Leggett-Garg inequalities for testing quantumness of gravity

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In this study, we determine a violation of the Leggett-Garg inequalities due to gravitational interaction in a hybrid system consisting of a harmonic oscillator and a spatially localized superposed particle. The violation of the Leggett-Garg inequalities is discussed using the two-time quasiprobability in connection with the entanglement negativity generated by gravitational interaction. It is demonstrated that the entanglement suppresses the violation of the Leggett-Garg inequalities when one of the two times of the quasiprobability t_1 is chosen as the initial time. Further, it is shown that the Leggett-Garg inequalities are generally violated due to gravitational interaction by properly choosing the configuration of the parameters, including t_1 and t_2, which are the times of the two-time quasiprobability. The feasibility of detecting violations of the Leggett-Garg inequalities in hybrid systems is also discussed.

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I. INTRODUCTION

Unifying quantum mechanics and gravity is one of the most fundamental issues in physics. Feynman discussed the possibility of testing whether gravity follows the framework of quantum mechanics [1], which has been reexamined because of recent developments in quantum information theory and quantum technologies [2]. The proposal to test the quantumness of gravity [3–5], called the BMV experiment, has garnered more attention and has stimulated many studies (e.g., [6–8] and references therein). The BMV experiment relies on the entanglement generated by the gravitational interaction, which is a quantum smoking-gun feature used to characterize the nonlocal quantum interaction. Optomechanical systems are also promising for detecting the quantum entanglement generated by gravitational interaction [9–16].

Other possible approaches to detect the quantumness of gravity have been discussed in the literature. One possible approach is a non-Gaussian feature of the quantum state generated through the quantum force of gravity in Bose-Einstein condensates [17]. The authors of Ref. [18] showed the visibility function of interference in a hybrid system consisting of an oscillator and a particle in a spatially localized superposition state (see Fig. 1). Based their study [18], they concluded that the revival in the oscillating feature of the visibility function reflects the nonseparable feature of the gravitational interaction, which generates the entanglement in the hybrid system (see also [19–22]). Therefore, the revival of the visibility function provides a unique approach to test the quantumness of gravitational interaction.

In this study, we propose a different approach to test the quantumness of gravity: We employ the Leggett-Garg inequalities, which were proposed to test the macrorealism in Ref. [23] (see also [24] for a review). Macrorealism involves characterizing classical systems, in which a macroscopic system is in a definite state at any given time for different available states, and the state can be measured without any effect on the system. The Leggett-Garg inequalities are temporal correlations, which might be realized in a way similar to the spatial nonlocal correlation described by Clauser-Horne-Shimony-Holt inequalities. Quantum systems may violate the predictions of macrorealism represented by the Leggett-Garg inequalities. The violation of the Leggett-Garg inequalities has been theoretically investigated and experimentally verified in many systems (Refs. [25,26] and references therein). In this study, we apply the two-time quasiprobability introduced in Ref. [27] and explored in [28–31] for the hybrid system described in Ref. [18] to probe the quantumness of gravitational interaction.

The remainder of this study is organized as follows. In Sec. II, we briefly review the Leggett-Garg inequalities based on the two-time quasiprobability and the hybrid system in Ref. [18]. In Sec. III, we apply the formalism to a hybrid system, and the behavior of the two-time quasiprobability is examined. The feasibility of detecting the violation of the Leggett-Garg inequalities is also mentioned. In Sec. IV, the prediction within the Newton-Schrödinger approach is presented. Section V includes a summary and conclusions. The origin of the violation of the Leggett-Garg inequalities due to gravitational interaction is also discussed. In the Appendix, a deviation of Eq. (A5) is described. Note that we adopt units of $\hbar = 1$ unless noted otherwise.

II. FORMULATION

A. Leggett-Garg inequalities

We begin by briefly reviewing the two-time quasiprobability function [27–31]. We introduce a dichotomic variable $\hat{Q} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{n}$ is a unit vector and $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the Pauli spin matrix. As the dichotomic variable is regarded as a

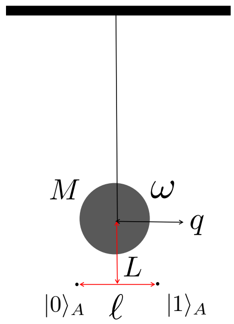

FIG. 1. System consisting of an oscillator and a particle. The particle is in a superposition state of two spatially localized states denoted by $|0\rangle_A$ and $|1\rangle_A$. The position of the oscillator is denoted by q, and its mass and angular frequencies are M and ω, respectively. L is the distance between the oscillator and the particle in a superposition state, and ℓ is the distance between the positions of the two spatially localized states.

spin, $\hat{Q}$ is the quantum variable that gives spin values of ± 1 by measuring in the direction $\boldsymbol{n}$. Therefore, $|\boldsymbol{n}\cdot\boldsymbol{\sigma}|^2 = 1$. The measurement operator of the dichotomic variable to obtain the measurement result $a = \pm 1$ is defined as

$$\hat{M}_a = \tfrac{1}{2}(\mathbf{1} + a\boldsymbol{n}\cdot\boldsymbol{\sigma}), \tag{1}$$

which satisfies $\hat{M}_a = \hat{M}_a^\dagger = \hat{M}_a^2$.

Assuming the initial state ρ_0, the probability that a is obtained through a measurement at t_1 is given by

$$P_1(a) = \mathrm{Tr}[\hat{M}_a\hat{U}(t_1)\rho_0\hat{U}^\dagger(t_1)\hat{M}_a^\dagger] = \mathrm{Tr}[\hat{M}_a(t_1)\rho_0\hat{M}_a^\dagger(t_1)], \tag{2}$$

where we define

$$\hat{M}_a(t) = \hat{U}^\dagger(t)\hat{M}_a\hat{U}(t) \tag{3}$$

and $\hat{U}(t)$ is the unitary operator of the time evolution of the system, in which we assume the time-translation invariance. Then, the expectation value of the dichotomic variable $\hat{Q}$ at t is

$$\langle\hat{Q}(t)\rangle = \sum_{a=\pm1} aP_1(a) = \mathrm{Tr}[\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1)\rho_0], \tag{4}$$

where $\boldsymbol{\sigma}(t) = U^\dagger(t)\boldsymbol{\sigma}U(t)$.

Similarly, the probability that the measurement results a and b are obtained via measurements at t_1 and t_2 ($\geqslant t_1$) with measurement axis $\boldsymbol{n}$

$$\begin{aligned} P_{12}(a,b) &= \mathrm{Tr}\big[\hat{M}_b\hat{U}(t_2-t_1)\hat{M}_a\hat{U}(t_1)\rho_0\hat{U}^\dagger(t_1)M_a^\dagger\hat{U}(t_2-t_1)\hat{M}_b\big] \\ &= \mathrm{Tr}[\hat{M}_b(t_2)\hat{M}_a(t_1)\rho_0\hat{M}_a^\dagger(t_1)\hat{M}_b^\dagger(t_2)] = \mathrm{Tr}[\hat{M}_b(t_2)\hat{M}_a(t_1)\rho_0\hat{M}_a^\dagger(t_1)], \end{aligned} \tag{5}$$

where

$$\hat{M}_b = \tfrac{1}{2}(\mathbf{1} + b\boldsymbol{n}\cdot\boldsymbol{\sigma}). \tag{6}$$

The two-time correlation function is introduced as

$$C(t_1,t_2) = \sum_{a,b=\pm1} abP_{12}(a,b), \tag{7}$$

which reduces to

$$C(t_1,t_2) = \tfrac{1}{2}\mathrm{Tr}[\{\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1), \boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\}\rho_0], \tag{8}$$

where $\{\cdot,\cdot\}$ denotes an anticommutator.

In the theory of macrorealism, the corresponding variables $Q_1 = Q(t_1)$ and $Q_2 = Q(t_2)$ take definite values of ± 1, implying that

$$(1 + s_1Q_1)(1 + s_2Q_2) \geqslant 0, \tag{9}$$

where $s_1, s_2 = \pm 1$. Following the framework of macrorealism, a joint probability distribution exists for the measurements results. The existence of such a joint probability distribution means that we can simply average the above formula and obtain the two-time Leggett-Garg inequalities [27,29]

$$1 + s_1\langle Q\rangle + s_2\langle Q_2\rangle + s_1s_2\langle Q_1Q_2\rangle \geqslant 0. \tag{10}$$

In the quantum mechanics, the corresponding expression can be discussed with the two-time quasiprobability defined by

$$q_{s_1,s_2}(t_1,t_2) = \tfrac{1}{4}[1 + s_1\langle\hat{Q}(t_1)\rangle + s_2\langle\hat{Q}(t_2)\rangle + s_1s_2C(t_1,t_2)], \tag{11}$$

which is equivalently written as [29]

$$q_{s_1s_2}(t_1,t_2) = \tfrac{1}{4}\mathrm{Tr}\big[\tfrac{1}{2}\{1 + s_1\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1), 1 + s_2\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\}\rho_0\big] = \mathrm{Re}\{\mathrm{Tr}[M_{s_2}(t_2)M_{s_1}(t_1)\rho_0]\}. \tag{12}$$

Note that the two-time quasiprobability produces the relations [29]

$$\langle\hat{Q}(t_1)\rangle = \sum_{a,b=\pm1} aq_{a,b}(t_1,t_2), \quad \langle\hat{Q}(t_2)\rangle = \sum_{a,b=\pm1} bq_{a,b}(t_1,t_2), \quad C_{1,2}(t_1,t_2) = \sum_{a,b=\pm1} abq_{a,b}(t_1,t_2). \tag{13}$$

However, it may take negative values, which means a violation of the Leggett-Garg inequalities.

B. Hybrid system

We consider a hybrid system consisting of an oscillator and a particle (see Fig. 1). An oscillator with a mass M is described by the coordinate variable q, whose oscillation is characterized by the angular frequency ω. A particle with mass m is in a superposition of the two spatially localized states denoted by $|0\rangle_A$ and $|1\rangle_A$. Here, we assume that ℓ is the distance between the positions of the two spatially localized states, and L is the distance between the oscillator and the particle. This model was introduced in Ref. [18], and the authors investigated the effects of gravitational interaction between the oscillator and the particle on the visibility function owing to the interference of the particle's state. Reference [18] demonstrated that the revival of a visibility function owing to the interference is the result of the entanglement because of the gravitational interaction, which can be tested as a signature of the quantumness of gravity. Furthermore, the nonseparable evolution owing to gravitational interaction is more fundamental to their argument to generate entanglement [18,20].

We investigate the Leggett-Garg inequalities in a hybrid system, whose Hamiltonian is given by

$$H = \Omega\sigma^z + \omega a^\dagger a + H_{\rm grav}, \tag{14}$$

where $\omega a^\dagger a$ is a free Hamiltonian of the oscillator with the creation (annihilation) operator $a\,(a^\dagger)$ and the last term on the right-hand side of Eq. (14) describes the gravitational interaction between the oscillator and the particle. The eigenstates of σ^z describe the two spatially localized states of the particle, and the first term in Eq. (14), $\Omega\sigma^z$, causes the phenomenon corresponding to the Larmor precession in the two states, which is not included in the analysis of Ref. [18]. Following the configuration shown in Fig. 1, the gravitational potential of the system can be written as

$$H_{\rm grav} = -\frac{GMm}{\sqrt{L^2+(q+\sigma^z\ell/2)^2}} \simeq \frac{GMmq\ell\sigma^z}{2\sqrt{L^2+\ell^2/4}^3} + \text{const}, \tag{15}$$

where G is the Newton constant and the approximate expression is obtained by assuming that q is small compared to L and ℓ. Introducing constant g and nondimensional variable $\tilde{q}$ via

$$g = \frac{GMm\ell}{2\sqrt{L^2+\ell^2/4}^3}\frac{1}{\sqrt{2M\omega}}, \quad q = \frac{1}{\sqrt{2M\omega}}\sqrt{2}\tilde{q}, \tag{16}$$

the Hamiltonian of the gravitational interaction reduces to

$$H_{\rm grav} = g\sigma^z\sqrt{2}\tilde{q}. \tag{17}$$

The unitary operator of the Hamiltonian is written as

$$U(t) = e^{-iHt} = e^{-i(\Omega\sigma^z+\omega a^\dagger a)t}\, T\exp\left[-i\int_0^t dt' g\sigma^z\sqrt{2}\tilde{q}_I(t)\right], \tag{18}$$

where $\tilde{q}_I$ denotes $\tilde{q}$ in the interaction picture,

$$\tilde{q}_I(t) = e^{i\omega a^\dagger a t}\tilde{q}e^{-i\omega a^\dagger a t} = \frac{1}{\sqrt{2}}(e^{-i\omega t}a + e^{i\omega t}a^\dagger). \tag{19}$$

Note that σ^z in the interaction picture is $\sigma^z_I(t) = e^{i\Omega\sigma^z t}\sigma^z e^{-i\Omega\sigma^z t} = \sigma^z$. Using the following relation (see also [14]), we have

$$\begin{aligned} T\exp\left[-i\int_0^t dt' g\sigma^z\sqrt{2}\tilde{q}_I(t')\right] &= \exp\left[-i\int_0^t dt' g\sigma^z\sqrt{2}\tilde{q}_I(t') - g^2\int_0^t dt'\int_0^{t'} dt''[\tilde{q}_I(t'),\tilde{q}_I(t'')]\right] \\ &= e^{g\sigma^z[\alpha(t)a-\alpha^*(t)a^\dagger]+ig^2\beta(t)}. \end{aligned} \tag{20}$$

We used the following relations to derive the second equality:

$$-i\int_0^t \sqrt{2}\tilde{q}_I(t')dt' = \alpha(t)a - \alpha^*(t)a^\dagger, \tag{21}$$

$$\int_0^t dt'\int_0^{t'} dt''[\tilde{q}_I(t'),\tilde{q}_I(t'')] = -i\beta(t), \tag{22}$$

and we defined

$$\alpha(t) = \frac{e^{-i\omega t}-1}{\omega}, \qquad \beta(t) = \frac{1}{\omega}\left(t - \frac{\sin\omega t}{\omega}\right). \tag{23}$$

Except for the total phase, the unitary operator of the time evolution of the system is written as

$$U(t) = e^{-i(\Omega\sigma^z+\omega a^\dagger a)t}e^{g\sigma^z[\alpha(t)a-\alpha^*(t)a^\dagger]}. \tag{24}$$

C. Two-time quasiprobability

We determine the two-time quasiprobability for the particle in the hybrid system above when the initial state is prepared as

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\otimes|0\rangle, \tag{25}$$

where $|0\rangle$ is the ground state of the oscillator. Using the unitary operator (24), the state at time t is

$$|\psi(t)\rangle = U(t)|\psi_0\rangle = \frac{e^{-i\omega a^\dagger a t}}{\sqrt{2}}[e^{-i\Omega t}|0\rangle_A|-g\alpha^*(t)\rangle_C + e^{i\Omega t}|1\rangle_A|+g\alpha^*(t)\rangle_C],$$
$$= \frac{1}{\sqrt{2}}[e^{-i\Omega t}|0\rangle_A|g\alpha(t)\rangle_C + e^{i\Omega t}|1\rangle_A|-g\alpha(t)\rangle_C], \tag{26}$$

where the oscillation is in the coherent state $|\xi\rangle_C$ defined by $|\xi\rangle_C = e^{\xi a^\dagger - \xi^* a}|0\rangle$. In deriving the second line of the equation, we used the expression of the coherent states in the Fock basis,

$$|\xi\rangle_C = e^{-|\xi|^2/2}\sum_{m=0}^{\infty}\frac{\xi^m}{\sqrt{m!}}|m\rangle,$$

where $|m\rangle$ is the mth energy excited state of the oscillator.

Now, we determine the expression of the two-time quasiprobability function (11). Hereafter we consider the case

$$\boldsymbol{n} = (\cos\varphi, \sin\varphi, 0), \tag{27}$$

unless otherwise stated. For the initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ with (25), from straightforward computations, we obtain

$$\langle\hat{Q}_1\rangle = \mathrm{Tr}[\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1)\rho_0] = \cos(2\Omega t_1 - \varphi)e^{-8\lambda^2\sin^2\frac{\omega t_1}{2}}, \tag{28}$$

$$\langle\hat{Q}_2\rangle = \mathrm{Tr}[\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\rho_0] = \cos(2\Omega t_2 - \varphi)e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}}, \tag{29}$$

and

$$C(t_2, t_1) = \tfrac{1}{2}\mathrm{Tr}[\{\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1), \boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\}\rho_0]$$
$$= \cos\Theta(t_2, t_1)\cos[2\Omega(t_2 - t_1)]e^{-8\lambda^2\sin^2\frac{\omega(t_2-t_1)}{2}}, \tag{30}$$

where we defined

$$\Theta(t_2, t_1) = 4\lambda^2[\sin\omega(t_2 - t_1) - \sin\omega t_2 + \sin\omega t_1]$$
$$= 16\lambda^2\sin\frac{\omega(t_2 - t_1)}{2}\sin\frac{\omega t_2}{2}\sin\frac{\omega t_1}{2} \tag{31}$$

and

$$\lambda = \frac{g}{\omega}. \tag{32}$$

Then, the expression for the two-time quasiprobability is written as

$$q_{s_1 s_2}(t_1, t_2) = \tfrac{1}{4}\{1 + s_1\cos(2\Omega t_1 - \varphi)e^{-8\lambda^2\sin^2\frac{\omega t_1}{2}}$$
$$+ s_2\cos(2\Omega t_2 - \varphi)e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}}$$
$$+ s_1 s_2\cos\Theta(t_2, t_1)\cos[2\Omega(t_2 - t_1)]$$
$$\times e^{-8\lambda^2\sin^2\frac{\omega(t_2-t_1)}{2}}\}. \tag{33}$$

III. BEHAVIOR OF TWO-TIME QUASIPROBABILITY

A. Case of $t_1 = 0$ and $\Omega \neq 0$

In this section, we investigate the behavior of the two-time quasiprobability. We first consider the cases imposing that t_1 is the initial time, $t_1 = 0$, and $\Omega \neq 0$ in Eq. (33). In this case, we show that gravitational interaction suppresses the violation of the Leggett-Garg inequalities. Imposing $t_1 = 0$ on the two-time quasiprobability (33), we have

$$q_{s_1,s_2}(0, t_2) = \tfrac{1}{4}\big[1 + s_1\cos\varphi + s_2\cos(2\Omega t_2 - \varphi)e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}}$$
$$+ s_1 s_2\cos(2\Omega t_2)e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}}\big]. \tag{34}$$

Assuming that the Leggett-Garg inequalities are violated when the gravitational interaction is switched off by setting $\lambda = 0$,

$$1 + s_1\cos\varphi + s_2\cos(2\Omega t_2 - \varphi) + s_1 s_2\cos(2\Omega t_2) < 0. \tag{35}$$

This inequality holds, depending on the parameters, unless we consider the case $\varphi = 0$. Under this condition, we have

$$s_2\cos(2\Omega t_2 - \varphi) + s_1 s_2\cos(2\Omega t_2) < 0 \tag{36}$$

because $1 + s_1\cos\varphi \geqslant 0$ is always satisfied. Then, the quasiprobability is rewritten as

$$q_{s_1,s_2}(0, t_2) = \tfrac{1}{4}[1 + s_1\cos\varphi + s_2\cos(2\Omega t_2 - \varphi) + s_1 s_2\cos(2\Omega t_2)] - \tfrac{1}{4}(1 - e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}})[s_2\cos(2\Omega t_2 - \varphi) + s_1 s_2\cos(2\Omega t_2)]. \tag{37}$$

The terms with the factor $(1 - e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}})$ of Eq. (37), which originates from gravitational interaction, are always positive from Eq (36). This means that gravitational interaction always suppresses the violation of the Leggett-Garg inequalities in this case.

Because the gravitational interaction generates the entanglement between the oscillator and the particle, the above argument is rephrased using the entanglement. To quantify the entanglement of a given density matrix ρ_{12} of a bipartite system, we use the entanglement negativity [32],

$$N = \sum_{\lambda_i<0}|\lambda_i|, \tag{38}$$

where λ_i is the eigenvalue of the partial transpose $\rho_{12}^{T_1}$ with the elements ${}_1\langle i|_2\langle j|\rho_{12}^{T_1}|k\rangle_1|\ell\rangle_2 = {}_1\langle k|_2\langle j|\rho_{12}|i\rangle_1|\ell\rangle_2$. The evolved state $|\psi(t)\rangle$ is rewritten as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}[e^{-i\Omega t}|0\rangle_A|g\alpha(t)\rangle_C + e^{i\Omega t}|1\rangle_A|-g\alpha(t)\rangle_C]$$
$$= \frac{1}{\sqrt{2}}\left[e^{-i\Omega t}|0\rangle_A\left(\frac{\sqrt{N_+}}{2}|+\rangle_C + \frac{\sqrt{N_-}}{2}|-\rangle_C\right) + e^{i\Omega t}|1\rangle_A\left(\frac{\sqrt{N_+}}{2}|+\rangle_C - \frac{\sqrt{N_-}}{2}|-\rangle_C\right)\right], \quad (39)$$

where $|\pm\rangle_C = 1/\sqrt{N_\pm}[|g\alpha(t)\rangle_C \pm |-g\alpha(t)\rangle_C]$ and $N_\pm = 2 \pm 2e^{-2g^2|\alpha(t)|^2}$. Hence, $|\psi(t)\rangle$ is regarded as a two-qubit state with the basis $\{|0\rangle_A|+\rangle_C, |0\rangle_A|-\rangle_C, |1\rangle_A|+\rangle_C, |1\rangle_A|-\rangle_C\}$, and the density matrix $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ is a 4×4 matrix. This is due to the fact that the Schmidt rank of a pure hybrid state is always finite. From the partial transposed matrix $\rho^{T_A}(t)$, we obtain the following entanglement negativity:

$$N(t) = \frac{1}{2}\sqrt{1 - e^{-16\lambda^2\sin^2\frac{\omega t}{2}}}.$$

In, e.g., [33], the above procedure was performed for a pure hybrid qubit-Schrödinger-cat state.

The term of the gravitational interaction in (37) is expressed as

$$1 - e^{-8\lambda^2\sin^2\frac{\omega t}{2}} = 1 - \sqrt{1 - 4N^2(t)}. \quad (40)$$

Then, Eq. (37) can be written as

$$q_{s_1,s_2}(0,t_2) = \tfrac{1}{4}[1 + s_1\cos\varphi + s_2\cos(2\Omega t_2 - \varphi) + s_1 s_2\cos(2\Omega t_2)] - \tfrac{1}{4}[1 - \sqrt{1 - 4N^2(t_2)}][s_2\cos(2\Omega t_2 - \varphi) + s_1 s_2\cos(2\Omega t_2)]. \quad (41)$$

The negativity takes values $0 \leqslant N(t) \leqslant 1/2$, in which $1 - \sqrt{1 - 4N^2(t)}$ is the monotonic increasing function of $N(t)$. Therefore, this implies that the entanglement suppresses the violation of the Leggett-Garg inequalities of $q_{s_1,s_2}(0,t_2)$.

For the $\Omega = 0$ case, we can determine the relation between the quasiprobability function and the negativity in the limit of $\lambda \ll 1$. In this limit, we have

$$N(t) \simeq 2\lambda\left|\sin\frac{\omega t}{2}\right| \ll 1, \quad (42)$$

with which Eq. (41) reduces to

$$q_{s_1 s_2}(0,t_2) \simeq \tfrac{1}{4}[1 + s_1\cos\varphi + s_2\cos\varphi + s_1 s_2 - 2N^2(t)(s_2\cos\varphi + s_1 s_2)] \quad (43)$$

for $\Omega = 0$. Furthermore, for $s_1\cos\varphi = 1$, $s_1 = -s_2$, we have

$$q_{s_1 s_2}(0,t) \simeq N^2(t). \quad (44)$$

Thus, with the choice of suitable parameters, the quasiprobability reflects the evolution of the entanglement negativity directly.

B. Case of $t_1 \neq 0$ and $\Omega = 0$

Next, we consider the violation of the Leggett-Garg inequality due to the gravitational interaction by setting $\Omega = 0$. Here, we assume the case where $\varphi = 0$ for simplicity. Then, the two-time quasiprobability becomes

$$q_{s_1 s_2}(t_1,t_2) = \tfrac{1}{4}\Big[1 + s_1 e^{-8\lambda^2\sin^2\frac{\omega t_1}{2}} + s_2 e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}} + s_1 s_2\cos\Theta(t_2,t_1)e^{-8\lambda^2\sin^2\frac{\omega(t_2-t_1)}{2}}\Big]. \quad (45)$$

For $\lambda = 0$, the two-time quasiprobability satisfies $q_{s_1 s_2}(t_1,t_2) = \frac{1}{4}(1 + s_1 + s_2 + s_1 s_2) \geqslant 0$. Figure 2 demonstrates the region where the quasiprobability function (45) with $\lambda \neq 0$ takes negative values on the t_1 and t_2 planes, where we show the region satisfying $0 \leqslant t_1 \leqslant t_2$. Thus, the Leggett-Garg inequalities are violated because of the gravitational interaction.

When $\lambda \ll 1$, the contribution from the term $\cos\Theta(t_2,t_1)$ in (45) becomes the highest order of $O(\lambda^4)$. Then, up to the order of $O(\lambda^2)$, the quasiprobability (45) reduces to

$$q_{s_1 s_2}(t_1,t_2) \simeq \frac{1}{4}(1 + s_1 + s_2 + s_1 s_2) - 2\lambda^2\left(s_1\sin^2\frac{\omega t_1}{2} + s_2\sin^2\frac{\omega t_2}{2} + s_1 s_2\sin^2\frac{\omega(t_2 - t_1)}{2}\right), \quad (46)$$

which may take negative values when $(s_1,s_2) = (1,-1)$, $(-1,1)$, or $(-1,-1)$ owing to the gravitational interaction. The minimum value of the quasiprobability function is approximately

$$\min\{q_{s_1 s_2}(t_1,t_2)\} \simeq -\frac{\lambda^2}{2}, \quad (47)$$

which appears for $s_1 = 1, s_2 = -1$ when

$$\omega t_1 = \frac{2}{3}\pi + 2\pi n, \quad \omega t_2 = \frac{7}{3}\pi + 2\pi n \ \ (n = 0,1,2,\dots), \quad (48)$$

$$\omega t_1 = \frac{4}{3}\pi + 2\pi m, \quad \omega t_2 = \frac{5}{3}\pi + 2\pi m \ \ (m = 0,1,2,\dots), \quad (49)$$

for $s_1 = -1, s_2 = 1$ when

$$\omega t_1 = \frac{\pi}{3} + 2\pi n, \quad \omega t_2 = \frac{2}{3}\pi + 2\pi n \ \ (n = 0,1,2,\dots), \quad (50)$$

$$\omega t_1 = \frac{5}{3}\pi + 2\pi m, \quad \omega t_2 = \frac{10}{3}\pi + 2\pi m \ \ (m = 0,1,2,\dots), \quad (51)$$

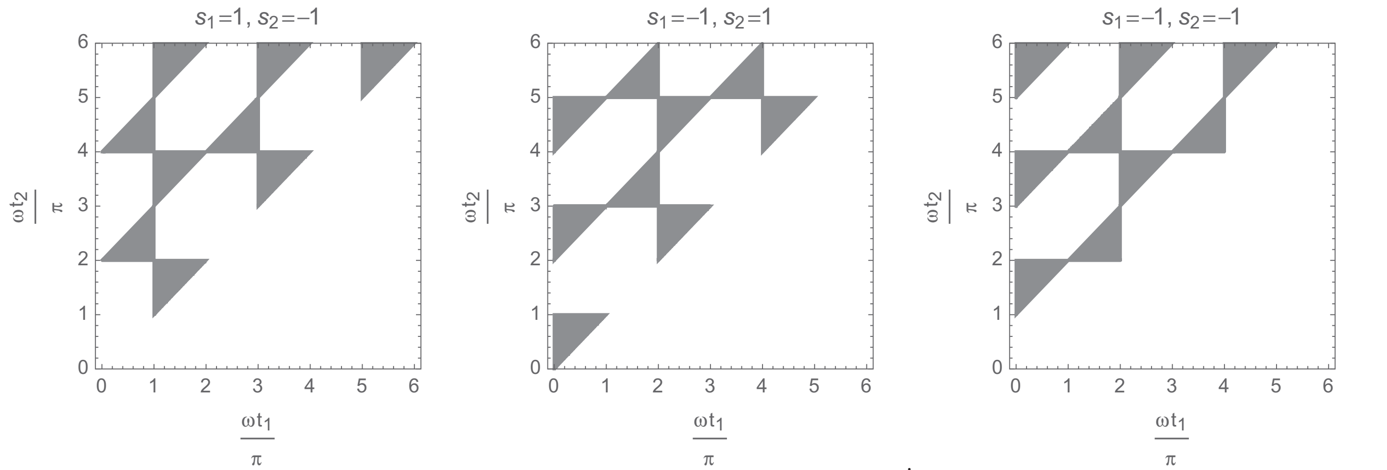

FIG. 2. Shaded regions show the regions where $q_{s_1s_2}(t_1,t_2)<0$ is satisfied on the $\omega t_1/\pi$ (horizontal axis) and $\omega t_2/\pi$ (vertical axis) planes. We adopted $\Omega=0$, $\varphi=0$, and $s_1=1,\ s_2=-1$ (left panel); $s_1=-1,\ s_2=1$ (middle panel); and $s_1=-1,\ s_2=-1$ (right panel). Here, we show only the region satisfying $0\leqslant t_1\leqslant t_2$, and we adopted $8\lambda^2=10^{-2}$.

and for $s_1=s_2=-1$ when

$$\omega t_1=\frac{\pi}{3}+2\pi n,\ \ \omega t_2=\frac{5}{3}\pi+2\pi n\ \ (n=0,1,2,\dots),\tag{52}$$

$$\omega t_1=\frac{5}{3}\pi+2\pi m,\ \ \omega t_2=\frac{7}{3}\pi+2\pi m\ \ (m=0,1,2,\dots).\tag{53}$$

Summarizing the result of the case, $\Omega=0$, the gravitational interaction is the unique interaction to evolve the particle's state. In this case, the Leggett-Garg inequalities are violated, except in the case $s_1=s_2=1$. The violation of the Leggett-Garg inequalities depends on the parameters s_1, s_2, t_1, t_2, and φ, which is not explicitly shown. The minimum value of the two-time quasiprobability is $-\lambda^2/2$. The violation further depends on the initial state, for which we adopted Eq. (25) in this section. Notably, in the case $\Omega=0$, the violation of the Leggett-Garg inequalities is derived from the gravitational interaction, and there appears to be no violation of the Leggett-Garg inequalities in the absence of the gravitational interaction.

C. Thermal state as the initial state for the oscillator

In this section, we consider the effects of the initial condition on the Leggett-Garg inequalities. Here, we adopt a thermal state for the initial state of the oscillator. The thermal state can be described by the density matrix in the Glauber P representation on the basis of the coherent state,

$$\rho_{\rm th}=\frac{1}{\pi\bar n}\int d^2\gamma\, e^{-|\gamma|^2/\bar n}|\gamma\rangle_{CC}\langle\gamma|,\tag{54}$$

where $\bar n$ is the mean occupation number, which is related to temperature T by $\bar n=k_BT/2\omega$, with k_B being the Boltzmann constant, and $|\gamma\rangle_C$ represents the coherent state. Using the expectation value with respect to the thermal state

$$\begin{aligned}\mathrm{Tr}[\rho_{\rm th}e^{\pm2g\{[\alpha(t_2)-\alpha(t_1)]a-[\alpha^*(t_2)-\alpha^*(t_1)]a^\dagger\}}]&=\frac{1}{\pi\bar n}\int d^2\gamma\, e^{-|\gamma|^2/\bar n}{}_C\langle\gamma|\exp[\pm2g\{[\alpha(t_2)-\alpha(t_1)]a-[\alpha^*(t_2)-\alpha^*(t_1)]a^\dagger\}]|\gamma\rangle_C\\&=\frac{1}{\pi\bar n}\int d^2\gamma\, e^{-|\gamma|^2/\bar n}\exp\{-2g^2|[\alpha(t_2)-\alpha(t_1)]|^2\}e^{\pm4ig\mathrm{Im}\{[\alpha(t_2)-\alpha(t_1)]\gamma\}}\\&=\exp\{-2(2\bar n+1)g^2|[\alpha(t_2)-\alpha(t_1)]|^2\},\end{aligned}\tag{55}$$

we find

$$\langle\hat Q_1\rangle=\mathrm{Tr}[\boldsymbol n\cdot\boldsymbol\sigma(t_1)\rho_{\rm th}]=\cos(2\Omega t_1-\varphi)e^{-8(2\bar n+1)\lambda^2\sin^2\frac{\omega t_1}{2}},\tag{56}$$

$$\langle\hat Q_2\rangle=\mathrm{Tr}[\boldsymbol n\cdot\boldsymbol\sigma(t_2)\rho_{\rm th}]=\cos(2\Omega t_2-\varphi)e^{-8(2\bar n+1)\lambda^2\sin^2\frac{\omega t_2}{2}},\tag{57}$$

and

$$C(t_2,t_1)=\tfrac12\mathrm{Tr}[\{\boldsymbol n\cdot\boldsymbol\sigma(t_1),\boldsymbol n\cdot\boldsymbol\sigma(t_2)\}\rho_{\rm th}]=\cos\Theta(t_2,t_1)\cos[2\Omega(t_2-t_1)]\exp\left(-8(2\bar n+1)\lambda^2\sin^2\frac{\omega(t_2-t_1)}{2}\right),\tag{58}$$

where $\Theta(t_2,t_1)$ is defined by Eq. (31). Thus, the quasiprobability with the thermal state as the oscillator's initial condition is given by

$$\begin{aligned}q_{s_1s_2}(t_1,t_2)=\tfrac14\big\{&1+s_1\cos(2\Omega t_1-\varphi)e^{-8(2\bar n+1)\lambda^2\sin^2\frac{\omega t_1}{2}}+s_2\cos(2\Omega t_2-\varphi)e^{-8(2\bar n+1)\lambda^2\sin^2\frac{\omega t_2}{2}}\\&+s_1s_2\cos\Theta(t_2,t_1)\cos[2\Omega(t_2-t_1)]e^{-8(2\bar n+1)\lambda^2\sin^2\frac{\omega(t_2-t_1)}{2}}\big\}.\end{aligned}\tag{59}$$

The difference between the ground state and the thermal state is the factor $(2\bar{n}+1)$ in the exponential function. Therefore, if λ is small, $\lambda \ll 1$, the minimum value of the quasiprobability function appears under the same condition as the ground state of the oscillator in the previous section with $\Omega = 0$, and the minimum value is approximately given by

$$\min\{q_{s_1s_2}(t_1,t_2)\} \simeq -\frac{\lambda^2}{2}(2\bar{n}+1). \tag{60}$$

D. Squeezed state as the initial state of the oscillator

Further, we consider the squeezed state as the initial state of the oscillator. The squeezed state can be obtained with

$$|\zeta\rangle_S = S(\zeta)|0\rangle, \tag{61}$$

with the squeezing operator $S(\zeta)$ defined by $S(\zeta) = e^{\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)}$. By using the mathematical formula $D(\xi)S(\zeta) = S(\zeta)D(\gamma)$, where $\gamma = \xi \cosh|\zeta| - \xi^* e^{i\theta}\sinh|\zeta|$, with $\zeta = |\zeta|e^{i\theta}$, we determine the expectation values with the squeezed state as the initial state for the oscillator,

$$\rho_{sq} = |\psi_{sq}\rangle\langle\psi_{sq}|, \quad |\psi_{sq}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)|\zeta\rangle_S, \tag{62}$$

as

$$\langle\hat{Q}_1\rangle = \mathrm{Tr}[\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1)\rho_{\rm sq}] = \cos(2\Omega t_1 - \varphi)\exp[-2\lambda^2|(e^{i\omega t_1}-1)\cosh|\zeta| - (e^{-i\omega t_1}-1)e^{i\theta}\sinh|\zeta||^2], \tag{63}$$

$$\langle\hat{Q}_2\rangle = \mathrm{Tr}[\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\rho_{\rm sq}] = \cos(2\Omega t_2 - \varphi)\exp[-2\lambda^2|(e^{i\omega t_2}-1)\cosh|\zeta| - (e^{-i\omega t_2}-1)e^{i\theta}\sinh|\zeta||^2], \tag{64}$$

and

$$\begin{aligned} C(t_2,t_1) &= \tfrac{1}{2}\mathrm{Tr}[\{\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1), \boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\}\rho_{\rm sq}] \\ &= \cos\Theta(t_2,t_1)\cos[2\Omega(t_2-t_1)]\exp[-2\lambda^2|(e^{i\omega t_2}-e^{i\omega t_1})\cosh|\zeta| - (e^{-i\omega t_2}-e^{-i\omega t_1})e^{i\theta}\sinh|\zeta||^2], \end{aligned} \tag{65}$$

where $\Theta(t_2,t_1)$ is defined by Eq. (31). In the $\Omega = 0$ and $\varphi = 0$ limits, the two-time quasiprobability reads

$$\begin{aligned} q_{s_1s_2}(t_1,t_2) = \tfrac{1}{4}\{&1 + s_1\exp[-2\lambda^2|(e^{i\omega t_1}-1)\cosh|\zeta| - (e^{-i\omega t_1}-1)e^{i\theta}\sinh|\zeta||^2] \\ &+ s_2\exp[-2\lambda^2|(e^{i\omega t_2}-1)\cosh|\zeta| - (e^{-i\omega t_2}-1)e^{i\theta}\sinh|\zeta||^2] \\ &+ s_1s_2\cos\Theta(t_2,t_1)\exp[-2\lambda^2|(e^{i\omega t_2}-e^{i\omega t_1})\cosh|\zeta| - (e^{-i\omega t_2}-e^{-i\omega t_1})e^{i\theta}\sinh|\zeta||^2]\}. \end{aligned} \tag{66}$$

When ζ takes a real number, Eq. (66) reduces to

$$\begin{aligned} q_{s_1s_2}(t_1,t_2) = \frac{1}{4}\bigg(&1 + s_1\exp\left[-8\lambda^2\sin^2\frac{\omega t_1}{2}[\cosh 2\zeta + \cos(\omega t_1)\sinh 2\zeta]\right] \\ &+ s_2\exp\left[-8\lambda^2\sin^2\frac{\omega t_2}{2}[\cosh 2\zeta + \cos(\omega t_2)\sinh 2\zeta]\right] \\ &+ s_1s_2\cos\Theta(t_2,t_1)\exp\left[-8\lambda^2\sin^2\frac{\omega(t_2-t_1)}{2}\{\cosh 2\zeta + \cos[\omega(t_1+t_2)]\sinh 2\zeta\}\right]\bigg). \end{aligned} \tag{67}$$

Figures 3 and 4 demonstrate the region where the two-time quasiprobability (67) takes negative values on the t_1 and t_2 planes, depending on the choice of s_1, s_2, and ζ. The minimum value of the quasiprobability function (67) is approximately of the order of

$$\min\{q_{s_1s_2}(t_1,t_2)\} \simeq -\frac{\lambda^2}{2}e^{2|\zeta|}. \tag{68}$$

In general, the squeezed initial condition boosts the signal of the Leggett-Garg inequality violation, except for the cases $s_1 = s_2 = 1$ and $s_1 = s_2 = -1$ with $\zeta < 0$.

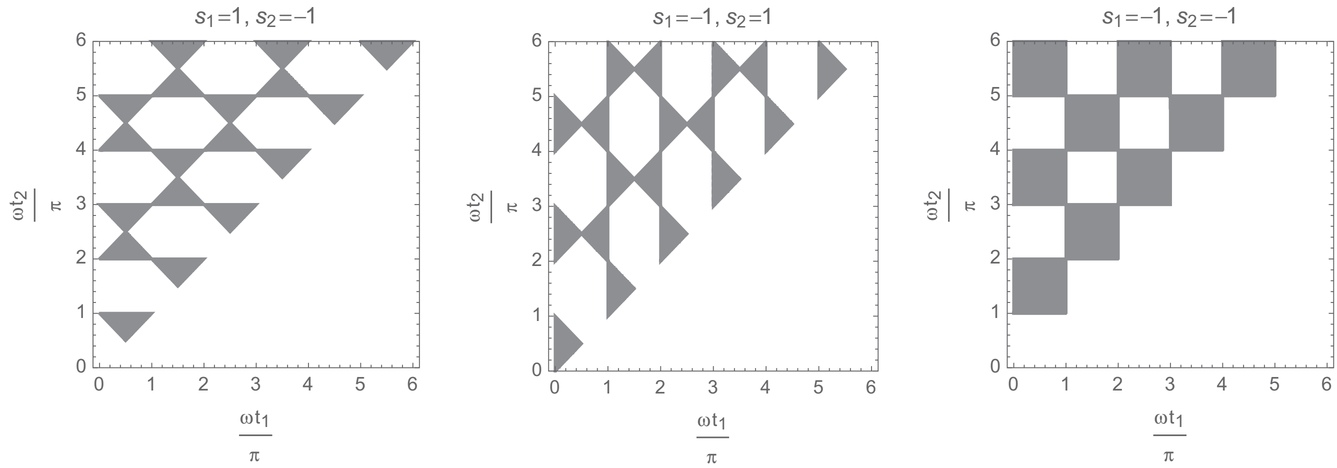

FIG. 3. Shaded regions show the regions satisfying $q_{s_1 s_2}(t_1, t_2) < 0$ in Eq. (67) with the squeezed state as the oscillator's initial condition on the $\omega t_1/\pi$ and $\omega t_2/\pi$ planes. We adopted $\Omega = 0$, $\varphi = 0$, and $s_1 = 1$, $s_2 = -1$ (left panel); $s_1 = -1$, $s_2 = 1$ (middle panel); and $s_1 = -1$, $s_2 = -1$ (right panel). Here, we adopted $8\lambda^2 = 10^{-4}$ and $\zeta = 5$. Here, we show only the region $0 \leqslant t_1 \leqslant t_2$. In this case, no violation of the Leggett-Garg inequalities appears for $s_1 = s_2 = 1$.

E. Connection with the experiment

Following Ref. [18], we discuss the feasibility of signal detection. We introduced the mass density ρ by $M = 4\pi\rho\ell^3/3$ for the oscillator, and with the approximation $L \sim \ell$, we have

$$\lambda^2 = \frac{g^2}{\omega^2} = \frac{G^2 m^2 M \ell^2}{8\omega^3\hbar\sqrt{L^2 + \ell^2/4}^3} \sim \frac{G^2 m^2 \rho}{\hbar\ell\omega^3}, \tag{69}$$

which is estimated as

$$\frac{G^2 m^2 \rho}{\hbar\ell\omega^3} = 1.7 \times 10^{-28}\left(\frac{m}{m_{\mathrm{Cs}}}\right)^2\left(\frac{\rho}{20\,\mathrm{g/cm^3}}\right)\left(\frac{\omega}{\omega_s}\right)^{-3}\left(\frac{\ell}{1\,\mathrm{mm}}\right)^{-1}, \tag{70}$$

where $m_{\mathrm{Cs}} = 2.2 \times 10^{-25}$ kg is the mass of a cesium atom and ω_s is defined as $\omega_s = 2\pi/\tau$, with $\tau = 10$ s. This was a significantly small signal, but when we assumed the initial thermal state for the oscillator, the effective coupling constant was boosted by the factor $\bar{n} = k_B T/2\hbar\omega$ as

$$\bar{n}\lambda^2 \sim 0.5 \times 10^{-14}\left(\frac{m}{m_{\mathrm{Cs}}}\right)^2\left(\frac{\rho}{20\,\mathrm{g/cm^3}}\right)\left(\frac{\omega}{\omega_s}\right)^{-4}\left(\frac{\ell}{1\,\mathrm{mm}}\right)^{-1}\left(\frac{T}{300\,\mathrm{K}}\right). \tag{71}$$

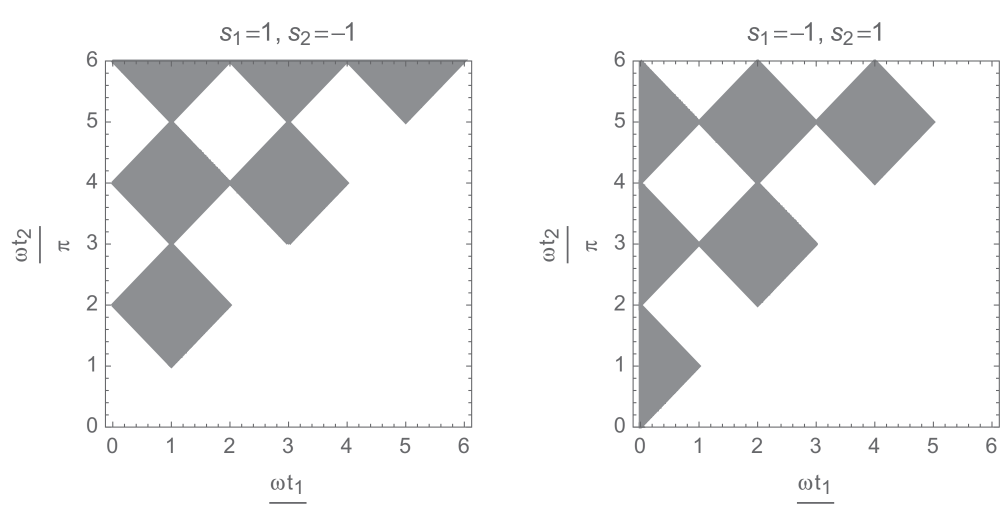

FIG. 4. Same as Fig. 3, but with $\zeta = -5$, $s_1 = 1$, $s_2 = -1$ (left panel) and $\zeta = -5$, $s_1 = -1$, $s_2 = 1$ (right panel). The other parameters are $\Omega = 0$, $\varphi = 0$, and $8\lambda^2 = 10^{-4}$. In this case, no violation of the Leggett-Garg inequalities appears for $s_1 = s_2 = 1$ and $s_1 = s_2 = -1$.

The amplitude of the signal was the same as that discussed in Ref. [18], in which the authors argued that the signal in the visibility function could be further amplified by using many atoms and coupling the oscillator with another two-state system.

For an experimental test of the violation of the Leggett-Garg inequalities, we need to measure the expectation values of $\langle\hat{Q}(t_j)\rangle = \mathrm{Tr}[\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_j)\rho_0]$, with $j = 1, 2$, and $C(t_2, t_1) = \frac{1}{2}\mathrm{Tr}[\{\boldsymbol{n}\cdot\boldsymbol{\sigma}(t_1), \boldsymbol{n}\cdot\boldsymbol{\sigma}(t_2)\}\rho_0]$. The simplest case with $\boldsymbol{n} = (1, 0, 0)$ and $\Omega = 0$ when we assume the initial thermal state for the oscillator is $\langle\hat{Q}(t_j)\rangle = e^{-8\lambda^2(2\bar{n}+1)\sin^2\omega(t_j/2)}$. This expression is the same as the visibility function in Ref. [18]. Therefore, the measurement of $\langle\hat{Q}(t_j)\rangle$ is the same as that of the visibility function, which is essentially obtained with the two-state interference. On the other hand, $C(t_2, t_1)$ is the correlation function, which requires a much larger number of measurements to detect the signal with a sufficient statistical significance. This is a disadvantage of our approach with the Leggett-Garg inequalities for testing the quantumness of gravity.

However, as discussed in Refs. [19–22], the collapse and revival of the visibility function in atomic interferometry can be generated by semiclassical models. The authors of Ref. [19] demonstrated that a local operations and classical communication channel between a harmonic oscillator and a particle in a double-well potential reproduces the collapse-and-revival dynamics in the interferometric signal. Similarly, the authors of Ref. [21] demonstrated that the periodic collapses and revivals of the visibility can appear even when the oscillator is fully classical. Therefore, the revival of the visibility is not necessarily the signature of the quantumness of gravity connected to the entanglement. The Leggett-Garg inequality cannot be violated in a classical system, which will be a unique method to test a quantum property of gravity. It will be helpful that the signal of the violation of the Leggett-Garg inequalities is boosted by preparing a squeezed initial state for the oscillator. The feasibility of detecting the signal against various noises is left for a future study.

IV. DISCUSSION

We consider the Newton-Schrödinger approach in the present system to compare the difference in the predictions in our theoretical model. In the Newton-Schrödinger approach, the gravitational potential Φ is given by expectation values of matter distributions with respect to the states. Explicitly, we may write the Newton-Schrödinger equations

$$i\frac{\partial|\psi(t)\rangle_A}{\partial t} = \left(\Omega\sigma^z + \frac{GMm\ell}{2\sqrt{L^2+\ell^2/4}^3}\langle q\rangle\sigma^z\right)|\psi(t)\rangle_A, \quad (72)$$

$$i\frac{\partial|\psi(t)\rangle_q}{\partial t} = \left(\frac{p^2}{2M} + \frac{M\omega^2}{2}q^2 + \frac{GMm\ell}{2\sqrt{L^2+\ell^2/4}^3}\langle\sigma^z\rangle q\right)|\psi(t)\rangle_q \quad (73)$$

for the state of the particle $|\psi(t)\rangle_A$ and the state of the oscillator $|\psi(t)\rangle_q$, respectively, with which $\langle q\rangle$ and $\langle\sigma^z\rangle$ are defined as $\langle q\rangle = {}_q\langle\psi(t)|q|\psi(t)\rangle_q$ and $\langle\sigma^z\rangle = {}_A\langle\psi(t)|\sigma^z|\psi(t)\rangle_A$, respectively. Here, p is the conjugate momentum of q.

For the initial state of the oscillator and the particle adopted in our analysis [for example, the initial state given by (25)], the gravitational interaction vanishes, i.e., $\langle q\rangle = \langle\sigma^z\rangle = 0$, because of the symmetry of the system. When the Larmor precessionlike frequency vanishes, $\Omega = 0$, there are no violation of the Leggett-Garg inequalities in the Newton-Schrödinger approach. The violation of the Leggett-Garg inequalities which appears via the gravitational interaction in the previous section can be regarded as a consequence of the quantum nature of the gravitational interaction.

V. CONCLUSIONS

We investigated the violation of the Leggett-Garg inequalities due to the gravitational interaction in the hybrid system [18] using a two-time quasiprobability. With the initial time $t_1 = 0$, we first discussed the role of the gravitational interaction in the violation of the Leggett-Garg inequalities of the two-time quasiprobability in connection to the entanglement generated by the gravitational interaction. In the case $\Omega \neq 0$, the Larmor precessionlike behavior appears, and we can assume the parameters so that the Leggett-Garg inequalities are violated when the gravitational interaction is switched off. This violation of the Leggett-Garg inequalities is due to the quantum property of the particle system itself. In this setup, with $t_1 = 0$ and $\Omega \neq 0$, we demonstrated that the entanglement, induced by the gravitational interaction being switched on, suppresses the violation of the Leggett-Garg inequalities. Furthermore, in some parameter settings, the quasiprobability equals the square of the entanglement negativity.

When the Larmor precessionlike behavior in the two spatially localized states was switched off, i.e., $\Omega = 0$, we demonstrated that the quasiprobability took negative values due to the gravitational interaction, in general, depending on the choice of the parameters and the initial conditions. For the realistic situation $g/\omega \ll 1$, the minimum value of the two-time quasiprobability was of the order of $-g^2/2\omega^2$ when the initial state of the oscillator was in the ground state, while it was of the order of $-g^2\bar{n}/\omega^2$ when the initial state of the oscillator was in the thermal state, where $\bar{n} = k_B T/2\omega$. As discussed in Ref. [18], the choice of the initial thermal state significantly increases the signal of the quasiprobability owing to gravitational interaction. We also demonstrated that squeezing the initial state of the oscillator significantly boosts the amplitude of the signal of the Leggett-Garg inequality violation.

Here, we discussed the origin of the violation of the Leggett-Garg inequalities due to gravity in the hybrid system that was determined in Secs. III B, III C, and III D. The violation of the Leggett-Garg inequalities in the case where $\Omega = 0$ originates from the gravitational interaction; otherwise, no evolution arises in the system of the particle. Gravitational interaction generates an entangled hybrid cat state [Eq. (26)]; therefore, entanglement plays an important role in the Leggett-Garg inequality violation. In the Leggett-Garg inequality violation, the terms $\langle Q(t_1)\rangle$ and $\langle Q(t_2)\rangle$ play a crucial role in making the two-time quasiprobability negative values. For the simplest case, $\boldsymbol{n} = (1, 0, 0)$, we have $\langle Q(t)\rangle = \mathrm{Tr}[\boldsymbol{n}\cdot\sigma(t)\rho_0] = e^{-8\lambda^2\sin^2(\omega t/2)}$, which is a visibility function addressed in Ref. [18]. Based on Refs. [18,20], the oscillatory behavior of the visibility function originates from nonseparable evolution of the state owing to the gravitational interaction, which causes the entanglement of the system. For the case $\Omega = 0$, the Leggett-Garg inequalities are not violated when the oscillator and the particle undergo separable unitary evolution with the separable initial state. Therefore, it can be concluded that the Leggett-Garg inequality violation for the case $\Omega = 0$ is derived from the nonseparable property of the gravitational interaction.

However, the origin of the violation of the Leggett-Garg inequalities may still have room for discussion. For the case $t_1 = 0$ and $\Omega \neq 0$, the gravitational interaction causes the entanglement, which always suppresses the violation of the Leggett-Garg inequalities caused by the quantum nature of the particle system itself. For the case $\Omega = 0$, the gravitational interaction causes only the evolution in the particle system, which causes the entanglement between the particle and the oscillator as long as $\lambda \neq 0$. Therefore, we concluded that the origin of the violation of the Leggett-Garg inequalities is the gravitational interaction and the entanglement induced by the gravitational interaction. This is supported by the result in Sec. IV that the Newton-Schrödinger approach does not cause the violation of the Leggett-Garg inequalities in which the gravitational interaction causes no entanglement. However, the gravitational entanglement has two effects, i.e., violation and holding of the Leggett-Garg inequalities depending on the parameters t_1 and t_2. This can be understood from Eq. (46). Namely, the two-time quasiprobability is expressed by the latter term in proportion to λ^2 in Eq. (46) when s_1 and s_2 are the same as those in Fig. 2. When the two-time quasiprobability takes negative (positive) values, the Leggett-Garg inequalities are violated (satisfied). We have not clarified how these different aspects of the entanglement due to the gravitational interaction appear in the violation and holding of the Leggett-Garg inequalities in an intuitive manner. Furthermore, the particle is equipped with quantum properties. Therefore, it might be difficult to exclude the possibility that the violation of the Leggett-Garg inequalities comes from the quantumness of the particle system itself.

In general, it is interesting to test quantum properties of macroscopic systems to determine the boundary between quantum systems and classical systems. Our research, which is motivated by testing quantum properties of the gravitational interaction, can be regarded as a test of the quantum aspects of a gravitational potential as a macroscopic system through the Leggett-Garg inequalities. The Leggett-Garg inequalities were originally developed on the basis of macroscopic realism and noninvasive measurability, which are tested by a measurement of the violation of the inequalities. In our system, a superposition state of the macroscopic oscillator is generated by the superposition state of the particle initially prepared. When the initial state of the oscillator is prepared as a superposition state of coherent states by some method, e.g., $(|\xi_0\rangle_C + |\xi_1\rangle_C)/\sqrt{2}$ with coherent parameters ξ_0 and ξ_1, an entangled state between the oscillator and the particle will appear, as is shown in the Appendix. The result in Eq. (A5) means that the particle system could be used as a probe of the superposition state of the oscillator by measuring the interference of the particle state caused by the entanglement. When the particle and the oscillator interact through a different force, the factor will be written in a corresponding form reflecting the different interaction. Therefore, a particle in a superposition state could be a probe of a quantum state of the macroscopic oscillator and the quantum nature of the interaction when the interaction between them is well understood. It is interesting to investigate the violation of the Leggett-Garg inequalities in the particle's state as a probe of the quantum aspects of macroscopic oscillators and their interaction, which is left for future investigations.

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APPENDIX: RESULT WITH OTHER INITIAL STATE FOR OSCILLATOR

When the initial state of the system is prepared as

$$|\psi_0\rangle = \tfrac{1}{2}(|0\rangle_A + |1\rangle_A) \otimes (|\xi_0\rangle_C + |\xi_1\rangle_C), \tag{A1}$$

where $|\xi_j\rangle_C$ for $j = 0, 1$ is a coherent state of the oscillator, the state will evolve as

$$|\psi(t)\rangle = U(t)|\psi_0\rangle = e^{-i(\Omega\sigma^z + \omega a^\dagger a)t} e^{g\sigma^z[\alpha(t)a - \alpha^*(t)a^\dagger]}|\psi_0\rangle. \tag{A2}$$

Using the formula, $D(-g\sigma^z\alpha^*(t))D(\xi_j) = e^{g\sigma^z[-\alpha^*(t)\xi_j^* + \alpha(t)\xi_j]/2}D(-g\sigma^z\alpha^*(t) + \xi_j)$, we have

$$e^{-i\omega a^\dagger a}D - g\sigma^z\alpha^*(t))D(\xi_j)|0\rangle = e^{g\sigma^z[-\alpha^*(t)\xi_j^* + \alpha(t)\xi_j]/2} e^{-i\omega a^\dagger a} D - g\sigma^z\alpha^*(t) + \xi_j)|0\rangle = e^{g\sigma^z[-\alpha^*(t)\xi_j^* + \alpha(t)\xi_j]/2}|g\sigma^z\alpha(t) + \xi_j e^{-i\omega t}\rangle_C, \tag{A3}$$

which leads to

$$\begin{aligned}|\psi(t)\rangle = \tfrac{1}{2}(&e^{g[-\alpha^*(t)\xi_0^* + \alpha(t)\xi_0]/2}|0\rangle_A|g\alpha(t) + \xi_0 e^{-i\omega t}\rangle_C + e^{-g[-\alpha^*(t)\xi_0^* + \alpha(t)\xi_0]/2}|1\rangle_A| - g\alpha(t) + \xi_0 e^{-i\omega t}\rangle_C \\ &+ e^{g[-\alpha^*(t)\xi_1^* + \alpha(t)\xi_1]/2}|0\rangle_A|g\alpha(t) + \xi_1 e^{-i\omega t}\rangle_C + e^{-g[-\alpha^*(t)\xi_1^* + \alpha(t)\xi_1]/2}|1\rangle_A| - g\alpha(t) + \xi_1 e^{-i\omega t}\rangle_C).\end{aligned} \tag{A4}$$

When $g/\omega \ll |\xi_j|$ for $j = 0, 1$, the state can be approximately written as

$$\begin{aligned}|\psi(t)\rangle \simeq \tfrac{1}{2}[&e^{g[-\alpha^*(t)\xi_0^* + \alpha(t)\xi_0]/2}|0\rangle_A|\xi_0 e^{-i\omega t}\rangle_C + e^{-g[-\alpha^*(t)\xi_0^* + \alpha(t)\xi_0]/2}|1\rangle_A|\xi_0 e^{-i\omega t}\rangle_C \\ &+ e^{g[-\alpha^*(t)\xi_1^* + \alpha(t)\xi_1]/2}|0\rangle_A|\xi_1 e^{-i\omega t}\rangle_C + e^{-g[-\alpha^*(t)\xi_1^* + \alpha(t)\xi_1]/2}|1\rangle_A|\xi_1 e^{-i\omega t}\rangle_C].\end{aligned} \tag{A5}$$

We note that the result is an entangled state between the oscillator and the particle with the factor $e^{\pm g[-\alpha^*(t)\xi_j^*+\alpha(t)\xi_j]/2}$, with $j = 0, 1$, which comes from the gravitational interaction between them.

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