Error-free interconversion of nonlocal boxes

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Understanding the structure of nonlocal correlations is important in many fields ranging from fundamental questions of physics to device-independent cryptography. We present a protocol that can convert extremal two-party–two-input nonlocal no-signaling boxes of any type into any other extremal two-party–two-input nonlocal no-signaling box perfectly. Our results are exact, and even though the number of required boxes cannot be determined in advance, their expected number is finite. Our protocol is adaptive and demonstrates for the first time the usefulness of using no-signaling boxes in different causal orders by the parties.

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I. INTRODUCTION

Nonlocality is amongst the most intriguing features of nature. Since the seminal paper of Einstein, Podolsky, and Rosen [1] and Bell's quantification [2], the structure of nonclassical correlations has been studied extensively [3], with implications on communication theory [4], cryptography [5], or game theory [6].

One possible way to study nonlocal correlations is to introduce a device (a so-called box), which has two separated, noninteracting parts, one at Alice and one at Bob. Alice chooses an input x from a set of possible inputs and receives a result a from a result set. Similarly, Bob's input is y, resulting in an output b. The behavior of the box is fully described by the conditional probability distribution $p_{ab|xy}$. If $p_{ab|xy}$ does not correspond to a statistical mixture of boxes with two parts that operate independently in parallel on Alice's and Bob's side, then the box is called a nonlocal box.

An important class of correlations is the one whose elements obey the no-signaling condition, compatible with the theory of special relativity. Mathematically, the no-signaling conditions can be formulated as

$$\forall x_1, x_2, b, y \quad \sum_a p_{ab|x_1 y} = \sum_a p_{ab|x_2 y},$$
$$\forall y_1, y_2, a, x \quad \sum_b p_{ab|x y_1} = \sum_b p_{ab|x y_2}. \tag{1}$$

These equations imply the existence of local marginals and, together with the normalization of probabilities, define the no-signaling polytope in the space of the conditional probabilities $p_{ab|xy}$. They are necessary and sufficient for a box not to be useful for direct communication [7]. In what follows, we will refer to nonlocal no-signaling boxes simply as "nonlocal boxes."

Correlations realized by quantum systems form a convex subset of the no-signaling polytope, which can be characterized by a series of semidefinte programs [8,9]. The no-signaling polytope is a mathematically simpler structure which includes supraquantum behaviors that cannot be described in the framework of quantum mechanics. Understanding the complete structure of nonlocal boxes is of fundamental importance. Notably, the extremal points, i.e., the vertices of the polytope are of special interest. The most frequently mentioned example is the Popescu-Rohrlich (PR) box [10,11], which is the extremal point of the no-signaling polytope in the two-input two-output case. Such "maximally nonlocal" correlations would enable incredible communicational and computational power [12,13]. On the other hand, somewhat surprisingly, they appear to underperform quantum correlations in randomness certification [14] and in correlation-assisted multiprover interactive proofs [15]. Networks of PR boxes were used very recently to study the structure of three-partite correlations [16]. These examples indicate that it is possible to study nonlocality from a resource theory point of view [17].

Regarding nonlocal correlations as a resource, it becomes important to know what kind of other correlations can be obtained if one has access to a given type of correlation. This corresponds to the question of how different nonlocal boxes can be interconverted, i.e., how boxes with certain input and output sets and behaviors can be used together to implement another box with different input and/or output sets and behavior. Barrett *et al.* [18] enumerated all extremal bipartite nonlocal boxes with two inputs and arbitrary number of outputs. In addition, they proved that extremal two-input nonlocal boxes of a given type can be converted to any other type, with an arbitrarily small error. More precisely, they showed that $\forall \varepsilon > 0$ there exists a number of d-boxes n so that these boxes can simulate a d'-box with an error probability of at

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most ε. Jones and Masanes [19] presented a protocol to exactly simulate any binary-output nonlocal box with PR boxes. Forster and Wolf [20] solved the general case of converting any type of extremal nonlocal boxes to any other type, also with arbitrarily small error.

There is a similar concept related to manipulating nonlocal boxes, namely, nonlocal correlation distillation, in which the target box has the same input-output arrangement as the (not necessarily extremal) resource box [21,22]. Very recently, Karvonen [23] studied the question of interconverting noncontextual and nonlocal resources in the context of generalized resource theory. In particular, he showed that the independent use of an ancillary correlated resource cannot catalyze any interconversion of correlations, which is an important structural property.

It was also pointed out in Ref. [23] that *adaptive* protocols, that is, when nonlocal boxes are used in a way that the input of one box can depend on another box's output, were studied only to a limited extent thus far. Indeed, the no-signaling conditions allow nonlocal boxes to be used asynchronously by the parties. Hence, it is possible that given two boxes, Alice uses box 1 first and her input to box 2 depends on the output, while Bob uses box 2 first, and then uses box 1 with an input depending on the previous output he received. The possibility of such a "crossed wiring" is prevalently known (e.g., it is also mentioned in Ref. [18] as a side remark), but to our knowledge there are no protocols which exploit this.

In fact, "crossed wiring" means that the boxes are used in a different (even though definite) causal order by the two parties. The question of causal order is deeply related to separability [24]. Indefinite causal order has been recognized as a resource in quantum communication [25,26] and computing [27]. It was verified experimentally [28] and recently was also studied in the context of general relativity.

The question of causal order was also studied in the device-independent context, which covers supraquantum (including extremal no-signaling) correlations [29]. Although "crossed wiring" does not realize an indefinite causal order, it is an unusual causal structure that can potentially have implications in this direction.

In this paper, we present a protocol which relies on the "different causal order" application of nonlocal boxes, enabling a perfect (error-free) interconversion of extremal two-input nonlocal boxes. The paper is organized as follows. In Sec. II A we present relevant prior work on the interconversion of nonlocal boxes, then, in Sec. II B our error-free protocol. In Sec. III some modified versions of the error-free protocol are given with which one can extend the directly reachable range of the output boxes. We conclude in Sec. IV.

II. INTERCONVERSION PROTOCOL

A. Prior work

Our protocol can be considered an extension of the results of Barrett *et al.* [18]. Let us recapitulate their main results. First, they showed that every extremal nonlocal box is equivalent to a d-box for some integer d. In a d-box the input of both parties are binary: $x, y \in \{0, 1\}$ and the box outputs for them a and b with values $\{0, 1, \dots, d-1\}$. The nonzero $p_{ab|xy}$ probabilities are uniform (all equal to $1/d$) for inputs and outputs which satisfy $(b-a) \mod d = xy$, and zero otherwise. Two nonlocal boxes are considered equivalent if one can be converted into the other by exchanging the roles of Alice and Bob, or permuting the inputs of Alice, permuting the inputs of Bob, permuting the outputs of an input, or deleting an input, where the output is deterministic. Furthermore, in Ref. [18] three protocols were presented to perform interconversions between different boxes. (We note that in these algorithms x and y denote the inputs that Alice and Bob wish to enter into the yet-to-be simulated box.) The three protocols are the following.

Protocol 1. Given a d_1 and a d_2-box, a d_1d_2-box can be simulated (without error). Alice enters x into the d_1-box. If the output a_1 is $d_1 - 1$, she enters x into the d_2-box, otherwise enters 0 to the d_2-box. Her overall output is computed as $a_2d_1 + a_1$. Bob enters y into both boxes. His overall output is computed as $b_2d_1 + b_1$.

Protocol 2. Given a d_1d_2-box, a d_1-box can be simulated (without error). Both parties enter their original input into the d_1d_2-box, and take the output modulo d_1.

Protocol 3. Given n pieces of a d_1-box, a d_2-box can be simulated provided that $d_2 \leqslant d_1^n$ (with arbitrarily small error by increasing n). Alice and Bob simulate a d_1^n-box using Protocol 1, and take the output modulo d_2. Note that this protocol is not error free: although the zero probabilities remain zero, the nonzero probabilities will deviate a little from the uniform distribution.

With the help of Protocols 1 to 3 of Ref. [18] any d-box can be simulated using d'-boxes, however, there is still a large class of boxes for which a nonzero error is unavoidable. This is the case for incommensurable d and d'.

B. Error-free interconversion protocol

In what follows, we will show that one can construct a protocol which can operate with our error. Our protocol requires a specific causal order in the use of the boxes: the parties have to use certain boxes in opposite order, so that the inputs on the box used later depends on the output of the box that is used first. We assume that a nonlocal box can be queried only once, so that, e.g., when the actions of the parties are repeated twice, we assume the use of two boxes of the same type, and not to query the same box twice. We speak of the "number of boxes" in this sense.

Lemma 1. Given two d-boxes Alice and Bob can convert them into one $(d+1)$-box with probability $(d^2-1)/d^2$ or get a specific output on both sides which signifies an unsuccessful conversion attempt, and this happens with probability $1/d^2$.

The conversion can be carried out using a single round of Protocol 4 below. Before introducing the protocol and proving Lemma 1, let us state our main result first.

Theorem 1. Given an infinite supply of d-boxes Alice and Bob can realize one $(d+1)$-box with probability 1 and the expected number of actually consumed boxes is $2d^2/(d^2-1)$.

Proof of Theorem 1. Repeating the rounds of Protocol 4 will eventually lead to success. As the probability that the protocol does not halt in the current round is $1/d^2$, the expected number of rounds can be computed by summing the series $(1 - 1/d^2)\sum_{k=1}^{\infty} k(1/d^2)^{k-1}$.

TABLE I. The joint probabilities pertaining to the inputs and outputs of two 2-boxes utilized according to Protocol 4. The order within the bit pairs corresponds to the temporal order of the boxes on Alice's side. The probabilities are determined by multiplying the probabilities of the individual 2-boxes, which are given as $p_{a_ib_i|x_iy_i} = 1/2$ if $b_i \oplus a_i = x_iy_i$ and 0 otherwise, where the subscripts $i = 1, 2$ refer to the first and second boxes, respectively. The notion of the colors and output pairs displayed in boldface is explained in the caption of Fig. 1.

x ↓	y→	0				1			
	in → out (in ↓ out)	00 ↓ 00	00 ↓ 01	00 ↓ 10	00 ↓ 11	01 ↓ **00**	11 ↓ **01**	01 ↓ **10**	11 ↓ **11**
0	00→ 00	1/4	0	0	0	1/4	0	0	0
	00→ 01	0	1/4	0	0	0	1/4	0	0
	00→ 10	0	0	1/4	0	0	0	1/4	0
	00→ 11	0	0	0	1/4	0	0	0	1/4
1	10→ **00**	1/4	0	0	0	1/4	0	0	0
	10→ **01**	0	1/4	0	0	0	0	0	1/4
	11→ **10**	0	0	1/4	0	0	1/4	0	0
	11→ **11**	0	0	0	1/4	0	0	1/4	0

Protocol 4. In a single round the parties consume two d-boxes. Alice inputs x (the value which would be the input of the box to be simulated) into the *first* box. If the result is 0, she inputs 0 to the second box, otherwise she inputs x to the second box as well. If the overall result is not 00, then Alice terminates the protocol and the output is a_1 if $a_1 \leqslant a_2$, and $a_1 + 1$ if $a_1 > a_2$. If her overall result is 00, then she starts a new round repeating these steps using two fresh d-boxes. On the other side, Bob inputs y to the *second* box (note the inverted causal order as compared to Alice's side, i.e., the "crossed wiring"). If the result is 0, he inputs 0 to the first box, otherwise he inputs y to the first box as well. If the overall result is not 00, then Bob terminates the protocol and his output is b_1 if $b_1 \leqslant b_2$, and $b_1 + 1$ if $b_1 > b_2$. If his overall result is 00 then he starts a new round using two new d-boxes (similarly to Alice's procedure).

We note that since the 00 result can only be obtained by Alice and Bob in coincidence in the same round there is no need for them to communicate classically in order to start a new round of the protocol.

Before proving the correctness of Protocol 4 for arbitrary d, let us illustrate, as a simple example, how it can convert two 2-boxes (PR-boxes) into a 3-box. The joint probabilities for a single round of the protocol are presented in Table I. The inputs xy for the target box divides the table into four blocks. Although there are 16 possible [in, out] combinations for each user in each block, we tabulate only those four that appear in the protocol with nonzero probability. Additionally in each 4×4 block there is only a single nonzero entry in each row and column. For instance, in the $(x, y) = (1, 1)$ block the probability pertaining to row 3 and column 3 means that Bob had entered 1 to the second box, received 0, then entered 0 to the first box and received 1. Meanwhile, Alice had entered 1 to the first box, received 1, therefore entered 1 also to the second box, and received 0. Because of the inputs, the outputs of the first box should be correlated $(x_1 + y_1 \mod 2 = 0)$ and those of the second box should be anticorrelated $(x_2 + y_2 \mod 2 = 1)$. But both outputs are correlated ($a_1 = b_1$, $a_2 = b_2$), therefore, this case is impossible, so the matrix entry is 0. Observe that the upper left entries of the blocks (corresponding to the outputs 00 for both parties) always have probability 1/4. According to the protocol this is the indication that the round fails and must be repeated. As the 00 outputs can only occur in coincidence, both parties recognize this failure without the need for any communication. The remaining lower right 3×3 submatrices of the blocks in the table are equivalent to a 3-box (presented in Table II), by the following relabeling of the outputs (independent of the inputs): 01 → 0, 10 → 2, 11 → 1. Overall, each round of the protocol succeeds with probability 3/4 and fails otherwise.

TABLE II. The conditional probability distribution of the targeted 3-box.

x↓	y→ / a↓ b→	0: 0	0: 1	0: 2	1: 0	1: 1	1: 2
0	0	1/3	0	0	1/3	0	0
	1	0	1/3	0	0	1/3	0
	2	0	0	1/3	0	0	1/3
1	0	1/3	0	0	0	1/3	0
	1	0	1/3	0	0	0	1/3
	2	0	0	1/3	1/3	0	0

The equivalence with the 3-box can also be seen by ordering the possible nonzero-probability outputs into a table, as shown in Fig. 1. This representation reveals that [apart from a trivial one-cycle (00)] there exists a length-3 cycle among the nontrivial output pairs. The relabelling of the outputs is then straightforward.

Proof of Lemma 1. The proof can be accomplished by analyzing Protocol 4 for general inputs. If any of x or y is 0, then Alice and Bob receive identical outputs, so the matrix in these blocks is proportional to the identity matrix, therefore it is sufficient to analyze the $x = y = 1$ case only. As both parties make deterministic steps, the $d^2 \times d^2$ matrix of possible outputs will still have only d^2 nonzero entries, one in each row and each column, so we can consider it as a permutation matrix, similar to the one in Fig. 1. Studying the the cycle structure of this permutation matrix reveals the type of boxes

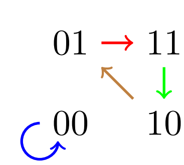

FIG. 1. Nontrivial output pairs of the parties and their interdependence when applying Protocol 4 on two boxes with $d = 2$. Pairs of numbers represent output pairs $a_1 a_2$ or $b_1 b_2$ when $x = y = 1$ (highlighted in bold in Table I). If there is a nonzero probability for a given pair of outputs, then an arrow is drawn from one pair to the other in the sense that Alice's output stands at the base, while Bob's output stands at the point of the arrow. The colors correspond to the colored entries of Table I. Due to the nature of the d-boxes, the correlations are unique, i.e., $a_1 a_2$ determines $b_1 b_2$ and vice versa. Therefore, there is only one arrow starting and ending at every point.

hiding in the result. The conditional probability distribution of any d-box can be transformed to a form similar to that in Fig. 1, in which three blocks are $1/d$ times the identity matrix, and the fourth block is $1/d$ times a permutation matrix with a single d cycle (cyclic shift of each element one step to the right). As the different cycles of the joint probability distributions of a single round of the protocol divide the blocks into different submatrices each containing a cycle with a certain length, after receiving their outputs Alice and Bob can identify the respective submatrix without communication. If they find that their output does not correspond to the desired submatrix, they can start a new round and can eventually reach the targeted submatrix: the one which simulates the d'-box (with d' equals the length of the cycle in this submatrix). Thus the cycle structure determines the types of boxes that can be simulated by this protocol.

The permutation matrix is a permutation of the set with elements of the form (c_1, c_2), where $0 \leqslant c_i < d$ are integers. If (a_1, a_2), (b_1, b_2) is a possible simultaneous output of Alice and Bob, then the permutation corresponding to the matrix takes the element (a_1, a_2) to (b_1, b_2). [Note that, for every output of Alice, (a_1, a_2) there is exactly one possible output of Bob (b_1, b_2)].

There are 4 (2×2) cases.

(1) If $a_1 = b_2 = 0$, then $a_2 = b_2$, and $a_1 = b_1$, therefore, the element (0,0) goes to (0,0).

(2) If $a_1 = 0$ and $b_2 \neq 0$, then $a_2 = b_2$, and $(a_1 + 1) \mod d = b_1$, therefore, the elements of the form $(0, a_2)$ go to $(1, a_2)$ $(a_2 \neq 0)$.

(3) If $a_1 \neq 0$ and $b_2 = 0$, then $(a_2 + 1) \mod d = b_2$, and $a_1 = b_1$, therefore the elements of the form $(a_1, d - 1)$ go to $(a_1, 0)$ (where $a_1 \neq 0$).

(4) If $a_1 \neq 0$ and $b_2 \neq 0$, then $(a_2 + 1) \mod d = b_2$, and $(a_1 + 1) \mod d = b_1$, therefore, the elements of the form (a_1, a_2) go to $(a_1 + 1 \mod d, a_2 + 1 \mod d)$ (where $a_1 \neq 0$ and $a_2 \neq d - 1$).

To obtain the cycle structure, let us examine the orbits of the elements $(0, a_2)$. If $a_2 = 0$, then (0,0) does not move, the orbit has one element, thus, this is a 1-cycle. Otherwise, $(0, a_2)$ first moves to $(1, a_2)$, then $(1 + s, a_2 + s)$ for $1 \leqslant s \leqslant d - a_2 - 1$, then to $(d - a_2, 0)$ then to $(d - a_2 + s, s)$ for $1 \leqslant s \leqslant a_2$, then returns to $(0, a_2)$. These are $d + 1$ steps altogether, which is a $(d + 1)$ cycle. The cycles do not overlap because the first coordinate cannot become 0 before returning, therefore, we get $d - 1$ $(d + 1)$ cycles, and these cases cover all pairs because $d^2 = (d - 1)(d + 1) + 1$. It is easy to see that the output function labels the consecutive elements of every $(d + 1)$ cycle from 0 to d. This completes the proof of Lemma 1.

As an illustration of the proof, the case $d = 5$ is displayed in Fig. 2. If all four outputs are different from 0, then the outputs of both boxes differ by one, so in the most part of the matrix, the arrow is upward diagonal. If, however, the first output of Alice is one, then the second outputs must coincide, therefore horizontal arrows start from the first column, and similarly, vertical arrows from the first row. Alice and Bob can get 00 at the same time, therefore, a circular arrow is drawn into the corresponding cell. The cycles are moving mainly diagonally, but at the first column they jump one position to the left, and at the first row, jump back. So nearly all diagonals

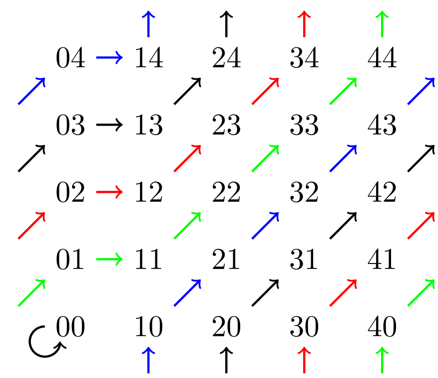

FIG. 2. Nontrivial output pairs of the parties and their interdependence when applying Protocol 4 on two boxes with $d = 5$. Pairs of numbers represent output pairs $a_1 a_2$ or $b_1 b_2$. If there is a correlation between a given pair of outputs, then an arrow is drawn from one pair to the other in the sense that Alice's output stands at the base, while Bob's output stands at the point of the arrow. The four possible six-cycles are highlighted in red, green, blue, and black.

correspond to some six-cycle, while one of them disappeared like in the so-called "vanishing leprechaun" puzzle [30]. As can be seen, there is a one-cycle (00) and four six-cycles, so a single round converts two 5-boxes into a 6-box with probability 24/25, and is unsuccessful with probability 1/25. Alice and Bob can unambiguously identify this latter case, and continue with the protocol.

III. GENERALIZATIONS

Protocol 4 can be slightly modified to simulate other boxes as well, not only $d + 1$ ones. To achieve this one needs to change the number of cases when Alice or Bob enters 0 into their respective "second" box. The case when Bob enters 0 to the first box as a result of getting either 0 or 1 as the output of the second box is illustrated in Fig. 3: the diagonally moving cycle jumps left once and right twice, so after traversing the matrix once it restarts at the next diagonal, covering the whole matrix (except for the two stationary points). Thus, it contains two one-cycles and a single 23-cycle. In general, one can say that, if one party enters 0 in one case and the other one enters 0 in two cases, then a d-box can be converted into

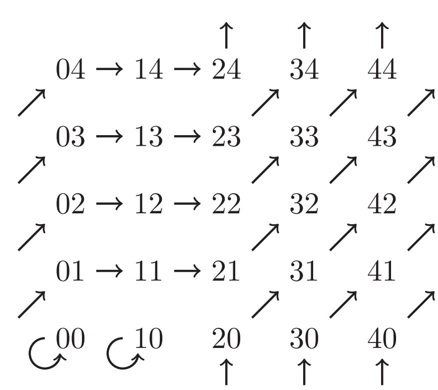

FIG. 3. Nontrivial output pairs of the parties and their interdependence when applying the modified version of Protocol 4 in the case of $d = 5$.

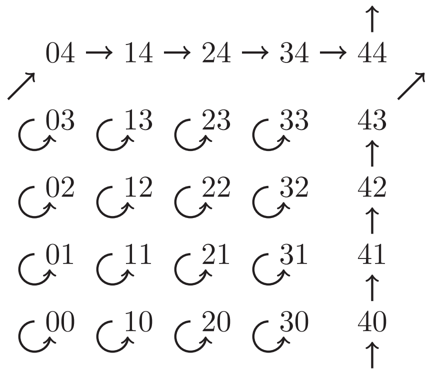

FIG. 4. Nontrivial output pairs of the parties and their interdependence when applying the second modified version of Protocol 4 in the case of $d = 5$ and $s = 4$.

a $(d^2 - 2)$-box. The success probability of a single round is $(d^2 - 2)/d^2$.

Another possible modification is when both parties enter 0 to their respective "second box" if their "first" output is smaller than some value s, i.e., if $a_1 < s$, $b_2 < s$ (where $1 \leqslant s < d$). In this case they can get a $(d + s)$ box, but with decreasing success probability. The permutation then has s^2 one-cycles ($a_1 < s$ and $b_2 < s$) while the $(d + s)$ cycles contain four different sections: (i) $(0, a_2)$ moves to (q_1, a_2), where $1 \leqslant q_1 \leqslant s$; (ii) $(s + q_2, a_2 + q_2)$, where $1 \leqslant q_2 \leqslant d - a_2 - 1$; (iii) $(d - a_2 - 1, q_3)$, where $1 \leqslant q_3 \leqslant s$; and finally (iv) $(d - a_2 - 1 + q_4, s + q_4)$, where $1 \leqslant q_4 \leqslant d - s - 1$. Thus, the success probability of a single round is $(d^2 - s^2)/d^2$. As a simple example one can choose $d = 5$ and $s = 4$, in which case a single round can simulate a 9-box with a success probability of 9/25 (see Fig. 4).

IV. CONCLUSION

We presented a protocol (and its relevant modified versions), which, together with Protocols 1 and 2 of Ref. [18], enable the conversion of any d-boxes into any other d'-box without error. In the other similar protocols known so far the parties have to agree on the number of turns to go below a fixed error and they need to communicate if they want to further improve on it. Our protocol, on the other hand, allows for unlimited number of iterations in principle, with a halting condition that can be verified without communication, and an error-free conversion. The expected number of required iterations is finite. There may be other possibilities to modify our protocol, such as, combining two boxes of different size.

Our conversion protocol is the first one to utilize the fact that Alice and Bob are allowed to query their parts of the boxes in *different causal order*. It is an open question whether there exists a protocol for realizing error-free interconversion of nonlocal boxes without "crossed wiring." It would be also interesting to find a useful protocol in which there are three boxes involved, and the order in which certain boxes are used depends on the output of some other boxes. This could potentially demonstrate the use of indefinite causal order in the present device-independent context.

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