-\frac{\gamma}{2}]} \left(\frac{(1-\gamma)p_j x_+ + \frac{p_j\gamma}{2}}{(1-\gamma)[p_j x_+ + (1-p_j)x_-] + \frac{\gamma}{2}}\right) = \max_{x_+ \in [0,1]} \max_{p_j \in [\frac{\gamma}{2}, 1-\frac{\gamma}{2}]} \left(\frac{(1-\gamma)p_j x_+ + \frac{p_j\gamma}{2}}{(1-\gamma)p_j x_+ + \frac{\gamma}{2}}\right) \\ &= \max_{p_j \in [\frac{\gamma}{2}, 1-\frac{\gamma}{2}]} \left(\frac{(1-\gamma)p_j + \frac{p_j\gamma}{2}}{(1-\gamma)p_j + \frac{\gamma}{2}}\right) = \frac{(1-\gamma)\left(1-\frac{\gamma}{2}\right) + \frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)}{(1-\gamma)\left(1-\frac{\gamma}{2}\right) + \frac{\gamma}{2}} = \frac{\left(1-\frac{\gamma}{2}\right)^2}{\left(1-\frac{\gamma}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2}. \end{aligned}$$

Thus,

$$P_{\mathrm{s,LG}}\left(\frac{1}{2},\rho_{\pm}\right) \leqslant p^{*}_{j+1} \leqslant \frac{\left(1-\frac{\gamma}{2}\right)^2}{\left(1-\frac{\gamma}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2}.$$

Finally, we note that the bound on p^{*}_{j+1} is tight and is achieved when $p_j = 1-\frac{\gamma}{2}$ and $\rho^{(j)}_{\pm} = (1-\gamma)|\pm\frac{\pi}{4}\rangle\langle\pm\frac{\pi}{4}| + \frac{\gamma}{2}\mathbb{I}$.

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